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ORIGINAL PAPER

High Order Numerical Scheme for Generalized Fractional Diffusion Equations

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Abstract
In this paper, a higher order finite difference scheme is proposed for generalized fractional diffusion equations (GFDEs). The fractional diffusion equation is considered in terms of the generalized fractional derivatives (GFDs) which uses the scale and weight functions in the definition. The GFD reduces to the Riemann–Liouville, Caputo derivatives and other fractional derivatives in a particular case. Due to importance of the scale and the weight functions in describing behaviour of real-life physical systems, we present the solutions of the GFDEs by considering various scale and weight functions. The convergence and stability analysis are also discussed for the finite difference scheme (FDS) to validate the proposed method. We consider test examples for numerical simulation of FDS to justify the proposed numerical method.

Keywords Caputo fractional derivative · Generalized time fractional derivative · Finite difference scheme · Generalized fractional diffusion equations

Introduction

Fractional derivative is the generalization of integer order derivative, where order of derivative is a fraction. Due to its wide applications in solving real world problems, it has become a centre of attention for researchers working in the field of fractional partial differential equations. The subject that deals with fractional diffusion equations (FDEs) is given by replacing the integer time and/or space derivatives with fractional derivatives. Applications of FDEs include problems based on particle tracking [1], ion channel gating [2], population dynamics [3], option prices in markets [4] and other science and engineering problems namely

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electromagnetic [5], viscosity [6] etc. In literature, different methods for solving FDEs and other fractional models are studied since the introduction of the fractional derivatives [7–9]. The finite difference method (FDM) is one of the popular methods to provide the numerical solution with higher-order accuracy. Nowadays, initial and boundary value problems for FDEs have been discussed thoroughly in various diffusion systems [10–28].

In [29], Agrawal presented the generalized fractional derivatives (GFD) which depend upon scale and weight functions. For the special choices of scale and weight functions, GFD reduces to Caputo, Riemann–Liouville, Riesz, Hadamard and other types of fractional derivatives. The scale function is useful for adjusting simulation time via stretching and compressing for the better demonstration of the solution. The weight functions allow us to assess simulation phenomena differently at different times as it extends the kernel of the fractional operator to provide high level of flexibility in formulation of the problems.

We consider the time fractional generalized diffusion equation (TFGDEs) defined in terms of GFD as follows:

$$\begin{cases} {}^*\partial_t^\alpha u(x,t) = \delta \frac{\partial^2 u(x,t)}{\partial^2 x} + f(x,t) & (x,t) \in [a,b] \times (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = f_1(t), u(b,t) = f_2(t), & t > 0, \end{cases} \tag{1}$$

where $T > 0$, $0 < \alpha < 1$, $\delta > 0$ is real numbers, $f(x,t)$, $f_1(t)$ and $f_2(t)$ are known functions. And, ${}^*\partial_t^\alpha u(x,t)$ denotes the generalized fractional derivative operator, which is defined for function u with the help of scale function $z(t)$ and weight function $w(t)$ as,

$${}^*\partial_t^\alpha u(x,t) = \frac{[w(t)]^{-1}}{\Gamma(1-\alpha)} \int_0^t \frac{(w(\tau)u(x,\tau))\prime}{(z(t)-z(\tau))^\alpha} d\tau, \quad 0 < \alpha < 1. \tag{2}$$

The presence of scale and weight functions in the fractional derivative term generalizes the problem, hence it becomes difficult to find analytical solution of FDEs. By changing the scale and weight function in the generalized fractional derivative it's reduces the most of existing fractional derivative as well as integer order derivative. Its importance could be visualized as follows. Some of the physical phenomena happens for few seconds or one percent of time and although, some for longer time, for instance, models need to be computed in hundreds or thousands of years. Both situations handle by taking stretching scale function $z(t)$ to dipict the numerical results of phenomena and taking contracting type scale function. However, weight function $w(t)$ play more flexibility to change kernel of the fractional derivative. For example, heavy weight may require modelling of the memory of child at a current instant of time and more weight may require older person in the past. Thus, the scale function plays important role to change the consider domain, accordingly the physical process happens. In [29], author discuss solution of many integral equations by using the scale and weight function. Hence numerical solution is necessary in order to solve such type of diffusion equations. In literature, very few research works are available related with numerical solution of GFDEs [30–35] with higher order of convergence. Motivated by the research works presented in [30–34], we present the numerical solution of TFGDEs as given in Eq. (1) with high order of convergence. For readers interested in the field of fractional calculus, we refer to the published articles [36–44]

Rest of the work is structured in following manner: In "Computational algorithm of problem" section, problem formulation as well as computational algorithm is presented for TFGDEs using finite difference scheme. In "Order of approximation for the generalized fractional derivative" section depicts the approximation of the GFD by using interpolating polynomials and order of error of approximation. The stability analysis and order of convergence of proposed FDS are derived in "Stability and convergence analysis" section. In "Numerical results and their analysis" section, we present some illustrative examples to validate the proposed numerical scheme for TFGDEs by considering different scale and weight functions. In "Conclusion" section concludes our work.

Computational Algorithm of Problem

To find the numerical solution firstly, we divide space domain $[a, b]$ and time domain $[0, T]$ into N and M equal parts, respectively. Suppose $\mathcal{P}_N = \{a = x_0 < x_1 < x_2 < \cdots < x_N = b\}$ be the partition of $[a, b]$ into N equal parts with spacing $\Delta x = \frac{b-a}{N}$, and $\mathcal{P}_M = \{0 = t_0 < t_1 < t_2 < \cdots < t_M = T\}$ be the partition of $[0, T]$ into M equal parts with spacing $\Delta t = \frac{T}{M}$. For convenience, we denote $u\big(x_i, t_j\big) = u_j^i$, $w\big(t_j\big) = w_j$ and $z\big(t_j\big) = z_j$. We assume $w(t) > 0$, and $z(t)$ is strictly increasing then $s = z^{-1}(v)$ by taking transformation $v = z(s)$. For numerical solution, we approximate each term of GFDE given by Eq. (1). The Caputo type generalized fractional derivative of function $u(x, t)$ can be approximated at node point $\big(x_i, t_{j+1}\big)$. Firstly, we approximate of first term of Eq. (1) as follow,

$$\begin{aligned}
\left[{}^*\partial_t^\alpha u(x,t)\right]_{(x_i,t_{j+1})} &= \frac{\left[w\big(t_{j+1}\big)\right]^{-1}}{\Gamma(1-\alpha)} \int\limits_0^{t_{j+1}} \frac{\frac{\partial}{\partial s}[w(s)u(x_i,s)]}{\left[z\big(t_{j+1}\big)-z(s)\right]^\alpha} ds = \frac{\left[w\big(t_{j+1}\big)\right]^{-1}}{\Gamma(1-\alpha)} \sum_{k=0}^{j} \int\limits_{t_k}^{t_{k+1}} \frac{\frac{\partial}{\partial s}[w(s)u(x_i,s)]}{\left[z\big(t_{j+1}\big)-z(s)\right]^\alpha} ds \\
&= \frac{\left[w\big(t_{j+1}\big)\right]^{-1}}{\Gamma(1-\alpha)} \int\limits_{t_0}^{t_1} \frac{\frac{\partial}{\partial s}[w(s)u(x_i,s)]}{\left[z\big(t_{j+1}\big)-z(s)\right]^\alpha} ds + \frac{\left[w\big(t_{j+1}\big)\right]^{-1}}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \int\limits_{t_k}^{t_{k+1}} \frac{\frac{\partial}{\partial s}[w(s)u(x_i,s)]}{\left[z\big(t_{j+1}\big)-z(s)\right]^\alpha} ds \\
&= \frac{\left[w\big(t_{j+1}\big)\right]^{-1}}{\Gamma(1-\alpha)} \int\limits_{z_0}^{z_1} \frac{1}{\left[z\big(t_{j+1}\big)-v\right]^\alpha} \frac{d\left[w\big(z^{-1}(v)\big)u\big(x_i,z^{-1}(v)\big)\right]}{dz^{-1}(v)} dz^{-1}(v) + \qquad (3)
\end{aligned}$$

$$\frac{\left[w\big(t_{j+1}\big)\right]^{-1}}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \int\limits_{z_k}^{z_{k+1}} \frac{1}{\left[z\big(t_{j+1}\big)-v\right]^\alpha} \frac{d\left[w\big(z^{-1}(v)\big)u\big(x_i,z^{-1}(v)\big)\right]}{dz^{-1}(v)} dz^{-1}(v) \qquad (4)$$

