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ORIGINAL PAPER

Fractional Integral and Derivative Formulae for Multi-index Wright Generalized Bessel Function

Raghib Nadeem[1] · Mohd. Saif[2,3] · Nabiullah Khan[2]

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Abstract
This article deals with the study of the multi-index Wright generalized Bessel function (or Bessel–Maitland function) $\mathbb{J}^{(\alpha_j)_m,\ \varsigma,\ c}_{(\beta_j)_m,\ k,\ b}(.)$ where $(j = 1, 2, \ldots, m)$ with the relation of pathway fractional integral operator and with extended Caputo fractional derivative operator which plays a remarkable contribution in the physical sciences, and various engineering disciplines, can be represented as the generalized Wright hypergeometric function ${}_r\Psi_s[z]$. We also discuss some special cases of our main result by choosing some particular values of the parameters in $\mathbb{J}^{(\alpha_j)_m,\ \varsigma,\ c}_{(\beta_j)_m,\ k,\ b}(z)$.

Keywords Bessel–Maitland function · Pathway fractional integral operator · Extended Caputo fractional derivative operator

Mathematics Subject Classification 2010 · 33C20 · 33B15

Introduction and Preliminaries

The Bessel function has grown noteworthiness due to its involvement in the solution of the problem in a conduction of heat, propagation of electromagnetic waves along wires, small vibration of gas, and variable flow of heat in a sphere, the stability of vertical wire, torsional vibration of a vertical cylinder, etc. (see [1]). Many authors of various fields have investigated

✉ Mohd. Saif
usmanisaif153@gmail.com

Raghib Nadeem
raghibmaths2015@gmail.com

Nabiullah Khan
nukhanmath@gmail.com

1 Department of Applied Sciences, Maulana Mukhtar Ahmad Nadvi Technical Campus, Malegaon, Nashik 423203, India

2 Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202002, India

3 Department of Mathematics, Madanpalle Institute of Technology and Science, Madanpalle, Andhra Pradesh 517325, India

several generalizations and a large number of integral transform (see [17, 19, 20, 24]). In [18] the authors developed different aspects of the generalized multi-index Bessel function, followed by significant unified integrals formulas containing the generalized multi-index Bessel function. The following definitions are considered in the present paper:

Definition 1 The generalized Wright hypergeometric function (Fox–Wright function) is represented by ${}_{r}\Psi_{s}[x]$ and is defined by (see [9, 10]):

$$ {}_{r}\Psi_{s}[x] = {}_{r}\Psi_{s}\left[\begin{matrix}(\varsigma_1,\acute{\varsigma}_1),\ldots,(\varsigma_r,\acute{\varsigma}_r);\\ (l_1,\acute{l}_1),\ldots,(l_s,\acute{l}_s);\end{matrix}\; x\right] $$

$$ = \sum_{k=0}^{\infty}\frac{\Gamma(\varsigma_1+\acute{\varsigma}_1 k),\ldots,\Gamma(\varsigma_r+\acute{\varsigma}_r k)}{\Gamma(l_1+\acute{l}_1 k),\ldots,\Gamma(l_s+\acute{l}_s k)}\frac{x^k}{k!} \tag{1} $$

$$ = H^{1,r}_{r,s+1}\left[-x\left|\begin{matrix}(1-\varsigma_1,\acute{\varsigma}_1),\ldots,(1-\varsigma_r,\acute{\varsigma}_r)\\ (0,1),(1-l_1,\acute{l}_1),\ldots,(1-l_s,\acute{l}_s)\end{matrix}\right.\right], \tag{2} $$

where $H^{1,r}_{r,s+1}[.]$ is a Fox-H function (see [5]) and the coefficients $\acute{\varsigma}_1,\ldots,\acute{\varsigma}_r,\acute{l}_1,\ldots,\acute{l}_s \in R^+$, such that $1+\sum_{j=1}^{s}\acute{l}_j-\sum_{i=1}^{r}\acute{\varsigma}_i$ for suitable value of $|x|$. On taking $\acute{\varsigma}_1=,\cdots,=\acute{\varsigma}_r=1$, $\acute{l}_1=,\cdots,=\acute{l}_s=1$ in (2), Fox–Wright function reduces to the generalized hypergeometric function ${}_{r}F_{s}$ (see [10]),

$$ {}_{r}\Psi_{s}\left[\begin{matrix}(\varsigma_1,\acute{\varsigma}_1),\ldots,(\varsigma_r,\acute{\varsigma}_r);\\ (l_1,\acute{l}_1),\ldots,(l_s,\acute{l}_s);\end{matrix}\; x\right] = \frac{\Gamma(\varsigma)_1,\ldots,\Gamma(\varsigma)_r}{\Gamma(l)_1,\ldots,\Gamma(l)_s}\,{}_{r}F_{s}(\varsigma_1,\ldots,\varsigma_r;l_1,\ldots,l_s;x). \tag{3} $$

We now present and investigate the generalised multi-index Bessel–Maitland function, also known as extended multi-index Bessel–Maitland function:

