Int. J. Appl. Comput. Math (2022) 8:168
https://doi.org/10.1007/s40819-022-01360-y

ORIGINAL PAPER

Some Properties of Generalized Srivastava's Triple Hypergeometric Function $H_{C,p,q}(\cdot)$

S. A. Dar[1] · M. Kamarujjama[1] · M. Daud[1]

Accepted: 23 May 2022 / Published online: 24 June 2022

Abstract

In this paper, we obtain a $(p, q)-$generalization of Srivastava's triple hypergeometric function $H_C(\cdot)$, along with its integral representations by using extended Beta function $B_{p,q}(x, y)$ introduced in 2014 by Choi et al. Also, we discuss some of its main fundamental properties such as the Mellin transform, derivative formula, recursive identity, and a bounded inequality. In addition, we obtain an integral form of $H_{C,p,q}(\cdot)$ function involving Laguerre polynomials.

Keywords Gauss hypergeometric function · Srivastava's triple hypergeometric functions · Beta and Gamma functions · bounded inequality · Laguerre polynomials

Mathematics Subject Classification 33C05 · 33C70 · 33B15 · 33C45

Introduction, definitions and preliminaries

Hypergeometric functions have a long history in a wide variety of fields of mathematical physics, Statistics, Economics etc. For $l_1, l_2 \in \mathbf{C}, l_3 \in \mathbf{C}\backslash\mathbf{Z}_0^-$ [12], the Gauss hypergeometric function is defined as

$$ {}_2F_1\left(\begin{matrix} l_1, l_2 \\ l_3 \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(l_1)_n (l_2)_n}{(l_3)_n} \frac{z^n}{n!} \qquad (|z| < 1). \tag{1.1}$$

This hypergeometric funcion extensions includes l_j $(1 \leq j \leq p, q)$. which also has so many wide applications; see [17].

In the availablle literature on hypergeometric series, this series and its generalizations appear in various branches of mathematics associated with applications. This type of series appears very naturally in quantum field theory. In particular in the computaion of analytic

✉ S. A. Dar
showkatjmi34@gmail.com

M. Kamarujjama
mdkamarujjama@rediffmail.com

M. Daud
daud.ali387@gmail.com

1 Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202002, India

expressions for Feyman integrals. On the other hand, the application of known relations for triple hypergeometric series may lead to simplificatons, help to solve problems or lead to greater insight in quantum field theory. Srivastava and Karlsson [16, Chapter 3] introduced and explored a table of distinct 205 triple hypergeometric functions. Some complete triple hypergeometric functions denoted as H_A, H_B and H_C of the second order are introduced by Srivastava, see [13, 14]. It is known that H_B and H_C are generalizations of the Appell hypergeometric function F_1 and F_2, while H_A is generalization of both F_1 and F_2.

In this paper, we study Srivastava's hypergeometric function H_C of three variables given by [16, p. 43], [13] and [15, p. 68]

$$\begin{aligned} &H_C(l_1, l_2, l_3; l_4; z_1, z_2, z_3) \\ &:= \sum_{m,n,k=0}^{\infty} \frac{(l_1)_{m+k}(l_2)_{m+n}(l_3)_{n+k}}{(l_4)_{m+n+k}} \frac{z_1^m}{m!} \frac{z_2^n}{n!} \frac{z_3^k}{k!}, \qquad (1.2) \\ &= \sum_{h,m,n=0}^{\infty} \left[\frac{(l_2)_{h+m}(l_3)_{m+n}}{(l_4)_m} \frac{B(l_1+h+n, l_4+m-l_1)}{B(l_1, l_4+n-l_1)} \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!} \right]. \qquad (1.3) \end{aligned}$$

where $|z_1| < 1$, $|z_2| < 1$, $|z_3| < 1$. This triple hypergeometric function H_C is very useful in analytic continuation. Its analytic continuation formula was obtained by Srivastava [20, p.104] which is the solution of the system of partial differential equations satisfied the triple hypergeometric function H_C.

Here $(u)_v$ $(u, v \in \mathbb{C})$ is the Pochhammer's symbol defined as $(1)_n = n!$ and

$$(u)_v := \frac{\Gamma(u+v)}{\Gamma(u)} = \begin{cases} 1, & (v = 0\,;\; u \in \mathbb{C}\backslash\{0\}) \\ u(u+1)...(u+n-1), & (v = n \in \mathbb{N}\,;\; u \in \mathbb{C}), \end{cases} \tag{1.4}$$

and Beta function $B(u, v)$ is defined by [9, (5.12.1)]

$$B(u, v) = \begin{cases} \int_0^1 t^{u-1}(1-t)^{v-1}dt & (\Re(u) > 0, \Re(v) > 0) \\ \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, & (\Re(u) < 0, \Re(v) < 0), \quad (u, v) \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases} \tag{1.5}$$

For convenience, we can add parameters r and s into $H_C(\cdot)$ in the form

$$\begin{aligned} &H_C^{(r,s)}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) \\ &:= \sum_{h,m,n=0}^{\infty} \frac{(l_2)_{h+m}(l_3)_{m+n}}{(l_4)_m} \frac{B(l_1+r+h+n, l_4+s+m-l_1)}{B(l_1, l_4+n-l_1)} \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!}. \qquad (1.6) \end{aligned}$$

