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Best Proximity Points for Weak Proximal Contractions

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Abstract In this article, we introduce a new class of non-self mappings, called weak proximal contractions, which contains the proximal contractions as a subclass. Existence and uniqueness results of a best proximity point for weak proximal contractions are obtained. Also, we provide sufficient conditions for the existence of common best proximity points for two non-self mappings in metric spaces having appropriate geometric property. Examples are given to support our main results.

Keywords Best proximity point · Common best proximity point · Proximal contraction · P-property

Mathematics Subject Classification 47H10 · 47H09

1 Introduction and Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a contraction mapping if there exists a constant $\alpha \in [0, 1)$ such that $d(Tx, Ty) \leq \alpha d(x, y)$, for all $x, y \in A$. If A is a complete subset of X and T is a contraction self map, then by the Banach contraction principle, the fixed point equation $Tx = x$ has exactly one solution.

In general, for the non-self mapping $T : A \to B$, the fixed point equation $Tx = x$ may not have a solution. Thus, it is contemplated to find an approximate solution $x \in A$ such that the error $d(x, Tx)$ is minimum. Indeed, best approximation theory has been derived from this idea.

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Definition 1.1 Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ be a non-self mapping. A point $p \in A$ is called best proximity point of T if $d(p, Tp) = \text{dist}(A, B)$, where

$$\text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

In fact, best proximity point theorems have been studied to find necessary conditions such that the minimization problem

$$\min_{x \in A} d(x, Tx) \tag{1}$$

has at least one solution.

One can refer to [1,3–10,12,15,17,19]) for best proximity point theorems for various classes of non-self mappings.

Let us consider the mappings $T : A \to B$ and $S : A \to B$, where (A, B) is pair of nonempty subsets of a metric space (X, d). The natural question is whether one can find a solution for the minimization problem

$$\min_{x \in A} d(x, Tx) \quad \& \quad \min_{x \in A} d(x, Sx). \tag{2}$$

Since $d(x, Tx), d(x, Sx) \geq \text{dist}(A, B)$, the optimal solution to the problem of minimizing the real valued functions $x \mapsto d(x, Tx)$ and $x \mapsto d(x, Sx)$ over the domain A of the mappings S, T will be the one for which the value $\text{dist}(A, B)$ is attained.

Definition 1.2 Let (A, B) be nonempty pair of a metric space (X, d) and $S : A \to B, T : A \to B$ be two non-self mappings. A point $x^* \in A$ is called a common best proximity point of the mappings S, T if

$$d(x^*, Tx^*) = d(x^*, Sx^*) = \text{dist}(A, B).$$

Let A and B be two nonempty subsets of a metric space (X, d). In this work, we adopt the following notations and definitions.

$$\begin{aligned} A_0 &:= \{x \in A : d(x, y) = \text{dist}(A, B), \quad \text{for some} \quad y \in B\}, \\ B_0 &:= \{y \in B : d(x, y) = \text{dist}(A, B), \quad \text{for some} \quad x \in A\}, \\ D(x, B) &:= \inf\{d(x, y) : y \in B\}, \quad \text{for all} \quad x \in X. \end{aligned}$$

In [13], Sadiq Basha introduced the notion of *proximal contractions* as follows.

Definition 1.3 ([13]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a proximal contraction if there exists a non-negative real number $\alpha < 1$ such that, for all $u_1, u_2, x_1, x_2 \in A$,

$$\begin{cases} d(u_1, Tx_1) = \text{dist}(A, B) \\ d(u_2, Tx_2) = \text{dist}(A, B) \end{cases} \Rightarrow d(u_1, u_2) \leq \alpha d(x_1, x_2).$$

Definition 1.4 ([13]) Let A, B be two nonempty subsets of a metric space (X, d).A is said to be approximatively compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y, x_n) \to D(y, A)$ for some $y \in B$ has a convergent subsequence.

The next theorem is a main result of [13].

Theorem 1.1 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a proximal contraction such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

The following notion of a geometric property in metric spaces was introduced by Sankar Raj in [16].

Definition 1.5 ([16]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have the P-property if and only if

$$\begin{cases} d(x_1, y_1) = \text{dist}(A, B) \\ d(x_2, y_2) = \text{dist}(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 1.1 ([16]) Let A, B be two nonempty closed convex subsets of a Hilbert space $\mathbb{H}$. Then (A, B) has the P-property.

