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Conservation Laws of Fractional Classical Fields

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Abstract

This paper presents a formulation of Noether's theorem for fractional classical fields. We extend the variational formulations for fractional discrete systems to fractional field systems. By applying the variational principle to a fractional action S, we obtain the fractional Euler–Lagrange equations of motion. Considerations of the Noether's variational problem for discrete systems whose action is invariant under gauge transformations will be extended to fractional variational problems for classical fields. The conservation laws associated with fractional classical fields are derived. As an example we present the conservation laws for the fractional Dirac fields.

Keywords Fractional calculus · Fractional variational principle · Noether's theorem

Introduction

Fractional calculus generalized the classical calculus and it has many important applications in various fields of science and engineering [1–6]. Also, it plays an important rule in the understanding of both conservative and non conservative systems [7, 8]. Particularly,the fractional and anomalous dynamics has experienced to be a framework in the theory of stochastic processes [9]. The physical and geometrical meaning of the fractional derivatives has been investigated by several authors [10, 11]. It is known what is the physical interpretation of the Stieltjes integral,namely the Stieltjes integral can be interpreted as the real distance passed by a moving object. Within this physical interpretation the Riemann–Liouville and Caputo have the same interpretation and therefore can be used successfully, for example,within fractional variational principles [7, 8, 12–21]. On the other hand, by making use of examples from viscoelasticity it was shown that it is possible to attribute physical meaning to initial conditions within Riemann–Liouville fractional derivatives [22]. The fractional operators are particular

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cases of non-local operators and during the last decade several interesting research was done in this direction [23–26].

Conservation laws play an important role in theoretical physics. The conservation of energy, momentum, and angular momentum are fundamental laws that any theory has to guarantee if it is to give a valid description of nature. Besides, physical systems often possess further conserved quantities, such as charge, isospin. Noether's theorem [27] is a central result of the calculus of variation and explain all the conservation laws based upon the action principle. It is a very general result, asserting that "the conservation laws are natural consequences of the symmetry properties of a system. For each continues transformation of the coordinates and/or the fields under which "the action is invariant" the existence of a conserved quantity can be deduced". For example, the conservation of energy, momentum, and angular momentum are based on the invariance of the action S under temporal and spatial translations and under rotation in space. Similarly the conservation of charge follows from an invariance of the action S under the phase transformations.

Following the above progress, one can naturally ask: "Can we apply the Noether's variational problem to Lagrangian systems with fractional derivatives". This is exactly what is done in [28, 29]. Frederic and Torres [28, 29] examine the Noether's variational problem to Lagrangian systems with finite degrees of freedom. At this stage, one may ask, "Can we extend the Noether's variational problem to fractional classical fields". To address this, we first define the variational problem for classical fields [21]. A *variational problem* is a problem that requires finding the extremum of a functional which may be subjected to algebraic and/or dynamic constraints. If either the functional and/or the algebraic/dynamic constraint(s) contain at least one fractional derivative term, then the problem is called a *fractional variational problem*. The *Fractional Variational Calculus* is an extension of the ordinary Variational Calculus that deals with finding the solution of a fractional variational problem [7, 8, 12–20]. We will define Noether's variational problem to fractional Lagrangian systems with infinite degrees of freedom.

An important point to be specified here is that, several definitions have been proposed of a fractional derivative, among those Riemann–Liouville and Caputo fractional derivatives are most popular. The differential equations defined in terms of Riemann–Liouville derivatives require fractional initial conditions whereas the differential equations defined in terms of Caputo derivatives require regular boundary conditions. For this reason, Caputo fractional derivatives are popular among scientists and engineers. Accordingly, we shall also develop our formulations in terms of Caputo fractional derivatives. However, most of the approach will also be applicable to problems defined using other derivatives.

The plan of this paper is as follows: In Sect. 2, we present the Euler-Lagrange equations for a fractional field. Section 3 presents the Noether's variational theorem for fractional classical fields. In Sect. 4, we define a fractional Lagrangian density function and use the theories developed in Sects. 2 and 3 to derive the conservation laws for fractional Dirac fields. Finally, Sect. 5 presents conclusions.