The region of convergence for $H_C(\cdot)$ function is given in [7, p.243] as $|z_1| < A$, $|z_2| < B$, $|z_3| < C$, where

$$B + A + C - 2\sqrt{(1-B)(1-A)(1-C)} < 2. \tag{1.7}$$

The simple Laguerre polynomials of order $m (m \in N_0)$ is defined by [12, p.213, eq(1-2)]

$$L_m(x) = {}_1F_1\left(\begin{matrix}-m \\ 1\end{matrix}; x\right) = \sum_{r=0}^{m} \frac{(-1)^r m!}{(m-r)!(r!)^2} x^r. \tag{1.8}$$

An integral representation of ${}_2F_1(\cdot)$ is given by [15, Eq.(11)] and [17]

$$ {}_2F_1\left(\begin{matrix} l_1, l_2 \\ l_3 \end{matrix}; z\right) = \frac{\Gamma(l_3)}{\Gamma(l_2)\Gamma(l_3-l_2)} \int_0^1 t^{l_2-1}(1-t)^{l_3-l_2-1}(1-zt)^{-l_1}dt, \tag{1.9} $$

where $\Re(l_3) > \Re(l_2) > 0, \quad |\arg(1-z)| < \pi$.
A Beta function $B(u, v, p)$ is given by Chaudhry et al. in 1997 [1, p.20, Eq.(1.7)]

$$ B(u, v; p) = \int_0^1 t^{u-1}(1-t)^{v-1} e^{\left[\frac{-p}{t(1-t)}\right]} dt, \qquad (\Re(p) > 0). \tag{1.10} $$

Further, Chaudhry et al. [2] utilise (1.10) to extend the Gauss hypergeometric series ${}_2F_1(\cdot)$ and its integral form. Choi *et al.* [5] extended the Beta function in the following way:

$$ B(u, v; p, q) \equiv B_{p,q}(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1}\, e^{\left\{-\frac{p}{t}-\frac{q}{1-t}\right\}} dt, \ \Re(p) > 0, \ \Re(q) > 0. \tag{1.11} $$

If $p = q$ then function becomes $B(u, v; p)$. A different generalization of the Beta function has been given in [11].
The Appell hypergeometric function $F_1(\cdot)$ is given by

$$ F_1(l_1, l_2, l_3; l_4; u, v) := \sum_{n,m=0}^{\infty} \frac{(l_2)_n (l_3)_m \ B(l_1+m+n, l_4-l_1)}{B(l_1, l_4-l_1)} \frac{u^m}{m!} \frac{v^n}{n!}, \ |u| < 1, |v| < 1, \tag{1.12} $$

and this function has been expanded by Özarslan and Özergin [10] .

Inspired by these extensions of special functions (as given above), the integral representations of the functions $H_C(\cdot)$ have been studied by many authors ; see [3, 4]. In this paper, we have investigated a generalised Srivastava Hypergeometric function of three variables in (1.2), which is represented by $H_{C,p,q}(\cdot)$, and investigate certain identities of this generalized function $H_{C,p,q}(\cdot)$ systematically.

Generalized Srivastava's triple hypergeometric function $H_{C,p,q}(\cdot)$

Srivastava investigated hypergeometric function of three variables $H_C(\cdot)$, associated with integral expressions in [13] and [14]. Here, we investigate a generalised Srivastava's hypergeometric function of three variables, which is expressed by $H_{C,p,q}(\cdot)$ based on generalised beta function $B_{p,q}(x, y)$ in (1.11)

$$ \begin{aligned} &H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) \\ &= \sum_{h,m,n=0}^{\infty} \left[\frac{(l_2)_{h+m}(l_3)_{m+n}}{(l_4)_m} \frac{B_{p,q}(l_1+h+n, l_4+m-l_1)}{B(l_1, l_4+m-l_1)} \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!} \right], \end{aligned} \tag{2.1} $$

where the parameters $l_1, l_2, l_3 \in \mathbb{C}$ and $l_4 \in \mathbb{C} \setminus \mathbb{Z}_0^-$. The region of convergence for this series is $|z_1| < A, |z_2| < B, |z_3| < C$, satisfying Eq. (1.7). This definition implies the original classical function (1.3) if $p = 0 = q$.

Theorem 1 *The integral representations of the function $H_{C,p,q}(\cdot)$ holds for $\Re(p), \Re(q), \Re(l_j) > 0\ (1 \leq j \leq 3)$ and $\Re(l_4 - l_1) > 0$:*

$$H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) = \frac{\Gamma(l_4)}{\Gamma(l_1)\Gamma(l_4 - l_1)} \int_0^1 \Bigg[t^{l_1-1}(1-t)^{l_4-l_1-1}(1-z_1t)^{-l_2}(1-z_3t)^{-l_3} \times e^{\left\{-\frac{p}{t}-\frac{q}{1-t}\right\}} {}_2F_1\left(\begin{matrix} l_2, l_3; \\ l_4 - l_1; \end{matrix}\ X \right) \Bigg] dt, \tag{2.2}$$

where

$$X := \frac{z_2(1-t)}{(1-z_1t)(1-z_3t)}$$

$$H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) = \frac{\Gamma(l_4)}{\Gamma(l_1)\Gamma(l_4 - l_1)} \int_0^\infty \Bigg[\mu^{l_1-1}(1+\mu)^{l_2+l_3-l_4}\{\Omega_1\}^{-l_2}\{\Omega_2\}^{-l_3} \times \exp\left\{ -\frac{p(1+\mu)}{\mu} - q(1+\mu) \right\} {}_2F_1\left(\begin{matrix} l_2, l_3; \\ l_4 - l_1; \end{matrix}\ \Omega z_2 \right) \Bigg] d\mu, \tag{2.3}$$

