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ORIGINAL PAPER

An Extended Analytical and Numerical Study the Nonlocal Boundary Value Problem for the Functional Integro-Differential Equation with the Different Conditions

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Abstract
In this work, we present a study of the nonlocal functional integro-differential equation with the nonlocal conditions. This study is different from the rest of the previous studies as we do the analytical study by studying the existence and uniqueness of the solution and also the continuous dependence of the proposed system. In addition, we apply all results of the analytical studying to some examples and find the exact solutions for them using the modified decomposition method. Also, we offer a numerical study of this system, unless it has been previously studied for the method of solving the proposed examples numerically using the finite difference-Simpson's method. Some comparisons of numerical solutions are given with exact solutions to show the accuracy of the methods used, in addition to some figures that illustrate this.

Keywords Nonlocal problem · Existence of solutions · Continuous dependence · Finite difference · Simpson's method · Modified decomposition method

Introduction

Nonlocal boundary value problems (NBVP) for nonlinear differential equations have attracted great research efforts worldwide, as they arise from the study of many important problems in various such as engineering, mechanics, mathematical physics, vehicular traffic theory, queuing theory, fluid flows, electrical networks, rheology, biology and chemical physics. In

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practical applications and also several real world problems, it is important to establish the conditions for the existence solutions. Hence, many authors have investigated the existence solutions for various functional differential equation NBVP, "such as Srivastava et al. studied a class of nonlinear boundary value problems for an arbitrary fractional-order with the nonlocal integral and infinite-point boundary conditions [29], El-Sayed et al. discussed many various types for functional differential equations see [14–17]. Also, El-Owaidy et al. studied on an integro-differential equation of arbitrary (fractional) orders [13]. Moreover, different igniters have been studied for the differential equations by several researchers [1–11, 19–22, 24–26, 30–32]". In this paper we study the NBVP for the functional integro-differential equation:

$$u''(x) = f\left(x, u(x), \int_a^b g(x, t, u'(t)dt)\right), \quad x \in [a, b], \tag{1}$$

with

$$\sum_{k=1}^{m} a_k u(\tau_k) = u_0, \quad u'(a) = \zeta, \quad a_k \geq 0, \quad \tau_k \in [a, b]. \tag{2}$$

The existence of solutions $u \in C[a, b]$ will be studied. The continuous dependence of the unique solution on u_0, ζ and a_k will be proved.

As applications, the nonlocal problem of Eq. (1) with the integral condition

$$\int_a^b u(s)d\mu(s) = u_0, \tag{3}$$

will be studied.

In this paper, we discuss the NBVP (1) with (2) and (3). Also, we find the analytical and numerical solutions for Eq. (1) using the modified decomposition method [33] and finite difference-Simpson's method since we apply the Simpson's rule on an integral part and finite difference method [12, 27, 28] on the derivative part and therefore the equation will be converted into a system of nonlinear algebraic equations which can be solved together to get the unknown function, we apply the proposed method to some problems. In addition, we present some figures that show the accuracy of the proposed method. The form of the proposed equation has not been studied analytically or numerically before, therefore what we have presented is a clear contribution to this point. Also, most researchers deal with the topic only analytically, with some examples being given, but these examples are not dealt with numerically or analytically.

This paper is organized as follows: In "Integral Representation" section, we discuss the integral representation of the problem. In "Existence of Solution" section, the existence of a solution will be discussed. In "Uniqueness of the Solution" section, the uniqueness of the solution will be discussed. In "Continuous Dependence" section, we study the continuous dependence on the problem. In "Derivation of the Analytical and Numerical Methods" section, the derivation of the analytical and numerical methods introduce. In "Application" section, some examples are presented and we made a comparison between the exact solution to demonstrate the applicability of the method. Finally, we give a conclusion section.

Integral Representation

Consider the NBVP (1)–(2) with the following assumption:

1. $f : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies Carathéodory condition. There exist a function $c_1 \in L_1[a, b]$ and a constant $d_1 > 0$, such that
$$|f(s, \eta, \phi)| \le c_1(s) + d_1|\eta| + d_1|\phi|.$$
2. $g : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory condition. There exist a function $c_2 : [a, b] \times [a, b] \rightarrow \mathbb{R}, \ \ c_2 \in L_1[a, b]$ and a constant $d_2 > 0$, such that
$$|g(s, t, \eta)| \le c_2(s, t) + d_2|\eta|.$$
3. $$\sup_{x\in[a,b]} \int_a^x c_1(s)ds \le M_1, \quad \sup_{x\in[a,b]} \int_a^x c_2(\theta, t)dt \le M_2.$$
4. $(2d_1b^2 + d_1d_2b^2) < 1$.

Lemma 1 *Let $B = \sum_{k=1}^m a_k \neq 0$, the solution of the NBVP (1)–(2), if it exist, then it can be represented by the integral equation*
$$u(x) = B^{-1}\Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^x v(s)ds, \tag{4}$$
where,
$$v(x) = \zeta + \int_a^x f\Big(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\Big)d\theta. \tag{5}$$

Proof Integrating both sides of (1), we get
$$u'(x) = \zeta + \int_a^x f\Big(\theta, u(\theta), \int_a^b g(\theta, t, u'(t))dt\Big)d\theta, \quad x \in [a, b]. \tag{6}$$
Let $u'(x) = v(x)$ in (6), we obtain
$$v(x) = \zeta + \int_a^x f\Big(\theta, u(\theta), \int_a^b g(\theta, t, v(t))dt\Big)d\theta, \quad x \in [a, b], \tag{7}$$
where
$$u(x) = u(a) + \int_a^x v(s)ds, \quad x \in [a, b], \tag{8}$$
using the condition (2), we obtain
$$\sum_{k=1}^m a_k u(\tau_k) = u(a)\sum_{k=1}^m a_k + \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds, \tag{9}$$
then,
$$u(a) = B^{-1}\Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\Big], \tag{10}$$
from (7), (8) and (10), we get
$$u(x) = B^{-1}\Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^x v(s)ds,$$

where,

$$v(x) = \zeta + \int_a^x f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right)d\theta.$$

□

Existence of Solution

Theorem 2 *Let the assumptions 1–4 be satisfied. Then the NBVP (1)–(2) has at least one solution $u \in C[a, b]$.*

Proof Define the operator G associated with the integral equation (5) by

$$Gv(x) = \zeta + \int_a^x f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right)d\theta.$$

Let $Q_r = \{v \in R : ||v||_C \le r\}$, where $r = \frac{|\zeta| + M_1 + d_1 b B^{-1}|u_0| + d_1 b M_2}{1 - (2d_1 b^2 + d_1 d_2 b^2)}$.
Then we have, for $v \in Q_r$.

$$\begin{aligned}
|Gv(x)| &= \left|\zeta + \int_a^x f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right)d\theta\right| \\
&\le |\zeta| + \int_a^x \left|f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right)\right|d\theta \\
&\le |\zeta| + \int_a^x \left[c_1(\theta) + d_1 B^{-1}\left|u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\right| + d_1 \int_a^\theta |v(s)|ds \right.\\
&\quad \left. + d_1 \int_a^b |g(\theta, t, v(t))|dt\right]d\theta \\
&\le |\zeta| + M_1 + \int_a^x \left[d_1 B^{-1}|u_0| + d_1 B^{-1} \sum_{k=1}^m a_k \int_a^{\tau_k} |v(s)|ds + d_1 \int_a^\theta |v(s)|ds \right.\\
&\quad \left. + d_1 \int_a^b |c_2(\theta, t)|dt + d_1 d_2 \int_a^b |v(t)|dt\right]d\theta \\
&\le |\zeta| + M_1 + \int_a^x [d_1 B^{-1}|u_0| + d_1 b||v|| + d_1 b||v|| + d_1 M_2 + d_1 d_2 b||v||]d\theta \\
&\le |\zeta| + M_1 + d_1 b B^{-1}|u_0| + 2d_1 b^2 r + d_1 b M_2 + d_1 d_2 b^2 r = r.
\end{aligned}$$

This is proves that $G : Q_r \to Q_r$ and the class of functions $\{Gv\}$ is uniformly bounded in Q_r.

Now, let $x_1, x_2 \in [a, b]$ such that $|x_2 - x_1| < \delta$, then

$$\begin{aligned}
&|Gv(x_2) - Gv(x_1)| \\
&= \left|\zeta + \int_a^{x_2} f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right)d\theta \right.\\
&\quad \left. - \zeta - \int_a^{x_1} f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right)d\theta\right|
\end{aligned}$$

$$= \Big| \int_a^{x_1} f\Big(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^{\theta} v(s)ds, \int_a^b g(\theta, t, v(t))dt\Big)d\theta$$

$$+ \int_{x_1}^{x_2} f\Big(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^{\theta} v(s)ds, \int_a^b g(\theta, t, v(t))dt\Big)d\theta$$

$$- \int_a^{x_1} f\Big(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^{\theta} v(s)ds, \int_a^b g(\theta, t, v(t))dt\Big)d\theta\Big|$$

$$\leq \int_{x_1}^{x_2} \Big| f\Big(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^{\theta} v(s)ds, \int_a^b g(\theta, t, v(t))dt\Big)\Big| d\theta$$

$$\leq \int_{x_1}^{x_2} \Big[c_1(\theta) + B^{-1}d_1|u_0| + B^{-1}d_1 \sum_{k=1}^{m} a_k \int_a^{\tau_k} |v(s)|ds + d_1 \int_a^{\theta} |v(s)|ds$$

$$+ d_1 \int_a^b |g(\theta, t, v(t))|dt\Big]d\theta$$

$$\leq \int_{x_1}^{x_2} \Big[c_1(\theta) + B^{-1}d_1|u_0| + d_1\|v\| + d_1\|v\| + d_1 \int_a^b |c_2(\theta, t) + d_2|v(t)|dt\Big]d\theta$$

$$\leq \int_{x_1}^{x_2} c_1(\theta)d\theta + (B^{-1}d_1u_0 + 2d_1br + d_1M_2 + d_1d_2br)\delta.$$

This is proves that the class of functions $\{Gv\}$ is equi-continuous in Q_r. □

Let $v_n \in Q_r$, $v_n \to v (n \to \infty)$, then from the continuity of the two functions f and g, we get $f(x, \eta_n, \phi_n) \to f(x, \eta, \phi)$ and $g(x, \eta_n, \phi_n) \to g(x, \eta, \phi)$ as $n \to \infty$.

Also,

$$\lim_{n\to\infty} Gv_n(x) = \lim_{n\to\infty}\Big[\zeta + \int_a^x f(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v_n(s)ds\Big] + \int_a^{\theta} v_n(s)ds, \int_a^b g(\theta, t, v_n(t))dt)d\theta\Big]. \tag{11}$$

Using assumptions 1-2 and Lebesgue dominated convergence Theorem [23], we obtain

$$\lim_{n\to\infty} Gv_n(x) = \zeta + \int_a^x \lim_{n\to\infty} f \Big(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v_n(s)ds\Big] + \int_a^{\theta} v_n(s)ds, \int_a^b g(\theta, t, v_n(t))dt\Big)d\theta = Gv(x). \tag{12}$$

Then $Gv_n \to Gv$ as $n \to \infty$. This mean that the operator G is continuous in Q_r. Then by Schauder Theorem [18] there exist at least one solution $v \in C[a, b]$ of the Eq. (5). Thus, based on the Lemma 1, the NBVP (1)–(2) possess a solution $u \in C[a, b]$.