Example 1.2 Let A, B be two nonempty subsets of a metric space (X, d) such that $A_0 \neq \emptyset$ and $\text{dist}(A, B) = 0$. Then (A, B) has the P-property.

Example 1.3 ([2]) Let A, B be two nonempty bounded, closed and convex subsets of a uniformly convex Banach space X. Then (A, B) has the P-property.

In the current paper, we introduce a new class of non-self mappings, called *weak proximal contractions*, which contains the proximal contractions as a subclass. For such mappings, we obtain existence and uniqueness results of best proximity points. Moreover, we prove the existence of a common best proximity point for two non-self mappings in a metric spaces with the P-property.

2 Weak Proximal Contractions

To establish our results of this section, we introduce the following new class of non-self mappings.

Definition 2.1 Define a strictly decreasing function η from $[0, 1)$ onto $(\frac{1}{2}, 1]$ by

$$\eta(r) = \frac{1}{1+r}.$$

Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A non-self mapping $T : A \to B$ is said to be a weak proximal contraction if there exists $r \in [0, 1)$ such that, for all $u, v, x, y \in A$ with

$$d(u, Tx) = \text{dist}(A, B) \quad \& \quad d(v, Ty) = \text{dist}(A, B),$$

we have

$$\eta(r)d^*(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(u, v) \leq rd(x, y), \tag{3}$$

where $d^*(a, b) := d(a, b) - \text{dist}(A, B)$, for all $(a, b) \in A \times B$.

Let us state our main result of this section.

Theorem 2.1 *Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a weak proximal contraction non-self mapping such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

Proof Let $x_0 \in A_0$. Since $Tx_0 \in B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = \text{dist}(A, B)$. Again, since $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = \text{dist}(A, B)$. Thus, we have a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = \text{dist}(A, B), \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{4}$$

We now have

$$d(x_0, Tx_0) \leq d(x_0, x_1) + d(x_1, Tx_0) = d(x_0, x_1) + \text{dist}(A, B),$$

which implies that

$$\eta(r)d^*(x_0, Tx_0) \leq d^*(x_0, Tx_0) \leq d(x_0, x_1).$$

Since T is weak proximal contraction,

$$d(x_1, x_2) \leq rd(x_0, x_1).$$

Similarly, we can see that $\eta(r)d^*(x_1, Tx_1) \leq d(x_1, x_2)$ and by the fact that T is weak proximal contraction, we must have

$$d(x_2, x_3) \leq rd(x_1, x_2) \leq r^2 d(x_0, x_1).$$

Continuing this process, we obtain

$$d(x_n, x_{n+1}) \leq r^n d(x_0, x_1).$$

Thus $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. So, $\{x_n\}$ is a Cauchy sequence and by the completeness of X and since A_0 is closed, there exists $p \in A_0$ such that $x_n \to p$. We claim that

$$d^*(p, Tx) \le rd(p, x) \quad \text{for all} \quad x \in A_0 \quad \text{with} \quad x \ne p. \tag{5}$$

Let $x \in A_0$ and $x \ne p$. Since $T(A_0) \subseteq B_0$, there exists $y \in A_0$ such that $d(y, Tx) = \text{dist}(A, B)$. As regards $x_n \to p$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, p) \le \frac{1}{3} d(x, p) \quad \text{for all} \quad n \ge N_1.$$

We now have

$$\begin{aligned} \eta(r) d^*(x_n, Tx_n) &\le d^*(x_n, Tx_n) = d(x_n, Tx_n) - \text{dist}(A, B) \\ &\le d(x_n, p) + d(p, x_{n+1}) + d(x_{n+1}, Tx_n) - \text{dist}(A, B) \\ &= d(x_n, p) + d(p, x_{n+1}) \\ &\le \frac{2}{3} d(x, p) = d(x, p) - \frac{1}{3} d(x, p) \\ &\le d(x, p) - d(x_n, p) \le d(x_n, x). \end{aligned}$$

Thus,

$$\begin{cases} d(x_{n+1}, Tx_n) = \text{dist}(A, B) \\ d(y, Tx) = \text{dist}(A, B) \end{cases} \quad \& \quad \eta(r) d^*(x_n, Tx_n) \le d(x_n, x).$$