where $\Omega_1 = 1 + \mu - z_1\mu,\ \Omega_2 = 1 + \mu - z_3\mu,\ \Omega = \frac{(1+\mu)}{\Omega_1\Omega_2}$,

$$\begin{aligned} &H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) \\ &= \frac{2\Gamma(l_4)}{\Gamma(l_1)\Gamma(l_4 - l_1)} \int_0^{\frac{\pi}{2}} \Bigg[(sin^2\mu)^{l_1-\frac{1}{2}}(cos^2\mu)^{l_4-l_1-\frac{1}{2}}(\vartheta_1)^{-l_2}(\vartheta_2)^{-l_3} \\ &\times \exp\left\{ -\frac{p}{sin^2\mu} - \frac{q}{cos^2\mu} \right\} {}_2F_1\left(\begin{matrix} l_2, l_3; \\ l_4 - l_1; \end{matrix}\ \frac{z_2cos^2\mu}{\vartheta_1\vartheta_2} \right) \Bigg] d\mu, \end{aligned} \tag{2.4}$$

where $\vartheta_1 = 1 - z_1sin^2\mu$, *and* $\vartheta_2 = 1 - z_3sin^2\mu$

$$\begin{aligned} H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) = &\frac{\Gamma(l_4)}{\Gamma(l_1)\Gamma(l_4 - l_1)} \frac{(B-C)^{l_1}(A-C)^{l_4-l_1}}{(B-A)^{l_4-l_2-l_3-1}} \\ &\times \int_A^B \Bigg[\frac{(\mu-A)^{l_1-1}(B-\mu)^{l_4-l_1-1}}{(\mu-C)^{l_4-l_2-l_3}}\{\sigma_1\}^{-l_2}\{\sigma_2\}^{-l_3} \\ &\times \exp\{-p\sigma_3 - q\sigma_4\}{}_2F_1\left(\begin{matrix} l_2, l_3; \\ l_4 - l_1; \end{matrix}\ \sigma z_2 \right) \Bigg] d\mu, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \sigma_1 &= [(B-A)(\mu-C) - z_1(B-C)(\mu-A)] \\ \sigma_2 &= [(B-A)(\mu-C) - z_3(B-C)(\mu-A)], \\ \sigma_3 &= \frac{(B-A)(\mu-C)}{(B-C)(\mu-A)} \quad and \quad \sigma_4 = \frac{(B-A)(\mu-C)}{(A-C)(B-\mu)}, \\ \sigma &= \frac{(A-C)(B-\mu)}{\sigma_1\sigma_2}, \end{aligned}$$

$$\begin{aligned} &H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) \\ &= \frac{\Gamma(l_4)(1+\lambda)^{l_1}}{\Gamma(l_1)\Gamma(l_4 - l_1)} \int_0^1 \Bigg[\frac{(\mu)^{l_1-1}(1-\mu)^{l_4-l_1-1}}{(1+\lambda\mu)^{l_4-l_2-l_3}}\{\nabla_1\}^{-l_2}\{\nabla_2\}^{-l_3} \\ &\times \exp\left\{ \frac{-p(1+\lambda\mu)}{\mu(1+\lambda)} - \frac{q(1+\lambda\mu)}{1-\mu} \right\} {}_2F_1\left(\begin{matrix} l_2, l_3; \\ l_4 - l_1; \end{matrix}\ \nabla z_2 \right) \Bigg] d\mu, \end{aligned} \tag{2.6}$$

where

$$\nabla_1 = [1 + \lambda\mu - z_1(1+\lambda)\mu],$$
$$\nabla_2 = [1 + \lambda\mu - z_3(1+\lambda)\mu],$$
$$\nabla = \frac{(1-\mu)(1+\lambda\mu)}{\nabla_1\nabla_2}; \quad \lambda > -1.$$

Proof We can prove first integral (2.2) by using the extended beta function from equation (1.11) in (2.1) and then changing order of integration and summation (since integral is uniform convergent) and finally using Gauss hypergeometric function (1.1). Then we get after simplification the right-hand side of (2.2). Furthermore,we can prove the integrals represented by (2.3)–(2.6), by using below transformations

$$t = \frac{\mu}{1+\mu}, \quad \frac{dt}{d\mu} = \frac{1}{(1+\mu)^2}, \tag{2.7}$$

$$t = \sin^2\mu, \quad \frac{dt}{d\mu} = 2\sin\mu\cos\mu, \tag{2.8}$$

$$t = \frac{(B-C)(\mu-A)}{(B-A)(\mu-C)}, \quad \frac{dt}{d\mu} = \frac{(B-A)(B-C)(A-C)}{(B-A)^2(\mu-C)^2}, \tag{2.9}$$

$$t = \frac{(1+\lambda)\mu}{1+\lambda\mu}, \quad \frac{dt}{d\mu} = \frac{(1+\lambda)}{(1+\lambda\mu)^2}, \tag{2.10}$$

in turn in (2.2) we obtained R.H.S. of respective results. □

Theorem 2 *The integral expression of $H_{C,p,q}(\cdot)$ function associated with Laguerre polynomials holds for $p, q > 0$ and $\Re(l_4) > \Re(l_1) > 0$.*

$$\begin{aligned} &H_{C,p,q}(l_1,\ l_2,\ l_3; l_4; z_1, z_2, z_3) \\ &= \frac{e^{-p-q}\Gamma(l_4)}{\Gamma(l_1)\Gamma(l_4-l_1)} \sum_{n,m=0}^{\infty} L_n(p)L_m(q) \\ &\times \int_0^1 t^{l_1+m}(1-t)^{l_4-l_1+n}(1-z_1t)^{-l_2}(1-z_3t)^{-l_3}{}_2F_1\left(\begin{matrix} l_2, l_3; \\ l_4-l_1; \end{matrix}\ X\right) dt, \end{aligned} \tag{2.11}$$

