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Semidualizing and Tilting Adjoint Pairs, Applications to Comodules

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Abstract The aim of this paper is to introduce the concept of right and left semidualizing adjoint pair of functors and study its main properties. This concept generalizes the concept of semidualizing module and allows one to consider semidualizing comodules, graded modules, etc. We also study tilting adjoint pair of functors as a particular case. We show generalized tilting theorem in this general setting and give some applications to tilting theory in the category of comodules over a coalgebra.

Keywords Semidualizing comodule · Tilting object · Adjoint functors

Mathematics Subject Classification Primary 16T15 · Secondary 18E15

1 Introduction and Preliminaries

Semidualizing modules were introduced by Foxby [19], Golod [17], and Vasconcelos [41] under different names (Foxby called them PG-modules of rank one, Vasconcelos called them Spherical modules and Golod suitable modules). They generalize dualizing modules over Cohen–Macaulay rings and projective modules of rank one. More recently Christensen [9] used the term "semidualizing" to describe this kind of modules

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(and complexes) for commutative noetherian rings. The theory has been developed quickly by other authors in the last two decades (see for example papers by Enochs et al. [16], Enochs and Yassemi [15], Holm and Jorgensen [21], etc.). Araya et al. extend the theory to the non commutative noetherian setting and investigate semidualizing bimodules in [2]. They are studied by Holm and White in [22] for arbitrary associative rings. These modules extend dualizing modules over local noetherian commutative ring and have a great importance in the theory of Cohen–Macaulay rings. Associated to a (semi) dualizing module there exist two classes of modules, the Auslander class and the Bass class [16] (also called Foxby classes for dualizing modules), that are equivalent as full subcategories of modules.

In 1980, Brenner and Butler in [4] and Happel and Ringel in [20] generalized the classical Morita theory of equivalences by introducing the notion of a tilting module over a finite dimensional Artin algebra. In order to obtain equivalences between subcategories of the module category, tilting modules were assumed to be of projective dimension at most one. Later, Miyashita [31] considered tilting modules of finite projective dimension and studied the equivalences induced by them. Colby and Fuller in [6] extended the setting to arbitrary rings. Keller in [23] introduced the notion of tilting adjoint pair of functors in module categories over associative rings and a description of one-tilting in terms of adjoint functors was given in [35].

Takeuchi in [39] characterized equivalences of comodule categories over fields, dualizing Morita Theorem on equivalences of module categories. As a generalization of the Takeuchi result, cotilting comodules are studied in order to obtain equivalences between certain subcategories of comodules. Mingyi [29] proved the tilting theorem for classical cotilting comodules over semiperfect conoetherian coalgebras over fields. More recently, Liu and Zhang extend the results of [29] and get a Brenner–Butler Theorem for semiperfect coalgebras [26, Theorem 5.2]. Simson in [36] investigates cotilting comodules and gets different interesting examples in coalgebras over quivers.

In this paper, we unify the concept of semidualizing object by extending this to the framework of adjoint functors. The advantage of this new point of view is that we get results valid for different abelian categories like those of comodules, graded modules, etc. We also connect this concept with the concept of a tilting adjoint pair given by Keller in [23]. In this way, we get Brenner–Butler Theorem for a large class of examples.

The paper is structured as follows. We start by introducing semidualizing adjoint pair in Sect. 2. We prove several results about the Auslander and Bass classes in this general setting, generalizing several results of Enochs and Yassemi in [15] on the classes $\mathcal{U}$ and $\mathcal{W}$ associated to a right and left semidualizing adjoint pair, respectively. Under some conditions, it is also proved that Gorenstein projective objects are in the Aulander class and Gorenstein injective in the Bass class.

Next, in Sect. 3, we investigate tilting adjoint pairs. We relate semidualizing and tilting adjoint pairs with a generalization of Brenner–Butler Theorem. Finally, in Sects. 4 and 5, we apply this general theory to the concrete case of categories of comodules. We prove that if two coalgebras C and D have a semidualizing bicomodule of finite injective dimension, then the coalgebras have isomorphic Grothendieck groups and we also show that C is finite dimensional if and only if D is so. We also give a relation among the concepts of semidualizing comodule and n-cotilting comodule [36]. We show that over a quasi-co-Frobenius coalgebras C every n-cotilting comodules is trivial (that is, the coendomorphism coalgebra of any n-cotilting comodule is Morita–Takeuchi equivalent to C).

Now, we give the terminology used in this paper. For concepts about categories and derived functors, we refer the reader to [37] or [42]. Concerning coalgebras and comodules, we use the notation of [1,38] or [10].

Let C be a k coalgebra, k a field. For two right C-comodules M and N, the k-space of all comodules maps from M to N will be written as $Com_C\ (M, N)$ and $\mathcal{M}^C$ will denote the category of right C-comodules. In the same way, we can construct the category ${}^C\mathcal{M}$ of left C-comodules. The category $\mathcal{M}^C$ is isomorphic to the category of rational left C^* modules, *Rat* (C^*-*Mod*). In fact, we have a left exact functor *Rat*: $C^* - Mod \to C^* - Mod$ which is a preradical.

If C and D are two coalgebras, a (C, D)-bicomodule is a left C-comodule and a right D-comodule M, such that the C-comodule structure map $\rho_M^- : M \to C \otimes M$ is a D-comodule map, or equivalently the D-comodule structure map $\rho_M^+ : M \to M \otimes D$ is a C-comodule map.

It is well known that $\mathcal{M}^C$ is an abelian category (see [1,10,38]). In fact, $\mathcal{M}^C$ is a Grothendieck category.

Let C be a k-coalgebras and M a right C-comodule. We denote by $Cogen(M)$ the full subcategory of right C-comodules cogenerated by M, that is subcomodules of products of copies of M. By $Prop(M)$ [resp. $add(M)$], we denote the full subcategory of direct summands of an arbitrary products (resp. finite direct sums) of copies of M, and $Add(M)$ denote the full subcategory of direct summands of an arbitrary direct sum of copies of M.

We recall from [39] that a right comodule M over a coalgebra C is quasi-finite if it contains only a finite number of copies of each simple right C-comodule S or equivalently if $Com_C(S, M)$ is finite dimensional for every simple S.

Now let M be a quasi-finite right C-comodule and consider the covariant right additive functor $h_C(M, -)$ form $\mathcal{M}^C$ to $\mathcal{M}_k$. If C is right semiperfect [24], then for every $N \in \mathcal{M}^C$ we may consider a projective resolution

$$\mathbf{P_N} : \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0.$$

Therefore, we can obtain the left derived functors of $h_C(M, -)$ denoted by $ext^i_C(M, -)$. Their main properties are listed in [30, Proposition 4.1].

On the other hand, $M \in \mathcal{M}^C$ is quasi-finite if and only if its injective envelope $E(M)$ is quasi-finite. Now, let us suppose that C is right semiperfect and let S be a simple right C-comodule. Then, there exists a finite dimensional projective right C-comodule P such that $P \to S \to 0$ is exact. From the exact sequences

$$0 \to Com_C(S, E(M)/M) \to Com_C(P, E(M)/M)$$

and

$$Com_C(P, M) \to Com_C(P, (E(M)/M) \to 0$$

it follows that $E(M)/M$ is quasi-finite, since $Com_C(P, M)$ is finite dimensional. Therefore, if C is right semiperfect and $M \in \mathcal{M}^C$, then there exists an injective resolution

$$\mathbf{E_M} : 0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots$$

where every E^i $i \geq 0$ is quasi-finite.