Definition 2 Let $\alpha_j,\ \beta_j,\ \varsigma,\ b,\ c\in\mathbb{C},(1\le j\le m)$ be such that $\sum_{j=1}^{m}\mathbb{R}(\alpha_j) > \max\{0;\mathbb{R}(k)-1\}; k>0, \mathbb{R}(\beta_j)>0, \mathbb{R}(\varsigma)>0$ then,

$$ \mathbb{J}^{(\alpha_j)_m,\ \varsigma,\ c}_{(\beta_j)_m,\ k,\ b}(z) = \sum_{n=0}^{\infty}\frac{(c)^n\ (\varsigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j n+\frac{1+b}{2})}\frac{z^n}{n!}, \qquad (m\in\mathbb{N}). \tag{4} $$

where $(\varsigma)_{kn}$ is the Pochhammer symbol given by:

$$ (\varsigma)_{kn} = \varsigma(\varsigma+k)(\varsigma+2k)\ldots,(\varsigma+(n-1)k), $$
$$ = \frac{\Gamma(\varsigma+nk)}{\Gamma(\varsigma)}\quad (\varsigma\in\mathbb{C},k\in\mathbb{R},n\in\mathbb{N}^+.) $$

Special Cases: The special cases of the multi-index Bessel–Maitland function $\mathbb{J}^{(\alpha_j)_m,\ \varsigma,\ c}_{(\beta_j)_m,\ k,\ b}(z)$ are listed below, with their respective parameter $(\alpha_j,\beta_j,\varsigma,k,b,c,(j=1,2,\ldots,m))$ values:

(i) On setting $b=c=1$ and $z\to -z$ in Eq. (4), then we get generalized multi-index Bessel function where $\alpha_j,\beta_j,\varsigma\in\mathbb{C}\ (j=1,2,\ldots,m)$, $\sum_{j=1}^{m}\mathbb{R}(\alpha_j) > \max\{0,\mathbb{R}(k)-1; k>0,\ \mathbb{R}(\beta)>-1\ \mathbb{R}(\varsigma)>0\}$

$$ \mathbb{J}^{(\alpha_j)_m,\ \varsigma,\ 1}_{(\beta_j)_m,\ k,\ 1}(z) = \mathbb{J}^{(\alpha_j)_m,\ \varsigma}_{(\beta_j)_m,\ k}(z). \tag{5} $$

(ii) On setting $\mathtt{b} = \mathtt{c} = 1$ in Eq. (4) then we get generalized multi-index Mittag–Leffler function (see [21]) which is given by

$$\mathbb{J}^{(\alpha_j)_m,\ \varsigma,\ 1}_{(\beta_j)_m,\ k,\ 1}(z) = \mathbb{E}^{\varsigma,k}_{(\alpha_j,\beta_j)^m_{j=1}} \tag{6}$$

where $\alpha_j, \beta_j, \varsigma, k, z \in \mathbb{C}, \mathbb{R}(\beta_j > 0),\ \mathbb{R}(\sum_{j=1}^{m} \alpha_j) >\ \max\{0, \mathbb{R}(k) - 1\}$.

(iii) Let $\mathtt{m} = \mathtt{c} = 1, \mathtt{b} = -1$, then multi-indices Bessel function reduces to Srivastava and Tomovski functions (see [14]).

$$\mathbb{J}^{(\alpha,\ \varsigma,\ 1)}_{(\beta,\ k,\ 1)}(z) = E^{\varsigma,k}_{\alpha,\beta}(z). \tag{7}$$

(iv) On substituting $\mathtt{m} = \alpha = 1, k = 0, \beta_1 = \nu$, and $z \to \frac{z^2}{4}$ in Eq. (4), we get the Bessel function (see [13])

$$\mathbb{J}^{1,\varsigma}_{\nu,0}\left(\frac{z^2}{4}\right) = \left(\frac{2}{z}\right)\mathbb{J}_{\nu}(z). \tag{8}$$

The theory of integrals and derivatives of any order $\alpha \in \mathbb{R}$ (even if $\alpha \in \mathbb{C}$) of a function is called a fractional calculus. It contributed significantly in the area of fractional calculus. As a result of expanding the scope of applications, it is one of the fastest-growing fields in mathematical research.For example, fractional differential equations are used for the pollution model [3]. Also, it has a various applications in a real world applications. For a detailed account of diversified applications of fractional differential equations, readers may refer to the literature (see, [6–8, 25])

Firstly, Riemann–Liouville defined fractional derivative operator of order ζ is given by:

$$D^{\zeta}_{x}g(x) = \frac{1}{\Gamma(n-\zeta)}\left(\frac{d}{dx}\right)^{n}\int_0^x \frac{g(\tau)}{(x-\tau)^{\zeta+1-n}}d\tau$$

when $n - 1 < \zeta < n$.

Note that if order of fractional derivative operator equals to n, $(n \in \mathbb{N})$, it reduces to simple n-th order derivative as $\frac{d^n}{dt^n}g(x)$. Italian mathematician Caputo in 1967 introduced the fractional derivative operator of order $\zeta > 0, n-1 < \zeta < n;\ n = 1, 2, \ldots$ is defined as:

$$D^{\zeta}g(x) = \frac{1}{\Gamma(n-\zeta)}\int_a^x \frac{g^{(n)}(\tau)}{(x-\tau)^{\zeta+1-n}}d\tau \ \text{ for } n-1 < \zeta < n\ , (n = 1, 2, \ldots).$$

Definition 3 The Caputo fractional derivative of order ζ is defined as

$$D^{\mu}_{z}\{g(z) : p, \nu\} = \sqrt{\frac{2pz^2}{\pi}}\frac{1}{\Gamma(m-\mu)}\int_0^z t^{-\frac{1}{2}}(z-t)^{m-\mu-\frac{3}{2}}K_{\nu+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right)\frac{d^m}{dt^m}g(t)dt$$

where $\Re(p) > 0, \nu \geq 0,\ m - 1 < \Re(u) < m, (m = 1, 2, \ldots)$ (see [11, 12]).