Lagrangian Formulation of Field Systems with Caputo Fractional Derivatives

Consider a function f depending on n variables, $x_1, \ldots, x_n$ defined over the domain $\Omega = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Following the convention used in physics, we defined the left and the right partial Riemann–Liouville and Caputo fractional derivatives of order α_k, $0 < \alpha_k < 1$

with respect to x_k as [21]

$$(_+\partial_k^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha_k)} \partial x_k \int_{a_k}^{x_k} \frac{f(x_1,\ldots,x_{k-1},u,x_{k+1},\ldots,x_n)}{(x_k-u)^{\alpha_k}} du, \quad (2.1)$$

$$(_-\partial_k^\alpha f)(x) = \frac{-1}{\Gamma(1-\alpha_k)} \partial x_k \int_{x_k}^{b_k} \frac{f(x_1,\ldots,x_{k-1},u,x_{k+1},\ldots,x_n)}{(u-x_k)^{\alpha_k}} du, \quad (2.2)$$

$$(_+^C\partial_k^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha_k)} \int_{a_k}^{x_k} \frac{\partial_u f(x_1,\ldots,x_{k-1},u,x_{k+1},\ldots,x_n)}{(x_k-u)_k^{\alpha}} du, \quad (2.3)$$

and

$$(_-^C\partial_k^\alpha f)(x) = \frac{-1}{\Gamma(1-\alpha_k)} \int_{x_k}^{b_k} \frac{\partial_u f(x_1,\ldots,x_{k-1},u,x_{k+1},\ldots,x_n)}{(u-x_k)^{\alpha_k}} du, \quad (2.4)$$

where $\partial x_k g$ is the partial derivatives of g with respect to the variable x_k. Here, in $_+\partial_k^\alpha$, $_-\partial_k^\alpha$, $_+^C\partial_k^\alpha$, and $_-^C\partial_k^\alpha$, the meaning of various subscripts and superscripts need to be made clear. The subscript k and the superscript α indicate that the derivative is taken with respect to the variable x_k and it is of order α_k (note that we write only α for α_k, and the subscript k to ∂ also represents the subscript to α), the subscripts $+$ and $-$ prior to the symbol ∂ represent the left and the right fractional derivatives respectively, and accordingly the limits of integrations are taken as $[a_k, x_k]$ and $[x_k, b_k]$. Further, no superscript and the superscript C prior to the symbol ∂ represent the Riemann–Liouville fractional derivative and the Caputo fractional derivative, respectively. Superscript α is necessary here as a reminder that the operator ∂^α represents a fractional derivative. When α is equal to 1, the superscript α can be neglected. Although, our aim in this section is to present the action principle for systems defined in terms of Caputo fractional derivatives, the Riemann–Liouville fractional derivatives are also defined here because they naturally arise in the formulation.

To develop the action principle for field systems described in terms of fractional derivatives, consider a Lagrangian density $\mathcal{L}$ depends on N distinct fields ϕ_r, $(r = 1, ..., N)$, and their fractional derivatives written as $\mathcal{L} = \mathcal{L}\left(\phi_r(x_k), (_+^C\partial_k^\alpha)\phi_r(x_k), (_-^C\partial_k^\beta)\phi_r(x_k), x_k\right)$. The functional $S(\phi)$ is defined as

$$S(\phi) = \int_\Omega \mathcal{L}\left(\phi_r(x_k), (_+^C\partial_k^\alpha)\phi_r(x_k), (_-^C\partial_k^\beta)\phi_r(x_k), x_k\right)(dx_k). \quad (2.5)$$

Here we have used Goldstein's [30] notation. Accordingly, x_k represents n variables x_1 to x_n, $\phi_r(x_k) \equiv \phi_r(x_1,\ldots,x_n)$, $\mathcal{L}(*, _+^C\partial_k^\alpha, *, *) \equiv \mathcal{L}(*, _+^C\partial_1^\alpha, \ldots, _+^C\partial_n^\alpha, *, *)$, $(dx_k) \equiv dx_1 \ldots dx_n$, and the integration is taken over the entire domain Ω. Other terms are defined accordingly.

To find the necessary condition for extremum of the action functional defined above, consider a one parameter family of possible functions $\phi_r(x_k; \epsilon)$ as follows,

$$\phi_r(x_k; \epsilon) = \phi_r(x_k; 0) + \epsilon \eta_r(x_k), \quad (2.6)$$

where $\phi_r(x_k; 0)$ are the correct functions which satisfy the Hamilton's principle for the fractional system, $\eta_r(x_k)$ are well-behaved functions that vanish at the endpoints, and ϵ is an arbitrary parameter. Note that $S[\phi_r(x_k; \epsilon)]$ is extremum at $\epsilon = 0$. Substituting Eq. (2.6) into Eq. (2.5), differentiating the resulting expression with respect to ϵ, and then setting ϵ to 0, we obtain,