Now let $N \in \mathcal{M}^C$ and consider the right additive contravariant functor $h_C(-, N)$ from the full subcategory of quasi-finite right C-comodules to $\mathcal{M}_k$. $h_C(-, N)$ is right exact and, by the above, if C is right semiperfect, we can consider left derived functors of $h_C(-, N)$ using injective resolutions on the first variable. These functors will be denoted by $\overline{ext}^i_C(-, N)$, $i \geq 0$.

Now, by [13, Remark 8.2.15] or simply, from the fact that

$$h_C(M, N)^* \cong Com_C(N, M)$$

we get the following result.

Corollary 1.1 *Let C be right semiperfect and $M, N \in \mathcal{M}^C$. If M is* quasi-finite*, then*

$$\overline{ext}^i_C(M, N) \cong ext^i_C(M, N), \quad ext^i_C(M, N)^* = Ext^i_C(N, M) \quad \forall i \geq 0.$$

The following Lemma will be useful.

Lemma 1.2 [10, Proposition 2.3.7] *There exists an isomorphism of left D-comodules*

$$M \square_C N^* \cong Com_C(N, M)$$

for every finite dimensional right C-comodule N and any $(D$–$C)$-bicomodule M.

Analogously, there exists an isomorphism of right D-comodules

$$K^* \square_C P \cong Com_C(P, K)$$

for every finite dimensional left C-comodule K and any $(C$–$D)$-bicomodule P.

2 Semidualizing Adjoint Pairs

Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories with arbitrary coproducts and products. Suppose that $\mathcal{C}$ has a projective generator, and $\mathcal{D}$ has an injective cogenerator.

Let (F, G) be an adjoint situation with $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$. Then, F is right exact and preserves arbitrary colimits, and G is left exact and preserves arbitrary limits. Their left and right derived functors will be denoted by $\mathbf{L}_i F$ and $\mathbf{R}^i G$ $\forall i \geq 0$, respectively.

We define the Auslander class of $\mathcal{C}$ relative to F, denoted by $\mathcal{A}(\mathcal{C})$, consisting of the objects X satisfying:

(i) $\mathbf{L}_i F(X) = 0 \ \forall i \geq 1$
(ii) $\mathbf{R}^i G(F(X)) = 0 \ \forall i \geq 1$
(iii) the unit $u_X : X \to GF(X)$ is an isomorphism.

Analogously, the Bass class of $\mathcal{D}$ relative to G denoted by $\mathcal{B}(\mathcal{D})$, consisting of the objects Y satisfying:

(i) $\mathbf{R}^i G(Y) = 0; \forall i \geq 1$
(ii) $\mathbf{L}_i F(G(Y)) = 0; \forall i \geq 1$
(iii) the counit $c_Y : FG(Y) \to Y$ is an isomorphism.

Proposition 2.1 *The functors F and G define an equivalence between $\mathcal{A}(\mathcal{C})$ and $\mathcal{B}(\mathcal{D})$.*

Proof We only need to show that if $X \in \mathcal{A}(\mathcal{C})$ then $F(X)$ is in $\mathcal{B}(\mathcal{D})$. Now, the first condition for $F(X)$ to be in the Bass class becomes the second condition for X to be in the Auslander class. Moreover, $\mathbf{L}_i F(G(F(X))) = \mathbf{L}_i F(X) = 0$ by the third condition. The equality $F(u_X) \circ c_{F(X)} = id_{FGF(X)}$ shows that $c_{F(X)}$ is an isomorphism. □

We now introduce the following definition.

Definition 2.1 (1) We say that the adjoint pair (F, G) is right semidualizing if the class of injective objects is contained in $\mathcal{B}(\mathcal{D})$.
(2) We say that the adjoint pair (F, G) is left semidualizing if the class of projective objects is contained in $\mathcal{A}(\mathcal{C})$.

Example 2.2 Let R and S be two rings and M be a semidualizing $(R$–$S)$-bimodule ([2] or [22]), then the adjoint pairs $(M \otimes_S -, Hom_R(M, -))$ and $(- \otimes_R M, Hom_S(M, -))$ are left semidualizing. Under some finiteness conditions both notions coincide.

In particular, if R and S are right and left noetherian rings, and V is a semidualizing $(R$–$S)$-bimodule, then $V[[x]]$ is a semidualizing $(R[[x]]$–$S[[x]])$-bimodule (see [14, Example 3.5] for V a dualizing bimodule).

Example 2.3 Let $\mathcal{F}$ be a Gabriel topology on the ring R, $\mathbf{Mod} - (R, \mathcal{F})$ the quotient category of $\mathbf{Mod} - R$ with respect to $\mathcal{F}$, i.e., the full subcategory of $\mathbf{Mod} - R$ consisting of $\mathcal{F}$-closed modules, and $a : \mathbf{Mod} - R \to \mathbf{Mod} - (R, \mathcal{F})$ the localization functor. It is well known [37, Proposition IX.1.11] that a is a left adjoint of the inclusion functor $i : \mathbf{Mod} - (R, \mathcal{F}) \to \mathbf{Mod} - R$ and a is exact. Hence condition ii) of the Bass class is trivially satisfied. On the other hand, $ai(X) \to X$ is an isomorphisms for any object X in the quotient category and trivially $\mathbf{R}^i i(E) = 0$ for any injective object in the quotient category. Hence (a, i) is right semidualizing.

The Auslander class will consist of all right R-modules such that $M \to ai(M)$ is an isomorphism, i.e., M is $\mathcal{F}$-closed and $\mathbf{R}^i i(a(M)) = 0$ and the Bass class is $\mathcal{F}$-closed and such that $\mathbf{R}^i i(Y) = 0$.

Example 2.4 Let $\mathbf{T}$ be the torsion class associated to an hereditary torsion theory $(\mathbf{T}, \mathbf{F})$. There is an adjoint pair of functors (i, t), where i is the inclusion functor and t is the torsion radical. In this case, this adjoint is left semidualizing and the Auslander and Bass class coincide. They consist of all torsion modules X with $\mathbf{R}^i t(X) = 0$, $i \geq 1$. An interesting case is $\mathbf{T} = \mathcal{M}^C = Rat(C^* - Mod)$.

Example 2.5 Let R and S be two G-graded rings, with G a group. If M is a graded R–S-bimodule then we have the adjoint pairs

$$\Big(M \otimes_S -, HOM_R(M,-)\Big) \text{ and } \Big(-\otimes_R M, HOM_S(M,-)\Big)$$

between the corresponding categories of graded modules (see [34, Proposition I.2.14]). The bimodule M is called a graded semidualizing bimodule if the mentioned adjoint pairs are left semidualizing. The results of this paper could be applied to this setting.

Example 2.6 Let $\mathcal{A}$ be a Grothendieck category and D an object in $\mathcal{A}$. By [8, Proposition 1.1], if we consider $R = End_{\mathcal{A}}(D)$, then there is an adjoint situation

$$\Big(T_D, Hom_{\mathcal{A}}(D,-)\Big), \quad T_D : Mod - R \to \mathcal{A}, \quad Hom_{\mathcal{A}}(D,-) : \mathcal{A} \to Mod - R$$

and $T_D(R) = D$. It can be seen that this adjoint pair is left semidualizing if and only if D is a semidualizing object $\big[$in the sense that $Ext^i_{\mathcal{A}}\left(D, D^{(I)}\right) = 0$ and $Hom_{\mathcal{A}}\left(D, D^{(I)}\right) \cong Hom_{\mathcal{A}}(D,D)^{(I)}$ for any index set $I\big]$. Analogously, we get conditions on D that guaranteeing that the adjoint pair is right semidualizing.