Nonlocal Integral Condition

Theorem 3 *Let the assumption 1–4 be satisfied, then the NBVP (1), (3) has at least one solution given by*

$$u(x) = \frac{1}{\mu(b) - \mu(a)}(u_0 - \int_a^b \int_a^\theta v(s)dsd\mu(\theta) + \int_a^x v(s)ds, \tag{13}$$

where

$$v(x) = \zeta + \int_a^x f\Big(\theta, \frac{1}{\mu(b) - \mu(a)}\Big(u_0 - \int_a^b \int_a^\theta v(s)dsd\mu(\theta) + \int_a^\theta v(s)ds\Big), \int_a^b g(\theta, t, v(t))dt\Big)d\theta. \tag{14}$$

Proof Let $v \in C[a, b]$ be the solution of Eq. (5). Let $a_k = \mu(x_k) - \mu(x_{k-1})$, μ is increasing function, $\tau_k \in (x_{k-1}, x_k)$, $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_m = b$ then, as $m \to \infty$ the condition (2) will be

$$\sum_{k=1}^{m} (\mu(x_k) - \mu(x_{k-1}))u(\tau_k) = u_0. \tag{15}$$

And

$$\lim_{m\to\infty} \sum_{k=1}^{m} (\mu(x_k) - \mu(x_{k-1}))u(\tau_k) = \int_a^b u(s)d\mu(s) = u_0. \tag{16}$$

As $m \to \infty$, the solution of the NBVP (1)–(2) will be

$$\begin{aligned} u(x) &= \lim_{m\to\infty} \Bigg[\frac{1}{\sum_{k=1}^m a_k}[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds] + \int_a^x v(s)ds\Bigg] \\ &= \frac{1}{\mu(b) - \mu(a)}\Bigg[u_0 - \sum_{k=1}^{m} \int_a^{\tau_k} v(s)ds(\mu(x_k) - \mu(x_{k-1})\Bigg] + \int_a^x v(s)ds \\ &= \frac{1}{\mu(b) - \mu(a)}\Bigg[u_0 - \int_a^b \int_a^{\tau_k} v(s)dsd\mu(\theta)\Bigg] + \int_a^x v(s)ds, \end{aligned}$$

where

$$v(x) = \zeta + \int_a^x f\Big(\theta, \frac{1}{\mu(b) - \mu(a)}\Big(u_0 - \int_a^b \int_a^\theta v(s)dsd\mu(\theta) + \int_a^\theta v(s)ds\Big), \int_a^b g(\theta, t, v(t))dt\Big)d\theta.$$

□

Uniqueness of the Solution

Let f and g satisfy the following assumptions

(i) $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in x for any $\eta, \phi \in R$ and satisfies the Carathéodory condition

$$|f(s, \eta, \phi) - f(s, w, z)| \le d_1|\eta - w| + d_1|\phi - z|,$$

(ii) $g : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is measurable in x for any $\eta, \phi \in R$ and satisfies the Carathéodory condition

$$|g(s, t, \eta) - g(s, t, \phi)| \le d_2|\eta - \phi|.$$

Theorem 4 *Let the assumptions* $(i)-(ii)$ *be satisfied, then the solution of the NBVP* (1)–(2) *is unique.*

Proof From assumption (i) we have f is measurable in x for any $\eta, \phi \in R$ and satisfies the Carathéodory condition, then it is continuous in $\eta, \phi \in R$ for all $x \in [a, b]$, and

$$|f(\theta, \eta, \phi)| \leq d_1|\eta| + d_1|\phi| + |f(\theta, 0, 0)|.$$

Then the first condition is satisfied. Also by the same we can see that the second condition is satisfied by assumption (ii). Now, from Theorem 2 the solution of the Eq. (5) exists. Let v, w be two solutions of the Eq. (5), then

$$\begin{aligned}
|v(x) - w(x)| \leq & \int_a^x \left| f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds + \int_a^\theta v(s)ds\right], \int_a^b g(s, t, v(t))dt\right)\right. \\
& \left. - f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} w(s)ds\right] + \int_a^\theta w(s)ds, \int_a^b g(s, t, w(t))dt\right)\right| d\theta \\
\leq & \int_a^x \left[d_1 \left| B^{-1} \sum_{k=1}^m a_k \int_a^{\tau_k} (w(s) - v(s))ds \right.\right. \\
& \left.\left. + \int_a^\theta (v(s) - w(s))ds \right| + d_1 \left| \int_a^b (g(\theta, t, v(t)) - g(\theta, t, w(t)))dt \right| \right] d\theta \\
\leq & \, d_1 \int_a^x \left[B^{-1} \sum_{k=1}^m a_k \int_a^{\tau_k} |w(s) - v(s)|ds \right. \\
& \left. + \int_a^\theta |w(s) - v(s)|ds + \int_a^b |g(\theta, t, v(t)) - g(\theta, t, w(t)))|dt \right] d\theta \\
\leq & \, d_1||w - v||b^2 + d_1||w - v||b^2 + d_1 \int_a^x \int_a^b d_2|v(t) - w(t)|dtd\theta \\
\leq & \, 2d_1||w - v||b^2 + d_1 d_2 b^2||w - v|| \\
\leq & \, (2d_1 b^2 + d_1 d_2 b^2)||w - v||.
\end{aligned}$$

Hence

$$[1 - (2d_1 b^2 + d_1 d_2 b^2)]||w - v|| \leq 0.$$

Since $2d_1 b^2 + d_1 d_2 b^2 < 1$, then $w(x) = v(x)$ and the solution of the Eq. (5) is unique. Thus, based on the Lemma 1, the NBVP (1)–(2) possess a unique solution $u \in C[a, b]$. □

Continuous Dependence

Continuous Dependence on u_0

Definition 5 The solution $u \in C[a, b]$ of the NBVP (1)–(2) depends continuously on u_0, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad |u_0 - u_0^*| < \delta \Rightarrow \|u - u^*\| < \epsilon,$$

where u^* is the solution of the NBVP

$$u^{*''}(x) = f\left(x, u^*(x), \int_a^b g(x, t, u^{*'}(t)dt)\right), \quad x \in [a, b], \tag{17}$$

with the condition

$$\sum_{k=1}^{m} a_k u^*(\tau_k) = u_0^*, \quad u^{*\prime}(a) = \zeta, \quad a_k \geq 0, \quad \tau_k \in [a, b]. \tag{18}$$

Theorem 6 *Let the assumption of the Theorem 4 be satisfied, then the solution of the NBVP (1)–(2) depends continuously on u_0.*

Proof Let u, u^* be two solutions of the NBVP (1)–*(2) and (17)–(18) respectively. Then

$$\begin{aligned}
&|v(x) - v^*(x)| \\
&= \Bigg|\zeta + \int_a^x \Bigg[f\Bigg(\theta, B^{-1}\Bigg[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\Bigg] + \int_a^{\theta} v(s)ds, \int_a^b g(\theta, t, v(t))dt\Bigg) \\
&\quad - \zeta - f\Bigg(\theta, B^{-1}\Bigg[u_0^* - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v^*(s)ds\Bigg] + \int_a^{\theta} v^*(s)ds, \int_a^b g(\theta, t, v^*(t))dt\Bigg)\Bigg]d\theta\Bigg| \\
&\leq \int_a^x \Bigg|f\Bigg(\theta, B^{-1}\Bigg[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\Bigg] + \int_a^{\theta} v(s)ds, \int_a^b g(\theta, t, v(t))dt\Bigg)d\theta \\
&\quad - f\Bigg(\theta, B^{-1}\Bigg[u_0^* - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v^*(s)ds\Bigg] + \int_a^{\theta} v^*(s)ds, \int_a^b g(\theta, t, v^*(t))dt\Bigg)\Bigg|d\theta \\
&\leq \int_a^x \Bigg[d_1\Bigg|B^{-1}(u_0 - u_0^*) + B^{-1}\sum_{k=1}^{m} a_k \int_a^{\tau_k} (v^*(s) - v(s))ds + \int_a^{\theta} (v(s) - v^*(s))ds\Bigg| \\
&\quad + d_1\Bigg|\int_a^b (g(\theta, t, v(t)) - g(\theta, t, v^*(t)))dt\Bigg|\Bigg]d\theta \\
&\leq \int_a^x \Bigg[d_1 B^{-1}|u_0 - u_0^*| + d_1 B^{-1}\sum_{k=1}^{m} a_k \int_a^{\tau_k} |v^*(s) - v(s)|ds + d_1\int_a^{\theta} |v(s) - v^*(s)|ds \\
&\quad + d_1\int_a^b |g(\theta, t, v(t)) - g(\theta, t, v^*(t))|dt\Bigg]d\theta \\
&\leq \int_a^x [d_1 B^{-1}|u_0 - u_0^*| + d_1 B^{-1}\sum_{k=1}^{m} a_k \int_a^{\tau_k} |v^*(s) - v(s)|ds + d_1\int_a^{\theta} |v(s) - v^*(s)|ds \\
&\quad + d_1 d_2\int_a^b |v(t) - v^*(t)|dt]d\theta \\
&\leq d_1 B^{-1}|u_0 - u_0^*|b + d_1||v - v^*||b^2 + d_1||v - v^*||b^2 \\
&\quad + d_1\int_a^x \int_a^b d_2|v(t) - v^*(t)|dtd\theta \\
&\leq d_1 b B^{-1}\delta + 2d_1||v - v^*||b^2 + d_1 d_2 b^2||v - v^*||.
\end{aligned}$$

Hence

$$||v - v^*|| \leq \frac{d_1 b B^{-1}\delta}{1 - (2d_1 b^2 + d_1 d_2 b^2)}.$$

And

$$|u(x) - u^*(x)| = |B^{-1}[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds] + \int_a^x v(s)ds$$

$$-B^{-1}[u_0^* - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v^*(s)ds] + \int_a^x v^*(s)ds|$$
$$\leq B^{-1}|u_0 - u_0^*| + 2b\|v - v^*\|.$$

Hence

$$\|u - u^*\| \leq B^{-1}\delta + \frac{2d_1 b^2 B^{-1}\delta}{1 - (2d_1 b^2 + d_1 d_2 b^2)} = \epsilon.$$

Therefor the solution of the NBVP (1)–(2) depends continuously on u_0. □

Continuous Dependence on ζ

Definition 7 The solution $u \in C[a, b]$ of the NBVP (1)–(2) depends continuously on ζ, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad |\zeta - \zeta^*| < \delta \Rightarrow \|u - u^*\| < \epsilon,$$

where u^* is the solution of the NBVP

$$u^{*\prime\prime}(x) = f\left(x, u^*(x), \int_a^b g(x, t, u^{*\prime}(t)dt)\right), \quad x \in [a, b], \tag{19}$$

with the condition

$$\sum_{k=1}^{m} a_k u^*(\tau_k) = u_0, \quad u^{*\prime}(a) = \zeta^*, \quad a_k \geq 0, \quad \tau_k \in [a, b]. \tag{20}$$