Since T is weal proximal contraction,

$$d(x_{n+1}, , y) \le rd(x_n, x). \tag{6}$$

Therefore, by (6) we conclude that

$$\begin{aligned} d(p, Tx) &= \lim_{n\to\infty} d(x_n, Tx) \\ &\le \lim_{n\to\infty} [d(x_n, x_{n+1}) + d(x_{n+1}, y) + d(y, Tx)] \\ &\quad \times \lim_{n\to\infty} [d(x_n, x_{n+1}) + rd(x_n, x) + d(y, Tx)] \\ &= rd(p, x) + \text{dist}(A, B), \end{aligned}$$

and hence $d^*(p, Tx) \le rd(p, x)$. Then

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, p) + d(p, Tx_n) \\ &\le d(x_n, p) + rd(p, x_n) + \text{dist}(A, B), \end{aligned}$$

which implies that $d^*(x_n, Tx_n) \le (1 + r) d(x_n, p)$, and hence

$$\frac{1}{1+r} d^*(x_n, Tx_n) = \eta(r) d^*(x_n, Tx_n) \le d(x_n, p).$$

On the other hand, since $p \in A_0$ and $T(A_0) \subseteq B_0$, there exists $q \in A_0$ such that $d(q, Tp) = \text{dist}(A, B)$. We have

$$\begin{cases} d(x_{n+1}, Tx_n) = \text{dist}(A, B) \\ d(q, Tp) = \text{dist}(A, B) \end{cases} \quad \& \quad \eta(r)d^*(x_n, Tx_n) \leq d(x_n, p).$$

As T is a weak proximal contraction, we obtain

$$d(x_{n+1}, q) \leq rd(x_n, p) \to 0.$$

This implies that $x_n \to q$. Thus $p = q$, that is, $d(p, Tp) = \text{dist}(A, B)$. We conclude the proof by showing that the best proximity point of T is unique. Suppose that $\acute{p} \in A_0$ is another best proximity point of the mapping T. We have

$$\begin{cases} d(p, Tp) = \text{dist}(A, B) \\ d(\acute{p}, T\acute{p}) = \text{dist}(A, B) \end{cases} \quad \& \quad \eta(r)d^*(p, Tp) = 0 \leq d(p, \acute{p}).$$

Then we must have $d(p, \acute{p}) \leq rd(p, \acute{p})$ which implies that $p = \acute{p}$. □

Example 2.1 Consider $X = \mathbb{R}^2$ and define the metric d on X by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|, \quad \forall\, (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2.$$

We know, (X, d) is a complete metric space. Suppose

$$A := \{(0, 0), (4, 5), (5, 4)\} \quad \text{and} \quad B = \{(0, 0), (0, 4), (4, 0)\}.$$

Define a non-self mapping $T : A \to B$ as follows:

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if} \quad x_1 \leq x_2, \\ (0, x_2) & \text{if} \quad x_2 < x_1. \end{cases}$$

We claim that T satisfies the condition (3). If $(x, y) \neq ((4, 5), (5, 4))$ and $(x, y) \neq ((5, 4), (4, 5))$, it is easy to see that $d(Tx, Ty) \leq \frac{4}{9}d(x, y)$. If $(x, y) = ((4, 5), (5, 4))$, we have

$$d(Tx, Ty) = d((4, 0), (0, 5)) = 9 > 2 = d(x, y),$$

which implies that T is not a contraction. Besides,

$$\eta(r)d(x, Tx) = \frac{1}{1+r}d((4, 5), T(4, 5)) = \frac{5}{1+r} > 2 = d(x, y)$$

for every $r \in [0, 1)$. That is, (3) holds. It now follows from Theorem 2.1 that T has a unique best proximity point.

The following results follow from Theorem 2.1, immediately.

Corollary 2.1 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is closed. Assume that $T : A \to B$ is a proximal contraction such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

Example 2.2 Suppose that $X = \mathbb{R}$ with the usual metric. Suppose that

$$A := [-2, -1] \cup \{4\}, \quad B := [1, 2].$$

Note that $\text{dist}(A, B) = 2$. Let $T : A \to B$ be a mapping defined as

$$T(x) = \begin{cases} x + 3 & \text{if} \quad x \neq 4, \\ 2 & \text{if} \quad x = 4. \end{cases}$$

We claim that T is a weak proximal contraction non-self mapping.