Let $\mathcal{L} \subset \mathcal{C}$ be a full subcategory of $\mathcal{C}$. As usual, we denote by $add(\mathcal{L})$ (resp. $Add(\mathcal{L})$) the full subcategory consisting of direct summands of finite direct sums (resp. arbitrary direct sums) of objects in $\mathcal{L}$. Analogously, we will define $prod(\mathcal{L})$ and $Prod(\mathcal{L})$ for products.

Lemma 2.7 *Let (F, G) be a right semidualizing adjoint pair. Then,*

$$add(G(Y)) \subseteq \mathcal{A}(\mathcal{C})$$

for every object $Y \in \mathcal{B}(\mathcal{D})$.

If, in addition, G commutes with arbitrary coproducts, then

$$Add(G(Y)) \subseteq \mathcal{A}(\mathcal{C})$$

for every object $Y \in \mathcal{B}(\mathcal{D})$.

The dual result with F and products also holds.

Proof Easy. □

The following result is easy to prove by the adjoint property and the condition (iii) of the definitions of the classes $\mathcal{A}(\mathcal{C})$ and $\mathcal{B}(\mathcal{D})$.

Lemma 2.8 *If $X, Y \in \mathcal{B}(\mathcal{D})$, then $Hom_{\mathcal{C}}(G(X), G(Y)) \cong Hom_{\mathcal{D}}(X, Y)$. If $X', Y' \in \mathcal{A}(\mathcal{C})$, then $Hom_{\mathcal{D}}(F(X'), F(Y')) \cong Hom_{\mathcal{C}}(X', Y')$.*

We can also have the following lemma.

Lemma 2.9 (1) *Let W be an injective cogenerator in $\mathcal{D}$. Then, (F, G) is right semidualizing if and only if $W \in \mathcal{B}(\mathcal{D})$.*

(2) *Let Q be a projective generator Q in $\mathcal{C}$. Then, (F, G) is left semidualizing if and only if $Q \in \mathcal{A}(\mathcal{C})$.*

Our aim now is to characterize the classes $\mathcal{A}(\mathcal{C})$ and $\mathcal{B}(\mathcal{D})$. We will do so in terms of two other natural classes of objects. Now let (F, G) be a right semidualizing adjoint pair and let $\mathcal{U} \subseteq \mathcal{C}$ be the class of objects U such that $U \cong G(E)$ for E injective object in $\mathcal{D}$.

Lemma 2.10 *Let $(F.G)$ be a right semidualizing adjoint pair. Then, every $X \in \mathcal{C}$ has an $\mathcal{U}$-preenvelope. If $\mathcal{D}$ is a Grothendieck category then every $X \in \mathcal{C}$ has an $\mathcal{U}$-envelope.*

Proof Let $X \in \mathcal{C}$ and consider $i : F(X) \to E$ a monomorphism with E injective (which exists since $\mathcal{D}$ has enough injectives). Then, the composition $X \stackrel{u_X}{\to} GF(X) \stackrel{G(i)}{\to} G(E)$ is the desired preenvelope: let $U' \in \mathcal{U}$ and morphism $X \to U'$. Then, we have $F(X) \to F(U') \cong FG(E') \cong E'$ for some injective E', and so there is a morphism $E \to E'$ such that the preceeding morphism $F(X) \to E'$ factors $F(X) \to E \to E'$. But then, we get $G(E) \to G(E')$ such that $X \to G(E) \to G(E')$ is the morphism $X \to G(E') \cong U'$.

Now, if $\mathcal{D}$ is Grothendieck, then we can take $i : F(X) \to E$ an injective envelope. Let $f : G(E) \to G(E)$ such that $f \circ \phi = \phi$, where $\phi = G(i) \circ u_X$. Then $F(f) \circ F(\phi) = F(f) \circ FG(i) \circ F(u_X) = F(\phi)$. But since $F(\phi) = FG(i)F(u_X) = i$ is an injective envelope, it follows that $F(f)$ is an automorphism, so $f \cong GF(f)$ [because $G(E) \in \mathcal{A}(\mathcal{C})$] is an automorphism. □

Theorem 2.11 *Let (F,G) be a right semidualizing adjoint pair and $X \in \mathcal{C}$. The following assertions are equivalent*:

(i) $X \in \mathcal{A}(\mathcal{C})$.
(ii) *There is an exact sequence*

$$\cdots P_1 \to P_0 \to U^0 \to U^1 \to \cdots$$

in $\mathcal{C}$, where every P_i is projective and every $U^i \in \mathcal{U}$, such that $X = Ker\left(U^0 \to U^1\right)$ and which remains exact when F is applied.
(iii) *There is an exact sequence*

$$0 \to X \to U^0 \to U^1 \to \cdots$$

in $\mathcal{C}$, where every $U^i \in \mathcal{U}$ such that remains exact when $Hom_{\mathcal{C}}(-, U)$ is applied for every $U \in \mathcal{U}$ and F leaves exact every projective resolution of X.

Proof (i)⇒(ii) Let

$$\cdots \to P_1 \to P_0 \to X \to 0$$

be a projective resolution of X in $\mathcal{C}$. Then, $\mathbf{L}_i F(X) = 0 \quad \forall i \geq 1$ gives that

$$\cdots \to F(P_1) \to F(P_0) \to F(X) \to 0$$

is exact. Now let

$$0 \to F(X) \to E^0 \to E^1 \to \cdots$$

be an injective resolution of $F(X)$. If we denote $U^i = G(E^i)$ then we get that the complex

$$0 \to X \cong GF(X) \to U^0 \to U^1 \to \cdots$$

is exact since $\mathbf{R}^i GF(X) = 0 \quad \forall i \geq 1$. Finally, the sequence also remains exact when F is applied by its construction.

(ii) $\Leftrightarrow$ (iii)

$$0 \to X \to U^0 \to U^1 \to \cdots$$

remains exact when $Hom_{\mathcal{C}}(-, U)$ for every $U \in \mathcal{U}$ is applied if and only if it remains exact when $Hom_{\mathcal{C}}(-, G(E))$ is applied for every injective $E \in \mathcal{D}$, and this assertion is equivalent to $Hom_{\mathcal{D}}(F(-), E)$ leaves the sequence exact for every injective $E \in \mathcal{D}$, and so, this is equivalent to that F leaves the sequence exact (since $\mathcal{D}$ has an injective cogenerator by hypothesis). Finally, F leaves

$$\cdots \to P_1 \to P_0 \to X \to 0$$

exact if and only if F leaves exact any projective resolution of X. Here, we use the fact that $\mathbf{L}_i F(X) = 0 \quad \forall i \geq 1$.