$$\frac{dS}{d\epsilon}\,|_{\epsilon=0} = \int \left[\frac{\partial \mathcal{L}}{\partial \phi_r}\eta_r + \sum_{k=1}^{n} \frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi_r)}(_+^C \partial_k^\alpha \eta_r) + \sum_{k=1}^{n} \frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi_r)}(_-^C \partial_k^\beta \eta_r) \right](dx_k) = 0.$$

$$(2.7)$$

Finally, using the formula for integration by part [14], the fact that $\eta(x_k)$ is zero at the boundary, and a lemma from Calculus of Variations, we obtain

$$\frac{\partial \mathcal{L}}{\partial \phi_r} + \sum_{k=1}^{n} {}_-\partial_k^\alpha \frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi_r)} + \sum_{k=1}^{n} {}_+\partial_k^\beta \frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi_r)} = 0.$$

$$(2.8)$$

Equation (2.8) is the Euler–Lagrange equation for the fractional field system. For α_k, $\beta_k \rightarrow 1$, Eq. (2.8) gives the usual Euler-Lagrange equations for classical fields.

It is worthwhile to mention that the same approach can be used to find the Euler-Lagrange equations for functionals defined in terms of Riemann–Liouville or mixed (Caputo and Riemann–Liouville) derivatives [31].

Noether's Variational Problem for Fractional Classical Fields

Like discrete systems and integer order fields, it is possible to develop Noether's variational problem for fields defined in terms of fractional derivatives. Let us consider the following space-time and fields transformations respectively as

$$x_k' = x_k + \delta x_k, \tag{3.1}$$

$$\phi_r'(x_k') = \phi_r(x_k) + \delta\phi_r(x_k). \tag{3.2}$$

Under the transformation (3.1) and (3.2), the resulting change in the Lagrangian density $\mathcal{L}\left(\phi_r(x_k), (_+^C \partial_k^\alpha)\phi_r(x_k), (_-^C \partial_k^\beta)\phi_r(x_k), x_k\right)$ will be

$$\mathcal{L}'\left(\phi_r'(x_k'), (_+^C \partial_k^\alpha)\phi_r'(x_k'), (_-^C \partial_k^\beta)\phi_r'(x_k'), x_k'\right)$$

Now let us study the consequences that follow if the transformations (3.1) and (3.2) leave the action integral invariant, i.e., we demand

$$\delta S(\phi) \equiv \int_{\Omega'} \mathcal{L}'\left(\phi_r'(x_k'), (_+^C \partial_k^\alpha)\phi_r'(x_k'), (_-^C \partial_k^\beta)\phi_r'(x_k'), x_k'\right)(dx_k')$$

$$- \int_{\Omega} \mathcal{L}\left(\phi_r(x_k), (_+^C \partial_k^\alpha)\phi_r(x_k), (_-^C \partial_k^\beta)\phi_r(x_k), x_k\right)(dx_k) = 0, \tag{3.3}$$

here Ω' denotes the volume of integration expressed in terms of the new coordinates x_k'. This will introduce a Jacobi determinant in the first order as

$$dx_k' = \left| \frac{\partial(x_k')}{\partial(x_k)} \right| dx_k. \tag{3.4}$$

The conservations laws in (3.3) are related to the symmetry transformations (9, 10). However, in the case of gauge transformations in field theory, where the field have internal symmetry, we are concerned with transformations of the fields only (i.e., the dependent variables (3.2)), and not transformations of the space-time coordinates (the independent variables (3.1)) and hence we ignore the terms in δx_k. In this case, we have $\phi_r'(x_k) = \phi_r(x_k) + \delta\phi_r(x_k)$ and (3.3)

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becomes

$$\sum_{r=1}^{N} \frac{\partial \mathcal{L}}{\partial \phi_r} \delta \phi_r + \sum_{k,r} \frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi_r)} \delta (_+^C \partial_k^\alpha \phi_r) + \sum_{k,r} \frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi_r)} \delta (_-^C \partial_k^\beta \phi_r) = 0. \tag{3.5}$$