On the first interval $[z_0, z_1]$, the linear interpolation $\big(p_u^1(v)\big)\prime$ is used for approximating the unknown function and for the other subintervals $(k \geq 1)$, the quadratic interpolation function $\big(p_u^2(v)\big)\prime$ for the three points $\big(z_{k-1}, u_{k-1}^i\big)$, $\big(z_k, u_k^i\big)$, $\big(z_{k+1}, u_{k+1}^i\big)$ is applied such that,

$$\big(p_u^1(v)\big)\prime = \frac{w_1 u_1^i - w_0 u_0^i}{z_1 - z_0}, \qquad (5)$$

$$\begin{aligned}
\big(p_u^2(v)\big)\prime = \Bigg(&\frac{(v-z_k)(v-z_{k+1})}{(z_k-z_{k-1})(z_{k+1}-z_{k-1})} w_{k-1}u_{k-1}^i - \frac{(v-z_{k-1})(v-z_{k+1})}{(z_k-z_{k-1})(z_{k+1}-z_k)} w_k u_k^i \\
&+ \frac{(v-z_{k-1})(v-z_k)}{(z_{k+1}-z_{k-1})(z_{k+1}-z_k)} w_{k+1}u_{k+1}^i\Bigg)\prime. \qquad (6)
\end{aligned}$$

$$\begin{aligned}
\big(p_u^2(v)\big)\prime = &\frac{(2v-z_k-z_{k+1})}{(z_k-z_{k-1})(z_{k+1}-z_{k-1})} w_{k-1}u_{k-1}^i - \frac{(2v-z_{k-1}-z_{k+1})}{(z_k-z_{k-1})(z_{k+1}-z_k)} w_k u_k^i \\
&+ \frac{(2v-z_{k-1}-z_k)}{(z_{k+1}-z_{k-1})(z_{k+1}-z_k)} w_{k+1}u_{k+1}^i. \qquad (7)
\end{aligned}$$

Again, we consider first term of Eq. (4)

$$\begin{aligned}
&\frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{1}{\left[z\left(t_{j+1}\right)-v\right]^{\alpha}} \frac{d\left[w\left(z^{-1}(v)\right)u\left(x_i, z^{-1}(v)\right)\right]}{dz^{-1}(v)} dz^{-1}(v), \\
&= \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(2-\alpha)} \left[\frac{w_1 u_1^i - w_0 u_0^i}{z_1 - z_0}\right] \left[\left(z_{j+1}-z_0\right)^{1-\alpha} - \left(z_{j+1}-z_1\right)^{1-\alpha}\right] \\
&= \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(2-\alpha)} \left[\frac{\left(z_{j+1}-z_0\right)^{1-\alpha} - \left(z_{j+1}-z_1\right)^{1-\alpha}}{z_1 - z_0}\right] w_1 u_1^i \\
&\quad - \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(2-\alpha)} \left[\frac{\left(z_{j+1}-z_0\right)^{1-\alpha} - \left(z_{j+1}-z_1\right)^{1-\alpha}}{z_1 - z_0}\right] w_0 u_0^i. \qquad (8)
\end{aligned}$$

And second term of Eq. (4)

$$\begin{aligned}
&\frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \int_{z_k}^{z_{k+1}} \frac{1}{\left[z\left(t_{j+1}\right)-v\right]^{\alpha}} \frac{d\left[w\left(z^{-1}(v)\right)u\left(x_i, z^{-1}(v)\right)\right]}{dz^{-1}(v)} dz^{-1}(v), \\
&= \sum_{k=1}^{j} a_k^j u_{k-1}^i - b_k^j u_k^i + c_k^j u_{k+1}^i. \qquad (9)
\end{aligned}$$

$$a_k^j = \frac{w_{k-1}}{(z_k - z_{k-1})(z_{k+1} - z_{k-1})} \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(1-\alpha)} \left[\frac{\left(2z_{j+1} - z_k - z_{k+1}\right)}{1-\alpha} p_k^j - \frac{2}{2-\alpha} q_k^j\right]. \qquad (10)$$

$$b_k^j = \frac{w_k}{(z_k - z_{k-1})(z_{k+1} - z_k)} \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(1-\alpha)} \left[\frac{\left(2z_{j+1} - z_{k-1} - z_{k+1}\right)}{1-\alpha} p_k^j - \frac{2}{2-\alpha} q_k^j\right]. \qquad (11)$$

$$c_k^j = \frac{w_{k+1}}{(z_{k+1} - z_{k-1})(z_{k+1} - z_k)} \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(1-\alpha)} \left[\frac{\left(2z_{j+1} - z_{k-1} - z_k\right)}{1-\alpha} p_k^j - \frac{2}{2-\alpha} q_k^j\right]. \qquad (12)$$

$$p_k^j = \left(z_{j+1} - z_k\right)^{1-\alpha} - \left(z_{j+1} - z_{k+1}\right)^{1-\alpha}. \qquad (13)$$

$$q_k^j = \left(z_{j+1} - z_k\right)^{2-\alpha} - \left(z_{j+1} - z_{k+1}\right)^{2-\alpha}. \qquad (14)$$

Finally, we get approximation of first term of Eq. (1) as,

$$\begin{aligned}
\left[{}^*\partial_t^{\alpha} u(x,t)\right]_{(x_i t_{j+1})} &\approx \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(2-\alpha)} \left[\frac{p_0^j}{z_1 - z_0}\right] \left[w_1 u_1^i - w_0 u_0^i\right] + \sum_{k=1}^{j} \left(a_k^j u_{k-1}^i - b_k^j u_k^i + c_k^j u_{k+1}^i\right) \\
&= p_j u_1^i - q_j u_0^i + \sum_{k=1}^{j} \left(a_k^j u_{k-1}^i - b_k^j u_k^i + c_k^j u_{k+1}^i\right), \qquad (15)
\end{aligned}$$

where

$$p_j = \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(2-\alpha)} \left[\frac{p_0^j w_1}{z_1 - z_0}\right] \qquad (16)$$

$$q_j = \frac{\left[w\left(t_{j+1}\right)\right]^{-1}}{\Gamma(2-\alpha)} \left[\frac{p_0^j w_0}{z_1 - z_0}\right] \qquad (17)$$

For second order derivative of $u(x,t)$ in the spatial direction to appear in Eq. (1), we use approximation as

$$\left[\frac{\partial^2 u(x,t)}{\partial^2 x}\right]_{(x_i t_{j+1})} \approx \frac{u(x_{i+1},t_{j+1}) - 2u(x_i,t_{j+1}) - u(x_{i-1},t_{j+1})}{(\Delta x)^2} = \frac{u^{i+1}_{j+1} - 2u^i_{j+1} + u^{i-1}_{j+1}}{(\Delta x)^2}. \tag{18}$$

Hence, from Eqs. (15), (18) and (1), the scheme takes the form:

$$p_j u^i_1 - q_j u^i_0 + \sum_{k=1}^{j}\left(a^j_k u^i_{k-1} - b^j_k u^i_k + c^j_k u^i_{k+1}\right) = \delta\left[\frac{u^{i+1}_{j+1} - 2u^i_{j+1} + u^{i-1}_{j+1}}{(\Delta x)^2}\right] + f^i_{j+1}. \tag{19}$$

For simplicity point of view Eq. (19) can be think as,

$$\mu\left(u^{i+1}_{j+1} - 2u^i_{j+1} + u^{i-1}_{j+1}\right) = p_j u^i_1 - q_j u^i_0 + \sum_{k=1}^{j}\left(a^j_k u^i_{k-1} - b^j_k u^i_k + c^j_k u^i_{k+1}\right) - f^i_{j+1} \tag{20}$$

where

$$\mu = \frac{\delta}{(\Delta x)^2}, \tag{21}$$

$$\begin{aligned}\mu u^{i+1}_{j+1} - \left(2\mu + c^j_j\right)u^i_{j+1} + \mu u^{i-1}_{j+1} &= p_j u^i_1 - q_j u^i_0 + \left(a^j_j u^i_{j-1} - b^j_j u^i_j\right)\\ &\quad + \sum_{k=1}^{j-1}\left(a^j_k u^i_{k-1} - b^j_k u^i_k + c^j_k u^i_{k+1}\right) - f^i_{j+1},\end{aligned} \tag{22}$$

where $0 \le j \le M-1$ and $1 \le i \le N-1$.

Using, $L^i_j = -\left(2\mu + c^j_j\right)$, Eq. (19) takes the matrix form for computation of u^i_{j+1}, $j = 0, 1, 2, \ldots, M-1$,

$$K_{j+1}U_{j+1} = F_{j+1}, 0 \le j \le M-1, \tag{23}$$

where

$$K_{j+1} = \begin{pmatrix} L^1_j & \mu & & & \\ \mu & L^2_j & \mu & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & L^{N-2}_j & \mu \\ & & & \mu & L^{N-1}_j \end{pmatrix}, \tag{24}$$

$$U_{j+1} = \left[u^1_{j+1}, u^2_{j+1}, \ldots, u^i_{j+1}, \ldots, u^{N-1}_{j+1}\right]^T, \tag{25}$$

$$F_{j+1} = \left[F^1_{j+1}, F^2_{j+1}, \ldots, F^i_{j+1}, \ldots, F^{N-1}_{j+1}\right]^T, \tag{26}$$

and,

$$F^i_{j+1} = \begin{cases} -q_0 u^i_0 - f^i_1, & for\ j = 0 \\ p_j u^i_1 - q_j u^i_0 + \left(a^j_j u^i_{j-1} - b^j_j u^i_j\right) + \cdots & \\ + \sum_{k=1}^{j-1}\left(a^j_k u^i_{k-1} - b^j_k u^i_k + c^j_k u^i_{k+1}\right) - f^i_{j+1}, & for\ 1 \le j \le M-1 \end{cases}. \tag{27}$$

Order of Approximation for the Generalized Fractional Derivative

To analyze error of approximation for generalized time fractional derivative, we use notation for simplicity, let $g(s) = w(s)f(s)$ and $s = z^{-1}(v)$. Hence, we find $g(v) = w\big(z^{-1}(v)\big)f\big(z^{-1}(v)\big)$, $w(t_k) = w_k$ and $z(t_k) = z_k$. According to Newton interpolation method, we interpolate $g(v)$ on $[z_0, z_1]$ and $\big[z_{k-1}, z_{k+1}\big]$ by $p_g^1(v)$ and $p_g^2(v)$ respectively such that

$$g(v) - p_g^1(v) = \frac{g''(\xi_1)}{2!}(v - z_0)(v - z_1), \quad \xi_1 \in [z_0, z_1]. \tag{28}$$

$$g(v) - p_g^2(v) = \frac{g'''(\xi_2)}{3!}(v - z_{k-1})(v - z_k)(v - z_{k+1}), \quad \xi_2 \in \big[z_{k-1}, z_{k+1}\big], \tag{29}$$

where $p_g^1(v)$ and $p_g^2(v)$ are piecewise linear and quadratic interpolation polynomials using the nodes (z_0, g_0), (z_1, g_1) and (z_{k-1}, g_{k-1}) (z_k, g_k) (z_{k+1}, g_{k+1}) respectively.