(iii) $\Rightarrow$ (i) The fact that F leaves exact any projective resolution of X implies that $\mathbf{L}_i F(X) = 0 \quad \forall i \geq 1$. On the other hand, $U^i \cong G(E^i)$ for some $E^i \in \mathcal{D}$ injective, and so $F(U^i) \cong FG(E^i) \cong E^i$. Therefore, the natural morphism $U^i \to GF(U^i)$ is an isomorphism and

$$0 \to F(X) \to F(U^0) \to F(U^1) \to \cdots$$

is an injective resolution of $F(X)$, since $\mathcal{D}$ has an injective cogenerator by hypothesis. But then, the complex

$$0 \to GF(X) \to GF(U^0) \to GF(U^1) \to \cdots$$

is isomorphic to the exact sequence

$$0 \to X \to U^0 \to U^1 \to \cdots ,$$

which shows that $X \cong GF(X)$ and $\mathbf{R}^i GF(X) = 0 \quad \forall i \geq 1$. □

Proposition 2.12 *Let (F, G) be a right semidualizing adjoint pair and let $0 \to X' \to X \to X'' \to 0$ be exact in $\mathcal{C}$. If any two of X', X and X'' are in $\mathcal{A}(\mathcal{C})$, then so is the third.*

Proof If $X'' \in \mathcal{A}(\mathcal{C})$, then $\mathbf{L}_1 F(X'') = 0$ and so

$$0 \to F(X') \to F(X) \to F(X'') \to 0$$

is exact. If $X \in \mathcal{A}(\mathcal{C})$, then we have an exact sequence

$$0 \to \mathbf{L}_1 F(X'') \to F(X') \to F(X) \to F(X'') \to 0$$

and so

$$0 \to G\mathbf{L}_1 F(X'') \to GF(X') \to GF(X)$$

is exact. Now, if $X, X' \in \mathcal{A}(\mathcal{C})$, we get that

$$0 \to G\mathbf{L}_1 F(X'') \to X' \to X$$

is exact and therefore, since G is left exact, $\mathbf{L}_1 F(X'') = 0$. In this way, if two of X', X and X'' are in $\mathcal{A}(\mathcal{C})$, then

$$0 \to F(X') \to F(X) \to F(X'') \to 0$$

is exact, which is equivalent to

$$0 \to Hom_{\mathcal{D}}(F(X''), E) \to Hom_{\mathcal{D}}(F(X), E) \to Hom_{\mathcal{D}}(F(X'), E) \to 0$$

is exact for every $E \in \mathcal{D}$ injective and so

$$0 \to Hom_{\mathcal{C}}(X'', G(E)) \to Hom_{\mathcal{C}}(X, G(E)) \to Hom_{\mathcal{C}}(X', G(E)) \to 0$$

is exact for every $E \in \mathcal{D}$ injective. Therefore

$$0 \to Hom_{\mathcal{C}}(X'', U) \to Hom_{\mathcal{C}}(X, U) \to Hom_{\mathcal{C}}(X', U) \to 0$$

is exact for every $U \in \mathcal{U}$. Now, by [13, Lemma 8.2.1], we can get an $\mathcal{U}$-resolution $0 \to X \to U^0 \to U^1 \to \cdots$ such that it becomes exact when $Hom_{\mathcal{C}}(-, U)$ is applied for every $U \in \mathcal{U}$ from such $\mathcal{U}$-resolutions for X' and X'', as well as a projective resolution. If we paste the two resolutions over $0 \to X' \to X \to X'' \to 0$ we get an exact sequence of complexes such that remains exact when we apply F. Now if two of these complexes are exact, then so is the third and so the result follows using the preceeding theorem. □

To get a dual characterization for $\mathcal{B}(\mathcal{D})$, and following what was done for $\mathcal{A}(\mathcal{C})$, we consider now the natural class $\mathcal{W}$ of objects W in $\mathcal{D}$ such that $W \cong F(P)$, where P is a projective object in $\mathcal{C}$. Also, let $\overline{\mathcal{W}}$ be the class of objects $\overline{W}$ in $\mathcal{D}$ such that $\overline{W} \cong F(\lim_i P_i)$, where $\lim_i P_i$ is a direct colimit of projective objects in $\mathcal{C}$.

Lemma 2.13 *Let (F, G) be a left semidualizing adjoint pair. Then, every $Y \in \mathcal{D}$ has a $\mathcal{W}$-precover.*

If $\mathcal{C}$ is a Grothendieck category and $\mathbf{R}^i G$ preserves arbitrary direct colimits for all $i \geq 0$, then every $Y \in \mathcal{D}$ has a $\overline{\mathcal{W}}$-cover.

Proof The proof of the first assertion is dual to that of Lemma 2.10 where we consider $P \to G(Y) \to 0$ exact with $P \in \mathcal{C}$ projective.

The second assertion is also dual to Lemma 2.10 by considering $P \to G(Y) \to 0$ a flat cover, which exists by [12, Theorem 3.2] and taking into account that the class of flat objects (i.e., direct colimits of projectives objects) in $\mathcal{C}$ is contained in $\mathcal{A}(\mathcal{C})$ (by the hypothesis). □

A dual proof to that of Theorem 2.11 gives the following.

Theorem 2.14 *Let (F,G) be a left semidualizing adjoint pair. The following assertions are equivalent for $Y \in \mathcal{D}$:*

(i) $Y \in \mathcal{B}(\mathcal{D})$.
(ii) *There is an exact sequence*

$$\cdots \to W_1 \to W_0 \to E^0 \to E^1 \to \cdots$$

in $\mathcal{D}$, where every E^i is injective and $W \in \mathcal{W}$, such that G leaves it exact and $Y = Ker(E^0 \to E^1)$.
(iii) *There is an exact $\mathcal{W}$-resolution*

$$\cdots \to W_1 \to W_0 \to Y \to 0$$

such that remains exact when $Hom_{\mathcal{D}}(W, -)$ is applied for every $W \in \mathcal{W}$ and G leaves exact every injective resolution of Y.

Proposition 2.15 *In the conditions of the preceeding theorem, let $0 \to Y' \to Y \to Y'' \to 0$ be exact in $\mathcal{D}$. If any of two Y', Y and Y'' are in $\mathcal{B}(\mathcal{D})$, then so is the third.*

Remark 1 As an immediate consequence of the fact that projective objects are in $\mathcal{A}(\mathcal{C})$ when (F, G) is left semidualizing and the last result we get that objects of finite projective dimension in $\mathcal{C}$ belong to $\mathcal{A}(\mathcal{C})$. A dual result holds for objects of finite injective dimension in $\mathcal{D}$.

We also can give information about Gorenstein projective and injective objects. We denote by $\mathcal{GP}$ (resp. $\mathcal{GI}$) the class of Gorenstein projective object in $\mathcal{C}$ (resp. Gorenstein injective objects in $\mathcal{D}$). For these definitions see [13]. The proof of the following proposition is similar to [16, Proposition 1.3], but we give it for completeness.

Proposition 2.16 *Let (F,G) be an adjoint pair.*

(i) *If (F,G) is left semidualizing and $\mathcal{D}$ has an injective cogenerator W such that $G(W)$ has finite projective dimension, then $\mathcal{GP} \subseteq \mathcal{A}(\mathcal{C})$.*

(ii) *If* (F,G) *is right semidualizing and* $\mathcal{C}$ *has a projective generator* Q *such that* $F(Q)$ *has finite injective dimension, then* $\mathcal{GI} \subseteq \mathcal{B}(\mathcal{D})$.