Definition 4 In [4, 22] the classical Caputo derivative of fractional order μ is given by

$$D^{\mu}_{z}\{g(z)\} = \frac{1}{\Gamma(n-\mu)}\int_0^z (z-t)^{n-\mu-1}\frac{d^n}{dt^n}g(t)dt,$$

where $n - 1 < \mu < n$, $(n = 1, 2, \ldots)$.

Definition 5 An extended form of Caputo derivative operator is defined as

$$D_{\mathrm{z}}^{\mu,p}\{\mathrm{g(z)}\}=\frac{1}{\Gamma(\mathrm{n}-\mu)}\int_0^{\mathrm{z}}(\mathrm{z}-t)^{\mathrm{n}-\mu-1}exp\left(\frac{-pz^2}{t(z-t)}\right)\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{d}t^{\mathrm{n}}}\mathrm{g}(t)dt$$

where $\Re(p)>0, \mu>0, \mathrm{n}-1<\Re(\mu)<\mathrm{n}, (\mathrm{n}=1,2,\ldots)$(see [16]).

Lemma 1 *For* $\mathrm{n}-1<\Re(\mu)$ *and* $\Re(\mu)<\Re(\eta), \Re(\mu)>0$ *the formula is as follows (see* [16]*):*

$$D_{\mathrm{z}}^{\mu,\nu}\left\{\mathrm{z}^{\eta}\right\}=\frac{\Gamma(\eta+1)\beta_{\nu}(\eta-\mathrm{n}+1,\mathrm{n}-\mu)}{\Gamma(\eta-\mu+1)\beta(\eta-\mathrm{n}+1,\mathrm{n}-\mu)}\mathrm{z}^{\eta-\mu}. \tag{9}$$

Many authors have recently considered pathway fractional integral operators involving various special functions. (see [13, 15, 16, 23]).

Definition 6 Let $\mathrm{g(x)}$ is Lebesgue measurable function and for $\eta\in\mathbb{C}, \Re(\eta)>0, \mathrm{d}>0$ and pathway parameter $\lambda<1$ then the pathway fractional integration operator [23] is defined as:

$$\left(P_{0^+}^{(\eta,\lambda,\mathrm{d})}\mathrm{g}\right)(\mathrm{x})=\mathrm{x}^{\eta}\int_0^{\left[\frac{\mathrm{x}}{\mathrm{d}(1-\lambda)}\right]}\left[1-\frac{\mathrm{d}(1-\lambda)t}{\mathrm{x}}\right]^{\frac{\eta}{1-\lambda}}\mathrm{g}(t)dt. \tag{10}$$

Let $[\mathrm{a},\mathrm{b}]\subseteq\mathbb{R}$, the left sided and right sided Riemann–Liouville integral $I_{\mathrm{a}+}^{\eta}\mathrm{g}$ and $I_{\mathrm{b}-}^{\eta}\mathrm{g}$ of order $\eta\in\mathbb{C}$, $(\mathbb{R}(\eta)>0)$ are defined respectively by

$$\left(I_{\mathrm{a}+}^{\eta}\mathrm{g}\right)(\mathrm{x})=\frac{1}{\Gamma(\eta)}\int_{\mathrm{a}}^{\mathrm{x}}\frac{\mathrm{g}(t)\mathrm{d}t}{(\mathrm{x}-t)^{1-\eta}}\quad(\mathrm{x}>\mathrm{a},\mathbb{R}(\eta)>0),$$

and

$$\left(I_{\mathrm{b}-}^{\eta}\mathrm{g}\right)(\mathrm{x})=\frac{1}{\Gamma(\eta)}\int_{\mathrm{x}}^{\mathrm{b}}\frac{\mathrm{g}(t)\mathrm{d}t}{(t-\mathrm{x})^{1-\eta}}\quad(\mathrm{x}<\mathrm{b},\mathbb{R}(\eta)>0).$$

Note: On setting $\lambda=0, \mathrm{a}=1$ and $\eta\to\eta-1, \mathbb{R}(\eta)>0$, then the pathway fractional integration operator (10) reduces to left-sided Riemann–Liouville fractional integral given as follows:

$$\left(P_{0^+}^{\eta-1,0,1}\mathrm{g}\right)(t)=\Gamma(\eta)\left(I_{0+}^{\eta}\mathrm{g}\right)(t).$$

For $0\le\mathrm{a}<t<\mathrm{b}\le\infty, \mathbb{R}(\eta)>0, \sigma>0, \zeta\in\mathbb{C}$, one of the Erdélyi–Kober type fractional integral operator (see [2]) defined as:

$$\left(I_{\mathrm{a}^+,\sigma,\zeta}^{\eta}\mathrm{g}(t)\right):=\frac{\sigma(t)^{-\sigma(\eta+\zeta)}}{\Gamma(\eta)}\int_{\mathrm{a}}^{t}\frac{\tau^{\sigma\mathrm{a}+\sigma-1}\mathrm{g}(\tau)\mathrm{d}\tau}{(t^{\sigma-\tau^{\sigma}})^{1-\eta}}.$$

Pathway fractional integration operator is closely related to Erdélyi–Kober operator which is given as:

$$\left(P_{0^+}^{\eta-1,0,1}\mathrm{g}\right)(t)=\Gamma(\eta)t^{\eta}\left(I_{0^+,1,0}^{\eta}\mathrm{g}\right)(t)\qquad(\mathbb{R}(\eta)>0).$$

Now on setting $\mathrm{g}(t)=t^{\beta-1}$ in Eq. (10), we get the following relation.