Case 1. If $(u, x) = (-1, -2)$ and $(v, y) = (4, -1)$ then

$$d(u, Tx) = d(v, Ty) = \text{dist}(A, B).$$

Also, for each $r \in [0, 1)$, we have

$$\eta(r)d^*(x, Tx) = \frac{1}{1+r} \times 2 > 1 = d(x, y).$$

That is, T satisfies the condition (3) in this case.

Case 2. If either $(u, x) = (-1, -2), (v, y) = (4, 4)$, or $(u, x) = (4, -1), (v, y) = (4, 4)$, then it is easy to see that T is proximal contraction in this case with the constant contraction $r \geq \frac{1}{6}$. It now follows from Theorem 2.1 that T has a unique best proximity point and this point is $p = 4$.

Note that the existence of best proximity point in the above example cannot be obtained from Theorem 1.1. Indeed, the non-self mapping T in Example 2.2 is not proximal contraction. Because, in Case 1, we have

$$d(Tx, Ty) = 1 > r \times 1 = rd(x, y)$$

for each $r \in [0, 1)$.

Remark 2.1 Note that Corollary 2.1 improves Theorem 1.1. Indeed, if (A, B) is a nonempty closed pair of subsets of a metric space (X, d) such that B is approximatively compact with respect to A, then A_0 is closed (see Proposition 3.1 of [11]).

Let us illustrate Remark 2.1 with the following example.

Example 2.3 Consider the complete metric space $X := \mathbb{R}^2$ with the metric d_∞ defined with

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\},$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Let $A := \{(0, x) : x \in [0, 1]\}$ and $B := \{(x, 0) : x \in [0, 1] \bigcap \mathbb{Q}\}$. We note that $A_0 := \{(0, 0)\}$, that is, A_0 is closed. Define a non-self mapping $T : A \to B$ by

$$T(0, x) = \begin{cases} (1, 0), & \text{if} \quad x \in \mathbb{Q}^c \bigcap [0, 1] \\ (0, 0), & \text{if} \quad x \in \mathbb{Q} \bigcap [0, 1]. \end{cases}$$

Clearly, T is not continuous. Besides, if $\mathbf{u} := (0, u), \mathbf{x} := (0, x) \in A$ be such that $d_\infty(\mathbf{u}, T\mathbf{x}) = 0$, then we must have $x \in \mathbb{Q}$ and so, $u = 0$. Thus, T is a proximal contraction. Therefore, by Theorem 2.1, T has a unique best proximity point which is a fixed point in this case. On the other hand, B is not approximatively compact with respect to A. Indeed, if $\mathbf{x} = (0, 1) \in A$ and we consider the sequence $\mathbf{y}_n = (y_n, 0)$ in B such that $\{y_n\}$ is an iteration sequence defined by

$$\begin{cases} y_1 = 1, \\ y_{n+1} = \frac{1}{4}(y_n + \frac{2}{y_n}), \quad \forall n \in \mathbb{N}, \end{cases}$$

then, we have $\lim_{n\to\infty} d_\infty(\mathbf{x}, \mathbf{y}_n) = 1 = D(\mathbf{x}, B)$ but the sequence $\{\mathbf{y_n}\}$ has no convergence subsequence in B. So, existence of the best proximity point for T cannot be obtained from Theorem 1.1.

The next result is an extension of Banach contraction principle.

Corollary 2.2 *Let A be a nonempty closed subset of a complete metric space (X, d). Suppose that $T : A \to A$ is a mapping such that*

$$\eta(r)d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leq rd(x, y), \tag{7}$$

for all $x, y \in A$. Then T has a unique fixed point.

Remark 2.2 In [18], Suzuki proved that if in Corollary 2.2, the function $\eta : [0, 1) \to (\frac{1}{2}, 1]$ is defined by

$$\eta(r) = \begin{cases} 1 & \text{if} \quad 0 \leq r \leq \frac{1}{2}(\sqrt{5} - 1), \\ \frac{1-r}{r^2} & \text{if} \quad \frac{1}{2}(\sqrt{5} - 1) \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if} \quad \frac{1}{\sqrt{2}} \leq r < 1, \end{cases} \tag{8}$$

then Corollary 2.2 is valid. But it is interesting to note that the function η defined in (8) is the best constant (see [18]). Motivated by Suzuki, we arise the following question.