where

$$X := \frac{z_2(1-t)}{(1-z_1t)(1-z_3t)}.$$

Proof We can get exponential factor representation in (1.11) including Laguerre polynomials by the generating function [12, p. 202]

$$e^{\left(-\frac{ut}{1-t}\right)} = (1-t)\sum_{n=0}^{\infty} t^n L_n(u), \quad -1 < t < 1,\ u > 0 \tag{2.12}$$

defines Laguerre polynomials $L_n(u)(n \in N_0)$
This implies us

$$e^{\left(-\frac{q}{1-t}\right)} = e^{-q}(1-t)\sum_{m=0}^{\infty} t^m L_m(q), \quad -1 < t < 1, \tag{2.13}$$

Substituting t for $1-t$, we get

$$e^{\left(-\frac{p}{t}\right)} = e^{-p}\, t \sum_{n=0}^{\infty} (1-t)^n L_n(p), \quad 0 < t < 2. \tag{2.14}$$

Comparison of above two equations implies

$$e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)} = e^{-p} e^{-q} \sum_{n,m=0}^{\infty} (1-t)^{n+1} t^{m+1} L_m(q) L_n(p), \;\; 0 < t < 1. \tag{2.15}$$

Using (2.15) in (2.2), we get the required result stated in (2.11). □

Mellin transforms for $H_{C,p,q}(\cdot)$

If $f(u, v)$ is a locally integrable function with indices r and s given in [8, p.193, sec.(2.1), Entry (1.1)] then the Mellin transform is given by

$$\Phi(r,s) = \mathscr{M}\{f(u,v)\}(r,s) = \int_0^{\infty}\int_0^{\infty} u^{r-1} v^{s-1} f(u,v)\, du dv, \tag{3.1}$$

which defines an analytic function in the strips of analyticity $A < \Re(r) < B$ and $C < \Re(s) < D$ The inverse Mellin transform is defined by

$$f(x,y) = \mathscr{M}^{-1}\{\Phi(r,s)\} = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty}\int_{d-i\infty}^{d+i\infty} x^{-r} y^{-s} \Phi(r,s)\, dr ds, \tag{3.2}$$

where $A < c < B,\; C < d < D.$

Theorem 3 *The Mellin transforms of the generalized Srivastava's triple hypergeometric function $H_{C,p,q}(\cdot)$ holds for $\Re(p), \Re(q) > 0$ and $\Re(r), \Re(s) > 0$ given by*

$$\begin{aligned} &\mathscr{M}\left\{H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)\right\}(r,s) \\ &= \int_0^{\infty}\int_0^{\infty} P^{r-1} q^{s-1} H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)\, dp dq, \qquad (3.3)\\ &= \Gamma(r)\Gamma(s) H_C^{(r,s)}(l_1,l_2,l_3;l_4;z_1,z_2,z_3), \qquad (3.4)\end{aligned}$$

where $\Re(l_1+r) > 0,\; \Re(l_2+s) > 0,\; l_4 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $H_C^{(r,s)}$ is given in (1.6).

Proof Using equation (2.1) in equation (3.3) and reversing order of integration, we can get

$$\begin{aligned} &\mathscr{M}\left\{H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)\right\}(r,s) \\ &= \sum_{m,n,k\geq 0} \left[\frac{(l_2)_{m+n}(l_3)_{n+k}}{(l_4)_n\, B(l_1, l_4+n-l_1)} \frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^k}{k!} \right. \\ &\left. \times \left\{ \int_0^{\infty}\int_0^{\infty} p^{r-1} q^{s-1} B_{p,q}(l_1+m+k, l_4+n-l_1) \right\} dp dq \right]. \qquad (3.5)\end{aligned}$$

Applying the double integral formula [5, eq.(2.1)]

$$\int_0^{\infty}\int_0^{\infty} p^{r-1} q^{s-1} B_{p,q}(u,v)\, dp dq = \Gamma(r)\Gamma(s)\, B(u+r, v+s), \tag{3.6}$$

where $\Re(p), \Re(q) > 0$; $\Re(s), \Re(r) > 0$; $\Re(u+r), \Re(v+s) > 0$ in the eq.(3.5). Then we obtain

$$\begin{aligned}\Phi(r,s) &= \mathscr{M}\left\{H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)\right\}(r,s)\\ &= \Gamma(r)\Gamma(s)\sum_{m,n,k=0}^{\infty}\left[\frac{(l_2)_{m+n}(l_3)_{n+k}}{(l_4)_n}\frac{B(l_1+r+m+k,l_4+s+n-l_1)}{B(l_1,l_4+n-l_1)}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^k}{k!}\right].\end{aligned} \tag{3.7}$$

Comparing above series $H_C^{(r,s)}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)$ in (1.6), we obtained R.H.S. of the Mellin transform as given in (3.4). □

Corollary 1 *The inverse Mellin transform of $H_{C,p,q}(\cdot)$ is given by:*

$$\begin{aligned}&H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3) = \mathscr{M}^{-1}\{\Phi(r,s)\}\\ &= \frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\int_{d-i\infty}^{d+i\infty}\left(\frac{1}{p}\right)^r\left(\frac{1}{q}\right)^s\Gamma(r)\Gamma(s)H_C^{(r,s)}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)\,drds.\end{aligned} \tag{3.8}$$

A derivative identity for $H_{C,p,q}(\cdot)$

Theorem 4 *The differentiation of $H_{C,p,q}(\cdot)$ gives the identity:*

$$\begin{aligned}&\frac{\partial^{L+J+K}}{\partial x^L\partial y^J\partial z^K}H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)\\ &= \frac{(l_1)_{L+K}(l_2)_{L+J}(l_3)_{J+K}}{(l_4)_{L+J+K}}\\ &\times H_{C,p,q}(l_1+L+K,l_2+L+J,l_3+J+K;l_4+L+J+K;z_1,z_2,z_3),\end{aligned} \tag{4.1}$$

where $L, J, K \in \mathbb{N}_0$.