Lemma 2 *Let* $\beta,\eta\in\mathbb{C}, \mathbb{R}(\eta), \mathbb{R}(\beta)>0,\ \zeta<1, \mathbb{R}(\frac{\eta}{1-\zeta})>-1$*, then we have the following result:*

$$\left\{P_{0^+}^{(\eta,\zeta)}t^{\beta-1}\right\}(\mathrm{x})=\frac{t^{\eta+\beta}\Gamma(\beta)\Gamma(1+\frac{\eta}{1-\zeta})}{[a(1-\zeta)]^{\beta}\Gamma(1+\frac{\eta}{1-\zeta}+\beta)}.$$

Lemma 3 *Let* $\beta, \eta \in \mathbb{C}, \mathbb{R}(\eta), \mathbb{R}(\beta) > 0, \lambda < 1, \mathbb{R}(\frac{\eta}{1-\zeta}) > -1$, *then we have the following result:*

$$\left\{ P_{0^+}^{(\eta,\lambda,\mathrm{d})}(t)^{\beta-1} \right\}(\mathrm{x}) = \frac{\mathrm{x}^{\eta+\beta}\Gamma(\beta)\Gamma(1+\frac{\eta}{1-\lambda})}{[\mathrm{d}(1-\lambda)]^{\beta}\Gamma(1+\frac{\eta}{1-\lambda}+\beta)}. \tag{11}$$

Pathway Fractional Integral of Multi-index Wright Generalized Bessel Function

We consider the composition of the pathway fractional integration operator given by Eq. (10) with the multi-index Bessel Maitland function (4) in this section, and obtain the findings involving with Fox–Wright function (1). Using some appropriate parameters, we also discuss some relevant and useful corollaries.

Theorem 1 *Let* $\zeta_{\mathrm{j}}, \beta_{\mathrm{j}}\ (\mathrm{j} = 1, 2, \ldots, \mathrm{m}), \varsigma, \eta, \mathrm{b}, \mathrm{c} \in \mathbb{C}, \mathbb{R}(\eta) > 0, \zeta < 1, \mathbb{R}(\beta) > 0, \mathbb{R}(\frac{\eta}{1-\zeta}) > -1$, *be such that* $\sum_{\mathrm{j}=1}^{\mathrm{m}} \mathbb{R}(\zeta_{\mathrm{j}}) > max\,\{0;\ \mathbb{R}(\mathrm{k}) - 1\}\,;\, \mathrm{k} > 0, \mathbb{R}(\beta_{\mathrm{j}}) > 0, \mathbb{R}(\varsigma) > 0$, *then following relation holds:*

$$\left\{ P_{0^+}^{\eta,\zeta,\mathrm{d}} \left(t^{\beta-1} \mathbb{J}_{(\beta_{\mathrm{j}})_{\mathrm{m}},q,\mathrm{b}}^{(\zeta_{\mathrm{j}})_{\mathrm{m}},\tau,\mathrm{c}}(\mathrm{z}t^{\nu}) \right) \right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[\mathrm{d}(1-\zeta)]^{\beta}}$$

$$\times\, {}_2\Psi_{\mathrm{m}+1}\left[\begin{array}{c} (\varsigma, \mathrm{k}),\ (\beta, \nu) \\ (\beta_{\mathrm{j}} + \frac{b+1}{2}, \zeta_{\mathrm{j}})_{\mathrm{j}=1}^{\mathrm{m}},\ (1 + \frac{\mathrm{n}}{1-\zeta} + \beta, \nu) \end{array} \Big|\ \frac{\mathrm{z}u\mathrm{c}}{[\mathrm{d}(1-\zeta)]} \right]. \tag{12}$$

Proof $\mathbb{I}$ stands for the left-hand side of the Eq. (12). On replacing $\mathrm{z} \to \mathrm{z}t^{\nu}$ with the definition of (4), we get

$$\mathbb{I} = \left\{ P_{0^+}^{\eta,\zeta,\mathrm{d}} \left(t^{\beta-1} \mathbb{J}_{(\beta_{\mathrm{j}})_{\mathrm{m}},q,\mathrm{b}}^{(\zeta_{\mathrm{j}})_{\mathrm{m}},\tau,\mathrm{c}}(\mathrm{z}t^{\nu}) \right) \right\}(u)$$

$$\mathbb{I} = \sum_{\mathrm{n}=0}^{\infty} \frac{\mathrm{c}^{\mathrm{n}}\,(\varsigma)_{\mathrm{kn}}}{\prod_{\mathrm{j}=1}^{\mathrm{m}} \Gamma(\zeta_{\mathrm{j}}\mathrm{n} + \beta_{\mathrm{j}}\mathrm{n} + \frac{\mathrm{b}+1}{2})} \frac{\mathrm{z}^{\mathrm{n}}}{\mathrm{n}!} \left\{ P_{0^+}^{\eta,\zeta,\mathrm{d}}(t^{\beta+n\nu-1}) \right\}.$$