Proof (i) Let X be a Gorenstein projective object in $\mathcal{C}$. Then, there exists an exact sequence

$$\cdots P_1 \stackrel{\delta_1}{\to} P_0 \stackrel{\delta_0}{\to} P_{-1} \to \cdots$$

where $X = Ker(\delta_0)$ and the sequence is $Hom_{\mathcal{C}}(-, P)$-exact for every projective object $P \in \mathcal{C}$. Then, $Hom_{\mathcal{C}}(-, G(W))$ leaves exact the above exact sequence ($G(W)$ has finite projective dimension), so $Hom_{\mathcal{D}}(F(-), W)$ also verifies it. Hence $F(-)$ leaves the sequence exact and $\mathbf{L}_i F(X) = 0$. Then,

$$0 \to F(X) \to F(P_0) \to F(P_{-1})$$

and thus

$$0 \to GF(X) \to GF(P_0) \to GF(P_{-1})$$

is also exact. But $P_i \in \mathcal{A}(\mathcal{C})$, then $X \cong GF(X)$ canonically. Now let $0 \to X \to P_0 \to Y \to 0$ be exact. Then, Y is also Gorenstein projective, so $Y \cong GF(Y)$ canonically. Thus

$$0 \to F(X) \to F(P_0) \to F(Y) \to 0$$

is exact, and applying G, we get

$$0 \to X \to P_0 \to Y \to \mathbf{R}^1 G(F(X)) \to \mathbf{R}^1 G(F(P_0)) = 0.$$

This implies that $\mathbf{R}^1 GF(X) = 0$. But then since we also have $\mathbf{R}^1 GF(Y) = 0$ and $\mathbf{R}^2 G(F(P_0)) = 0$, we get $\mathbf{R}^2 GF(X) = 0$. Now by induction, we get $\mathbf{R}^i GF(X) = 0$ for all $i \geq 1$.

Similar argument gives (ii). □

In the following results, we will study the orthogonal classes with respect to the Ext^1 of the classes $\mathcal{U}$ and $\overline{\mathcal{W}}$. Our aim is to characterize when the pairs $({}^{\perp}\mathcal{U}, \mathcal{U})$ and $(\overline{\mathcal{W}}, \overline{\mathcal{W}}^{\perp})$ are cotorsion theories (see [13] for the definitions). The proofs are similar to [15, Section 2], so they will be omitted.

Theorem 2.17 *Let* (F,G) *be a right semidualizing adjoint pair. The following assertions are equivalents*:

(i) $\left({}^{\perp}\mathcal{U}, \mathcal{U}\right)$ *is a cotorsion theory.*
(ii) *Every injective object is in* $\mathcal{U}$.
(iii) *Every* $\mathcal{U}$*-envelope* $X \to U$ *is a monomorphism.*
(iv) $u_X : X \to GF(X)$ *is a monomorphism for all* $X \in \mathcal{C}$.
(v) $u_E : E \to GF(E)$ *is a monomorphism for all* $E \in \mathcal{C}$ *injective.*

If $\left({}^{\perp}\mathcal{U}, \mathcal{U}\right)$ *is a cotorsion theory, then it is a perfect cotorsion theory.*

Theorem 2.18 *If (F, G) is a left semidualizing adjoint pair, $\mathcal{C}$ is a Grothendieck category and $\mathbf{R}^i G$ preserves arbitrary direct colimits for all $i \geq 0$, then the following assertions are equivalents:*

(i) $\left(\overline{\mathcal{W}}, \overline{\mathcal{W}}^{\perp}\right)$ *is a cotorsion theory.*
(ii) *Every projective object is in $\overline{\mathcal{W}}$.*
(iii) *Every $\overline{\mathcal{W}}$-cover $P \to Y$ is an epimorphism.*
(iv) $c_Y : FG(Y) \to Y$ *is an epimorphism for all $Y \in \mathcal{D}$.*
(v) $c_P : FG(P) \to P$ *is an epimorphism for all $P \in \mathcal{D}$ projective.*

If $\left(\overline{\mathcal{W}}, \overline{\mathcal{W}}^{\perp}\right)$ is a cotorsion theory, then it is a perfect cotorsion theory.

3 Tilting Adjoint Pairs

The aim of this section is to define a tilting adjoint pair and connect this definition with the definition of a semidualizing adjoint pair.

Let (F, G) be an adjoint pair as in Sect. 2. For $m \geq 0$ we will denote

$$KerF_m = \{X \in \mathcal{C} \mid \mathbf{L}_i F(X) = 0 \quad \forall i \neq m, \quad i \geq 0\},$$
$$KerG_m = \{X \in \mathcal{D} \mid \mathbf{R}^i G(X) = 0 \quad \forall i \neq m, \quad i \geq 0\}.$$

The following definition is due to Keller [23].

Definition 3.1 Suppose that F and G have finite cohomological dimension $\leq k$ and let m be an integer such that $0 \leq m \leq k$.

We will say that the adjoint pair (F, G) is a tilting adjoint pair if the categories $KerF_m$ and $KerG_m$ are equivalent under the functors $\mathbf{L}_m F$ and $\mathbf{R}^m G$, $\forall 0 \leq m \leq k$.

Lemma 3.1 *Let (F, G) be a right semidualizing adjoint pair. Suppose that F has finite cohomological dimension $\leq k$. If $X \in KerG_0$, then $G(X) \in KerF_0$ and $c_X : FG(X) \to X$ is an isomorphism.*

Proof Let $0 \to X \overset{d^0}{\to} E^0 \overset{d^1}{\to} E^1 \overset{d^2}{\to} \cdots \overset{d^{k-1}}{\to} E^{k-1} \overset{d^k}{\to} E^k \to \cdots$ be an injective resolution and let $L^i = Coker\left(d^{i-1}\right)$ $i \geq 1$. Then, we have an exact sequence

$$0 \to G(X) \to G\left(E^0\right) \to \cdots \to G\left(E^k\right) \to G\left(E^{k+1}\right)$$

If we denote $Y^i = G\left(L^i\right)$, since by the hypothesis it follows that $G\left(E^i\right) \in KerF_0$, we get that $G(X)$, Y^1 and Y^2 are in $KerF_0$ (F having finite cohomological dimension). Since, by hypothesis, $E^i \in \mathcal{B}(\mathcal{D})$, it follows that $c_{E^i} : FG(E^i) \to E^i$ are isomorphisms $\forall i \geq 0$. So, from the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & FG(X) & \longrightarrow & FG(E^0) & \longrightarrow & FG(E^1) \\
 & \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 \longrightarrow & X & \longrightarrow & E^0 & \longrightarrow & E^1
\end{array}$$

we get that $c_X : FG(X) \to X$ is an isomorphism. □

Dually we get:

Lemma 3.2 *Let (F, G) be a left semidualizing adjoint pair. Suppose that G has finite cohomological dimension $\leq k$. If $Y \in Ker F_0$ then $F(Y) \in Ker G_0$ and $u_Y : Y \to GF(Y)$ is an isomorphism.*

Theorem 3.3 *Let (F, G) be a right semidualizing adjoint pair. Suppose that F has finite cohomological dimension $\leq k$. Let $m \geq 0$ be an integer and let $X \in Ker G_m$. Then $\mathbf{R}^m G(X) \in Ker F_m$ and there is an isomorphism $\mathbf{L}_m F(\mathbf{R}^m G(X)) \cong X$.*

Proof By Lemma 3.1 we may suppose that $m \geq 1$. Now let

$$0 \to X \stackrel{d^0}{\to} E^0 \stackrel{d^1}{\to} E^1 \stackrel{d^2}{\to} \cdots \stackrel{d^{k-1}}{\to} E^{m-1} \stackrel{d^m}{\to} E^m \to \cdots$$

be an injective resolution and let $L^i = Coker\left(d^{i-1}\right)$. By general properties of right derived functors $\mathbf{R}^i G(L^m) \cong \mathbf{R}^{i+m} G(X) = 0\ \forall i \geq 1$ and so, $L^m \in Ker G_0$. Then, by Lemma 3.1 $G(L^m) \in Ker F_0$ and $c_{L^m} : FG(L^m) \to L^m$ is an isomorphism. If we apply G to the injective resolution, we get

$$0 \to G(X) \to G\left(E^0\right) \to \cdots \to G\left(E^{m-1}\right) \to G\left(E^m\right) \to Y \to 0,$$

where $\mathbf{R}^1 G\left(L^{n-1}\right) \cong \cdots \cong \mathbf{R}^n G(X) = Y$. Now $G(E^i) \in Ker F_0$ by hypothesis and so, from general properties of derived functors, it follows that $\mathbf{L}_i F(Y) = 0\ \forall i \geq m+1$ and therefore

$$0 \to FG(E^0) \to \cdots \to FG(E^m) \to FG(Y) \to 0 \qquad (*)$$

is exact.