Proof If we differentiate partially the series for $\mathscr{H} \equiv H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)$ in (2.1) with respect to z_1 we obtain

$$\frac{\partial\mathscr{H}}{\partial z_1} = \sum_{i=1}^{\infty}\sum_{j,k=0}^{\infty}\left[\frac{(l_2)_{i+j}(l_3)_{j+k}}{(l_4)_j}\frac{B_{p,q}(l_1+i+k,l_4+j-l_1)}{B(l_1,l_4+j-l_1)}\frac{z_1^{i-1}}{(i-1)!}\frac{z_2^j}{j!}\frac{z_3^k}{k!}\right], \tag{4.2}$$

using

$$B(l_1,l_4+j-l_1) = \frac{(l_4+j)}{l_1}B(l_1+1,l_4+j-l_1), \tag{4.3}$$

and algebraic property $(\delta)_{i+j} = (\delta)_i(\delta+i)_j$, we have upon setting $i \to i+1$

$$\frac{\partial\mathscr{H}}{\partial z_1} = \frac{l_1\,l_2}{l_4}\sum_{i,j,k=0}^{\infty}\left[\frac{(l_2+1)_{i+j}(l_3)_{j+k}}{(l_4+1)_j}\frac{B_{p,q}(l_1+1+i+k,l_4+j-l_1)}{B(l_1+1,l_4+j-l_1)}\frac{z_1^i}{i!}\frac{z_2^j}{j!}\frac{z_3^k}{k!}\right], \tag{4.4}$$

$$= \frac{l_1\,l_2}{l_4}H_{C,p,q}(l_1+1,\ l_2+1,\ l_3;\ l_4+1;z_1,z_2,z_3). \tag{4.5}$$

Repeated application of (4.5) then yields for L = 1,2,...

$$\frac{\partial^L \mathscr{H}}{\partial z_1^L} = \frac{(l_1)_L\ (l_2)_L}{(l_4)_L}\ H_{C,p,q}\left(l_1+L,\ l_2+L,\ l_3;\ l_4+L; z_1, z_2, z_3\right). \tag{4.6}$$

A similar reasoning shows that

$$\frac{\partial^{L+1} \mathscr{H}}{\partial z_1^L \partial z_2} = \frac{(l_1)_L\ (l_2)_L}{(l_4)_L} \sum_{i,j,k=0}^{\infty} \times\left[\frac{(l_2+L)_{i+j}(l_3)_{j+k}}{(l_4+L)_j}\frac{B_{p,q}(l_1+L+i+k, l_4+I+j-l_1)}{B(l_1+L, l_4+L+j-l_1)}\frac{z_1^i}{i!}\frac{z_2^j}{j!}\frac{z_3^k}{k!}\right], \tag{4.7}$$

$$= \frac{(l_1)_L\ (l_2)_{L+1}\ (l_3)}{(l_4)_{L+1}}\ H_{C,p,q}\left(l_1+L,\ l_2+L+1,\ l_3+1;\ l_4+L+1; z_1, z_2, z_3\right), \tag{4.8}$$

now replacing $j \to j+1$ and applying the Beta function properties in (1.5) and then differentiating (4.8) J times repeatedly with respect to z_2 we get

$$\frac{\partial^{L+J} \mathscr{H}}{\partial z_1^L \partial z_2^J} = \frac{(l_1)_L\ (l_2)_{L+J}\ (l_3)_J}{(l_4)_{L+J}}\ H_{C,p,q}\left(l_1+L,\ l_2+L+J,\ l_3+J;\ l_4+L+J; z_1, z_2, z_3\right). \tag{4.9}$$

Following same methods and differentiating with respect to z_3 we will get result (4.1). □

An upper bound for $H_{C,p,q}(\cdot)$

Theorem 5 *The inequality of $H_{C,p,q}(\cdot)$ function for parameters $l_4,\ l_j \geq 0\ (1 \leq j \leq 3)$ and complex variables $z_1, z_2, z_3 \in \mathbb{C}$ holds true*

$$\left|H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3)\right| < \varphi_E\ H_C(l_1, l_2, l_3; l_4; |z_1|,\ |z_2|,\ |z_3|), \tag{5.1}$$

where $\Re(p), \Re(q) > 0$ and $\varphi_E := exp[-\Re(p) - \Re(q) - 2\sqrt{\Re(p)\Re(q)}]$.