After simplifying and using (11), we get

$$\mathbb{I} = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[\mathrm{d}(1-\zeta)]^{\beta}} \sum_{\mathrm{n}=0}^{\infty} \frac{\mathrm{c}^{\mathrm{n}}(\varsigma)_{\mathrm{kn}}\Gamma(\beta+\mathrm{n})}{\Gamma(\zeta_{\mathrm{j}}\mathrm{n} + \beta_{\mathrm{j}}\mathrm{n} + \frac{\mathrm{b}+1}{2})\Gamma(1+\frac{\eta}{1-\zeta}+\beta+\mathrm{n})} \frac{\mathrm{z}^{\mathrm{n}}}{\mathrm{n}!} \left(\frac{\mathrm{c}u}{[\mathrm{d}(1-\zeta)]} \right)^{\mathrm{n}}.$$

After using the definition of (1), we obtain our findings. □

Corollary 1 *Let* $\eta, \zeta_{\mathrm{j}}, \beta_{\mathrm{j}}, \varsigma, \mathrm{b}, \mathrm{c} \in \mathbb{C}, \zeta < 1, \mathrm{d} = 1, \mathbb{R}(\beta), \mathbb{R}(\varsigma) > 0, \mathbb{R}(\frac{\eta}{1-\zeta}) > -1$, *be such that* $\mathbb{R}(\beta_{\mathrm{j}}) > 0$, *then following relation hold:*

$$\left\{ P_{0^+}^{\eta,\zeta} \left(t^{\beta-1} \mathbb{J}_{(\beta_{\mathrm{j}})_{\mathrm{m}},q,\mathrm{b}}^{(\zeta_{\mathrm{j}})_{\mathrm{m}},\tau,c}(\mathrm{z}t^{\nu}) \right) \right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[(1-\zeta)]^{\beta}}$$

$$\times\, {}_2\Psi_{\mathrm{m}+1}\left[\begin{array}{c} (\varsigma, \mathrm{k}),\ (\beta, \nu) \\ (\beta_{\mathrm{j}} + \frac{b+1}{2}, \zeta_{\mathrm{j}})_{\mathrm{j}=1}^{\mathrm{m}},\ (1 + \frac{n}{1-\zeta} + \beta, \nu) \end{array} \Big|\ \frac{\mathrm{z}u\mathrm{c}}{[(1-\zeta)]} \right].$$

Corollary 2 *Let* $\mathtt{c} = \mathtt{m} = 1, \mathtt{b} = -1, \eta, \zeta, \beta, \varsigma, \in \mathbb{C}, \mathbb{R}(\eta), \mathbb{R}(\varsigma), \mathbb{R}(\beta) > 0, \zeta < 1, , \mathbb{R}(\frac{\eta}{1-\zeta}) > -1$*, then following relation hold:*

$$\left\{P_{0^+}^{\eta,\zeta,\mathtt{d}}\left(t^{\beta-1}\mathbb{E}_{(\zeta,\beta,)}^{(\varsigma,\mathtt{k})}(\mathtt{z}t^{\nu})\right)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[\mathtt{d}(1-\zeta)]^{\beta}} \times {}_2\Psi_1\left[\begin{matrix}(\varsigma,\mathtt{k}),\ (\beta,\nu)\\ (\beta+\frac{b+1}{2},\zeta),\ (1+\frac{n}{1-\zeta}+\beta,\nu)\end{matrix}\,\Big|\,\frac{\mathtt{z}u}{[\mathtt{d}(1-\zeta)]}\right].$$

Corollary 3 *Let* $\mathtt{c} = 1, \mathtt{b} = -1, \eta, \zeta_{\mathtt{j}}, \beta_{\mathtt{j}}, \varsigma, \mathtt{b}, \mathtt{c} \in \mathbb{C},\ \mathbb{R}(\eta),\ \mathbb{R}(\beta),\ \mathbb{R}(\varsigma) > 0, \zeta < 1, \mathbb{R}(\frac{\eta}{1-\zeta}) > -1, \mathbb{R}(\beta_j) > 0$*, then*

$$\left\{P_{0^+}^{\eta,\zeta,\mathtt{d}}\left(t^{\beta-1}\mathbb{E}_{(\beta_{\mathtt{j}})_{\mathtt{m}},\mathtt{k}}^{(\zeta_{\mathtt{j}})_{\mathtt{m}},\varsigma}(\mathtt{z}t^{\nu})\right)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[\mathtt{d}(1-\zeta)]^{\beta}} \times {}_2\Psi_{\mathtt{m}+1}\left[\begin{matrix}(\varsigma,\mathtt{k}),\ (\beta,\nu)\\ (\beta_{\mathtt{j}},\zeta_{\mathtt{j}})_{\mathtt{j}=1}^{\mathtt{m}},\ (1+\frac{\mathtt{n}}{1-\zeta}+\beta,\nu)\end{matrix}\,\Big|\,\frac{\mathtt{z}u}{[d(1-\zeta)]}\right].$$

Corollary 4 *Let* $\mathtt{b} = \mathtt{c} = 1$ *and* $\mathtt{z} \to -\mathtt{z}, \eta, \zeta_j, \beta_{\mathtt{j}}, \varsigma, \mathtt{b}, \mathtt{c} \in \mathbb{C}, \mathbb{R}(\eta), \mathbb{R}(\beta), \mathbb{R}(\varsigma) > 0, \mathtt{a} < 1, \mathbb{R}(\frac{\eta}{1-\zeta}) > -1, \mathbb{R}(\beta_{\mathtt{j}}) > 0$*, then*

$$\left\{P_{0^+}^{\eta,\zeta,\mathtt{d}}\left(t^{\beta-1}\mathbb{J}_{(\beta_{\mathtt{j}})_{\mathtt{m}},q}^{(\zeta_{\mathtt{j}})_{\mathtt{m}},\tau}(\mathtt{z}t^{\nu})\right)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[\mathtt{d}(1-\zeta)]^{\beta}} \times {}_2\Psi_{\mathtt{m}+1}\left[\begin{matrix}(\varsigma,\mathtt{k}),\ (\beta,\nu)\\ (\beta_{\mathtt{j}}+1,\zeta_{\mathtt{j}})_{\mathtt{j}=1}^{\mathtt{m}},\ (1+\frac{\mathtt{n}}{1-\zeta}+\beta,\nu)\end{matrix}\,\Big|\,\frac{\mathtt{z}u}{[\mathtt{d}(1-\zeta)]}\right].$$