Let E^{j+1} be the truncation error given by

$$E^{j+1} = \frac{\big[w_{j+1}\big]^{-1}}{\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{\big[g(v) - p_g^1(v)\big]'}{\big[z_{j+1} - v\big]^{\alpha}} dv + \frac{\big[w_{j+1}\big]^{-1}}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \int_{z_k}^{z_{k+1}} \frac{1}{\big[z_{j+1} - v\big]^{\alpha}} \big[g(v) - p_g^2(v)\big]' dv. \tag{30}$$

Consider first term of Eq. (30)

$$\frac{\big[w_{j+1}\big]^{-1}}{\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{1}{\big[z_{j+1} - v\big]^{\alpha}} \big[g(v) - p_g^1(v)\big]' dv = \frac{\big[w_{j+1}\big]^{-1}}{2\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{g''(\xi_1)}{\big[z_{j+1} - v\big]^{\alpha}} \left[(v - z_0)(v - z_1)\right]' dv. \tag{31}$$

Using integration by parts, we get

$$\begin{aligned} &\frac{\big[w_{j+1}\big]^{-1}}{2\Gamma(1-\alpha)} \int_{z_0}^{z_1} g''(\xi_1)\big(z_{j+1} - v\big)^{-\alpha-1}(v - z_0)(v - z_1)dv \\ &\le \frac{\big[w_{j+1}\big]^{-1}}{2\Gamma(1-\alpha)} \max_{t_0 \le \eta_1 \le t_{j+1}} g''(\eta_1)\big(z_{j+1} - z_1\big)^{-\alpha-1} \int_{z_0}^{z_1} (v - z_0)(z_1 - v)dv, \\ &\le \frac{-\alpha \max_{t_0 \le \eta_1 \le t_{j+1}} g''(\eta_1)\big(z_{j+1} - z_1\big)^{-\alpha-1}(z_1 - z_0)^3}{w_{j+1}\Gamma(1-\alpha)12}. \end{aligned} \tag{32}$$

Consider the second term of Eq. (30)

$$\frac{\big[w_{j+1}\big]^{-1}}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \int_{z_k}^{z_{k+1}} \frac{1}{\big[z_{j+1} - v\big]^{\alpha}} \Big[g(v) - p_g^2(v)\Big]' dv.$$

Since, $g(v) - p_g^2(v) = \frac{g'''(\xi_2)}{3!}(v - z_{k-1})(v - z_k)(v - z_{k+1})$ and applying integration by parts, we get

$$\frac{\alpha[w_{j+1}]^{-1}}{\Gamma(1-\alpha)} \sum_{k=1}^{j} \int_{z_k}^{z_{k+1}} \frac{g'''(\xi_2)}{3!}(v - z_{k-1})(v - z_k)(v - z_{k+1})\left[z\left(t_{j+1}\right) - v\right]^{-\alpha-1} dv.$$

$$= \frac{\alpha[w_{j+1}]^{-1}}{\Gamma(1-\alpha)} \sum_{k=1}^{j-1} \int_{z_k}^{z_{k+1}} \frac{g'''(\xi_2)}{3!}(v - z_{k-1})(v - z_k)(v - z_{k+1})\left[z\left(t_{j+1}\right) - v\right]^{-\alpha-1} dv$$

$$+ \frac{\alpha[w_{j+1}]^{-1}}{\Gamma(1-\alpha)} \int_{z_j}^{z_{j+1}} \frac{g'''(\xi_2)}{3!}\left(v - z_{j-1}\right)\left(v - z_j\right)\left(v - z_{j+1}\right)\left[z\left(t_{j+1}\right) - v\right]^{-\alpha-1} dv. \quad (33)$$

Consider the first part of Eq. (33)

$$\frac{\alpha[w_{j+1}]^{-1}}{\Gamma(1-\alpha)} \sum_{k=1}^{j-1} \int_{z_k}^{z_{k+1}} \frac{g'''(\xi_2)}{3!}(v - z_{k-1})(v - z_k)(v - z_{k+1})\left[z\left(t_{j+1}\right) - v\right]^{-\alpha-1} dv,$$

$$\leq \frac{\alpha[w_{j+1}]^{-1}}{\Gamma(1-\alpha)} \varphi(z_{k-1}, z_k, z_{k+1}) \sum_{k=1}^{j-1} \int_{z_k}^{z_{k+1}} \frac{g'''(\xi_2)}{3!}\left[z\left(t_{j+1}\right) - v\right]^{-\alpha-1} dv, \quad (34)$$

$$= \frac{\alpha\left[w_{j+1}\right]^{-1}}{\Gamma(1-\alpha)} \varphi\left(z_{k-1}, z_k, z_{k+1}\right) \frac{\max_{t_0 \leq \eta_2 \leq t_j} \left|g'''\left(\eta_2\right)\right|}{3!} \int_{z_1}^{z_j} \left[z\left(t_{j+1}\right) - v\right]^{-\alpha-1} dv \quad (35)$$

where

$$\varphi(z_{k-1}, z_k, z_{k+1}) = \frac{1}{27}\varphi_1(z_{k-1}, z_k, z_{k+1})\varphi_2(z_{k-1}, z_k, z_{k+1})\varphi_3(z_{k-1}, z_k, z_{k+1}). \quad (36)$$

$$\varphi_1(z_{k-1}, z_k, z_{k+1}) = (z_k + z_{k+1} - 2z_{k-1}) - \sigma(z_{k-1}, z_k, z_{k+1}). \quad (37)$$

$$\varphi_2(z_{k-1}, z_k, z_{k+1}) = (z_{k+1} + z_{k-1} - 2z_k) - \sigma(z_{k-1}, z_k, z_{k+1}). \quad (38)$$

$$\varphi_3(z_{k-1}, z_k, z_{k+1}) = (z_{k-1} + z_k - 2z_{k+1}) - \sigma(z_{k-1}, z_k, z_{k+1}). \quad (39)$$

$$\sigma(z_{k-1}, z_k, z_{k+1}) = \sqrt{z_{k-1}(z_{k-1} - z_k) + z_k(z_k - z_{k+1}) + z_{k+1}(z_{k+1} - z_{k-1})} \quad (40)$$

Again,

$$\int_{z_1}^{z_j} \left[z\left(t_{j+1}\right) - v\right]^{-\alpha-1} dv = \frac{1}{\alpha}\left[\left(z_{j+1} - z_j\right)^{-\alpha} - \left(z_{j+1} - z_1\right)^{-\alpha}\right] \leq \left(z_{j+1} - z_j\right)^{-\alpha}. \quad (41)$$

Hence, from Eq. (35) and (41), we have first term of Eq. (33) is less than or equal to

$$\frac{\alpha[w_{j+1}]^{-1}}{\Gamma(1-\alpha)} \varphi(z_{k-1}, z_k, z_{k+1}) \frac{\max_{t_0 \leq \eta_2 \leq t_j} \left|g'''(\eta_2)\right|}{3!} \left(z_{j+1} - z_j\right)^{-\alpha}. \quad (42)$$