Now, for $i = 0, \ldots, m-2$, let

$$G\left(E^i\right) \to Y^{i+1} \to G\left(E^{i+1}\right)$$

be a factorization and consider $Y = Y^{m-1}$. From the exact sequence

$$0 \to Y^i \to G\left(E^i\right) \to Y^{i+1} \to 0,$$

we get an exact sequence

$$0 = \mathbf{L}_1 F\left(G\left(E^i\right)\right) \to \mathbf{L}_1 F\left(Y^{i+1}\right) \to F\left(Y^i\right) \to FG\left(E^i\right) \to F\left(Y^{i+1}\right) \to 0$$

for $i = 0, \ldots, m-2$. But the fact that $(*)$ is exact is equivalent to $F(Y_0) = 0$ and $\mathbf{L}_1 F(Y_i) = 0$ for $i = 0, \cdots, m-2$. Now, since $G\left(E^i\right) \in Ker F_0$ $i = 0, \ldots, m-2$, we get

$$\mathbf{L}_i F\left(Y^{m-1}\right) \cong \mathbf{L}_{i-1} F\left(Y^{m-2}\right) \cong \cdots \cong \mathbf{L}_1 F\left(Y^{m-i+1}\right) = 0$$

and so $\mathbf{L}_i F(Y) = 0,\ i \geq 0, i \neq m$, i.e., $Y = \mathbf{R}^m G(X) \in Ker F_m$.

Moreover,

$$\mathbf{L}_m F\left(\mathbf{R}^m G(X)\right) \cong Ker\left(FG\left(E^0\right) \to FG\left(E^1\right)\right) \cong Ker\left(E^0 \to E^1\right) = X.$$

□

Now, a dual proof using Lemma 3.2 gives the following result.

Theorem 3.4 *Let (F,G) be a left semidualizing adjoint pair. Suppose that G has finite cohomological dimension $\leq k$. Let $m \geq 0$ be an integer and let $X \in Ker F_m$. Then, $\mathbf{L}_m F(X) \in Ker G_m$ and there is an isomorphism $\mathbf{R}^m G(\mathbf{L}_m F(X)) \cong X$.*

Then, using Theorems 3.3 and 3.4, we get the following result which may be viewed as a generalization of Tilting Theorem.

Theorem 3.5 *Let (F,G) be an adjoint pair. Suppose that F and G have finite cohomological dimension $\leq k$.*

The following assertions are equivalent:

(1) *(F, G) is a right and left semidualizing adjoint pair.*
(2) *(F, G) is a tilting adjoint pair.*

Proof We only have to prove (2) $\Rightarrow$ (1). If $E \in \mathcal{D}$ is injective, then $E \in Ker G_0$ and so $G(E) \in Ker F_0$, therefore $\mathbf{L}_i F(G(E)) = 0$ $\forall i \geq 1$. Also, by the equivalence for $m = 0$, we get that $FG(E) \cong E$. Therefore $E \in \mathcal{B}(\mathcal{D})$.

The proof for projectives is dual. □

Remark 2 If $k = 0$, then $Ker G_0 = \mathcal{D}$, $Ker F_0 = \mathcal{C}$ and (F, G) are mutually inverse equivalences of categories.

If $k = 1$, then F and G give an equivalence between the categories of F-acyclic objects in $\mathcal{C}$ and G-acyclic objects in $\mathcal{D}$. Also $Ker F$ and $Ker G$ are equivalent via $\mathbf{L}_1 F$ and $\mathbf{R}^1 G$.

Another consequence of Theorems 3.3 and 3.4 is the following.

Corollary 3.6 *In the conditions of Theorem 3.5*:

(i) *If $X \in \mathcal{D}$ is such that $\mathbf{R}^i G(X) = 0$ for $i = 0, \ldots, k$, then $X = 0$.*
(ii) *Let $Y \in \mathcal{C}$. If $\mathbf{L}_i F(Y) = 0$ for $i = 0, \ldots, k$, then $Y = 0$.*

If $\mathcal{C}$ is an abelian category, we recall that the Grothendieck group of $\mathcal{C}$, denoted, $\mathcal{K}_0(\mathcal{C})$, is defined as follows: for each object $X \in \mathcal{C}$ there is a generator $[X]$; for each exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $\mathcal{C}$ there is a relation

$$[X] = \left[X'\right] + \left[X''\right].$$

The following result generalizes [31, Theorem 1.19]. Its proof follows by the same arguments.

Theorem 3.7 *Let (F, G) be an adjoint pair as above. Suppose that F and G have finite cohomological dimension $\leq k$. If (F, G) is a tilting adjoint pair, then the map*

$$\mathbf{LF} : \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{D}), \quad [X] \mapsto \sum_{i \geq 0} [\mathbf{L}_i F(X)],$$

is an isomorphism. The inverse is given by

$$\mathbf{RG} : \mathcal{K}_0(\mathcal{D}) \to \mathcal{K}_0(\mathcal{C}), \quad [X] \mapsto \sum_{i \geq 0} \left[\mathbf{R}^i G(X)\right].$$

4 Semidualizing Comodules

We start this section with an useful result that gives a characterization of semidualizing comodules.

Proposition 4.1 *Let C be a coalgebra, T a* quasi-finite *right C-comodule and $D = e_C(T)$.*

(i) *Suppose that $D = e_C(T)$ is a left semiperfect coalgebra and ${}_DT$ is a* quasi-finite *left D-comodule. Then, $ext^i_C(T, T) = 0 \ \forall i \geq 1$ if and only if the adjoint pair $\left(h_D(T, -), T\square_C -\right)$ is left semidualizing.*
(ii) *Suppose that ${}_DT$ is a* quasi-finite *left D-comodule and $C \cong e_D(T)$ canonically. Then,*
$ext^i_D(T, T) = 0 \quad \forall i \geq 1$ if and only if the adjoint pair $\left(h_D(T, -), T\square_C -\right)$ is right semidualizing.
(iii) *$ext^i_C(T, T) = 0 \quad \forall i \geq 1$ if and only if the adjoint pair $\left(h_C(T, -), -\square_D T\right)$ is right semidualizing.*
(iv) *If C is a right semiperfect coalgebra, then $ext^i_C(T, T) = 0 \quad \forall i \geq 1$ if and only if the adjoint pair $\left(h_C(T, -), -\square_D T\right)$ is left semidualizing.*

Proof (i) By hypothesis ${}^D\mathcal{M}$ has a projective generator and any projective left D-comodule is a direct sum of finite dimensional left D-comodules of the form E^*, where E is a finite dimensional injective right D-comodule. In particular, E^* is a direct summand of a finite direct sum of copies of D^*. Let P be a projective left D-comodule. In order to show that $P \in \mathcal{A}\left({}^D\mathcal{M}\right)$ it is enough to take $P = E^*$ a finite dimensional projective left D-comodule.
We have, by Lemma 1.2, that

$$T\square_C h_D(T, E^*) \cong T\square_C Com_D(E^*, T)^* \cong Com_C(Com_D(E^*, T), T).$$

In this way, the unit of the adjunction $u_{E^*} : E^* \to T\square_C h_D(T, E^*)$ is nothing but the evaluation map $E^* \to Com_C(Com_D(E^*, T), T)$. Since this last map is an isomorphism for D^* (by the same argument as above), we conclude that the same is true for E^*.