Theorem 8 *Let the assumption of the Theorem 4 be satisfied, then the solution of the NBVP (1)–(2) depends continuously on ζ.*

Proof Let u, u^* be two solutions of the NBVP (1)–(2) and (19)–(20) respectively. Then

$$\begin{aligned}
&|v(x) - v^*(x)| \\
&= \left|\zeta + \int_a^x \left[f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right)\right.\right. \\
&\quad \left.\left. - \zeta^* - f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v^*(s)ds\right] + \int_a^\theta v^*(s)ds, \int_a^b g(\theta, t, v^*(t))dt\right)\right] d\theta\right| \\
&\leq |\zeta - \zeta^*| \\
&\quad + \int_a^x \left| f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right) d\theta \right. \\
&\quad \left. - f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v^*(s)ds\right] + \int_a^\theta v^*(s)ds, \int_a^b g(\theta, t, v^*(t))dt\right)\right| d\theta \\
&\leq |\zeta - \zeta^*| + \int_a^x \left[d_1 \left| B^{-1} \sum_{k=1}^{m} a_k \int_a^{\tau_k} (v^*(s) - v(s))ds + \int_a^\theta (v(s) - v^*(s))ds \right| \right. \\
&\quad \left. + d_1 \left| \int_a^b (g(\theta, t, v(t)) - g(\theta, t, v^*(t)))dt \right| \right] d\theta \\
&\leq |\zeta - \zeta^*| + \int_a^x \left[d_1 B^{-1} \sum_{k=1}^{m} a_k \int_a^{\tau_k} |v^*(s) - v(s)|ds + d_1 \int_a^\theta |v(s) - v^*(s)|ds \right.
\end{aligned}$$

$$
\begin{aligned}
&+d_1 \int_a^b |g(\theta,t,v(t)) - g(\theta,t,v^*(t))|dt\Big] d\theta \\
&\leq |\zeta - \zeta^*| + \int_a^x \Big[d_1 B^{-1} \sum_{k=1}^m a_k \int_a^{\tau_k} |v^*(s) - v(s)|ds + d_1 \int_a^\theta |v(s) - v^*(s)|ds \\
&+d_1 d_2 \int_a^b |v(t) - v^*(t)|dt \Big] d\theta \\
&\leq |\zeta - \zeta^*| + d_1 ||v - v^*|| b^2 + d_1 ||v - v^*|| b^2 \\
&+ d_1 \int_a^x \int_a^b d_2 |v(t) - v^*(t)| dt d\theta \\
&\leq \delta + 2d_1 ||v - v^*|| b^2 + d_1 d_2 b^2 ||v - v^*||.
\end{aligned}
$$

Hence

$$
||v - v^*|| \leq \frac{\delta}{1 - (2d_1 b^2 + d_1 d_2 b^2)}.
$$

And

$$
\begin{aligned}
|u(x) - u^*(x)| &= |B^{-1} \Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds \Big] + \int_a^x v(s)ds \\
&\quad -B^{-1} \Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v^*(s)ds \Big] + \int_a^x v^*(s)ds| \\
&\leq 2b\|v - v^*\|.
\end{aligned}
$$

Hence

$$
\|u - u^*\| \leq \frac{2b\delta}{1 - (2d_1 b^2 + d_1 d_2 b^2)} = \epsilon.
$$

Therefor the solution of the NBVP (1)–(2) depends continuously on ζ. □

Definition 9 The solution $u \in C[a, b]$ of the NBVP (1)–(2) depends continuously on u_0 and ζ, if

$$
\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad |u_0 - u_0^*| < \delta_1 |\zeta - \zeta^*| < \delta_2 \Rightarrow \|u - u^*\| < \epsilon,
$$

where u^* is the solution of the NBVP

$$
u^{*''}(x) = f\left(x, u^*(x), \int_a^b g(x, t, u^{*'}(t)dt)\right), \quad x \in [a, b], \tag{21}
$$

with the condition

$$
\sum_{k=1}^m a_k u^*(\tau_k) = u_0^*, \quad u^{*'}(a) = \zeta^*, \quad a_k \geq 0, \quad \tau_k \in [a, b]. \tag{22}
$$

Theorem 10 *Let the assumption of the Theorem 4 be satisfied, then the solution of the NBVP (1)–(2) depends continuously on u_0 and ζ.*

Proof Let u, u^* be two solutions of the NBVP (1)–(2) and (21)–(22) respectively. Then

$$
\begin{aligned}
&|v(x) - v^*(x)| \\
&= \Big|\zeta + \int_a^x \Big[f\Big(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\Big) \\
&\quad - \zeta^* - f\Big(\theta, B^{-1}\Big[u_0^* - \sum_{k=1}^m a_k \int_a^{\tau_k} v^*(s)ds\Big] + \int_a^\theta v^*(s)ds, \int_a^b g(\theta, t, v^*(t))dt\Big)\Big] d\theta\Big| \\
&\le |\zeta - \zeta^*| \\
&\quad + \int_a^x \Big| f\Big(\theta, B^{-1}\Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\Big] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\Big) d\theta \\
&\quad - f\Big(\theta, B^{-1}\Big[u_0^* - \sum_{k=1}^m a_k \int_a^{\tau_k} v^*(s)ds\Big] + \int_a^\theta v^*(s)ds, \int_a^b g(\theta, t, v^*(t))dt\Big)\Big| d\theta \\
&\le |\zeta - \zeta^*| + d_1 B^{-1} |u_0 - u_0^*| b \\
&\quad + \int_a^x \Big[d_1 \Big| B^{-1} \sum_{k=1}^m a_k \int_a^{\tau_k} (v^*(s) - v(s))ds + \int_a^\theta (v(s) - v^*(s))ds \Big| \\
&\quad + d_1 \Big| \int_a^b (g(\theta, t, v(t)) - g(\theta, t, v^*(t)))dt \Big|\Big] d\theta \\
&\le |\zeta - \zeta^*| + d_1 B^{-1} |u_0 - u_0^*| b \\
&\quad + \int_a^x \Big[d_1 B^{-1} \sum_{k=1}^m a_k \int_a^{\tau_k} |v^*(s) - v(s)| ds + d_1 \int_a^\theta |v(s) - v^*(s)| ds \\
&\quad + d_1 \int_a^b |g(\theta, t, v(t)) - g(\theta, t, v^*(t))| dt \Big] d\theta \\
&\le |\zeta - \zeta^*| + d_1 B^{-1} |u_0 - u_0^*| b \\
&\quad + \int_a^x \Big[d_1 B^{-1} \sum_{k=1}^m a_k \int_a^{\tau_k} |v^*(s) - v(s)| ds + d_1 \int_a^\theta |v(s) - v^*(s)| ds \\
&\quad + d_1 d_2 \int_a^b |v(t) - v^*(t)| dt \Big] d\theta \\
&\le |\zeta - \zeta^*| + d_1 B^{-1} |u_0 - u_0^*| b + d_1 ||v - v^*|| b^2 + d_1 ||v - v^*|| b^2 \\
&\quad + d_1 \int_a^x \int_a^b d_2 |v(t) - v^*(t)| dt d\theta \\
&\le d_1 B^{-1} \delta_1 b + \delta_2 + 2 d_1 ||v - v^*|| b^2 + d_1 d_2 b^2 ||v - v^*||.
\end{aligned}
$$

Hence

$$||v - v^*|| \le \frac{d_1 B^{-1} \delta_1 b + \delta_2}{1 - (2 d_1 b^2 + d_1 d_2 b^2)}.$$

And

$$
\begin{aligned}
|u(x) - u^*(x)| = |B^{-1} &\Big[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds \Big] + \int_a^x v(s)ds \\
&- B^{-1} \Big[u_0^* - \sum_{k=1}^m a_k \int_a^{\tau_k} v^*(s)ds \Big] + \int_a^x v^*(s)ds|
\end{aligned}
$$

$$\leq B^{-1}|u_0 - u_0^*| + 2b\|v - v^*\|.$$

Hence

$$\|u - u^*\| \leq B^{-1}\delta_1 + \frac{2b(d_1 B^{-1}\delta_1 b + \delta_2)}{1 - (2d_1 b^2 + d_1 d_2 b^2)} = \epsilon.$$

Therefor the solution of the NBVP (1)–(2) depends continuously on u_0 and ζ. □

Continuous Dependence on a_k

Definition 11 The solution $u \in C[a, b]$ of the NBVP (1)–(2) depends continuously on a_k, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad |a_k - a_k^*| < \delta \Rightarrow \|u - u^*\| < \epsilon,$$

where u^* is the solution of the NBVP

$$u^{*''}(x) = f\left(x, u^*(x), \int_a^b g(x, t, u^{*'}(t)dt)\right), \quad x \in [a, b], \tag{23}$$

with the condition

$$\sum_{k=1}^{m} a_k^* u^*(\tau_k) = u_0, \quad u^{*'}(a) = \zeta, \quad a_k \geq 0, \quad \tau_k \in [a, b]. \tag{24}$$

Theorem 12 *Let the assumption of the Theorem 4 be satisfied, then the solution of the NBVP (1)–(2) depends continuously on a_k.*

Proof Let $B^* = \sum_{k=1}^m a_k^* \neq 0$ and v, v^* be two solutions of the NBVP (1)–(2) and (23)–(24) respectively. Then

$$|v(x) - v^*(x)|$$
$$\leq \int_a^x \left| f\left(\theta, B^{-1}\left[u_0 - \sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b g(\theta, t, v(t))dt\right)\right.$$
$$\left. - f\left(\theta, B^{*-1}\left[u_0 - \sum_{k=1}^m a_k^* \int_a^{\tau_k} v^*(s)ds\right] + \int_a^\theta v^*(s)ds, \int_a^b g(\theta, t, v^*(t))dt\right)\right| d\theta$$
$$\leq \int_a^x \left[d_1 |B^{-1}(u_0) - B^{*-1}(u_0) + B^{*-1}\sum_{k=1}^m a_k^* \int_a^{\tau_k} v^*(s)ds - B^{-1}\sum_{k=1}^m a_k \int_a^{\tau_k} v(s)ds \right.$$
$$\left. + \int_a^\theta v(s)ds - \int_a^\theta v^*(s)ds| + d_1 |\int_a^b (g(\theta, t, v(t)) - g(\theta, t, v^*(t)))dt \right] d\theta$$
$$\leq \int_a^x [d_1|B^{-1}(u_0) - B^{*-1}(u_0)| + d_1 B^{*-1}\sum_{k=1}^m a_k^* \int_a^{\tau_k} |v^*(s) - v(s)|ds$$
$$+ d_1 B^{*-1}\left(\sum_{k=1}^m |a_k^* - a_k|\right)\int_a^{\tau_k} |v(s)|ds + d_1 B^{-1} B^{*-1}\sum_{k=1}^m |a_k - a_k^*| \sum_{k=1}^m a_k \int_a^{\tau_k} |v(s)|ds$$
$$+ d_1 \int_a^\theta |v(s) - v^*(s)|ds + d_1 \int_a^b |g(\theta, t, v(t)) - g(\theta, t, v^*(t)))|dt]d\theta$$
$$\leq \int_a^x [d_1|B^{-1}(u_0) - B^{*-1}(u_0)| + d_1 B^{*-1}\sum_{k=1}^m a_k^* \int_a^{\tau_k} |v^*(s) - v(s)|ds$$

$$+ d_1 B^{*-1}\left(\sum_{k=1}^{m} |a_k^* - a_k|\right)\int_a^{\tau_k} |v(s)|ds + d_1 B^{-1} B^{*-1}\sum_{k=1}^{m} |a_k - a_k^*|\sum_{k=1}^{m} a_k \int_a^{\tau_k} |v(s)|ds$$
$$+ d_1 \int_a^{\theta} |v(s) - v^*(s)|ds + d_1 d_2 \int_a^{b} |v(t) - v^*(t)|dt]d\theta$$
$$\le d_1 B^{-1} B^{*-1} m\delta u_0 + d_1||v - v^*||b^2 + d_1 B^{*-1} m\delta||v||b^2 + d_1 B^{*-1} m\delta||v||b^2$$
$$+ d_1||v - v^*||b^2 + d_1 d_2 b^2||v - v^*||$$
$$\le d_1 B^{-1} B^{*-1} m\delta u_0 + (2d_1 b^2 + d_1 d_2 b^2)||v - v^*|| + 2d_1 B^{*-1} m\delta||v||b^2.$$