Consider the second term of Eq. (33)

$$\frac{\alpha\left[w_{j+1}\right]^{-1}}{\Gamma(1-\alpha)}\int\limits_{z_j}^{z_{j+1}}\frac{g'''(\xi_2)}{3!}\left(v-z_{j-1}\right)\left(v-z_j\right)\left(v-z_{j+1}\right)\left[z\left(t_{j+1}\right)-v\right]^{-\alpha-1}dv.$$

$$=-\frac{\alpha\left[w_{j+1}\right]^{-1}}{3!\,\Gamma(1-\alpha)}\int\limits_{z_j}^{z_{j+1}}\frac{g'''(\xi_2)}{1}\left(v-z_{j-1}\right)\left(v-z_j\right)\left[z\left(t_{j+1}\right)-v\right]^{-\alpha}dv. \tag{43}$$

$$=-\frac{\alpha\left[w_{j+1}\right]^{-1}\max_{t_0\le\eta_2\le t_j}\left|g'''\left(\eta_2\right)\right|\left(z_{j+1}-z_j\right)^{(2-\alpha)}}{3!\,\Gamma\left(1-\alpha\right)}\times\left[\frac{\left(z_j-z_{j-1}\right)}{\left(1-\alpha\right)\left(2-\alpha\right)}+\frac{2\left(z_{j+1}-z_j\right)}{\left(1-\alpha\right)\left(2-\alpha\right)\left(3-\alpha\right)}\right]. \tag{44}$$

$$=-\frac{\alpha\left[w_{j+1}\right]^{-1}}{3\Gamma(1-\alpha)}\frac{\max_{t_0\le\eta_2\le t_j}\left|g'''(\eta_2)\right|\left(z_{j+1}-z_j\right)^{(2-\alpha)}}{(1-\alpha)(2-\alpha)}\left[\frac{\left(z_j-z_{j-1}\right)}{2}+\frac{\left(z_{j+1}-z_j\right)}{(3-\alpha)}\right]. \tag{45}$$

From Eqs. (32), (42) and (45), we have

$$\begin{aligned}E^{j+1}\le&\frac{-\alpha\max_{t_0\le\eta_1\le t_{j+1}}g''(\eta_1)\left(z_{j+1}-z_1\right)^{-\alpha-1}(z_1-z_0)^3}{w_{j+1}\Gamma(1-\alpha)12}\\&+\frac{\alpha\left[w_{j+1}\right]^{-1}}{\Gamma(1-\alpha)}\varphi(z_{k-1},z_k,z_{k+1})\frac{\max_{t_0\le\eta_2\le t_j}\left|g'''(\eta_2)\right|}{3!}\left(z_{j+1}-z_j\right)^{-\alpha}\\&-\frac{\alpha\left[w_{j+1}\right]^{-1}}{3\Gamma(1-\alpha)}\frac{\max_{t_0\le\eta_2\le t_j}\left|g'''(\eta_2)\right|\left(z_{j+1}-z_j\right)^{(2-\alpha)}}{(1-\alpha)(2-\alpha)}\left[\frac{\left(z_j-z_{j-1}\right)}{2}+\frac{\left(z_{j+1}-z_j\right)}{(3-\alpha)}\right].\end{aligned} \tag{46}$$

Lemma 1 [45] *If scale function $z(t)$ is satisfying the Lipschitz condition with constant L that is $\left(z_{j+1}-z_j\right)\le L\Delta t$ then we have following relation.*

(i) $\sigma(z_{k-1},z_k,z_{k+1})\le\sqrt{3}L\Delta t$.

(ii) $\varphi_1(z_{k-1},z_k,z_{k+1})\le L\Delta t\left(3-\sqrt{3}\right)$.

(iii) $\varphi_2(z_{k-1},z_k,z_{k+1})\le -L\Delta t\left(\sqrt{3}\right)$.

(iv) $\varphi_3(z_{k-1},z_k,z_{k+1})\le L\Delta t\left(-3-\sqrt{3}\right)$.

(v) $\varphi(z_{k-1},z_k,z_{k+1})\le\left(\frac{2\sqrt{3}}{9}\right)L(\Delta t)^3\le\frac{1}{2}L(\Delta t)^3$.

It is clear from above Lemma 1 and Eq. (46), the truncation error E^{j+1} achieves the order of convergence $(\Delta t)^{3-\alpha}$. If the scale function is satisfying the Lipschitz condition with constant L then the truncation error E^{j+1} has the form

$$\begin{aligned}E^{j+1}\le&\frac{-\alpha\max_{t_0\le\eta_1\le t_{j+1}}g''(\eta_1)\left(t_{j+1}-t_1\right)^{-\alpha-1}L^{-\alpha}(\Delta t)^3}{w_{j+1}\Gamma(1-\alpha)12}+\frac{\alpha\left[w_{j+1}\right]^{-1}}{\Gamma(1-\alpha)}\frac{1}{12}L^{3-\alpha}\max_{t_0\le\eta_2\le t_j}\left|g'''(\eta_2)\right|(\Delta t)^{3-\alpha}\\&-\frac{\alpha\left[w_{j+1}\right]^{-1}}{3\Gamma(1-\alpha)}\frac{\max_{t_0\le\xi_2\le t_j}\left|g'''(\eta_2)\right|}{(1-\alpha)(2-\alpha)}\left[\frac{1}{2}+\frac{2}{(3-\alpha)}\right]L^{3-\alpha}(\Delta t)^{3-\alpha}.\end{aligned} \tag{47}$$

Lemma 2 *Suppose $z(t)$ is strictly increasing with $z(t) \geq 0$ and $w(t)$ is increasing with $w(t) > 0$. Then the following conditions hold for $\alpha \in (0, 1)$,*

(i) $$A_k^j = \alpha\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] - (2-\alpha)\Big\{\big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha} - \big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha}\Big\},$$

(ii) $$A_k^j = 2\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] - (2-\alpha)(z_{k+1} - z_k)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} + \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big],$$

(iii) $A_k^j > 0$,

(iv) $C_k^j > B_k^j > A_k^j > 0$,

(v) a_k^j,b_k^j *and* c_k^j *are positive,*

where

$$a_k^j = \frac{w_{k-1}}{(z_k - z_{k-1})(z_{k+1} - z_{k-1})} \frac{\big[w\big(t_{j+1}\big)\big]^{-1}}{\Gamma(3-\alpha)} A_k^j \tag{48}$$

$$b_k^j = \frac{w_k}{(z_k - z_{k-1})(z_{k+1} - z_k)} \frac{\big[w\big(t_{j+1}\big)\big]^{-1}}{\Gamma(1-\alpha)} B_k^j \tag{49}$$

$$c_k^j = \frac{w_{k+1}}{(z_{k+1} - z_{k-1})(z_{k+1} - z_k)} \frac{\big[w\big(t_{j+1}\big)\big]^{-1}}{\Gamma(3-\alpha)} C_k^j, \tag{50}$$

$$A_k^j = (2-\alpha)\big(2z_{j+1} - z_k - z_{k+1}\big) p_k^j - 2(1-\alpha) q_k^j, \tag{51}$$

$$B_k^j = (2-\alpha)\big(2z_{j+1} - z_{k-1} - z_{k+1}\big) p_k^j - 2(1-\alpha) q_k^j, \tag{52}$$

$$C_k^j = (2-\alpha)\big(2z_{j+1} - z_{k-1} - z_k\big) p_k^j - 2(1-\alpha) q_k^j, \tag{53}$$

Proof (i) Since $A_k^j = \Big[(2-\alpha)\big(2z_{j+1} - z_k - z_{k+1}\big) p_k^j - 2(1-\alpha) q_k^j\Big]$.

From the Eqs. (13) and (14), we have

$$\begin{aligned}
A_k^j &= (2-\alpha)\big(2z_{j+1} - z_k - z_{k+1}\big)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big] \\
&\quad - 2(1-\alpha)\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] \\
&= (2-\alpha)\big(z_{j+1} - z_k + z_{j+1} - z_{k+1}\big)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big] \\
&\quad - 2\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] + 2\alpha\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] \\
A_k^j &= (2-\alpha)\big(2z_{j+1} - z_k - z_{k+1}\big)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big] \\
&\quad - 2(1-\alpha)\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] \\
&= (2-\alpha)\big(z_{j+1} - z_k + z_{j+1} - z_{k+1}\big)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big] \\
&\quad - 2\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] + 2\alpha\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] \\
&= (2-\alpha)\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] \\
&\quad (2-\alpha)\Big[\big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha} - \big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big]
\end{aligned}$$

$$- 2\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big] + 2\alpha\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big]$$
$$= \alpha\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big]$$
$$- (2-\alpha)\Big[\big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha} - \big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha}\Big] \tag{54}$$

(ii) Since $2\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big]$

$$-(2-\alpha)(z_{k+1} - z_k)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} + \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big] \tag{55}$$
$$= 2\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big]$$
$$- (2-\alpha)\big(z_{j+1} - z_{j+1} + z_{k+1} - z_k\big)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} + \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big]$$
$$= 2\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big]$$
$$- (2-\alpha)\big(z_{j+1} - z_k\big)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} + \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big]$$
$$+ (2-\alpha)\big(z_{j+1} - z_{k+1}\big)\Big[\big(z_{j+1} - z_k\big)^{1-\alpha} + \big(z_{j+1} - z_{k+1}\big)^{1-\alpha}\Big]$$
$$= 2\big(z_{j+1} - z_k\big)^{2-\alpha} - 2\big(z_{j+1} - z_{k+1}\big)^{2-\alpha}$$
$$- (2-\alpha)\big(z_{j+1} - z_k\big)^{2-\alpha} - (2-\alpha)\big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha}$$
$$+ (2-\alpha)\big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha} + (2-\alpha)\big(z_{j+1} - z_{k+1}\big)^{2-\alpha}$$
$$= \alpha\big(z_{j+1} - z_k\big)^{2-\alpha} - \alpha\big(z_{j+1} - z_{k+1}\big)^{2-\alpha} - (2-\alpha)\big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha}$$
$$+ (2-\alpha)\big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha}$$
$$= \alpha\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big]$$
$$- (2-\alpha)\big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha} + (2-\alpha)\big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha}$$
$$= \alpha\Big[\big(z_{j+1} - z_k\big)^{2-\alpha} - \big(z_{j+1} - z_{k+1}\big)^{2-\alpha}\Big]$$
$$- (2-\alpha)\Big\{\big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha} - \big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha}\Big\}$$
$$? = A?_k^j \tag{56}$$