Question 2.1 *It is interesting to ask whether the function η defined in Theorem 2.1 is the best constant.*

3 Common Best Proximity Points

To establish our results of this section, we recall the following definitions which were introduced in [14], and were used to prove a common best proximity point theorem.

Definition 3.1 ([14]) The mappings $S : A \to B$ and $T : A \to B$ are said to commute proximally if they satisfy the following condition

$$[d(u, Sx) = d(v, Tx) = \text{dist}(A, B)] \Rightarrow Sv = Tu,$$

for all x, u, and v in A.

It is clear that the proximal commutativity of self mappings is just commutativity of the mappings.

Definition 3.2 ([14]) It is said that the mappings $S : A \to B$ and $T : A \to B$ can be swapped proximally if

$$[d(y, u) = d(y, v) = \text{dist}(A, B) \quad \& \quad Su = Tv] \Rightarrow Sv = Tu,$$

for all $u, v \in A$ and $y \in B$.

Remark 3.1 Let A, B be two nonempty subsets of a metric space (X, d) such that A_0 is nonempty. If (A, B) has the P-property, then every two non-self mappings $S : A \to B$ and $T : A \to B$ can be swapped proximally.

Here, we state the main result of [14].

Theorem 3.1 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A is approximatively compact with respect to B. Assume that A_0 and B_0 are nonempty. Let the non-self mappings $S : A \to B$ and $T : A \to B$ satisfy the following conditions:*

(a) There is a non-negative real number $\alpha < 1$ such that

$$d(Sx_1, Sx_2) \leq \alpha d(Tx_1, Tx_2),$$

for all $x_1, x_2 \in A$.

(b) S, T are continuous.

(c) S and T commute proximally.

(d) S and T can be swapped proximally.

(e) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.

Then, S and T have a common best proximity point.

Motivated by the main result of [14], we prove the following common best proximity point theorem.

Theorem 3.2 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and (A, B) has the P-property. Assume that the non-self mappings $S : A \to B$ and $T : A \to B$ satisfy the following conditions:*

(a) There is a non-negative real number $\alpha < 1$ such that

$$d(Sx_1, Sx_2) \leq \alpha d(Tx_1, Tx_2),$$

for all $x_1, x_2 \in A$.
(b) S, T are continuous.
(c) S and T commute proximally.
(d) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.

Then, S and T have a common best proximity point.

Proof Choose $x_0 \in A_0$. Since $S(A_0) \subseteq T(A_0)$, there exists $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Again, since $S(A_0) \subseteq T(A_0)$ and $x_1 \in A_0$, there exists $x_2 \in A_0$ such that $Sx_1 = Tx_2$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$Sx_{n-1} = Tx_n, \quad \text{for all} \quad n \in \mathbb{N}. \tag{9}$$

We have

$$d(Sx_n, Sx_{n+1}) \leq \alpha d(Tx_n, Tx_{n+1}) = \alpha d(Sx_{n-1}, Sx_n),$$

which implies that $\{Sx_n\}$ is a Cauchy sequence in B and hence converges to some $y \in B$. By (9), we must have $Tx_n \to y$. On the other hand, since $S(A_0) \subseteq B_0$, there exists $a_n \in A_0$ such that $d(Sx_n, a_n) = \text{dist}(A, B)$, for all $n \in \mathbb{N}$. From (9), we obtain

$$d(Tx_n, a_{n-1}) = d(Sx_{n-1}, a_{n-1}) = \text{dist}(A, B). \tag{10}$$

Since S and T are commuting proximally,

$$Sa_{n-1} = Ta_n, \quad \text{for all} \quad n \in \mathbb{N}. \tag{11}$$

Also, because of the fact that (A, B) has the P-property, we conclude that $d(a_n, a_{n-1}) = d(Tx_n, Sx_n)$. We now have

$$\begin{aligned} d(a_n, a_{n-1}) &= d(Tx_n, Sx_n) = d(Sx_{n-1}, Sx_n) \\ &\leq \alpha d(Tx_{n-1}, Tx_n) = \alpha d(Tx_{n-1}, Sx_{n-1}) = \alpha d(a_{n-1}, a_{n-2}) \\ &\leq \alpha[d(Tx_{n-1}, Sx_{n-2}) + d(Sx_{n-2}, Sx_{n-1})] = \alpha d(Sx_{n-2}, Sx_{n-1}) \\ &\leq \alpha^2 d(Tx_{n-2}, Tx_{n-1}) = \alpha^2 d(Tx_{n-2}, Sx_{n-2}) \\ &= \alpha^2 d(a_{n-2}, a_{n-3}) \leq \ldots \leq \alpha^{n-1} d(a_1, a_0). \end{aligned}$$