On the other hand, let

$$0 \to T \to I^0 \to I^1 \to \cdots$$

be an injective resolution of T in $\mathcal{M}^C$. The homology of the complex

$$0 \to I^0 \square_C h_D(T, E^*) \to I^1 \square_C h_D(T, E^*) \to \cdots$$

is precisely $Tor^i_C(T, h_D(T, E^*))$. But by the above

$$I^i \square_C h_D(T, E^*) \cong Com_C(Com_D(E^*, T), I^i)$$

naturally. Therefore $Tor^i_C(T, h_D(T, E^*)) \cong Ext^i_C(Com_D(E^*, T), T)$ which is a direct summand of

$$Ext^i_C(Hom_{D^*}((D^*)^n, T), T) \cong Ext^i_C(T, T)^n \cong (ext^i_C(T, T)^*)^n = 0.$$

(ii) Immediate from the facts that $C \in \mathcal{B}({}^D\mathcal{M})$, $T \square_C -$ preserves arbitrary direct sums, and any injective left C-comodule is a direct summand of a direct sum of copies of C.
The proofs of (iii) and (iv) are similar to (i) and (ii). □

Let C and D be right and left semiperfect coalgebras, respectively, over the same field k. Most of the results of this paper are valid for k a commutative ring such that C and D are k-flat; however, we do not make such a generalization at this time. Let T be a $(D–C)$-bicomodule such that $T_C \in \mathcal{M}^C$ and ${}_DT \in {}^D\mathcal{M}$ are quasi-finite. In that case, we have two adjoint situations, $\big(h_D(T, -), T \square_C -\big)$, where $h_D(T, -) : {}^D\mathcal{M} \to {}^C\mathcal{M}$, $T \square_C - : {}^C\mathcal{M} \to {}^D\mathcal{M}$ and similarly $\big(h_C(T, -), -{}_D\square T\big)$ on the right.

Definition 4.1 We will say that T is a semidualizing $(D–C)$-bicomodule if the adjoint pairs $\big(h_D(T, -), T \square_C -\big)$ and $\big(h_C(T, -), -{}_D\square T\big)$ are right semidualizing.

The following result is immediate from the definition.

Proposition 4.2 *A $(D–C)$-bicomodule ${}_DT_C$ is a semidualizing bicomodule if and only if it satisfies the following two conditions*:

(i) $h_D(T, T) \cong C$ *and* $h_C(T, T) \cong D$. (T *is balanced*)
(ii) $ext^i_D(T, T) = 0$ *and* $ext^i_C(T, T) = 0$ $\forall i \geq 1$. (T *is self-orthogonal*).

From a right C-comodule T, we can find conditions for ${}_DT_C$ to be semidualizing for $D = e_C(T)$.

Proposition 4.3 *Let T be a* quasi-finite *right C-comodule and let $D = e_C(T)$ its endomorphisms coalgebra. Suppose ${}_DT$ is* quasi-finite *left D-comodule, $ext^i_C(T, T) = 0$ $\forall i \geq 1$ and let $D = e_C(T)$ be left semiperfect. If there is an exact sequence*

$$0 \to T_n \to \cdots \to T_1 \to T_0 \to C \to 0$$

for some $n \geq 0$ with $T_i \in add(T_C)$ for $i = 0, \ldots, n$, then ${}_DT_C$ is semidualizing.

Proof Since $T \cong h_D(h_C(T,T),T)$, we get that $T_i \cong h_D(h_C(T_i,T),T)$ for $i = 0,\ldots,n$ and so we get a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & h_D(h_C(T_n,T),T) & \longrightarrow & \cdots & \longrightarrow & h_D(h_C(T_0,T),T) & \longrightarrow & h_D(T,T) & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_n & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & T_0 & \longrightarrow & C & \longrightarrow & 0, \end{array}$$

where the vertical arrows are isomorphisms. By hypothesis

$$0 \to T \cong h_C(C,T) \to h_C(T_0,T) \to \cdots \to h_C(T_n,T) \to 0$$

is exact and so is an injective resolution of ${}_DT$. Therefore $e_D(T) \cong C$ and $ext^i_D(T,T) = 0,\ \forall i \geq 1.$ □

5 *N*-Cotilting Comodules

In this section, we will consider a particular class of semidualizing comodules that give equivalence of categories and which generalize other classical results.

Following the definitions of [36, Section 3], we give a general definition of n-cotilting comodule and n-f-cotilting comodule.

Definition 5.1 Let T be a right C-comodule. We will say that T is a n-cotilting comodule if the following four conditions are satisfied.

(i) T is quasi-finite.
(ii) $inj.dim_C T \leq n$.
(iii) $Ext^i_C\left(T^I, T\right) = 0$ for any index set I and all $i \geq 1$.
(iv) There exists an exact sequence

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to C \to 0$$

in $\mathcal{M}^C$ with $T_i \in Prod(T)$ for all $i \geq 0$.

We define T to be an n-f-cotilting comodule if T satisfies (i) and the following three conditions:

(i') T admits a finite injective resolution

$$0 \to T \to E^0 \to E^1 \to \cdots \to E^n \to 0,$$

where the comodules E^i are quasi-finite injective and lie in $add(C)$ for every $i \geq 0$.
(ii') $Ext^i_C(T,T) = 0 \quad \forall i \geq 1$.
(iii') There exists an exact sequence

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to C \to 0$$

in $\mathcal{M}^C$ with $T_i \in add(T)$ for all $i \geq 0$.

We note that classical one-cotilting, [30], is different from one-cotilting in this new sense.

Example 5.1 (1) Let C be a Gorenstein coalgebra [3, Definition 2.1]. Then, C is right and left semiperfect and there is minimal projective resolution of quasi-finite comodules in $\mathcal{M}^C$ (see [28, Theorem 3.1])

$$0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to C \to 0.$$

By [3, Lemma 2.2], any projective right C-comodule has finite injective dimension. We easily can see that $T = \oplus_{i=0}^{k} P_i$ is a generalized f-cotilting right C-comodule.
(2) Examples of 1-f-cotilting comodules over quivers coalgebras can be found in [36].

In the following result, we show that n-cotilting comodules T over a quasi-co-Frobenius coalgebra C are trivial in the sense that $e_C(T)$ is Morita–Takeuchi equivalent to C.

Theorem 5.2 *If C is a* quasi-*co-Frobrenius coalgebra*, *every n-cotilting comodule is an injective cogenerator.*

Proof Let C be a quasi-co-Frobenius coalgebra, which is equivalent to that C is projective in $\mathcal{M}^C$ and let T be in $\mathcal{M}^C$ such that T is a n-cotilting comodule. Then, there is an exact sequence

$$0 \to T_n \to T_{n-1} \to \cdots \to T_0 \to C \to 0$$

with $T_i \in Prod(T)$. This sequence splits at C, so $C \oplus M = T^J$, for some $M \in \mathcal{M}^C$ and some set J. Therefore T is a cogenerator. Also there is an exact sequence

$$0 \to T \to I_0 \to I_1 \to \cdots \to I_n \to 0.$$

Since every injective is also projective, this sequence splits everywhere, so $T \oplus N = C^{(I)}$ for some $N \in \mathcal{M}^C$ and some set I. This implies that T is injective. □

Now, we show that duals of f-cotilting comodules are tilting modules in the sense of Miyashita [31].