Proof Assume $l_4 > 0,\ l_j > 0 (1 \leq j \leq 3)$, $\Re(p) > 0, \Re(q) > 0$ with $z_1, z_2, z_3 \in \mathbb{C}$. Then

$$\left|H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3)\right| \leq \sum_{m,n,k\geq 0} \frac{(l_2)_{m+n}(l_3)_{n+k}}{(l_4)_n}\frac{|B_{p,q}(l_1+m+k, l_4+n-l_1)|}{B(l_1, l_4+n-l_1)}\frac{|z_1|^m}{m!}\frac{|z_2|^n}{n!}\frac{|z_3|^k}{k!}, \tag{5.2}$$

Using definition of $B_{p,q}(A, B)$ in (1.11), with $U, V > 0$, we get

$$|B_{p,q}(U, V)| \leq \int_0^1 t^{U-1}(1-t)^{V-1}\ |E_{p,q}(t)| dt,\quad E_{p,q}(t) := e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)},$$
$$< \int_0^1 t^{U-1}(1-t)^{V-1}\ E_{\Re(p),\Re(q)}(t) dt,$$

Since, $E_{\Re(p),\Re(q)}(t)$ is maximum at $t^* = r/(1+r),\ r = \sqrt{\Re(p)/\Re(q)}$. We have

$$|B_{p,q}(U, V)| < \varphi_E\ B(U, V),\quad \varphi_E := exp[-\Re(p) - \Re(q) - 2\sqrt{\Re(p)\Re(q)}].$$

Further, the Eq. (5.2) implies

$$\begin{aligned}&\left|H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)\right|\\&\quad<\varphi_E\sum_{m,n,k\geq 0}\frac{(l_2)_{m+n}(l_3)_{n+k}}{(l_4)_n}\frac{B(l_1+m+k,l_4+n-l_1)}{B(l_1,l_4+n-l_1)}\frac{|z_1|^m}{m!}\frac{|z_2|^n}{n!}\frac{|z_3|^k}{k!},\end{aligned}\tag{5.3}$$

Comparison with Eq. (1.3), we get desired result (5.1).
Note that for $p=\ell=q>0$ we get $\varphi_E=exp(-4\ell)$. □

Recursion formulas for $H_{C,p,q}(\cdot)$

We have derived two recursive formulas of the generalized Srivastava hypergeometric function $H_{C,p,q}(\cdot)$ with triple complex variables in the following theorems:

Theorem 6 *The recursive formulas for $H_{C,p,q}(\cdot)$ involving numerator parameters l_2 and l_3 holds true*

$$\begin{aligned}&H_{C,p,q}(l_1,l_2+1,l_3;l_4;z_1,z_2,z_3)\\&\quad=H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)+\frac{z_1\,l_1}{l_4}H_{C,p,q}(l_1+1,l_2+1,l_3;\;l_4+1;z_1,z_2,z_3)\\&\qquad+\frac{z_2\,l_3}{l_4}H_{C,p,q}(l_1,l_2+1,l_3+1;l_4+1;z_1,z_2,z_3),\end{aligned}\tag{6.1}$$

$$\begin{aligned}&H_{C,p,q}(l_1,l_2,l_3+1;l_4;z_1,z_2,z_3)\\&\quad=H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)+\frac{z_2\,l_2}{l_4}H_{C,p,q}(l_1,l_2+1,l_3+1;l_4+1;z_1,z_2,z_3)\\&\qquad+\frac{z_3\,l_1}{l_4}H_{C,p,q}(l_1+1,l_2,l_3+1;l_4+1;z_1,z_2,z_3).\end{aligned}\tag{6.2}$$

Proof Using Eq. (2.1) and identity $(l_2+1)_{h+m}=(l_2)_{h+m}(1+h/l_2+m/l_2)$, implies us

$$\begin{aligned}&H_{C,p,q}(l_1,l_2+1,l_3;l_4;z_1,z_2,z_3)\\&\quad=\sum_{h,m,n=0}^{\infty}\left[\frac{(l_2+1)_{h+m}(l_3)_{m+n}}{(l_4)_m}\frac{B_{p,q}(l_1+h+n,l_4+m-l_1)}{B(l_1,l_4+m-l_1)}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^n}{n!}\right],\end{aligned}\tag{6.3}$$

$$\begin{aligned}&\quad=H_{C,p,q}(l_1,l_2,l_3;l_4;z_1,z_2,z_3)+\\&\qquad+\frac{z_1}{l_2}\sum_{h=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{(l_2)_{h+m}(l_3)_{m+n}}{(l_4)_m}\frac{B_{p,q}(l_1+h+n,l_4+m-l_1)}{B(l_1,l_4+m-l_1)}\frac{z_1^{h-1}}{(h-1)!}\frac{z_2^m}{m!}\frac{z_3^n}{n!}\right]+\\&\qquad+\frac{z_2}{l_2}\sum_{h=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left[\frac{(l_2)_{h+m}(l_3)_{m+n}}{(l_4)_m}\frac{B_{p,q}(l_1+h+n,l_4+m-l_1)}{B(l_1,l_4+m-l_1)}\frac{z_1^h}{h!}\frac{z_2^{m-1}}{(m-1)!}\frac{z_3^n}{n!}\right].\end{aligned}\tag{6.4}$$

In above Eq. (6.4) denote S_1 as first sum and replace $m \to m+1$ and applying formula $(z)_{n+1} = z(z+1)_n$, we obtain

$$S_1 = \frac{z_1}{l_2} \sum_{h,m,n=0}^{\infty} \left[\frac{(l_2)_{h+1+m}(l_3)_{m+n}}{(l_4)_m} \frac{B_{p,q}(l_1+1+h+n, l_4+m-l_1)}{B(l_1, l_4+m-l_1)} \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!} \right],$$
$$= z_1 \sum_{h,m,n=0}^{\infty} \left[\frac{(l_2+1)_{h+m}(l_3)_{m+n}}{(l_4)_m} \frac{B_{p,q}(l_1+1+h+n, l_4+m-l_1)}{B(l_1, l_4+m-l_1)} \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!} \right]. \tag{6.5}$$