Hence

$$||v - v^*|| \le \frac{d_1 m\delta u_0 + 2d_1 Bm\delta||v||b^2}{[1 - (2d_1 b^2 + d_1 d_2 b^2])BB^*}.$$

And

$$|u(x) - u^*(x)| = \left|\frac{1}{\sum_{k=1}^m a_k}\left[u_0 - \sum_{k=1}^{m} a_k \int_a^{\tau_k} v(s)ds\right] + \int_a^x v(s)ds\right.$$
$$\left. - \frac{1}{\sum_{k=1}^m a_k^*}\left[u_0 - \sum_{k=1}^{m} a_k^* \int_a^{\tau_k} v^*(s)ds\right] + \int_a^x v^*(s)ds\right|$$
$$\le \frac{m\delta|u_0|}{BB^*} + 2mb\delta B^{-1} r + 2b\|v - v^*\|.$$

Hence

$$\|u - u^*\| \le \frac{m\delta|u_0|}{BB^*} + 2mb\delta B^{-1} r + 2b\frac{d_1 m\delta u_0 + 2d_1 Bm\delta r b^2}{[1 - (2d_1 b^2 + d_1 d_2 b^2)]BB^*} = \epsilon.$$

Therefor the solution of the NBVP (1)–(2) depends continuously on a_k. □

Derivation of the Analytical and Numerical Methods

In this section, we present the methods used to study the proposed equation

A Brief Review of the Modified Decomposition Method

In this section we use the modified decomposition method to get the exact solution for nonlocal Fredholm integro differential equation. Firstly, we use the nonlocal condition to put Eq. (1) in the form

$$u(x) = \rho(x) + \lambda \int_a^b k(x,t)u'(t)dt, \tag{25}$$

where $\rho(x)$ is a known function, $k(x,t)$ is the kernel of the integro -differential equation, λ is any constant, $u(x)$ is the unknown function to be determined.
This method depends mainly on splitting the function $\rho(x)$ into two parts, therefore it cannot be used if the function $\rho(x)$ consists of only one terms. Now, we can express the procedure as follows

1 We substitute $u(x) = \sum_{l=0}^{\infty} u_l(x)$ into both sides of Eq. (25).
2 We set $\rho(x) = \rho_1(x) + \rho_2(x)$.

3 We use the following recurence relation

$$
\begin{aligned}
u_0(x) &= \rho_1(x), \\
u_1(x) &= \rho_2(x) + \lambda \int_a^b k(x,t)u_0'(t)dt, \\
u_{l+1}(x) &= \lambda \int_a^b k(x,t)u_l'(t)dt, \quad l \geq 1.
\end{aligned}
$$

If we make a proper choise of the function $\rho_1(x)$, $\rho_2(x)$, we can obtain the exact solution $u(x)$ by using few iterations, and sometimes by calculating only two components.

Derivation of Numerical Method

To obtain the numerical solution of Eq. (1), we divide the domain $[a, b]$ of Eq. (1) into N finite points as $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. Using uniform step length $h = (b-a)/N$ as $x_i = a + ih$, $i = 0, 1, 2, \ldots, N$. Then we approximate the integral part of (1) by using the composite Simpson's as follows

$$
\begin{aligned}
\int_a^b k(x,t)u'(t)dt \simeq \frac{h}{3}[&k(x,t_0)u'(t_0) + 4k(x,t_1)u'(t_1) + 2k(x,t_2)u'(t_2) + \cdots \\
&+ 2k(x,t_{N-2})u'(t_{N-2}) + 4k(x,t_{N-1})u'(t_{N-1}) + \cdots + k(x,t_N)u'(t_N)].
\end{aligned}
$$

By taking $u_i'' = u''(x_i)$, $u_i' = u'(x_i)$, $k(x_i, t_j) = k_{i,j}$, then (1) can be written as

$$
\begin{aligned}
u_i'' - u_i \simeq \rho_i + \frac{h}{3}[&k_{i0}(u_0') + 4k_{i1}(u_1') + 2k_{i2}(u_2') + \cdots + 2k_{iN-2}(u_{N-2}') \\
&+ 4k_{iN-1}(u_{N-1}') + \cdots + k_{iN}(u_N')].
\end{aligned} \tag{26}
$$

And, we use centeral difference to approximate the derivative part of (26) as:

$$
\begin{aligned}
u_i'' &\simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \\
u_i' &\simeq \frac{u_{i+1} - u_{i-1}}{2h}.
\end{aligned}
$$

Then Eq. (26) can be written as

$$
\begin{aligned}
\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - u_i \simeq \rho_i + \frac{h}{3}\Bigg[&k_{i0}\frac{u_1 - u_{-1}}{2h} + 4k_{i1}\frac{u_2 - u_0}{2h} + 2k_{i2}\frac{u_3 - u_1}{2h} \\
&+ \cdots + 2k_{iN-2}\frac{u_{N-1} - u_{N-3}}{2h} + 4k_{iN-1}\frac{u_N - u_{N-2}}{2h} \\
&+ \cdots + k_{iN}\frac{u_{N+1} - u_{N-1}}{2h}\Bigg], \quad i = 0, 1, 2, 3, \ldots N.
\end{aligned} \tag{27}
$$

From Eq. (27), we can generate a system of equations for $u_{-1}, u_0, u_1, u_2, \ldots, u_N, u_{N+1}$ which can be represent in a matrix form

$MU = W$

$$M = \begin{pmatrix} 2-C_{00} & D & 2+C_{00}-B_{02} & A_{01}-A_{03} & B_{02}-B_{04} & \cdots & B_{0N-2}+C_{0N} & A_{0N-1} & C_{0N} \\ -C_{10} & 2-A_{11} & Z & I & B_{12}-B14 & \cdots & B_{1N-2}+C_{1N} & A_{1N-1} & C_{1N} \\ -C_{21} & -A_{21} & 2+C_{20}-B_{22} & Y & E & \cdots & B_{2N-2}+C_{2N} & A_{2N-1} & C_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ -C_{N0} & -A_{N1} & C_{N0}-B_{N2} & A_{N1}-A_{N3} & B_{N2}-B_{N4} & \cdots & S & Q & 2+C_{NN} \end{pmatrix},$$

$$U = \begin{pmatrix} u_{-1} \\ u_0 \\ u_1 \\ \vdots \\ u_N \\ u_{N+1} \end{pmatrix},$$

$$W = \begin{pmatrix} 2h^2\rho_0 \\ 2h^2\rho_1 \\ 2h^2\rho_2 \\ \vdots \\ 2h^2\rho_N \\ 2h^2\rho_{N+1} \end{pmatrix},$$

noindent where $A_{ij} = \frac{-4h^2}{3}K_{ij}$, $B_{ij} = \frac{-2h^2}{3}K_{ij}$, $C_{ij} = \frac{-h^2}{3}K_{ij}$, $Z = -4 - 2h^2 + C_{10} - B_{12}$, $Y = -4 - 2h^2 + A_{21} - A_{23}$, $D = -4 - 2h^2 - A_{01}$, $Q = -4 - 2h^2 + A_{NN-1}$, $E = 2 + B_{22} - B_{24}$, $I = 2 + A_{11} - A_{13}$, $S = 2 + B_{NN-2} + C_{NN}$.

Error Estimation

Theorem 13 *Suppose that $\sigma_1, \sigma_2, \sigma_3 \in [a, b]$ such that the errors e_1 of Second order centeral, e_2 of first order central difference, e_3 of Simpson's rule respectively are given by $\frac{h^2}{12}u^{(4)}(\sigma_1)$, $\frac{h^2}{6}u^{(3)}(\sigma_2)$, and $\frac{(b-a)}{180}h^4u^{(4)}(\sigma_3)$. Then we obtain the error estimation for the Eq. (1) by*

$$e \leq \left| \frac{(b-a)^2}{12N^2}\mu \right|, \tag{28}$$

where $\mu = max\{u^{(4)}(\sigma_1), u^{(3)}(\sigma_2)\}$, and N is the number of subinterval.

Proof From Eq. (27), the exact solution for $i = 0, 1, 2, 3, \ldots, N$.

$$\begin{aligned} &\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - u_i + \frac{h^2}{12}u^{(4)}(\sigma_1) \\ &\quad = \rho_i + \frac{h}{3}\Bigg[k_{i0}\left(\frac{u_1 - u_{-1}}{2h}\right) + 4k_{i1}\left(\frac{u_2 - u_0}{2h}\right) + 2k_{i2}\left(\frac{u_3 - u_1}{2h}\right) \\ &\qquad + \cdots + 2k_{iN-2}\left(\frac{u_{N-1} - u_{N-3}}{2h}\right) + 4k_{iN-1}\left(\frac{u_N - u_{N-2}}{2h}\right) \\ &\qquad + k_{iN}\left(\frac{u_{N+1} - u_{N-1}}{2h}\right)\Bigg] + \frac{h^2}{6}u^{(3)}(\sigma_2) + \frac{(b-a)}{180}h^4u^{(4)}(\sigma_3)]. \end{aligned} \tag{29}$$

Substracting (27) from (29), we obtain the error terms as follows:

$$e = \left| \frac{h^2}{12}u^{(4)}(\sigma_1) - \frac{h^2}{6}u^{(3)}(\sigma_2) - \frac{(b-a)}{180}h^4u^{(4)}(\sigma_3) \right|, \leq \left| \frac{h^2}{12}u^{(4)}(\sigma_1) - \frac{h^2}{6}u^{(3)}(\sigma_2) \right|.$$

Let $\mu_1 = u^{(4)}(\sigma_1)$, $\mu_2 = u^{(3)}(\sigma_2)$, then

$$e \leq \left| \frac{h^2}{12}\mu_1 - \frac{h^2}{6}\mu_2 \right|,$$

if we take $\mu = max\{\mu_1, \mu_2\}$, then we have

$$e \leq \left| \frac{h^2}{12}\mu - \frac{h^2}{6}\mu \right| = \left| \frac{h^2}{12}\mu \right|. \tag{30}$$

Substituting $h = \frac{b-a}{N}$ in (30), we get

$$e \leq \left| \frac{(b-a)^2}{12N^2}\mu \right|.$$

Which is the error estimation. □

Application

In this section, the existence Theorem 2 will be applied on some examples of nonlocal Fredholm integro differential equation and we solve it analytically by using the modified decomposition method, numerically by using the finite difference Simpson approach. The results obtained are tabulated in Tables 1, 2, 3, 4, 5 and, 6, all results for these examples are performed by using Mathematica.