(iii) Since $\big(z_{j+1} - z_{k+1}\big) < \big(z_{j+1} - z_k\big)$

$$\Rightarrow \big(z_{j+1} - z_{k+1}\big)^{1-\alpha} < \big(z_{j+1} - z_k\big)^{1-\alpha}$$
$$\Rightarrow \big(2z_{j+1} - z_{k+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha} < \big(2z_{j+1} - z_{k+1} - z_k\big)\big(z_{j+1} - z_k\big)^{1-\alpha}$$
$$\Rightarrow \big(z_{j+1} - z_{k+1} + z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha} < \big(z_{j+1} - z_{k+1} + z_{j+1} - z_k\big)\big(z_{j+1} - z_k\big)^{1-\alpha}$$
$$\Rightarrow \big(z_{j+1} - z_{k+1}\big)^{2-\alpha} + \big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha}$$
$$< \big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha} + \big(z_{j+1} - z_k\big)^{2-\alpha}$$
$$\Rightarrow \big(z_{j+1} - z_{k+1}\big)^{2-\alpha} - \big(z_{j+1} - z_k\big)^{2-\alpha}$$
$$< \big(z_{j+1} - z_{k+1}\big)\big(z_{j+1} - z_k\big)^{1-\alpha} - \big(z_{j+1} - z_k\big)\big(z_{j+1} - z_{k+1}\big)^{1-\alpha} \tag{57}$$
$$\Rightarrow \alpha\Big\{\big(z_{j+1} - z_{k+1}\big)^{2-\alpha} - \big(z_{j+1} - z_k\big)^{2-\alpha}\Big\}$$

$$< (2-\alpha)\Big\{\big(z_{j+1}-z_{k+1}\big)\big(z_{j+1}-z_k\big)^{1-\alpha}-\big(z_{j+1}-z_k\big)\big(z_{j+1}-z_{k+1}\big)^{1-\alpha}\Big\} \tag{58}$$

$$\begin{aligned}\Rightarrow \alpha &\left\{\left(z_{j+1}-z_k\right)^{2-\alpha}-\left(z_{j+1}-z_{k+1}\right)^{2-\alpha}\right\}\\ &-(2-\alpha)\left\{\left(z_{j+1}-z_k\right)\left(z_{j+1}-z_{k+1}\right)^{1-\alpha}-\left(z_{j+1}-z_{k+1}\right)\left(z_{j+1}-z_k\right)^{1-\alpha}\right\}>0 \\ &\Rightarrow A_k^j>0\end{aligned} \tag{59}$$

(iv) Since

$$\begin{aligned}B_k^j-A_k^j &= (2-\alpha)\big(2z_{j+1}-z_{k-1}-z_{k+1}\big)p_k^j-2(1-\alpha)q_k^j\\ &\quad-(2-\alpha)\big(2z_{j+1}-z_k-z_{k+1}\big)p_k^j-2(1-\alpha)q_k^j\\ &=(2-\alpha)(z_k-z_{k-1})p_k^j>0\end{aligned} \tag{60}$$

Hence, from Eq. (60), $B_k^j > A_k^j$.
Similarly,

$$C_k^j - B_k^j = (2-\alpha)(z_{k+1}-z_k)p_k^j > 0. \tag{61}$$

Hence, from Eq. (61), $C_k^j > B_k^j$ and finally we get $C_k^j > B_k^j > A_k^j > 0$.

(v) According to assumption the weight function $w(t)$ is positive and the scale function $z(t)$ is nonnegative and strictly monotone increasing. Hence, from Eqs. (48), (49) and (50) that a_k^j,b_k^j and c_k^j are positive.

Stability and Convergence Analysis

To establish the convergence analysis through stability of the numerical scheme given in Eq. (22), the Lax–Richertmyer theorem [46] is used.

For the stability of the numerical scheme (22), we rewrite the Eq. (23) as:

$$K_{j+1}U_{j+1} = \left(c_{j-1}^j - b_j^j\right).U_j + V_j, \quad 1 \le j \le M-1, \tag{62}$$

where

$$U_j = \left[u_j^1, u_j^2, \ldots, u_j^i, \ldots, u_j^{N-1}\right]^T, \tag{63}$$

$$V_j = \left[v_j^1, v_j^2, \ldots, v_j^i, \ldots, v_j^{N-1}\right]^T, \tag{64}$$

and

$$v_j^i = \left(p_j u_1^i - q_j u_0^i\right) + a_{j-1}^j u_{j-2}^i + \left(a_j^j - b_{j-1}^j\right)u_{j-1}^i + \sum_{k=1}^{j-2}\left(a_k^j u_{k-1}^i - b_k^j u_k^i + c_k^j u_{k+1}^i\right) - f_{j+1}^i \tag{65}$$

For the stability of numerical scheme given in Eq. (23), we have the following stability theorem.

Theorem 1 *If coefficient c_j^j in obtained matrix K_{j+1} satisfies the monotonic increasing property, then the numerical scheme given in Eq. (23) is convergent as well as stable.*

Proof Since the coefficient matrix K_{j+1} is tridiagonal and strictly diagonally dominant hence it is invertible. Therefore Eq. (23) is solvable and iteration scheme is well-posed.

Now, we wish to show the boundedness of posteriori error. Suppose u_j represents the exact solution at $t = t_j$ and ϵ_j denotes the posteriori error that is $\epsilon_j = u_j - U_j$. Then, we have

$$\epsilon_{j+1} = K_{J+1}^{-1}\left(c_{j-1}^{j} - b_j^j\right)\epsilon_j + O\left((\Delta t)^{3-\alpha} + \Delta x^2\right), \quad \text{for } 1 \leq j \leq M-1. \tag{66}$$

By ([47], Theorem A], suppose $K_{j+1} = \left[k_{ni,nj}\right]_{(N-1)\times(N-1)}$, then

$$K_{J+1}^{-1} \leq \frac{1}{\min\limits_{1\leq ni\leq N-1}\left\{\left|k_{ni,nj}\right| - \sum_{nj=1,nj\neq ni}^{N-1}\left|k_{ni,nj}\right|\right\}} = \frac{1}{c_j^j}.$$

According to assumption given in Eq. (66), we have

$$K_{J+1}^{-1}\left(c_{j-1}^{j} - b_j^j\right) \leq 1 \tag{67}$$

Hence, Eqs. (66) and (67) suggest the existence of a positive constant C, such that

$$\begin{aligned}\epsilon_{j+1} &\leq K_{J+1}^{-1}\left|c_{j-1}^{j} - b_j^j\right| \cdot \epsilon_j + C\left(\Delta t^{3-\alpha} + \Delta x^2\right)\\ &\leq \epsilon_j + C\left(\Delta t^{3-\alpha} + \Delta x^2\right),\end{aligned}$$

which implies

$$\epsilon_{j+1} \leq \left(E_j + E_j^2 + \cdots + E_j^j\right)C\left(\Delta t^{3-\alpha} + \Delta x^2\right),$$

where $E_j = K_{J+1}^{-1}\left(c_{j-1}^{j} - b_j^j\right)$ is an amplification matrix. By using stability condition $E_j \leq 1$ given in Eq. (67) we have

$$\epsilon_{j+1} \leq jC\left(\Delta t^{3-\alpha} + \Delta x^2\right), \quad 1 \leq j \leq M-1,$$

which shows that the error is bounded, and $\epsilon_{j+1} \to 0$ as $\Delta t \to 0$, $\Delta x \to 0$. Hence, the numerical scheme given in Eq. (23) is stable and consistent as well. Hence, by the Lax–Richertmyer theorem [46], the convergence of numerical scheme is established.

Numerical Results and Their Analysis

In the present Section, the validity and numerical ability of our high order proposed finite difference numerical scheme for GFDEs are demonstrated via five examples. And also, the numerical results are compared with the known results in literature. In the first and fourth example, we solve with non-zero initial condition. Although, the other examples are solved by zero initial condition. Fifth example is solved by non-zero boundary condition while others are solved by zero boundary condition. The GFD is defined using the scale and weight function so we also present solution of example 4 with non-zero weight function. Here, we present the numerical results, exact results through figures and tables. The maximum absolute error (MAE) and convergence order (CO) are also presented through tables. Spatial as well as temporal direction errors of the numerical example are also presented through tables.