Applying (4.3), we then obtain

$$S_1 = \frac{z_1\, l_1}{l_4} \sum_{h,m,n=0}^{\infty} \left[\frac{(l_2+1)_{h+m}(l_3)_{m+n}}{(l_4+1)_m} \frac{B_{p,q}(l_1+1+h+n, l_4+m-l_1)}{B(l_1+1, l_4+m-l_1)} \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!} \right],$$
$$= \frac{z_1\, l_1}{l_4} H_{C,p,q}(l_1+1, l_2+1, l_3; l_4+1; z_1, z_2, z_3). \tag{6.6}$$

Applying same procedure for the other series sum in (6.4) and replacing $m \to m+1$ we can obtain

$$S_2 = \frac{z_2\, l_3}{l_4} H_{C,p,q}(l_1, l_2+1, l_3+1; l_4+1; z_1, z_2, z_3). \tag{6.7}$$

Combining Eqs. (6.6) and (6.7) and comparing with Eq. (6.4) gives the result in (6.1). Similarly, an expression (6.2) is obtained in the same way by interchanging l_3.

Corollary 2 *The Eq. (6.1) provide a recursive relation as:*

$$H_{C,p,q}(l_1, l_2+N, l_3; l_4; z_1, z_2, z_3) = H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3)$$
$$+ \frac{z_1\, l_1}{l_4} \sum_{\ell=1}^{N} H_{C,p,q}(l_1+1, l_2+\ell, l_3; l_4+1; z_1, z_2, z_3)$$
$$+ \frac{z_2 l_3}{l_4} \sum_{\ell=1}^{N} H_{C,p,q}(l_1, l_2+\ell, l_3+1; l_4+1; z_1, z_2, z_3), \quad N \in \{1, 2, 3, ...\} \tag{6.8}$$

Corollary 3 *The Eq. (6.2) provides another recursive relation as:*

$$H_{C,p,q}(l_1, l_2, l_3+N; l_4; z_1, z_2, z_3) = H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3)$$
$$+ \frac{z_2\, l_2}{l_4} \sum_{\ell=1}^{N} H_{C,p,q}(l_1, l_2+1, l_3+\ell; l_4+1; z_1, z_2, z_3)$$
$$+ \frac{z_3\, l_1}{l_4} \sum_{\ell=1}^{N} H_{C,p,q}(l_1+1, l_2, l_3+\ell; l_4+1; z_1, z_2, z_3), \tag{6.9}$$

for positive integer N.

Theorem 7 *The recursive formula of $H_{C,p,q}(\cdot)$ involving the denominator parameter l_4 is given by:*

$$H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) = H_{C,p,q}(l_1, l_2, l_3; l_4+1; z_1, z_2, z_3) + \frac{z_2\, l_2 l_3}{l_4(l_4+1)}$$
$$H_{C,p,q}(l_1, l_2+1, l_3+1; l_4+2; z_1, z_2, z_3). \tag{6.10}$$

Proof Take

$$\mathbb{H} :\equiv H_{C,p,q}(l_1, l_2, l_3; l_4 - 1; z_1, z_2, z_3), \tag{6.11}$$

and upon the use of fact that $(l_4 - 1)_n = (l_4)_n / \left\{1 + \frac{n}{l_4 - 1}\right\}$. Then

$$\begin{aligned}
\mathbb{H} &= \sum_{h,m,n=0}^{\infty} \left[\frac{(l_2)_{h+m}(l_3)_{m+n}}{(l_4-1)_m} \frac{B_{p,q}(l_1+h+n, l_4+m-l_1)}{B(l_1, l_4+m-l_1)} \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!} \right], \\
&= \sum_{h,m,n=0}^{\infty} \left[\frac{(l_2)_{h+m}(l_3)_{m+n}}{(l_4)_m} \frac{B_{p,q}(l_1+h+n, l_4+m-l_1)}{B(l_1, l_4+m-l_1)} \left(1 + \frac{m}{l_4-1}\right) \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!} \right], \\
&= H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) + \\
&\quad + \frac{z_2}{l_4-1} \sum_{h,n=0}^{\infty} \sum_{m=1}^{\infty} \left[\frac{(l_2)_{h+m}(l_3)_{m+n}}{(l_4)_m} \frac{B_{p,q}(l_1+h+n, l_4+m-l_1)}{B(l_1, l_4+m-l_1)} \frac{z_1^h}{h!} \frac{z_2^{m-1}}{(m-1)!} \frac{z_3^n}{n!} \right].
\end{aligned} \tag{6.12}$$

Replacing $m \to m + 1$, we get

$$\begin{aligned}
\mathbb{H} &= H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) + \\
&\quad + \frac{z_2 l_2 l_3}{l_4(l_4-1)} \sum_{h,m,n=0}^{\infty} \left[\frac{(l_2+1)_{h+m}(l_3+1)_{m+n}}{(l_4+1)_m} \frac{B_{p,q}(l_1+h+n, l_4+1+m-l_1)}{B(l_1, l_4+1+m-l_1)} \frac{z_1^h}{h!} \frac{z_2^m}{m!} \frac{z_3^n}{n!} \right], \\
&= H_{C,p,q}(l_1, l_2, l_3; l_4; z_1, z_2, z_3) + \frac{z_2\, l_2 l_3}{l_4(l_4-1)} H_{C,p,q}(l_1, l_2+1, l_3+1; l_4+1; z_1, z_2, z_3).
\end{aligned} \tag{6.13}$$

Finally changing l_4 by $l_4 + 1$ we get desired result in (6.10). □

Conclusions

We have introduced the generalized Srivastava's triple hypergeometric function given by $H_{C,p,q}(.)$ in (2.1), together with the integral representations. Also, we derived an integral representation of the generalised Srivastava's function $H_{C,p,q}(\cdot)$ associated with Laguerre polynomial. In addition, we established some properties of this function, namely the Mellin transforms, a differential formula, a bounded inequality and recursion relations. This work is continuation of earlier work [6] on Srivastava triple hypergeometric function H_B which supports the corresponding results of this paper.