Corollary 5 *Let* $\zeta = 0, \mathtt{a} = 1, \eta \to \eta-1, \zeta_{\mathtt{j}}, \beta_{\mathtt{j}}, b, \mathtt{c} \in \mathbb{C},\ \mathbb{R}(\varsigma),\ \mathbb{R}(\eta) > 0$*, then following relation hold:*

$$\left\{P_{0^+}^{\eta,0,1}\left(t^{\beta-1}\mathbb{J}_{(\beta_{\mathtt{j}})_{\mathtt{m}},q,\mathtt{b}}^{(\zeta_{\mathtt{j}})_{\mathtt{m}},\tau,\mathtt{c}}(\mathtt{z}t^{\nu})\right)\right\}(u) = \frac{u^{\eta+\beta-1}\Gamma(\eta)}{\Gamma(\varsigma)} \times {}_2\Psi_{\mathtt{m}+1}\left[\begin{matrix}(\varsigma,\mathtt{k}),\ (\beta,\nu)\\ (\beta_{\mathtt{j}}+\frac{\mathtt{b}+1}{2},\zeta_{\mathtt{j}})_{\mathtt{j}=1}^{\mathtt{m}},\ (\eta+\beta,\nu)\end{matrix}\,\Big|\,\mathtt{z}u\right].$$

Corollary 6 *Let* $\varsigma, \mathtt{k} = 1, \zeta_{\mathtt{j}} \to \frac{1}{\zeta_{\mathtt{j}}}\ (\mathtt{j} = 1, 2\ldots, \mathtt{m}), \eta, \zeta, \in \mathbb{C}, \mathbb{R}(\varsigma), \mathbb{R}(\beta) > 0$*, then:*

$$\left\{P_{0^+}^{\eta,\zeta,\mathtt{b}}\left(t^{\beta-1}\mathbb{E}_{(\beta_{\mathtt{j}})_{\mathtt{m}}}^{(\frac{1}{\zeta_{\mathtt{j}}})_{\mathtt{m}}}(\mathtt{z}t^{\nu})\right)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[\mathtt{d}(1-\zeta)]^{\beta}} \times {}_2\Psi_{\mathtt{m}+1}\left[\begin{matrix}(1,1),\ (\beta,\nu)\\ (\beta_{\mathtt{j}},\frac{1}{\zeta_{\mathtt{j}}})_{\mathtt{j}=1}^{\mathtt{m}},\ (1+\frac{\mathtt{n}}{1-\zeta}+\beta,\nu)\end{matrix}\,\Big|\,\frac{\mathtt{z}u}{[\mathtt{d}(1-\zeta)]}\right].$$

Corollary 7 *Let* $\mathtt{b} = \mathtt{c} = \mathtt{m} = 1, \mathtt{k} = 0, \zeta_1 = 1, \beta_1 = \nu$ *and* $\mathtt{z} \to \frac{\mathtt{z}^2}{4}, \mathbb{R}(\varsigma) > 0, \zeta < 1, \beta \in \mathbb{C}$*, then*

$$\left\{P_{0^+}^{\eta,\zeta,\mathtt{d}}\left(t^{\beta-1}\left(\frac{2}{\mathtt{z}}\right)^{\nu}\mathbb{J}_{\nu}(\mathtt{z}t^{\nu})\right)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[\mathtt{d}(1-\zeta)]^{\beta}} \times {}_2\Psi_{\mathtt{m}+1}\left[\begin{matrix}(\varsigma,0),\ (\beta,\nu)\\ (\beta+1,\zeta)(1+\frac{\mathtt{n}}{1-\zeta}+\beta,\nu)\end{matrix}\,\Big|\,\frac{\mathtt{z}u}{[\mathtt{d}(1-\zeta)]}\right].$$

Caputo Fractional Differential Operator of Multi-index Bessel–Maitland Function

We consider the composition of the fractional order of Caputo differential operator given by Eqs. (3) and (9) with the multi-index Bessel–Maitland function (4) in this section, and obtain the findings involving with Fox–Wright function (1). Also, using some specific parameter values, we discuss some useful results on these as provided in the corollaries.