To check the CO of approximation of GFD, we take $u(x,t) = t - t^3$. It is also clear from Tables 1 and 2 that the CO in the spatial direction is $(\Delta t)^{3-\alpha}$. The approximation results are

Table 1 MAE and CO in approximation of GFD for $u(x,t)=t-t^3$ with $z(t)=t^2, \alpha=0.5$, at $=0.6$ t

Δt	MAE	CO
1/10	6.84003×10^{-3}	
1/20	1.04274×10^{-3}	2.71362
1/40	1.70108×10^{-4}	2.61586
1/80	2.85787×10^{-5}	2.57344
1/160	4.87585×10^{-6}	2.55121

Table 2 MAE and CO in approximation of GFD for $u(x,t)=t-t^3$ with $z(t)=t^2, \alpha=0.2$, at $t=0.6$

	MAE	CO
1/10	1.48829×10^{-3}	
1/20	2.07611×10^{-4}	2.8417
1/40	2.99599×10^{-5}	2.79278
1/80	4.35344×10^{-6}	2.78281
1/160	6.31831×10^{-7}	2.78455

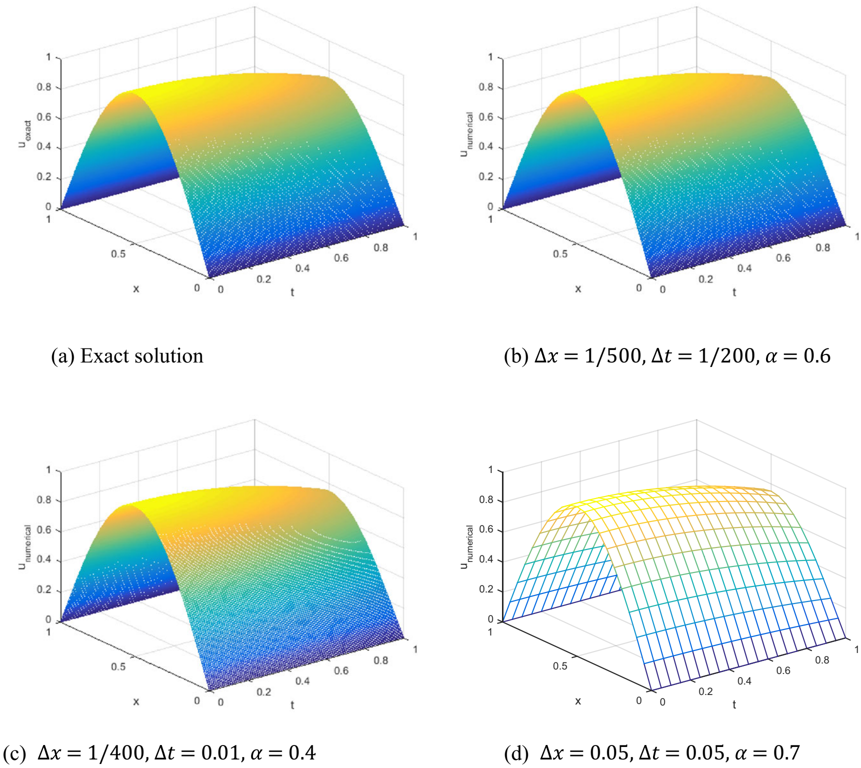
obtained for weight function $w(t)=1$ and it validates the theoretical results presented earlier. To calculate the MAE, and CO, we use the formula, CO = lg[MAE(Δt)/MAE($\Delta t/2$)]/lg(2), where MAE is obtained using maximum of absolute value of difference of exact solution and numerical solution at the node point.

Example 1 Consider GFDEs [30] defined by Eq. (1) as,

$$^*\partial_t^\alpha u(x,t)-\frac{\partial^2 u(x,t)}{\partial^2 x}=\frac{2x(x-1)t^{2-\alpha}}{\Gamma(3-\alpha)}-2t^2+\pi^2\sin(\pi x), x\in[0,1], t>0, \quad (68)$$

with the initial and boundary conditions as, $u(x,0)=\sin(\pi x), u(0,t)=u(1,t)=0$. We take $z(t)=t, w(t)=1$ and $0\le t\le 1$.

The exact solution of the Eq. (1) under above condition is given by $u(x,t)=x(x-1)t^2+\sin(\pi x)$. We solved Example 1 with finite difference scheme proposed in "Computational algorithm of problem" section for different $\alpha=0.6, 0.4$ and 0.7, and different step sizes $\Delta x=1/500,\ \Delta t=1/200$; $\Delta x=1/400,\ \Delta t=0.01$, and $\Delta x=0.05,\ \Delta t=0.05$. The numerical results are shown in the Fig. 1a–d. Figure 1a displays for exact solution while Fig. 1b–d present for numerical solutions. We also solved Example 1 with (1) varying both Δx and Δt with $\alpha=0.85$; (2) varying both Δx and Δt with $\alpha=0.6$; (3) varying Δx and fixed $\Delta t=1/600$ with $\alpha=0.85$ and (4) varying Δt and fixed $\Delta x=1/512$ with $\alpha=0.85$ which are shown through Tables 3, 4, 5 and 6 respectively. It is clear from the Tables 3 and 4 that the finite difference scheme is of second order of convergence. Table 5 also shows the numerical scheme having second order of convergence in spatial direction. Figure 1 shows that whenever we reduce the step size in spatial or temporal or both directions, the numerical results converge to the exact solution which validates the scheme is numerically convergent. Table 6 shows that MAE and CO for $\alpha=0.85$ and compares the results with reference [30] for Example 1. It is clear from the Tables 5 and 6 our proposed numerical scheme is of high order of convergence and achieve the good accuracy. The numerical scheme proposed in [30] is $O\left(\Delta x^2+\Delta t\right)$ while our numerical finite difference scheme of order $O\left(\Delta x^2+\Delta t^{3-\alpha}\right)$.



(a) Exact solution

(b) $\Delta x = 1/500$, $\Delta t = 1/200$, $\alpha = 0.6$

(c) $\Delta x = 1/400$, $\Delta t = 0.01$, $\alpha = 0.4$

(d) $\Delta x = 0.05$, $\Delta t = 0.05$, $\alpha = 0.7$

Fig. 1 Comparison of solutions for Example 1 with different parameters

Table 3 MAE and CO for Example 1 with $\alpha = 0.85$, $\delta = 1$ $z(t) = t$, $w(t) = 1$

Δt	Δx	MAE	CO
1/8	1/8	0.012682479250020	–
1/16	1/16	0.003154209795232	2.0075
1/32	1/32	0.000787641393682	2.0017
1/64	1/64	0.000196866793636	2.0003
1/128	1/128	0.000049216024556	2.0000

Table 4 MAE and CO for Example 1 with $\alpha = 0.6$, $\delta = 1$ $z(t) = t$, $w(t) = 1$

Δt	Δx	MAE	CO
1/8	1/8	0.012307702788089	–
1/16	1/16	0.003063419371486	2.0063
1/32	1/32	0.000765192639171	2.0012
1/64	1/64	0.000191279519140	2.0001
1/128	1/128	0.000047821744845	1.9999

Table 5 MAE and CO for Example 1 with $\alpha = 0.85$, $\delta = 1$ $z(t) = t$, $w(t) = 1$, and $\Delta t = 1/600$

Δx	MAE [30]	CO [30]	MAE	CO
1/8	0.05290053759936	–	0.012697453071990	–
1/16	0.01365644886944	1.9537	0.003156727061421	2.0080
1/32	0.00384054346854	1.8302	0.000788084897559	2.0020
1/64	0.00108689342074	1.8211	0.000196952333293	2.0005

Table 6 MAE and CO for Example 1 for $\alpha = 0.85$ with different Δt and fixing $\Delta x = 1/512$

Δt	MAE [30]	CO [30]	MAE	CO
1/8	0.06935363245950	–	0.001149644902625	–
1/16	0.03256819563577	1.0905	0.000393504352298	1.5467
1/32	0.01578117824480	1.0453	0.000123282907375	1.6744
1/64	0.00777362866697	1.0215	0.000035575711316	1.7930

Example 2 Consider GFDEs defined in Eq. (1) as,

$$^{*}\partial_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial^2 x} = \frac{2x(x-1)t^{2-\alpha}}{\Gamma(3-\alpha)} - 2t^2, x \in [0,1], t > 0, \tag{31}$$

with the initial and boundary conditions as $u(x, 0) = 0, u(0, t) = u(1, t) = 0$. We take $z(t) = t$, $w(t) = 1$ and $0 \leq t \leq 1$.