Example 1 Consider the equation:

$$u''(x) - \frac{1}{5}u(x) = 2 + \frac{1}{5}\left(-x^2 - x\right) + \frac{1}{25}\left(-x - \frac{7\cos(x)}{6} - \frac{1}{2}\right) + \frac{1}{25}\int_0^1 (t + x + t\cos(x)u'(t))dt, \tag{31}$$

$$u(0) + u(0.5) = \frac{3}{4}, \quad u'(0) = 1. \tag{32}$$

The exact solution of this equation is $u(x) = x + x^2$.

Firstly, we prove that this example has a continous solution,

$$f(x, u(x), \int_a^b g(x, t, u'(t)dt) = 2 + \frac{1}{5}\left(-x^2 - x\right) + \frac{1}{25}\left(-x - \frac{7\cos(x)}{6} - \frac{1}{2}\right) + \frac{1}{5}u(x) + \frac{1}{25}\int_0^1 (t + x + t\cos(x)u'(t))dt.$$

Then,

$$\left| f(x, u(x), \int_a^b g(x, t, u'(t)dt) \right| \leq \left| 2 + \frac{1}{5}\left(-x^2 - x\right) + \frac{1}{25}\left(-x - \frac{7\cos(x)}{6} - \frac{1}{2}\right) \right| + \frac{1}{5}|u(x)| + \frac{1}{5}\int_0^1 \frac{1}{5}|t + x + t\cos(x)u'(t)|dt,$$

and also

$$|g(x, t, u'(t)dt)| \leq \frac{1}{5}(x + t) + \frac{1}{5}|u'(t)|,$$

Table 1 The exact and numerical solution of Example 1

x_i	Approximate solution	Exact solution	Absolute error
0	0	0	4.9272 E−17
0.1	0.11	0.11	5.5511 E−17
0.2	0.24	0.24	5.5511 E−17
0.3	0.39	0.39	5.5511 E−17
0.4	0.56	0.56	1.1102 E−16
0.5	0.75	0.75	1.1102 E−16
0.6	0.96	0.96	2.2205 E−16
0.7	1.19	1.19	4.4409 E−16
0.8	1.44	1.44	8.8818 E−16
0.9	1.71	1.71	1.1102 E−15
1	2	2	1.3323 E−15

where $c_1(x) = 2 + \frac{1}{5}\left(-x^2 - x\right) + \frac{1}{25}\left(-x - \frac{7\cos(x)}{6} - \frac{1}{2}\right)$, $c_2(x,t) = \frac{1}{5}(x+t)$, $d_1 = \frac{1}{5}$, $d_2 = \frac{1}{5}$, $b = 1$, then $2d_1b^2 + d_1d_2b^2 = \frac{2}{5} + \frac{1}{25} = \frac{11}{25} < 1$. It is clear that the Assumption 1–4 of Theorem 2 is satisfied, therefore the given NBVP has a continuous solution. Then, we use the modified decomposition method to obtain the exact solution of this example. From Eq. (31) we get

$$u(x) \approx 0.038889\cos(x) + x + x^2 - 0.036056\cosh\left(\frac{x}{\sqrt{5}}\right) + \left(0.030905\cosh\left(\frac{x}{\sqrt{5}}\right) - 0.033333\cos(x)\right)\int_0^1 tu'(t)dt.$$

By using the modified decomposition method we can get the following recurrence relation

$$\begin{aligned} u_0(x) &= x + x^2, \\ u_1(x) &\approx 0.03889\cos(x) - 0.036056\cosh\left(\frac{x}{\sqrt{5}}\right) \\ &\quad + \left(0.030905\cosh\left(\frac{x}{\sqrt{5}}\right) - 0.033333\cos(x)\right)\int_0^1 tu_0'(t)dt \approx 0, \\ u_{l+1}(x) &= \int_0^1 K(x,t)u_l'(t)dt = 0, \quad l \geq 1. \end{aligned}$$

It is clear that each component of $u_l, l \geq 1$ is zero. This, in turn, gives the exact solution by

$$u(x) = x + x^2.$$

Now, we use the finite difference Simpson's approach to find the numerical solution of this example. Table 1 and Fig. 1 below give the approximate solution of this example and compare with tha exact solution.

Now, we study the continous dependence on u_0. If we take $u^*(0) + u^*(0.5) = 0.75001$, $u^{*\prime}(0) = 1.00001$. Then, the exact solution of Example 1 is given by

$$u^*(x) = x + x^2 + 2.44838 \times 10^{-6}\cosh\frac{x}{\sqrt{5}} + 2.23607 \times 10^{-5}\sinh\frac{x}{\sqrt{5}}.$$

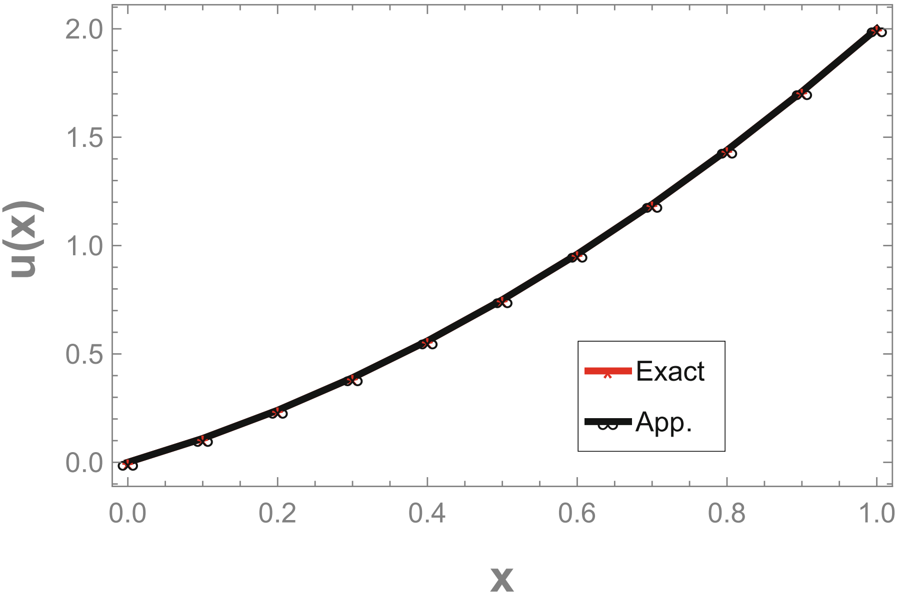

Fig. 1 Comparison between the approximate and exact solutions of Example 1

Then,

$$|u_0 - u_0^*| = 0.00001 \Longrightarrow \|u - u^*\| = 2.44838 \times 10^{-6} \cosh \frac{x}{\sqrt{5}} + 2.23607 \times 10^{-5} \sinh \frac{x}{\sqrt{5}} \leq 1.3034 \times 10^{-5}.$$

Then, Example 1 is a continous dependence on u_0. It is showing that in Fig. 2.

Now, we study the continous dependence on a_k. If we take $1.00001u^*(0)+1.00001u^*(0.5) = 0.75$, $u^{*\prime}(0) = 1$. Then, the exact solution of Example 1 is given by

$$u^*(x) = x + x^2 - 3.70348 \times 10^{-6} \cosh \frac{x}{\sqrt{5}}.$$

Then,

$$|a_k - a_k^*| = 0.00001 \Longrightarrow \|u - u^*\| = -3.70348 \times 10^{-6} \cosh \frac{x}{\sqrt{5}} \leq 4.08004 \times 10^{-6}.$$

Then, Example 1 is a continous dependence on a_k. It is showing that in Fig. 3.

Example 2 Consider the equation:

$$u''(x) + \frac{1}{3}u(x) = -\frac{2\sin(x)}{3} + \frac{1}{9}\int_{-1}^{1} (\sin(tx) + xtu'(t))dt, \tag{33}$$

$$u(-0.8) + 2u(-1) = \sin(-0.8) + 2\sin(-1), \quad u'(-1) = \cos(-1). \tag{34}$$

The exact solution of this equation is $u(x) = \sin(x)$.

Firstly we apply the assumption of Theorem 2 to prove that this example has a continuous solution:

$$f(x, u(x), \int_a^b g(x, t, u'(t)dt) = -\frac{2\sin(x)}{3} - \frac{1}{3}u(x) + \frac{1}{9}\int_{-1}^{1} (\sin(tx) + xtu'(t))dt.$$