Proposition 5.3 *Let C be a right semiperfect coalgebra and let $T \in \mathcal{M}^C$. If T_C is a k-f-cotilting comodule, then $T^*_{C^*} \in \mathcal{M}_{C^*}$ is a tilting module of projective dimension less or equal than k.*

Proof It is clear that $T^*_{C^*}$ is finitely generated. Now if we consider

$$0 \to T \to E^0 \to \cdots \to E^k \to 0$$

exact with E^i injective finitely cogenerated, we get that E^i is a direct summand of $C^{(n)}$ for some n and so, E^{i*} is a direct summand of $C^{(n)*}$, which gives that $pd(T^*) \leq k$.

On the other hand, from the exact sequence

$$0 \to T_k \to \cdots \to T_1 \to T_0 \to C \to 0$$

with $T_i \in add(T_C)$ for $i = 0, \ldots, k$ we get

$$0 \to C^* \to T_0^* \to \cdots \to T_k^* \to 0$$

with $T_i^* \in add(T^*)$ $i = 0, \ldots, k$.

Finally, from the exact sequence

$$0 \to T \to E^0 \to \cdots \to E^k \to 0$$

with E^i injective finitely cogenerated, we get by [30, Lemma 5.1] that the complexes

$$0 \to Com_C(T, T) \to Com_C(T, E^0) \to \cdots \to Com_C(T, E^k) \to 0$$

and

$$0 \to Hom_{C^*}(T^*, T^*) \to Hom_{C^*}(E^{0*}, T^*) \to \cdots \to Hom_{C^*}(E^{k*}, T^*) \to 0$$

are isomorphic and so, give isomorphic homology groups. Now since $ext^i_C(T, T) = 0,\ \forall i \geq 1$ we get that $Ext^i_C(T, T) = 0,\ \forall i \geq 1$ and so, $Ext^i_{C^*}(T^*, T^*) = 0,\ \forall i \geq 1$. □

Proposition 5.4 *Let C be right semiperfect and let $D = e_C(T)$ be left semiperfect. If T is a k-f-cotilting comodule, then ${}_DT$ is k-f-cotilting and $e_D(T) \cong C$.*

Proof By Proposition 4.3, we get that ${}_DT_C$ is semidualizing and $e_D(T) \cong C$. On the other hand, by the hypothesis, there is an exact sequence in $\mathcal{M}^C$

$$0 \to T \overset{d^0}{\to} E^0 \overset{d^1}{\to} \cdots \overset{d^k}{\to} E^k \to 0$$

where every E^i is a direct summand of $C^{(n)}$. Now consider $L_i = Coker\left(d^{i-1}\right)$, $i = 1, \ldots, k-1$. Then for $j = 1, \ldots k-2$, $ext^i_C(L_j, T) \cong ext^{i-1}_C(L_{j+1}, T)$. In particular, for $j = 1, \ldots, k-1$, we get that

$$ext^1_C(L_j, T) \cong ext^2_C\left(L_{j-1}, T\right) \cong \cdots \cong ext^{k+1}_C\left(L_{j-k}, T\right) = 0$$

taking into account that $Y_{-1} = T$, and so we get an exact sequence in ${}^D\mathcal{M}$

$$0 \to h_C\left(E^k, T\right) \to h_C\left(E^{k-1}, T\right) \to h_C\left(E^0, T\right) \to h_C(T, T) = D \to 0.$$

Now, since $h_C(C, T) \cong {}_DT$ and E^i are direct summands of $C^{(n)}$, we get that $h_C(E^i, T) \in add({}_DT)$ for $i = 0, \cdots, k$.

Finally, let

$$0 \to T_k \stackrel{d_k}{\to} T_{k-1} \stackrel{d_{k-1}}{\to} \cdots \stackrel{d_1}{\to} T_0 \stackrel{d_0}{\to} C \to 0$$

be exact in $\mathcal{M}^C$ and $T_i \in add(T_C)$. Since $ext^i_C(T, T) = 0,\ \forall i \geq 1$ it follows that $ext^i_C\left(T_j, T\right) = 0,\ \forall i \geq 1$ and $j = 0, \ldots, k$. By the same reasoning as above, we get an exact sequence in ${}^D\mathcal{M}$

$$0 \to T \cong h_C(C, T) \to h_C(T_0, T) \to \cdots \to h_C(T_k, T) \to 0$$

Now, the fact that for $j = 0, \ldots, k, h_C\left(T_j, T\right)$ is a direct summand of $h_C\left(T^{(n)}, T\right) \cong D^{(n)}$ gives that $id\left({}_DT\right) \leq k$. □

For the next Theorem, we need to introduce some notation.

Let k be a non-negative fixed integer. For $T \in \mathcal{M}^C$ and $m \geq 0$, we will denote
$Kerext_m(T_C) = \{M \in \mathcal{M}^C : ext^i_C(T, M) = 0 \text{ for } i = 0, \ldots, k+m \text{ and } i \neq m\}$
$KerTor_m(T_C) = \{M \in {}^C\mathcal{M} : Tor^i_C(T, M) = 0 \text{ for } i = 0, \ldots, k+m \text{ and } i \neq m\}$

For ${}_CT' \in {}^C\mathcal{M}$ $Kerext_m\left({}_CT'\right)$ and $KerTor_m\left({}_CT'\right)$ are defined analogously.

Now, by Theorem 3.5 we have:

Theorem 5.5 *Let C and D be right and left semiperfect coalgebras, respectively, and T a $(D$–$C)$-bicomodule such that T_C and ${}_DT$ are quasi-finite. Suppose:*

(i) *${}_DT_C$ is a semidualizing bicomodule.*
(ii) *$id\ (T_C) \leq k$ and $id\ ({}_DT) \leq k$.*

Let m be an integer such that $0 \leq m \leq k$.

Then, the categories $KerTor_m(T_C)$ and $Kerext_m\left({}_DT\right)$ are equivalent under the functors $Tor^m_C(T, -)$ and $ext^m_D(T, -)$.

At the end of Sect. 3, we have defined the Grothendieck group of an abelian category. For a coalgebra C, we denote by $\mathcal{K}_0(C)$ the Grothendieck group of ${}^C\mathcal{M}$ and by $K_0(C)$ the Grothendieck group of ${}^C\mathcal{M}_{f.d}$, the full subcategory of finitely dimensional left C-comodules.

Theorem 5.6 *Under conditions of Theorem 5.5, the Grothendieck groups of C and D are isomorphic, that is, $K_0(C) \cong K_0(D)$ and $\mathcal{K}_0(C) \cong \mathcal{K}_0(D)$. In particular, the number of isomorphism classes of simple left C-comodules is equal to the number of isomorphic classes of simple left D-comodules.*

Proof Follows from Theorem 3.7 noting that derived functors of $h_D(T, -)$ and $T\square_C-$ preserve finite dimensional comodules. □

Corollary 5.7 *Under conditions of Theorem 5.5, C is finite dimensional if and only if D is finite dimensional.*

Proof Follows by Theorem 5.6 and [10, Exercise. 3.3.13]. □

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