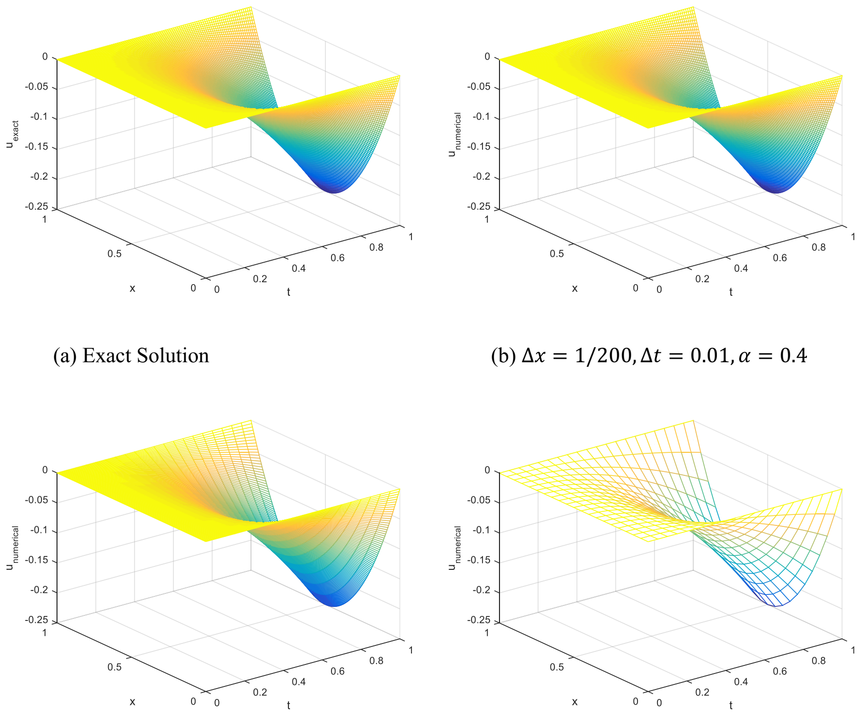
The exact solution of the Eq. (1) under above conditions is given by $u(x, t) = x(x-1)t^2$. We solved Example 2 with finite difference scheme proposed in "Computational algorithm of problem" section for different $\alpha = 0.4, 0.5$ and 0.8 and different step sizes $\Delta x = 1/200$, $\Delta t = 0.01$; $\Delta x = 1/200$, $\Delta t = 0.05$ and $\Delta x = 0.05$, $\Delta t = 0.05$. The numerical results are shown in the Fig. 2a–d. The Fig. 2a display for the exact solution while Fig. 2b–d present for numerical solution. We also solved Example 2 with (1) varying both Δx and Δt with $\alpha = 0.8$; and (2) varying Δt and fixed $\Delta x = 1/5000$ with $\alpha = 0.5$ which are shown through Tables 7 and 8 respectively. It is clear from the Table 7 that our finite difference of second order of convergence. Table 8 also shows that the numerical scheme having $3 - \alpha$ order of convergence in temporal direction. It is clear from the Tables 7 and 8 our proposed numerical scheme is high order of convergence and achieves the good accuracy.

Example 3 Consider GFDEs defined in Eq. (1) as,

$$^{*}\partial_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial^2 x} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \sin(\pi x) + \pi^2 t^2 \sin(\pi x), x \in [0,1], t > 0, \tag{34}$$

with the initial and boundary conditions as $u(x, 0) = 0, u(0, t) = u(1, t) = 0$.

The exact solution of the Eq. (1) under above conditions is given by $u(x, t) = t^2 \sin(\pi x)$. We solved Example 3 with finite difference scheme proposed in "Computational algorithm of problem" section with $z(t) = t$, $w(t) = 1$ for different $\alpha = 0.6$ and 0.8 and different step sizes $\Delta x = 0.001$, $\Delta t = 0.01$; $\Delta x = 0.01$, $\Delta t = 0.02$ and $\Delta x = 0.01$, $\Delta t = 0.025$.



(a) Exact Solution

(b) $\Delta x = 1/200, \Delta t = 0.01, \alpha = 0.4$

(c) $\Delta x = 1/200, \Delta t = 0.05, \alpha = 0.5$

(d) $\Delta x = 0.05, \Delta t = 0.05, \alpha = 0.8$

Fig. 2 Comparison of solutions of Example 2 with different parameters

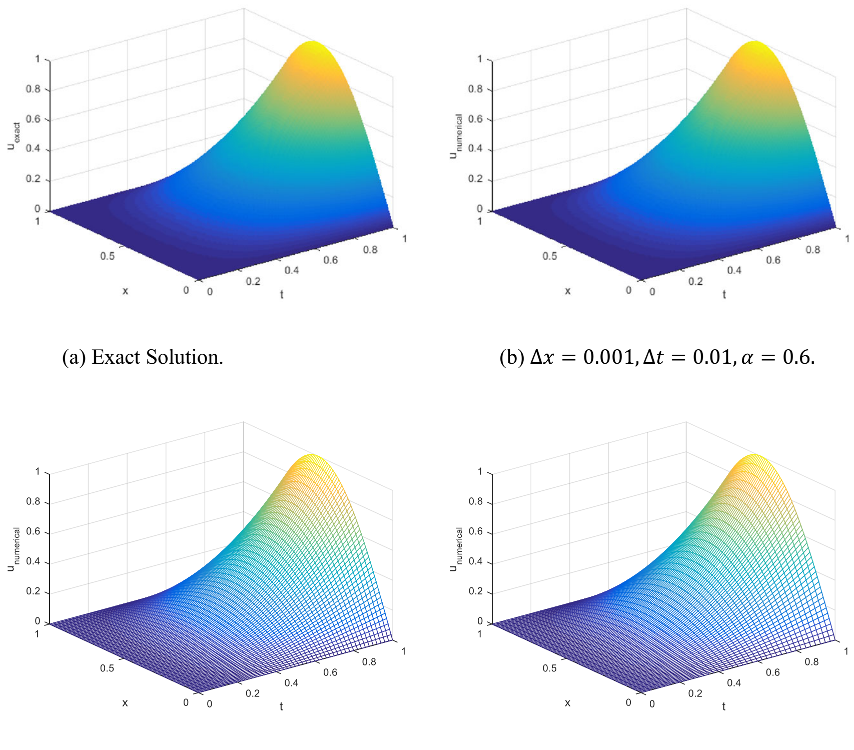
Table 7 MAE and CO for Example 2 with $\alpha = 0.8$, $\delta = 1$ $z(t) = t$, $w(t) = 1$

Δt	Δx	MAE	CO
1/10	1/10	7.0493e−04	–
1/20	1/20	2.3408e−04	1.5905
1/40	1/40	7.2255e−05	1.6958
1/80	1/80	2.0862e−05	1.7922
1/160	1/160	5.9576e−06	1.8081
1/320	1/320	1.6449e−06	1.8567
1/640	1/640	4.3554e−07	1.9172

The numerical results are shown in the Fig. 3a–d. Figure 3a displays the exact solution while Fig. 3b–d present for numerical solutions. Figure 3 shows that whenever we reduce the step size in spatial or temporal or both directions, the numerical results converges to exact solution which validates that the scheme is numerically convergent. We also solved Example 3 with (1) varying both Δx and Δt with $\alpha = 0.4$; and (2) varying Δx and fixed $\Delta t = 1/600$ with

Table 8 MAE and CO for Example 2 with $\alpha = 0.5, \Delta x = \frac{1}{5000}, \delta = 1$ for Example 2 for $T = 1$

Δt	MAE	CO
1/10	2.6296e−06	–
1/20	3.8045e−07	2.7891
1/40	5.8002e−08	2.7135
1/80	9.1278e−09	2.6678
1/160	1.4762e−09	2.6284
1/320	2.7062e−10	2.4475



(a) Exact Solution.

(b) $\Delta x = 0.001, \Delta t = 0.01, \alpha = 0.6$.

(c) $\Delta x = 0.01, \Delta t = 0.02, \alpha = 0.6$.

(d) $\Delta x = 0.01, \Delta t = 0.025,\ \alpha = 0.8$.

Fig. 3 Comparison of solutions of Example 3 with different parameters

$\alpha = 0.4$ which are shown through Tables 9 and 10 respectively. It is clear from the Tables 9 and 10 that our finite difference is of second order of convergence. Table 10 shows that the numerical scheme having second order of convergence in spatial direction. Tables 9 and 10 are presented for the MAE and CO with $\alpha = 0.4$. It is clear from the Tables 9 and 10 our proposed numerical scheme is high order of convergence and achieve the good accuracy. It is clear from this example that numerical results conclude the theoretical results.

Table 9 MAE and CO for Example 3 with $\alpha = 0.4, \delta = 1$ $z(t) = t, w(t) = 1$

Δt	Δx	MAE	CO
1/10	1/10	7.2313e−03	–
1/20	1/20	1.8017e−03	2.0049
1/40	1/40	4.4997e−04	2.0014
1/80	1/80	1.1246e−04	2.0005
1/160	1/160	2.8111e−05	2.0002
1/320	1/320	7.0274e−06	2.0001

Table 10 MAE and CO for Example 3 with $\alpha = 0.4, \delta = 1$ $z(t) = t, w(t) = 1$ $\Delta t = 1/600$

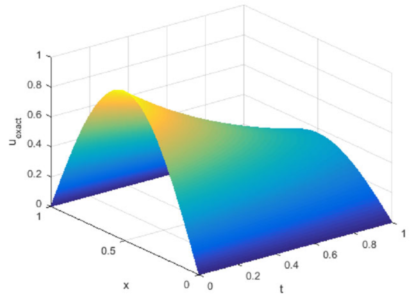
Δx	MAE	CO
1/10	7.2237e−03	–
1/20	1.8006e−03	2.0042
1/40	4.4983e−04	2.0011
1/80	1.1244e−04	2.0003
1/160	2.8108e−05	2.0001
1/320	7.0270e−06	2.0000

Example 4 Consider GFDEs defined in Eq. (1) as [33],

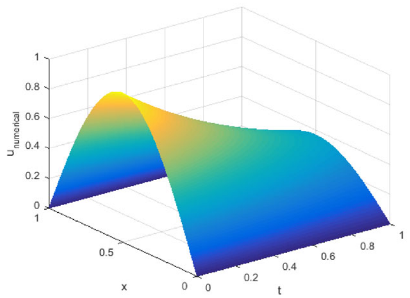
$$
\begin{aligned}
{}^{*}\partial_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial^2 x} &= \frac{1}{\Gamma(1-\alpha)} \sin(\pi x) \exp(-t) \left(\Gamma(1-\alpha) - \gamma(1-\alpha, t)\right) \\
&\quad + \pi^2 \sin(\pi x) \exp(-t),\ x \in [0,1],\ t > 0,
\end{aligned} \tag{35}
$$

with the initial and boundary conditions as $u(x,0) = \sin(\pi x)$, $u(0,t) = u(1,t) = 0$. Where $\gamma(s,t)$ denotes the incomplete Gamma function. We take $z(t) = t$, $w(t) = \exp(2t)$ and $0 \leq t \leq 1$.