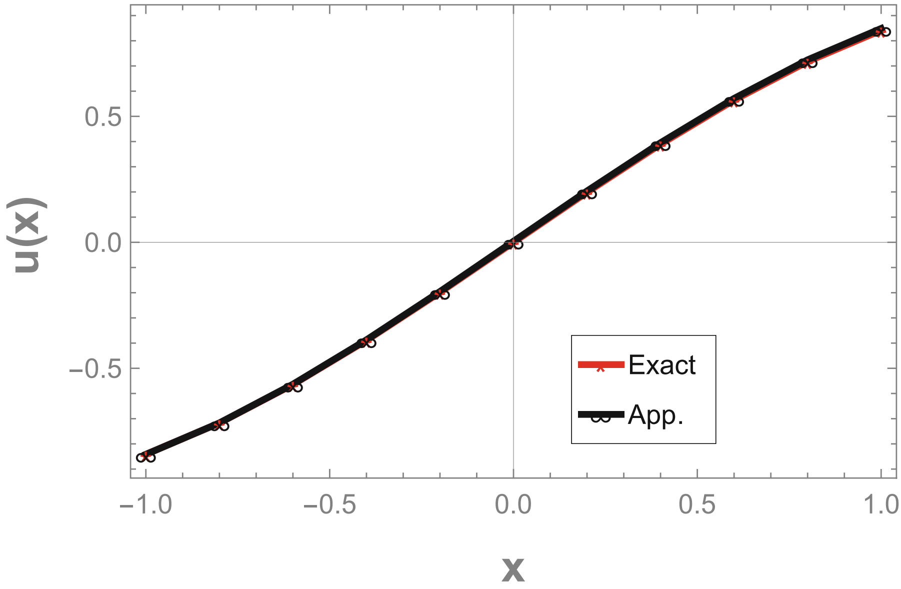

Fig. 2 Comparison between the exact solutions of u and u^*

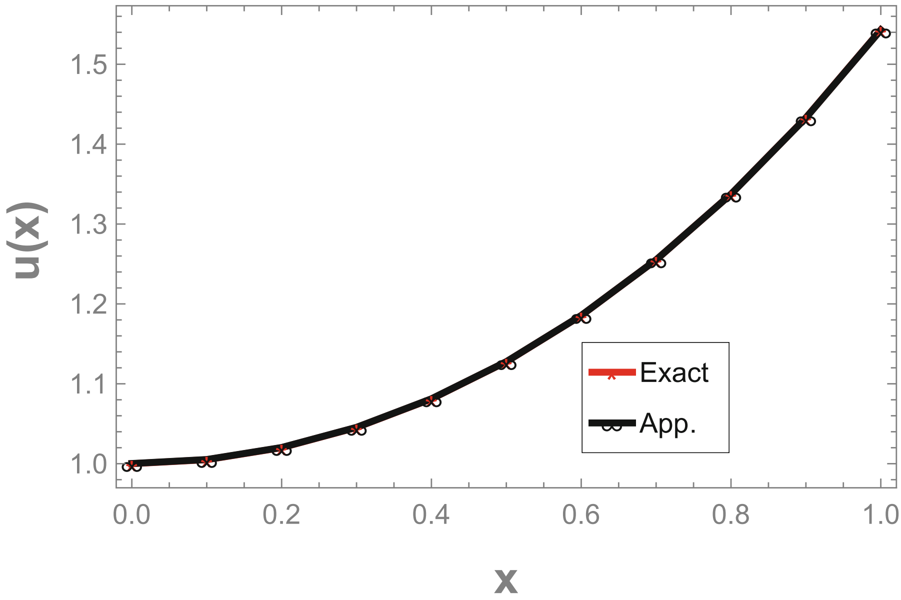

Fig. 3 Comparison between the exact solutions of u and u^*

Then,

$$|f(x, u(x), \int_a^b g(x, t, u'(t)dt)| \leq \frac{2\sin(x)}{3} + \frac{1}{3}|u(x)| + \frac{1}{3}\int_{-1}^{1} \frac{1}{3}|\sin(xt) + xtu'(t)|dt,$$

Table 2 The exact and numerical solution of Example 2

x_i	Approximate solution	Exact solution	Absolute error
−1	−0.84173	−0.841471	2.5958 E−4
−0.8	−0.71684	−0.717356	5.1915 E−4
−0.6	−0.56325	−0.564642	1.3897 E−3
−0.4	−0.38710	−0.389418	2.3193 E−3
−0.2	−0.19531	−0.198669	3.2714 E−3
0.	0.004207	0.	4.2072 E−3
0.2	0.203756	0.198669	5.0869 E−3
0.4	0.39529	0.389418	5.8715 E−3
0.6	0.571167	0.564642	6.5243 E−3
0.8	0.724369	0.717356	7.0126 E−3
1	0.84878	0.841471	7.3086 E−3

and also

$$|g(x,t,u'(t)dt)| \leq \frac{1}{3}\sin(xt) + \frac{1}{3}|u'(t)|,$$

where $c_1(x) = -\frac{2\sin(x)}{3}$, $c_2(x,t) = \frac{1}{3}\sin(xt)$, $d_1 = \frac{1}{3}$, $d_2 = \frac{1}{3}$, $b = 1$, $2d_1b^2 + d_1d_2b^2 = \frac{2}{3} + \frac{1}{3}\frac{1}{3} = \frac{7}{9} < 1$. It is clear that the Assumption 1–4 of Theorem 2 is satisfied, therefore the given NBVP has a continuous solution. Then, we use the modified decomposition to find the exact solution of this example. From Eq. (33) we get

$$u(x) \approx 1.64363 \times 10^{-16} \sin\frac{x}{\sqrt{3}} - 1.191 \times 10^{-17} \cos\frac{x}{\sqrt{3}} + \sin(x)$$
$$+ \left(0.333333x - 0.666082 \sin\left(\frac{x}{\sqrt{3}}\right) - 0.0352372 \cos\left(\frac{x}{\sqrt{3}}\right)\right) \int_{-1}^{1} tu'(t)dt.$$

By using the modified decomposition method we can get the following recurrence relation

$$u_0(x) = \sin(x),$$
$$u_1(x) \approx 1.64363 \times 10^{-16} \sin\frac{x}{\sqrt{3}} - 1.191 \times 10^{-17} \cos\frac{x}{\sqrt{3}}$$
$$+ \left(0.333333x - 0.666082 \sin\left(\frac{x}{\sqrt{3}}\right) - 0.0352372 \cos\left(\frac{x}{\sqrt{3}}\right)\right) \int_{-1}^{1} tu_0'(t)dt \approx 0,$$
$$u_{l+1}(x) = \int_{-1}^{1} K(x,t)u_l'(t)dt = 0, \quad l \geq 1.$$

It is clear that each component of u_l, $l \geq 1$ is zero. This in turn gives the exact solution by

$$u(x) = \sin(x).$$

Now, we use the finite diffrence Simpson's approach to find the numerical solution of this example. Table 2 and Fig. 4 below give the approximate solution of this example and compare it with the exact solution to show the accuracy of the method.

Now, we study the continous dependence on u_0. If we take

$$u^*(-0.8) + 2u^*(-1) = \sin(-0.8) + 2\sin(-1) + 0.00001, (u^*)'(-1) = \cos(-1) + 0.00001.$$

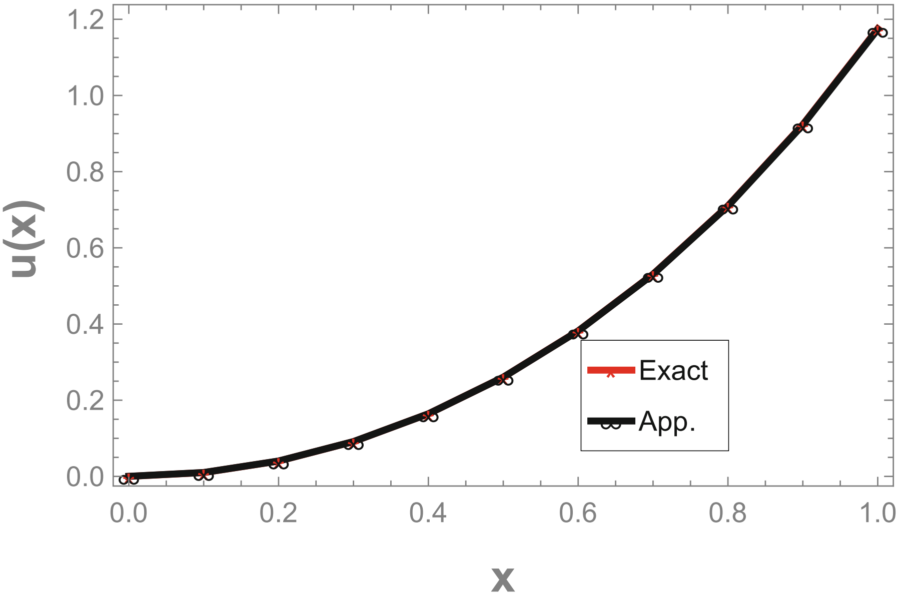

Fig. 4 Comparison between the approximate and exact solutions of Example 2

Then, the exact solution of Example 2 is

$$u^*(x) = 1.\sin(x) + 0.0000130535 \sin\left(\frac{x}{\sqrt{3}}\right) + 0.0000116943 \cos\left(\frac{x}{\sqrt{3}}\right).$$

Since

$$|u_0 - u_0^*| = 0.00001$$
$$\Longrightarrow \|u - u^*\| = 0.0000130535 \sin\left(\frac{x}{\sqrt{3}}\right) + 0.0000116943 \cos\left(\frac{x}{\sqrt{3}}\right) \leq 1.69235 \times 10^{-4}.$$

Then, Example 2 is a continous dependence on u_0. It is showing that in Fig. 5.
Now, we study the continous dependence on a_k. If we take

$$1.00001u^*(-0.8) + 2.00001u^*(-1) = \sin(-0.8) + 2\sin(-1),\ (u^*)'(-1) = \cos(-1).$$

Then, the exact solution of Example 2 is

$$u^*(x) = \sin(x) - 2.84235 \times 10^{-6} \sin\frac{x}{\sqrt{3}} + 4.36352 \times 10^{-6} \cos\frac{x}{\sqrt{3}}.$$

Since

$$|a_k - a_k^*| = 0.00001$$
$$\Longrightarrow \|u - u^*\| = -2.84235 \times 10^{-6} \sin\frac{x}{\sqrt{3}} + 4.36352 \times 10^{-6} \cos\frac{x}{\sqrt{3}} \leq 2.10488 \times 10^{-6}.$$

Then, Example 2 is a continous dependence on a_k. It is showing that in Fig. 6

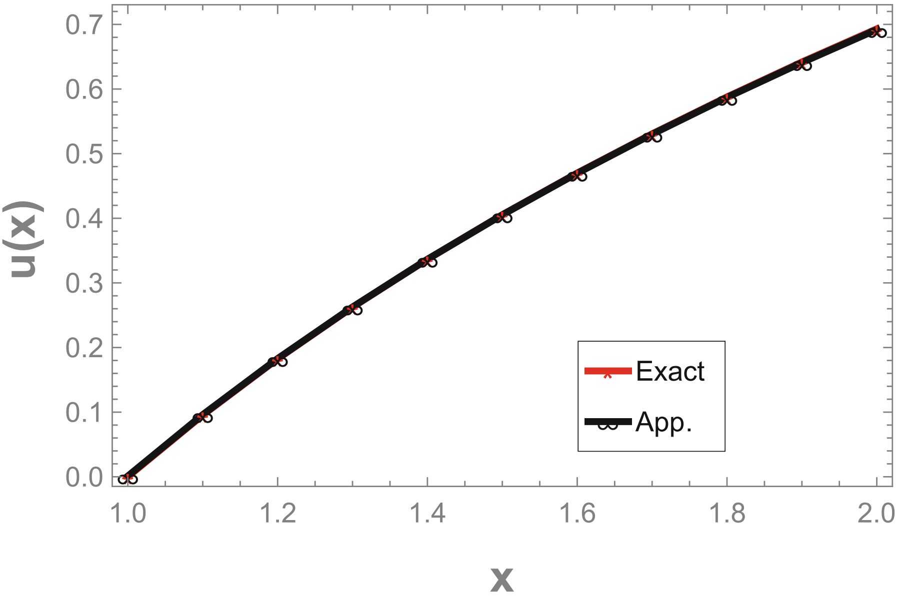

Fig. 5 Comparison between the exact solutions of u and u^*

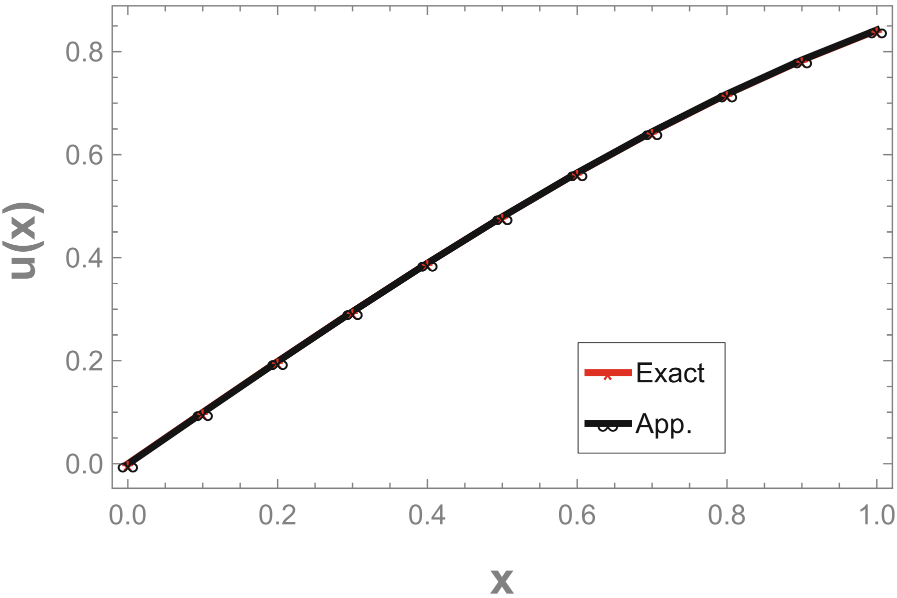

Fig. 6 Comparison between the exact solutions of u and u^*

Example 3 Consider the equation:

$$u''(x)-\frac{1}{7}u(x)=\frac{6\cosh(x)}{7}-\frac{(2+e)x}{28e}+\frac{1}{14}\int_0^1(tx+txu'(t))dt, \tag{35}$$

$$u(1)-u(0)=\cosh(1)-1,\quad u'(0)=0. \tag{36}$$

The exact solution of this equation is $u(x)=\cosh(x)$.