For motivating further research along the lines described in this paper, we choose to include a number of recent works [6, 22] which address generalized Srivastava's triple hypergeometric functions, and evaluation of some properties and inequalities. Also, since hypergeometric series are solutions of differential equatons, therefore this fact can be used to solve non-linear differential equations in future [23, 24] In addition, we can derive some results involving the fractional integral and derivative operators [10, 18, 19, 21].

Acknowledgements The Authors are grateful to the reviewers for their remarks which improved the earlier version of the paper.

Author Contributions All Authors contributed equally.

Funding This study was funded by University grants commission of India for the award of a Dr. D. S. Kothari Post Doctoral Fellowship(DSKPDF) (Grant number F.4-2/2006 (BSR)/MA/20-21/0061).

Availability of data and material Not applicable.

Code Availability Not applicable.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the author.

References

1. Chaudhry, M.A., Qadir, A., Rafique, M., Zubair, S.M.: Extension of Euler's beta function. J. Comput. Appl. Math. **78**(1), 19–32 (1997)
2. Chaudhry, M.A., Qadir, A., Srivastava, H.M., Paris, R.B.: Extended hypergeometric and confluent hypergeometric functions. Appl. Math. Comput. **159**(2), 589–602 (2004)
3. Choi, J., Hasanov, A., Srivastava, H.M., Turaev, M.: Integral representations for Srivastava's triple hypergeometric functions. Taiwanese J. Math. **15**(6), 2751–2762 (2011)
4. Choi, J., Hasanov, A., Turaev, M.: Integral representations for Srivastava's triple hypergeometric functions H_C. Honam Mathematical J. **34**(4), 473–482 (2012)
5. Choi, J., Rathie, A.K., Parmar, R.K.: Extension of extended Beta, hypergeometric function and confluent hypergeometric function. Honam Mathematical J. **36**(2), 357–385 (2014)
6. Dar, S.A., Paris, R.B.: A (p, q)-extension of Srivastava's triple hypergeometric function H_B and its properties. Journal of Computational and Applied Mathematics **348**, 237–245 (2019)
7. Karlsson, P.W.: Regions of convergences for hypergeometric series in three variables. Math. Scand **48**, 241–248 (1974)
8. Oberhettinger, F.: Tables of Mellin Transforms. Springer-Verlag, Berlin, Heidelberg, New York (1974)
9. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds.): NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge (2010)
10. Özarslan, M.A., Özergin, E.: Some generating relations for extended hypergeometric functions via generalized fractional derivative operator. Math. Comput. Modelling **52**, 1825–1833 (2010)
11. Özarslan, M.A.: Some remarks on extended hypergeometric, extended confluent hypergeometric and extended Appell's functions. J. Comput. Anal. Appl. **14**(6), 1148–1153 (2012)
12. Rainville, E. D. : Special Functions. Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York (1971)
13. Srivastava, H.M.: Hypergeometric functions of three variables. Ganita **15**, 97–108 (1964)
14. Srivastava, H.M.: Some integrals representing triple hypergeometric functions. Rend. Circ. Mat. Palermo (Ser. 2) **16**, 99–115 (1967)
15. Srivastava, H.M., Manocha, H.L.: A Treatise on Generating Functions Halsted Press (Ellis Horwood Limited, Chichester, U.K.) John Wiley and Sons, New york, Chichester, Brisbane and Toronto (1984)
16. Srivastava, H. M., Karlsson, P.W.: Multiple Gaussian Hypergeometric Series. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto (1985)
17. Slater, L.J.: Generalized Hypergeometric Functions. Cambridge University Press, Cambridge (1966)
18. Chaurasia, V. B. L., Gill, V.: New fractional calculus results involving Srivastava's general class of multivariable polynomials and The H-function, JAMSI, 11 (2015), No. 1 11(1), 19-32 (2015)
19. Chaurasia, V.B.L., Gill, V.: Generalized Elliptic-Type Integrals and Generating Functions with Aleph-Function. General Mathematics Notes **14**(1), 21–34 (2013)
20. Srivastava, H.M.: A note on certain hypergeometric differential equations. Mat. Vesnik **9**(24), 101–107 (1972)
21. Chandel, R.C.S., Vishwakarma, P.K.: Fractional derivatives of the multiple Hypergeometric functions of four variables. Jnanabha **26**, 83–86 (1996)
22. Dar, S.A., Paris, R.B.: $A(p, v)$ extension of the Appell function $F_1(.)$ and its properties. Journal of Computational and Applied Mathematics **358**, 12–19 (2019)

23. Sabir, Z.: Stochastic numerical investigations for nonlinear three-species food chain system. International Journal of Biomathematics **15**(04), 2250005 (2022)
24. Sabir, Z., Wahab, H.A., Javeed, S., Baskonus, H.M.: An Efficient Stochastic Numerical Computing Framework for the Nonlinear Higher Order Singular Models. Fractal Fract. **5**(4), 176 (2021)