Theorem 2 *Let* $\mathbb{R}(\mu), \mathbb{R}(\varsigma) > 0, \zeta_j, \beta_j \ (j = 1, 2, \ldots, m), b, c \in \mathbb{C}, m - 1 < \mathbb{R}(\mu) < m, \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\varsigma) > 0, \mathbb{R}(m - \mu) > 0$*, then following relation hold:*

$$D_z^{\mu}\left\{\left(\mathbb{J}_{(\beta_j)_m, q, b}^{(\zeta_j)_m, \tau, c}(z, p, \nu)\right)\right\}(u) = \frac{\beta_\nu(n - m + 1, m - \mu; p)}{\Gamma(\varsigma)\Gamma(m - \mu)}$$

$$\times \ {}_2\Psi_{m+1}\left[\begin{matrix}(\varsigma, k), \ (1, 1) \\ (\beta_j + \frac{b+1}{2}, \zeta_j)_{j=1}^m, (1 - m, 1)\end{matrix} \ \middle| \ z^{1-\mu}\right]. \tag{13}$$

Proof Let us denote left-hand side of Eq. (13) by $\mathbb{I}$ and applying the definition of (4), we get

$$\mathbb{I} = D_z^{\mu}\left\{\left(\mathbb{J}_{(\beta_j)_m, q, b}^{(\zeta_j)_m, \tau, c}(z, p, \nu)\right)\right\}(u)$$

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{c^n (\varsigma)_{kn}}{\prod_{j=1}^{m} \Gamma(\zeta_j n + \beta_j n + \frac{b+1}{2}) n!} \left\{D_z^{\mu}(z^n; p, \nu)\right\}.$$

Simplifying the above and using the result of Eq. (9), we get

$$\mathbb{I} = \frac{1}{\Gamma(\varsigma)} \sum_{n=0}^{\infty} \frac{c^n B_\nu(n - j + 1, m - \mu; p)\Gamma(\varsigma + kn)\Gamma(\beta + n)\Gamma(n + 1)}{\Gamma(n - m + 1)\Gamma(m - \mu)\Gamma(\zeta_j n + \beta_j n + \frac{b+1}{2})} \frac{z^{n-\mu}}{n!}.$$

In view of the definition (1) we get the required findings. □

Corollary 8 *Let* $\mathbb{R}(\varsigma) > 0, \ \mathbb{R}(m - \mu) > 0, \ m - 1 < \mathbb{R}(\mu) < m, \ \mathbb{R}(\mu) < \mathbb{R}(\eta),$ *and* $p = 1, \ \mathbb{R}(\mu) > 0, \ \zeta_j, \beta_j \ (j = 1, 2, \ldots, m), b, c \in \mathbb{C}$ *then following relation hold:*

$$D_z^{\mu}\left\{\left(\mathbb{J}_{(\beta_j)_m, q, b}^{(\zeta_j)_m, \tau, c}(z, \nu)\right)\right\}(u) = \frac{1}{\Gamma(\varsigma)} \, {}_2\Psi_{m+1}\left[\begin{matrix}(\varsigma, k), \ (1 - m, 1) \\ (\beta_j + \frac{b+1}{2}, \zeta_j)_{j=1}^m, (m, 1)\end{matrix} \ \middle| \ z^{1-\mu}\right].$$

Corollary 9 *Let* $m - 1 < \mathbb{R}(\mu) < m, \ \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\varsigma), \mathbb{R}(\mu), \mathbb{R}(m - \mu) > 0, b = 1, c = 1, \zeta_j, \beta_j \ \ (j = 1, 2, \ldots, m), b, c \in \mathbb{C}$ *then following relation hold:*

$$D_z^{\mu}\left\{\left(\mathbb{E}_{(\beta_j)_m, k}^{(\zeta_j)_m, \varsigma}(z, p, \nu)\right)\right\}(u) = \frac{\beta_\nu(n - m + 1, m - \mu; p)}{\Gamma(\varsigma)\Gamma(m - \mu)}$$

$$\times {}_2\Psi_{m+1}\left[\begin{matrix}(\varsigma, k), \ (1, 1) \\ (\beta_j, \zeta_j)_{j=1}^m, (1 - m, 1)\end{matrix} \ \middle| \ z^{1-\mu}\right].$$

Corollary 10 *Let* $m - 1 < \mathbb{R}(\mu) < m, \ \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\varsigma) > 0, \mathbb{R}(m - \mu) > 0,$ *and* $b = -1, c = 1, m = 1, \mathbb{R}(\mu) > 0, \zeta_j, \beta_j \ \ (j = 1, 2, \ldots, m), b, c \in \mathbb{C}$ *then following relation hold:*

$$D_z^{\mu}\left\{\left(\mathbb{E}_{(\beta, k)}^{(\zeta, \varsigma)}(z, p, \nu)\right)\right\}(u) = \frac{\beta_\nu(n - m + 1, m - \mu; p)}{\Gamma(\varsigma)\Gamma(m - \mu)}$$

$$\times {}_2\Psi_2\left[\begin{matrix}(\varsigma, k), \ (1, 1) \\ (\beta, \zeta), (1 - m, 1)\end{matrix} \ \middle| \ z^{1-\mu}\right].$$

Corollary 11 *Let* $\mathtt{m}-1<\mathbb{R}(\mu)<\mathtt{m}, \mathbb{R}(\mu)<\mathbb{R}\eta), \ \mathbb{R}(\varsigma)>0, \mathbb{R}(\mathtt{m}-\mu)>0, \ \mathtt{b}=\mathtt{c}=1, \mathtt{z}\rightarrow -\mathtt{z}, \mathbb{R}(\mu)>0, \zeta_{\mathtt{j}}, \beta_{\mathtt{j}} \quad (\mathtt{j}=1,2,\ldots,m), b, c \in \mathbb{C}$ *then following relation hold:*

$$D_{\mathtt{z}}^{\mu}\left\{\left(\mathbb{J}_{(\beta_{\mathtt{j}})_{\mathtt{m}},q}^{(\zeta_{\mathtt{j}})_{\mathtt{m}},\tau}(\mathtt{z},p,\nu)\right)\right\}(u)=\frac{\beta_{\nu}(n-\mathtt{m}+1,\mathtt{m}-\mu;p)}{\Gamma(\varsigma)\Gamma(\mathtt{m}-\mu)} \times {}_2\Psi_{\mathtt{m}+1}\left[\begin{matrix}(\varsigma,\mathtt{k}),\ (1,1)\\(\beta_{\mathtt{j}},\zeta_{\mathtt{j}})_{\mathtt{j}=1}^{\mathtt{m}},\ (1-\mathtt{m},1)\end{matrix}\mid \mathtt{z}^{1-\mu}\right].$$