Firstly, we apply the assumption of Theorem 2 to prove that this example has a continuous solution:

$$f(x,u(x),\int_a^b g(x,t,u'(t)dt)=\frac{6\cosh(x)}{7}-\frac{(2+e)x}{28e}+\frac{1}{7}u(x)+\frac{1}{14}\int_0^1(tx+txu'(t))dt.$$

Then,

$$|f(x,u(x),\int_a^b g(x,t,u'(t)dt)|\leq|\frac{6\cosh(x)}{7}-\frac{(2+e)x}{28e}|+\frac{1}{7}|u(x)|+\frac{1}{7}\int_0^1\frac{1}{2}|tx+txu'(t)|dt,$$

and also

$$|g(x,t,u'(t)dt)|\leq\frac{1}{2}tx+\frac{1}{2}|u'(t)|,$$

where $c_1(x)=\frac{6\cosh(x)}{7}-\frac{(2+e)x}{28e}$, $c_2(x,t)=\frac{1}{2}tx$, $d_1=\frac{1}{7}$, $d_2=\frac{1}{2}$, $b=1$, $2d_1b^2+d_1d_2b^2=\frac{2}{7}+\frac{1}{7}\frac{1}{2}=\frac{5}{14}<1$. It is clear that the Assumption 1–4 of Theorem 2 is satisfied, therefore the given NBVP has a continuous solution. Then, we use the modified decomposition method to find the exact solution of this example. From Eq. (35) we get

$$u(x) = \cosh(x)+\frac{e^{-\frac{x}{\sqrt{7}}-1}\left(e^{\frac{x}{\sqrt{7}}}x+2e^{\frac{x+1}{\sqrt{7}}}x+e^{\frac{x+2}{\sqrt{7}}}x-\sqrt{7}e^{\frac{2x}{\sqrt{7}}}-\left(\sqrt{7}+1\right)e^{\frac{2x+1}{\sqrt{7}}}+e^{\frac{1}{\sqrt{7}}}\left(\sqrt{7}-1\right)+e^{\frac{2}{\sqrt{7}}}\sqrt{7}\right)}{2\left(1+e^{\frac{1}{\sqrt{7}}}\right)^2}$$
$$-\frac{e^{-\frac{x}{\sqrt{7}}}\left(e^{\frac{x}{\sqrt{7}}}x+2e^{\frac{x+1}{\sqrt{7}}}x+e^{\frac{x+2}{\sqrt{7}}}x-\sqrt{7}e^{\frac{2x}{\sqrt{7}}}-\left(\sqrt{7}+1\right)e^{\frac{2x+1}{\sqrt{7}}}+e^{\frac{1}{\sqrt{7}}}\left(\sqrt{7}-1\right)+e^{\frac{2}{\sqrt{7}}}\sqrt{7}\right)}{2\left(1+e^{\frac{1}{\sqrt{7}}}\right)^2}\int_0^1 tu'(t)dt.$$

By using the modified decomposition method we can get the following recurence relation

$$u_0(x)=\cosh(x),$$
$$u_1(x)=\frac{e^{-\frac{x}{\sqrt{7}}-1}\left(e^{\frac{x}{\sqrt{7}}}x+2e^{\frac{x+1}{\sqrt{7}}}x+e^{\frac{x+2}{\sqrt{7}}}x-\sqrt{7}e^{\frac{2x}{\sqrt{7}}}-\left(\sqrt{7}+1\right)e^{\frac{2x+1}{\sqrt{7}}}+e^{\frac{1}{\sqrt{7}}}\left(\sqrt{7}-1\right)+e^{\frac{2}{\sqrt{7}}}\sqrt{7}\right)}{2\left(1+e^{\frac{1}{\sqrt{7}}}\right)^2}$$
$$-\frac{e^{-\frac{x}{\sqrt{7}}}\left(e^{\frac{x}{\sqrt{7}}}x+2e^{\frac{x+1}{\sqrt{7}}}x+e^{\frac{x+2}{\sqrt{7}}}x-\sqrt{7}e^{\frac{2x}{\sqrt{7}}}-\left(\sqrt{7}+1\right)e^{\frac{2x+1}{\sqrt{7}}}+e^{\frac{1}{\sqrt{7}}}\left(\sqrt{7}-1\right)+e^{\frac{2}{\sqrt{7}}}\sqrt{7}\right)}{2\left(1+e^{\frac{1}{\sqrt{7}}}\right)^2}$$
$$\int_0^1 tu_0'(t)dt=0,\quad u_{l+1}(x)=-\int_0^1 K(x,)u_l'(t)dt=0,\quad l\geq 1.$$

It is clear that each component of u_l, $l\geq 1$ is zero. This in turn gives the exact solution by

$$u(x)=\cosh(x).$$

Table 3 The exact and numerical solution of Example 3

x_i	Approximate solution	Exact solution	Absolute error
0	1.00022	1	2.1892 E−4
0.1	1.00522	1.005	2.1491 E−4
0.2	1.02027	1.02007	2.0285 E−4
0.3	1.04552	1.04534	1.8262 E−4
0.4	1.08123	1.08107	1.5401 E−4
0.5	1.12774	1.12763	1.1669 E−4
0.6	1.18554	1.18547	7.0258 E−5
0.7	1.25518	1.25517	1.4179 E−5
0.8	1.33738	1.33743	5.2183 E−5
0.9	1.43296	1.43309	1.2959 E−4
1	1.54286	1.54308	2.1892 E−4

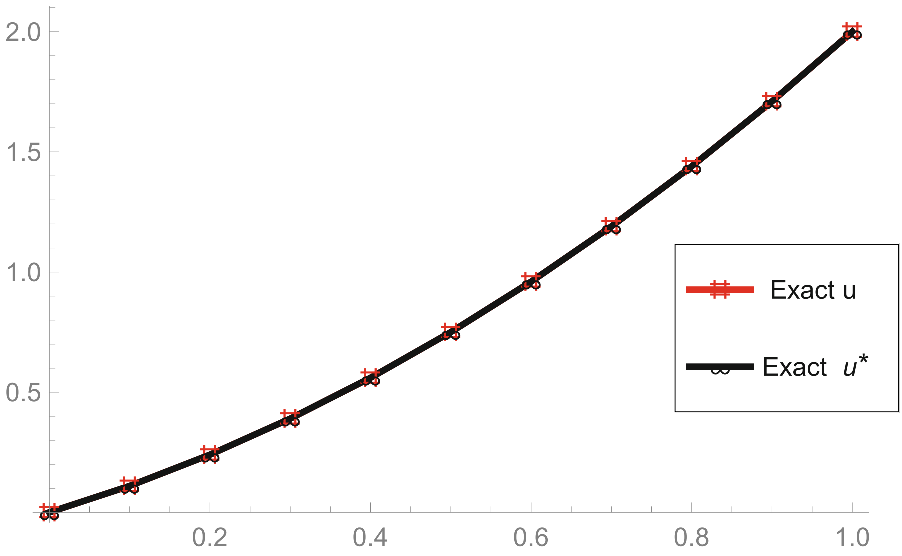

Fig. 7 Comparison between the approximate and exact solutions of Example 3

Now, we use the finite difference Simpson's approach to finding the numerical solution of this example.

Table 3 and Fig. 7 below give the approximate solution of this example and compare it with the exact solution to show the accuracy of the presented method.

Example 4 Consider the equation:

$$u''(x) - \frac{1}{4}u(x) = 2\cosh(x) + \frac{1}{8}(-4 - \sinh(1)) - \sinh(x) + \frac{3}{4}x\sinh(x) + \int_0^1 \left(t + \sinh(x) + \frac{1}{8}u'(t)\right) dt, \quad (37)$$

$$u(0.5) + u(0) = 0.5\sinh(0.5), \quad u'(0) = 0. \quad (38)$$

The exact solution of this equation is $u(x) = x\sinh(x)$.

Firstly, we apply the assumption of Theorem 2 to prove that this example has a continuous solution:

$$f(x, u(x), \int_a^b g(x, t, u'(t)dt) = 2\cosh(x) + \frac{1}{8}(-4 - \sinh(1)) - \sinh(x) + \frac{3}{4}x\sinh(x) + \frac{1}{4}u(x) + \int_0^1 \left(t + \sinh(x) + \frac{1}{8}u'(t)\right) dt.$$

Then

$$\begin{aligned}
&|f(x, u(x), \int_a^b g(x, t, u'(t)dt)| \\
&\leq |2\cosh(x) + \frac{1}{8}(-4 - \sinh(1)) - \sinh(x) + \frac{3}{4}x\sinh(x)| + \frac{1}{4}|u(x)| \\
&+ \frac{1}{4}\int_0^1 |4(t + \sinh(x)) + \frac{1}{2}u'(t)|dt,
\end{aligned}$$

and also

$$|g(x, t, u'(t)dt)| \leq 4(t + \sinh(x)) + \frac{1}{2}|u'(t)|,$$

where $c_1(x) = 2\cosh(x) + \frac{1}{8}(-4 - \sinh(1)) - \sinh(x) + \frac{3}{4}x\sinh(x), c_2(x,t) = 4(t + \sinh(x)), d_1 = \frac{1}{4}$,

$d_2 = \frac{1}{2}, b = 1, 2d_1b^2 + d_1d_2b^2 = 2\frac{1}{4} + \frac{1}{4}\frac{1}{2} = \frac{5}{8} < 1$. It is clear that the Assumption 1–4 of Theorem 2 is satisfied, therefore the given NBVP has a continuous solution. Then, we use the modified decomposition method to obtain the exact solution of this example. From Eq. (37) we get

$$u(x) \approx x\sinh(x) + 0.5876 - 0.5785\cosh\left(\frac{x}{2}\right) + \left(-0.5 + 0.4923\cosh\left(\frac{x}{2}\right)\right)\int_0^1 u'(t)dt.$$

By using modified decomposition method we can write the following recurence relation

$$\begin{aligned}
u_0(x) &= x\sinh(x), \\
u_1(x) &= 0.5876 - 0.5785\cosh\left(\frac{x}{2}\right) + \left(-0.5 + 0.4923\cosh\left(\frac{x}{2}\right)\right)\int_0^1 u'(t)dt, \\
u_{l+1}(x) &= \int_0^1 K(x,t)u_l(t)dt = 0, \quad l \geq 1.
\end{aligned}$$

It is clear that each component of $u_l, l \geq 1$ is zero. This in turn gives the exact solution by

$$u(x) = x\sinh(x).$$

Now, we use the finite difference Simpson's approach to finding the numerical solution of this example.