Using the Euler Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \phi_r} = -\sum_{k=1}^{n} {}_- \partial_k^\alpha \frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi_r)} - \sum_{k=1}^{n} {}_+ \partial_k^\beta \frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi_r)}, \tag{3.6}$$

and substituting Eq. (3.6) in Eq. (3.5), we obtain the conserved quantity for fractional classical field under the internal symmetry transformations as

$$\sum_{k,r} \left\{ \frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi_r)} {}_+^C \partial_k^\alpha (\delta \phi_r) - \left(-\partial_k^\alpha \frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi)} \right) \delta \phi_r \right\}$$

$$+ \sum_{k,r} \left\{ \frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi_r)} {}_-^C \partial_k^\beta (\delta \phi_r) - \left({}_+\partial_k^\beta \frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi)} \right) \delta \phi_r \right\} = 0. \tag{3.7}$$

Defining the new operators ${}_+^C D_k^\alpha$ and ${}_-^C D_k^\beta$ as left and right operators respectively as

$$_+^C D_k^\alpha [\mathbf{f}, \ \mathbf{g}] = \mathbf{f} (_+^C \partial_k^\alpha \mathbf{g}) - (-\partial_k^\alpha \mathbf{f}) \mathbf{g}, \tag{3.8}$$

$$_-^C D_k^\beta [\mathbf{f}, \ \mathbf{g}] = \mathbf{f} (_-^C \partial_k^\beta \mathbf{g}) - (_+\partial_k^\beta \mathbf{f}) \mathbf{g}, \tag{3.9}$$

then the conserved quantity (3.7) can be written as

$$\sum_{k,r} \left\{ {}_+^C D_k^\alpha \left[\frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi_r)}, \delta \phi_r \right] + {}_-^C D_k^\beta \left[\frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi_r)}, \delta \phi_r \right] \right\} = 0. \tag{3.10}$$

For a Lagrangian density $\mathcal{L} = \mathcal{L} \left(\phi_r(x_k), (_+^C \partial_k^\alpha) \phi_r(x_k), x_k \right)$ and in the limit $\alpha \to 1$, we obtain

$$\sum_{k,r} \partial_k \left(\frac{\partial \mathcal{L}}{\partial (\partial_k \phi_r)}, \delta \phi_r \right) = 0. \tag{3.11}$$

Now, consider the following infinitesimal internal symmetry transformations:

$$\phi_r'(x_k) = \phi_r(x_k) + i\epsilon \sum_s \lambda_{rs} \phi_s(x_k), \tag{3.12}$$

$$\delta x_k = 0, \tag{3.13}$$

the Noether's conserved quantity (3.7) reads as

$$i\epsilon \sum_{k,r,s} \left\{ \frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi_r)} \lambda_{rs} (_+^C \partial_k^\alpha \phi_s) - \left(-\partial_k^\alpha \frac{\partial \mathcal{L}}{\partial (_+^C \partial_k^\alpha \phi)} \right) \lambda_{rs} \phi_s \right\}$$

$$+ i\epsilon \sum_{k,r,s} \left\{ \frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi_r)} \lambda_{rs} (_-^C \partial_k^\beta \phi_s) - \left({}_+\partial_k^\beta \frac{\partial \mathcal{L}}{\partial (_-^C \partial_k^\beta \phi)} \right) \lambda_{rs} \phi_s \right\} = 0. \tag{3.14}$$

As an application of this concept, let us study a complex field described by a Lagrangian density which is invariant under the phase transformations

$$\phi' = \phi + i\epsilon\phi, \tag{3.15}$$

$$\phi'^* = \phi^* - i\epsilon\phi^*. \tag{3.16}$$

This leads to the conserved quantity

$$
i\epsilon \sum_k \left\{ \left(\frac{\partial\mathcal{L}}{\partial({}_+^C\partial_k^\alpha\phi)} \right) ({}_+^C\partial_k^\alpha\phi) - \left(\frac{\partial\mathcal{L}}{\partial({}_+^C\partial_k^\alpha\phi^*)} \right) ({}_+^C\partial_k^\alpha\phi^*) \right\}
$$

$$
- i\epsilon \sum_k \left\{ \left(-\partial_k^\alpha \frac{\partial\mathcal{L}}{\partial({}_+^C\partial_k^\alpha\phi)} \right) \phi - \left(-\partial_k^\alpha \frac{\partial\mathcal{L}}{\partial({}_+^C\partial_k^\alpha\phi^*)} \right) \phi^* \right\}
$$

$$
+ i\epsilon \sum_k \left\{ \left(\frac{\partial\mathcal{L}}{\partial({}_-^C\partial_k^\beta\phi)} \right) ({}_-^C\partial_k^\beta\phi) - \left(\frac{\partial\mathcal{L}}{\partial({}_-^C\partial_k^\beta\phi^*)} \right) ({}_-^C\partial_k^\beta\phi^*) \right\}
$$

$$
- i\epsilon \sum_k \left\{ \left(+\partial_k^\beta \frac{\partial\mathcal{L}}{\partial({}_-^C\partial_k^\beta\phi)} \right) \phi - \left(+\partial_k^\beta \frac{\partial\mathcal{L}}{\partial({}_-^C\partial_k^\beta\phi^*)} \right) \phi^* \right\} = 0. \tag{3.17}
$$