This implies that $\{a_n\}$ is a Cauchy sequence in A. Let $a_n \to p \in A$. By the continuity of S and T we obtain $Sa_n \to Sp$ and $Ta_n \to Tp$. From the (11), we must have $Sp = Tp$. Also, by using the relation (10), we obtain $d(y, p) = \text{dist}(A, B)$ and hence $p \in A_0$. Since $S(A_0) \subseteq B_0$, there exists $x^* \in A_0$ such that $d(x^*, Sp) = \text{dist}(A, B)$

and then $d(x^*, Tp) = \text{dist}(A, B)$. As S and T are commuting proximally, $Tx^* = Sx^*$. Therefore,

$$d(Sx^*, Sp) \leq \alpha d(Tx^*, Tp) = \alpha d(Sx^*, Sp),$$

which implies that $Sx^* = Sp = Tx^* = Tp$. Hence,

$$d(x^*, Tx^*) = \text{dist}(A, B) = d(x^*, Sx^*),$$

where x^* is a common best proximity point of S and T. □

We now conclude the next corollaries from Theorem 3.2, directly.

Corollary 3.1 *Let (A, B) be a nonempty closed pair of subsets of a metric space (X, d) such that* $\text{dist}(A, B) = 0$. *Assume that the non-self mappings $S : A \to B$ and $T : A \to B$ satisfy the conditions (a), (b), (c) and (d) of Theorem 3.4. Then, S and T have a common best proximity point.*

Corollary 3.2 *Let (A, B) be a nonempty closed convex pair in a Hilbert space $\mathbb{H}$. Assume that the non-self mappings $S : A \to B$ and $T : A \to B$ satisfy the conditions (a), (b), (c), and (d) of Theorem 3.4. Then, S and T have a common best proximity point.*

Corollary 3.3 *If in Corollary 3.5, (A, B) is a nonempty bounded closed convex pair in a uniformly convex Banach space X, then the result is valid.*

Remark 3.2 In the general case, Corollaries 3.1 and 3.2 cannot be obtained from Theorem 2.4. Because we have no information about the approximatively compactness of one set with respect to another set. The following example illustrates this reality.

Example 3.1 Let l^∞ be the Banach space consisting of all bounded real sequences with supremum norm and let $\{e_n\}$ be the canonical basis of l^∞. Suppose that e_0 is the zero of l^∞. Let

$$A := \{xe_{2n} : n \in \mathbb{N}, 0 \leq x \leq 1\} \quad \text{and} \quad B := \{xe_{2n-1} : n \in \mathbb{N}, 0 \leq x \leq 1\}.$$

We have $\text{dist}(A, B) = 0$ and $A_0 = B_0 = \{e_0\}$. Assume that $S : A \to B$ and $T : A \to B$ are defined as follows.

$$S(xe_{2n}) = \frac{x}{6} e_{2n-1} \quad \& \quad T(xe_{2n}) = \frac{x}{3} e_{2n-1}.$$

Clearly,

$$\|S(xe_{2n}) - S(ye_{2n})\| \leq \frac{1}{2} \|T(xe_{2n}) - T(ye_{2n})\|.$$

Also, if $\mathbf{u} := ue_{2n}, \mathbf{x} := xe_{2n} \in A$ are such that $\|\mathbf{u} - T\mathbf{x}\| = \text{dist}(A, B)$, then $\mathbf{u} = \mathbf{x} = e_0$. This implies that S, T are commute proximally. Hence, all conditions of

Theorem 3.2 hold. Therefore, S and T have a common best proximity point. Obviously, this point is e_0. It is easy to see that B is not approximatively compact with respect to A, that is, existence of a common best proximity point for non-self mappings S and T cannot be obtained from Theorem 3.1 due to Sadiq Basha ([14]).

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