Corollary 12 *Let* $\mathtt{m}-1<\mathbb{R}(\mu)<\mathtt{m}, \mathbb{R}(\mu)<\mathbb{R}(\eta), \mathbb{R}(\varsigma)>0, \mathbb{R}(\mathtt{m}-\mu)>0,$ *and* $\varsigma=\mathtt{k}=1, \zeta_{\mathtt{j}}\rightarrow\frac{1}{\zeta_{\mathtt{j}}}, (\mathtt{j}=1,2,\ldots,m), \mathbb{R}(\varsigma)>0, \mathbb{R}(\mu)>0, \zeta_{\mathtt{j}}, \beta_{\mathtt{j}}\in\mathbb{C}$ *then following relation holds:*

$$D_{\mathtt{z}}^{\mu}\left\{\left(\mathbb{E}_{(\beta_{\mathtt{j}})_{\mathtt{m}},1}^{\frac{1}{(\zeta_{\mathtt{j}})}_{\mathtt{m}},1}(\mathtt{z},p,\nu)\right)\right\}(u)=\frac{\beta_{\nu}(n-\mathtt{m}+1,\mathtt{m}-\mu;p)}{\Gamma(\mathtt{m}-\mu)} \times {}_2\Psi_{\mathtt{m}+1}\left[\begin{matrix}(1,1),\ (1,1)\\(\beta_{\mathtt{j}}+\frac{\mathtt{b}+1}{2},\zeta_{\mathtt{j}})_{\mathtt{j}=1}^{\mathtt{m}},(1-\mathtt{m},1)\end{matrix}\mid \mathtt{z}^{1-\mu}\right].$$

Remark 1 On setting $\varsigma=\mathtt{k}=1, \zeta_{\mathtt{j}}\rightarrow\frac{1}{\zeta_{\mathtt{j}}}, (\mathtt{j}=1,2,\ldots,\mathtt{m}), \mathbb{R}(\varsigma), \mathbb{R}(\mu)>0, \zeta_{\mathtt{j}}, \beta_{\mathtt{j}}\in\mathbb{C}, \mathtt{m}-1<\mathbb{R}(\mu)<\mathtt{m}, \mathbb{R}(\mu)<\mathbb{R}(\eta), \mathbb{R}(\varsigma)>0, \mathbb{R}(\mathtt{m}-\mu)>0$, then following relation hold:

$$D_{\mathtt{z}}^{\mu}\left\{\left(\mathbb{E}_{(\beta_{\mathtt{j}})_{\mathtt{m}},1}^{\frac{1}{(\zeta_{\mathtt{j}})}_{\mathtt{m}},1}(\mathtt{z},\nu)\right)\right\}(u)=\frac{1}{\Gamma(\varsigma)}\,{}_2\Psi_{\mathtt{m}+1}\left[\begin{matrix}(\varsigma,k),\ (1-\mathtt{m},1)\\(\beta_{\mathtt{j}}+\frac{\mathtt{b}+1}{2},\zeta_{\mathtt{j}})_{\mathtt{j}=1}^{\mathtt{m}},\ (\mathtt{m},1)\end{matrix}\mid \mathtt{z}^{1-\mu}\right].$$

Corollary 13 *Suppose* $\beta, \mathtt{b}, \mathtt{c}\in\mathbb{C}, \zeta<1, \mathbb{R}(\varsigma)>0, \mathtt{b}=\mathtt{c}=\mathtt{m}=\zeta_1=1, \mathtt{k}=0, \beta_1=\nu$ *and* $\mathtt{z}\rightarrow\frac{\mathtt{z}^2}{4}$, *then following relation hold:*

$$\left\{D_{\mathtt{z}}^{\mu}\left(t^{\beta-1}\left(\frac{2}{\mathtt{z}}\right)^{\nu}\mathbb{J}_{\nu}(\mathtt{z}t^{\nu})\right)\right\}(u)=\frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\zeta})}{\Gamma(\varsigma)\,[\mathtt{d}(1-\zeta)]^{\beta}} \times {}_2\Psi_2\left[\begin{matrix}(\varsigma,0),\ (\beta,\nu)\\(\beta+1,\zeta),\ (1+\frac{\mathtt{n}}{1-\zeta}+\beta,\nu)\end{matrix}\mid \frac{\mathtt{z}u}{[\mathtt{d}(1-\zeta)]}\right].$$

Concluding Remark

In this manuscript, we have introduced a new extended multi-index Bessel–Maitland function that is most general in nature; one can find many other impressive special functions involving Bessel functions, multi-index Mittag–Leffler functions, Srivastava and Tomovski functions, after appropriate parametric replacements (see [13, 14, 21]). Moreover, the results obtained in this manuscript yield the pathway fractional integration composition formula and the Caputo fractional integration operator involving Fox–Wright function. These results can also be expressed as Fox H-functions, Meijer G-functions, and other functions. Therefore, the results derived in this article would at once give way a large number of results involving a variety of special functions that appear in problems such as mathematical physics, science, and engineering.

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