Noether's Theorem for Dirac's Field with Caputo Fractional Derivatives

We will now apply Noether's method developed in Sects. 2 and 3 to derive fractional conservation laws for fractional Dirac field [32]. We shall limit our discussion to four dimensional system (the first three for space, x_1, x_2 and x_3, and the fourth for time, $x_4 = it$, where i is the imaginary unit. Note that we are considering the units that takes the speed of light equal to 1). The Greek indices μ, λ, ν etc. will range from 1 to 4, the Roman indices i, j, k etc. will range from 1 to 3, and unless specifically stated, the repeated indices will represent summation. Following this convention, we propose the following Lagrangian density field [21]

$$\mathcal{L} = m^\alpha \bar{\Psi}\Psi + \bar{\Psi}\gamma^\mu \, ({}_+^C\partial_\mu^\alpha\Psi) \tag{4.1}$$

where Ψ is a wave function (the bold character Ψ suggests that it could be a vector function. In literature, it is also called a spinor), m is the mass of the particle, α ($= \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$) is the order of the fractional derivative, γ^μ are matrices, $\bar{\Psi}$ is the adjoint wave function, The dimensions of Ψ, $\bar{\Psi}$ and γ^μ depend on the order of the derivatives and the way the algebra of matrices γ^μ is developed [21].

Using Eqs. (2.8) and (4.1), the Euler–Lagrange equations for variables $\bar{\Psi}$ and Ψ are given as

$$\gamma^\mu \, ({}_+^C\partial_\mu^\alpha\Psi) + m^\alpha\Psi = 0, \tag{4.2}$$

and

$$({}_-\partial_\mu^\alpha\bar{\Psi}) \, \gamma^\mu + m^\alpha\bar{\Psi} = 0. \tag{4.3}$$

The Lagrangian density (4.1) is invariant under infinitesimal internal symmetry transformations:

$$\Psi \to \Psi' = \Psi e^{i\theta}, \tag{4.4}$$

$$\bar{\Psi} \to \bar{\Psi}' = \bar{\Psi} e^{-i\theta}, \tag{4.5}$$

where θ is a constant. Using Eq. (3.17), the conserved quantity for fractional Dirac's field is given by

$$\bar{\Psi}\gamma^{\mu} \left({}^{C}_{+}\partial^{\alpha}_{\mu}\Psi \right) - {}_{-}\partial^{\alpha}_{k}(\bar{\Psi}\gamma^{\mu})\Psi = 0. \tag{4.6}$$

If we introduce the following operator D^{α}_{μ} as

$$D^{\alpha}_{\mu}[\mathbf{f}, \ \mathbf{g}] = \mathbf{f}({}^{C}_{+}\partial^{\alpha}_{\mu}\mathbf{g}) - ({}_{-}\partial^{\alpha}_{k}\mathbf{f})\mathbf{g}, \tag{4.7}$$

then the Dirac's conserved quantity can be written as

$$D^{\alpha}_{\mu}[\bar{\Psi}\gamma^{\mu}, \ \Psi] = 0. \tag{4.8}$$

It is worthwhile to mentio that for $\alpha \rightarrow 1$, then

$$D^{1}_{\mu}[\mathbf{f}, \ \mathbf{g}] = \partial_{\mu}(\mathbf{fg}). \tag{4.9}$$

In this case the Dirac's continuity equation reads as

$$\partial_{\mu} j^{\mu} = 0, \tag{4.10}$$

where

$$j^{\mu} = \bar{\Psi}\gamma^{\mu}\Psi, \tag{4.11}$$

is the familiar electric 4-current.

Conclusions

In this paper we have presented the formulation of Noether's theorem for fractional classical fields. We used the fractional variational principle to derive the equations of motion for fractional classical field as well as the conservation laws associated with the internal symmetry of gauge transformations of classical fields.

As an example we have derived the continuity equation for fractional Dirac's field of order α. Under the limit that $\alpha \rightarrow 1$, we obtain the conserved 4-current for the regular Dirac's field.

Data availability Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

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