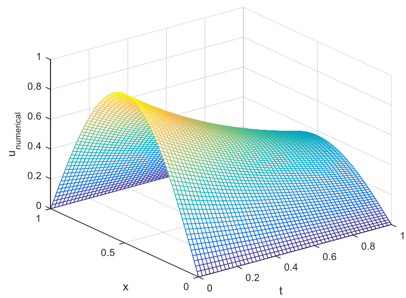
The exact solution of the Eq. (1) under above condition is given by $u(x,t) = \sin(\pi x)\exp(-t)$. We solved Example 4 with finite difference scheme proposed in "Computational algorithm of problem" section for $\alpha = 0.4$ and different step sizes $\Delta x = \Delta t = 0.001$ and $\Delta x = 0.01,\ \ \Delta t = 0.02$. The numerical results are shown in the Fig. 4a–c. The exact solution is displayed through Fig. 4a while Fig. 4b and c show the numerical solutions. We also solved Example 4 with (1) varying both Δx and Δt with $\alpha = 0.6$; (2) varying Δx and fixed $\Delta t = 1/600$ with $\alpha = 0.8$ and (3) varying Δt and fixed $\Delta x = 1/10000$ with $\alpha = 0.6$ which are shown through Tables 11, 12 and 13 respectively. It is clear from the Tables 11, 12 and 13 that our finite difference of second order of convergence. Tables 12 and 13 is also shows that the numerical scheme having second order of convergence in spatial direction and $3 - \alpha$ order of convergence in temporal direction respectively. Tables 11, 12 and 13 are presented for the MAE and CO with $\alpha = 0.6$ and 0.8. It is clear from the Tables 11, 12 and 13 our proposed numerical scheme is high order of convergence and achieves good accuracy in approximating numerical solutions.



(a) Exact Solution.



(b) $\Delta x = \Delta t = 0.001, \alpha = 0.4$.



(c) $\Delta x = 0.01, \Delta t = 0.02, \alpha = 0.4$.

Fig. 4 Comparison of solutions of Example 4 with different parameters

Table 11 MAE and CO for Example 4 with $\alpha = 0.6, \delta = 1$ $z(t) = t, w(t) = \exp(2t)$

Δt	Δx	MAE	CO
1/10	1/10	5.9274e−03	–
1/20	1/20	1.5408e−03	1.9437
1/40	1/40	3.7698e−04	2.0311
1/80	1/80	9.3308e−05	2.0144
1/160	1/160	2.3282e−05	2.0028
1/320	1/320	5.8117e−06	2.0022

Example 5 Consider GFDEs [28] defined in Eq. (1) as [28],

$$^{*}\partial_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial^2 x} = \exp(x) t^4 \left(\frac{\Gamma(5+\alpha)}{24} - t^{\alpha} \right), x \in [0,1], t > 0, \quad (35)$$

with the initial and boundary conditions as $u(x, 0) = 0$, $u(0, t) = t^{4+\alpha}$, $u(1, t) = \exp(1) t^{4+\alpha}$.

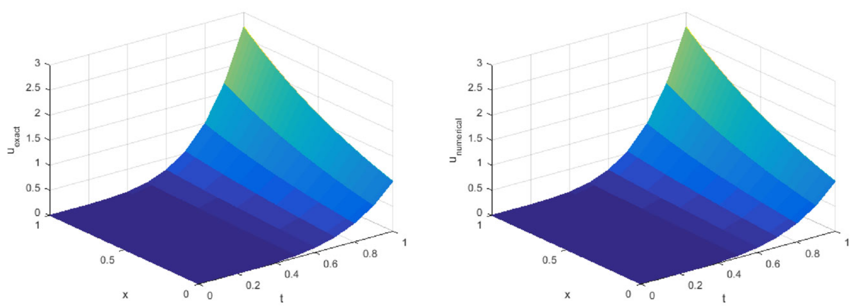
Table 12 MAE and CO for Example 4 with $\alpha = 0.8, \delta = 1$ $z(t) = t, w(t) = \exp(2t)$ $\Delta t = 1/600$

Δx	MAE	CO
1/10	5.6650e−03	–
1/20	1.4133e−03	2.0030
1/40	3.5316e−04	2.0006
1/80	8.8310e−05	1.9997
1/160	2.2107e−05	1.9980
1/320	5.5580e−06	1.9919

Table 13 MAE and CO for Example 4 with $\alpha = 0.6, \Delta x = \frac{1}{10000}, \delta = 1$ $z(t) = t, w(t) = \exp(2t)$. for $T = 1$

Δt	MAE	CO
1/10	2.0563e−05	–
1/20	4.0706e−06	2.3367
1/40	7.9332e−07	2.3593
1/80	1.5510e−07	2.3547
1/160	3.2244e−08	2.2661

The exact solution of the Eq. (1) under above condition is given by $u(x, t) = \exp(x)t^{4+\alpha}$. We take $z(t) = t$, $w(t) = 1$ and $0 \leq t \leq 1$. We solved Example 5 using the proposed method for $\alpha = 0.5$ and step sizes $\Delta x = 1/20000$, $\Delta t = 0.1$. The numerical results are shown in the Fig. 5a and b. The Fig. 5a shows the exact solution while Fig. 5b depicts numerical solution. We also solved Example 5 with (1) varying both Δx and Δt with $\alpha = 0.15$; (2) varying Δx and fixed $\Delta t = 1/2000$ with $\alpha = 0.4$ and (3) varying Δt and fixed $\Delta x = 1/20000$ with $\alpha = 0.5$ which are shown through Table 14, 15 and 16 respectively. Tables 14, 15 and 16 establish that the proposed scheme is of second order of convergence. Tables 15 and 16 show that the numerical scheme having second order of convergence in spatial direction and $3 - \alpha$ order of convergence in temporal direction respectively. Tables 14, 15 and 16 are presented



(a) Exact Solution. (b) $\Delta x = 1/20000, \Delta t = 0.1, \alpha = 0.5$.

Fig. 5 Comparison of numerical and exact solutions of Example 5

Table 14 MAE and CO for Example 5 with $\alpha = 0.15$, $\delta = 1$ $z(t) = t$, $w(t) = 1$

Δt	Δx	MAE	CO
1/10	1/10	3.6564e−04	–
1/20	1/20	7.2845e−05	2.3275
1/40	1/40	1.4938e−05	2.2858
1/80	1/80	3.2157e−06	2.2158
1/160	1/160	7.2456e−07	2.1499
1/320	1/320	1.6926e−07	2.0979

Table 15 MAE and CO for Example 5 with $\alpha = 0.4$, $\delta = 1$ $z(t) = t$, $w(t) = 1$ $\Delta t = 1/2000$

$\Delta \boldsymbol{x}$	MAE	CO
1/10	1.4634e−04	–
1/20	3.6972e−05	1.9848
1/40	9.2475e−06	1.9993
1/80	2.3135e−06	1.9990
1/160	5.7981e−07	1.9964
1/320	1.4638e−07	1.9859

Table 16 MAE and CO for Example 5 with $\alpha = 0.5$, $\Delta x = \frac{1}{20000}$, $\delta = 1$ $z(t) = t$, $w(t) = 1$ for $T = 1$

Δt	MAE [28]	CO
1/10	2.493639e−03	–
1/20	4.836648−04	2.37
1/40	9.015368e−05	2.42
1/80	1.644426e−05	2.45
1/160	2.972672e−06	2.47

for the MAE and CO with $\alpha = 0.15$, 0.4 and 0.5 respectively. It is clear from the Tables 14, 15 and 16 our proposed numerical scheme is of high order of convergence and achieve the good accuracy. This example also concludes that the theoretical results with the proposed numerical results.

Conclusion

We presented a high order scheme for solving the GFDEs defined in term of scale function and weight function in Caputo sense. We also proved the stability via convergence analysis of the proposed finite difference scheme. To check the performance of numerical scheme, we took five examples for performing numerical simulations. Simulation results show that numerical scheme is of high order and stable. We derived theoretically the stability of numerical scheme using the Fourier series method. Numerical results given in the form of Tables and Figures validate the presented numerical method. It was also noticed that the scheme works well with nonlinear scale and weight functions. The order of numerical scheme was investigated in temporal direction as $(\Delta t)^{3-\alpha}$ while in the spatial direction is $(\Delta x)^2$.

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Author contributions Kamlesh Kumar: Conceptualization, formal analysis ,Investigation, methodology. Awadhesh K Pandey: Validation, Originality Rajesh K. Pandey: Supervision ,Validation

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Data Availability Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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