Table 4 and Fig. 8 below give the approximate solution of this example and compare it with the exact solution to show the accuracy of the method.

Example 5 Consider the equation:

$$u''(x) - \frac{1}{15}u(x) = -\frac{1}{x^2} - \frac{\ln(x)}{15} + \frac{1}{60}(1 - \ln(8x)) + \frac{1}{60}\int_1^2 (\ln(xt) + u'(t))dt, \quad (39)$$

$$u(1) + u(2) = \ln(2), \quad u'(1) = 1. \quad (40)$$

Table 4 The exact and numerical solution of Example 4

x_i	Approximate solution	Exact solution	Absolute error
0	0.0001970	0	1.9700E−4
0.1	0.0101985	0.0100167	1.8182E−4
0.2	0.0404032	0.0402672	1.3600E−4
0.3	0.0914148	0.0913561	5.8668E−5
0.4	0.164249	0.164301	5.1642E−5
0.5	0.260351	0.260548	1.9700E−4
0.6	0.381612	0.381992	3.8013E−4
0.7	0.530404	0.531009	6.0443E−4
0.8	0.709611	0.710485	8.7402E−4
0.9	0.922671	0.923865	1.1938E−3
1	1.17363	1.1752	1.5696E−2

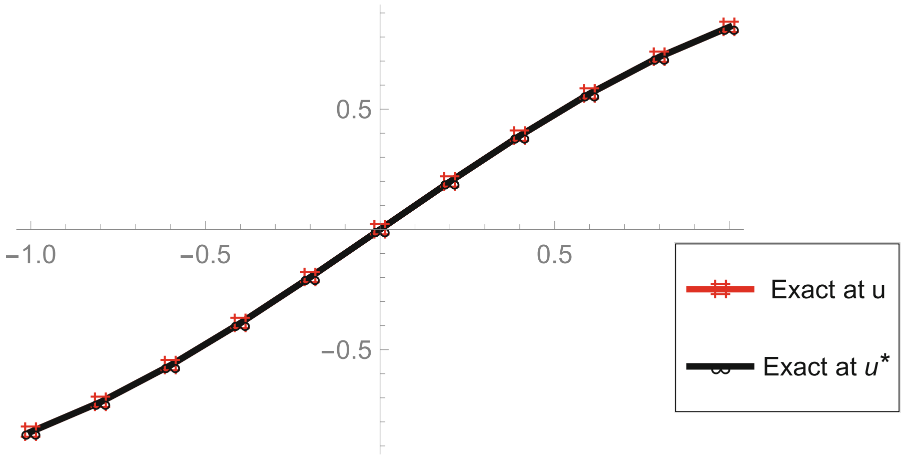

Fig. 8 Comparison between the approximate and exact solutions of Example 4

The exact solution of this equation is $u(x) = \ln(x)$.

Firstly, we apply the assumption of Theorem 2 to prove that this example has a continuous solution:

$$f(x, u(x), \int_a^b g(x, t, u'(t)dt) = -\frac{1}{x^2} - \frac{\ln(x)}{15} + \frac{1}{60}(1 - \ln(8x)) + \frac{1}{15}u(x) + \frac{1}{60}\int_1^2 (\ln(xt)) + u'(t)dt.$$

Then

$$|f(x, u(x), \int_a^b g(x, t, u'(t)dt)| \leq |-\frac{1}{x^2} - \frac{\ln(x)}{15} + \frac{1}{60}(1 - \ln(8x))| + \frac{1}{15}|u(x)| + \frac{1}{15}\int_1^2 \frac{1}{4}|(\ln(xt)) + u'(t)|dt,$$

and also

$$|g(x,t,u'(t)dt)| \leq \frac{1}{4}\ln(xt) + \frac{1}{4}|u'(t)|,$$

where $c_1(x) = -\frac{1}{x^2} - \frac{\ln(x)}{15} + \frac{1}{60}(1-\ln(8x))$, $c_2(x,t) = \frac{1}{4}\ln(xt)$, $d_1 = \frac{1}{15}$, $d_2 = \frac{1}{4}$, $b = 2$, $2d_1b^2 + d_1d_2b^2 = \frac{8}{15} + \frac{1}{15} = \frac{9}{15} < 1$. It is clear that the Assumption 1–4 of Theorem 2 is satisfied, therefore the given NBVP has a continuous solution. Then, we use the modified decomposition to find the exact solution of this example. From Eq. (39) we get

$$u(x) = \frac{e^{-\frac{x}{\sqrt{15}}}\left(e^{\frac{x}{\sqrt{15}}} - 2e^{\frac{2x}{\sqrt{15}}} + 2e^{\frac{x+1}{\sqrt{15}}} + e^{\frac{x+2}{\sqrt{15}}} - 2e^{\frac{2}{\sqrt{15}}}\right)\ln(2)}{4\left(1+e^{\frac{1}{\sqrt{15}}}\right)^2} + \ln(x)$$
$$- \frac{e^{-\frac{x}{\sqrt{15}}}\left(e^{\frac{x}{\sqrt{15}}} - 2e^{\frac{2x}{\sqrt{15}}} + 2e^{\frac{x+1}{\sqrt{15}}} + e^{\frac{x+2}{\sqrt{15}}} - 2e^{\frac{2}{\sqrt{15}}}\right)}{4\left(1+e^{\frac{1}{\sqrt{15}}}\right)^2}\int_1^2 u'(t)\,dt.$$

By using the modified decomposition method we can get the following recurence relation

$$u_0(x) = \ln(x),$$
$$u_1(x) = \frac{e^{-\frac{x}{\sqrt{15}}}\left(e^{\frac{x}{\sqrt{15}}} - 2e^{\frac{2x}{\sqrt{15}}} + 2e^{\frac{x+1}{\sqrt{15}}} + e^{\frac{x+2}{\sqrt{15}}} - 2e^{\frac{2}{\sqrt{15}}}\right)\ln(2)}{4\left(1+e^{\frac{1}{\sqrt{15}}}\right)^2}$$
$$- \frac{e^{-\frac{x}{\sqrt{15}}}\left(e^{\frac{x}{\sqrt{15}}} - 2e^{\frac{2x}{\sqrt{15}}} + 2e^{\frac{x+1}{\sqrt{15}}} + e^{\frac{x+2}{\sqrt{15}}} - 2e^{\frac{2}{\sqrt{15}}}\right)}{4\left(1+e^{\frac{1}{\sqrt{15}}}\right)^2}\int_1^2 u_0'(t)dt = 0,$$
$$u_{l+1}(x) = -\int_1^2 K(x,t)u_l(t)dt = 0, \quad l \geq 1.$$

It is clear that each component of u_l, $l \geq 1$ is zero. This in turn gives the exact solution by

$$u(x) = \ln(x).$$

Now, we use the finite difference Simpson's approach to finding the numerical solution of this example.

Table 5 and Fig. 9 below give the approximate solution of this example and compare it with the exact solution to show the accuracy of the method.

From the results in Table 5 we can say that the proposed method is effective.

Example 6 Consider the equation:

$$u''(x) - \frac{1}{5}u(x) = -\frac{1}{45}x\left(2x + 3\sin^2(1)\right) - \frac{6\sin(x)}{5} + \frac{2}{15}\int_0^1\Big((tx)^2 + x\sin(t)u'(t)\Big)dt, \tag{41}$$

$$\int_0^1 u(s)ds = 1-\cos(1), \quad u'(0) = 1. \tag{42}$$

Table 5 The exact and numerical solution of Example 5

x_i	Approximate solution	Exact solution	Absolute error
1.	0.00114268	0.	1.1427E−3
1.1	0.096143	0.0953102	8.3280E−4
1.2	0.182879	0.182322	5.5763E−4
1.3	0.262671	0.262364	3.0689E−4
1.4	0.336546	0.336472	7.3764E−5
1.5	0.405319	0.405465	1.4643E−4
1.6	0.469647	0.470004	3.5699E−4
1.7	0.530068	0.530628	5.6031E−4
1.8	0.587028	0.587787	7.5818E−4
1.9	0.640902	0.641854	9.5195E−4
2.	0.692004	0.693147	1.1427E−3

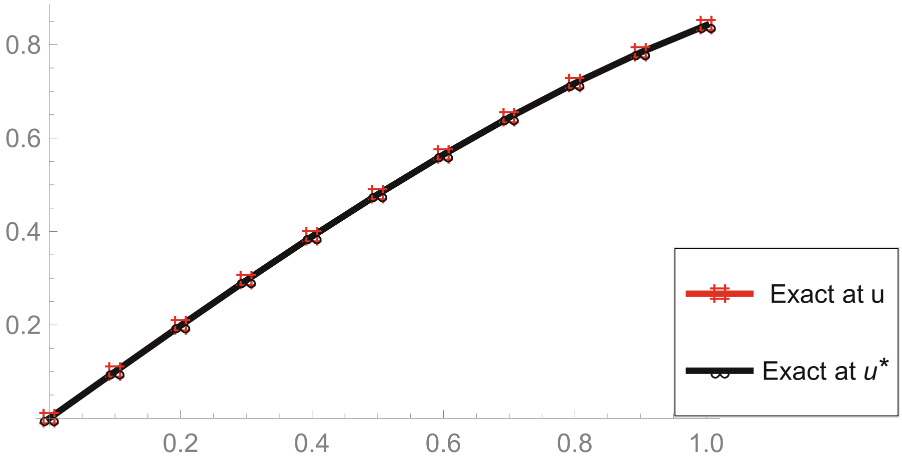

Fig. 9 Comparison between the approximate and exact solutions of Example 5

The exact solution of this equation is $u(x) = \sin(x)$.
Firstly we prove that this example has a continous solution:

$$f(x, u(x), \int_a^b g(x, t, u'(t)dt) = -\frac{1}{45}x\left(2x + 3\sin^2(1)\right) - \frac{6\sin(x)}{5} + \frac{1}{5}u(x) + \frac{2}{15}\int_0^1 \left((tx)^2 + x\sin(t)u'(t)\right)dt.$$

Then,

$$|f(x, u(x), \int_a^b g(x, t, u'(t)dt)| \leq \left|-\frac{1}{45}x\left(2x + 3\sin^2(1)\right) - \frac{6\sin(x)}{5}\right| + \frac{1}{5}|u(x)| + \frac{1}{5}\int_0^1 \frac{2}{3}|(tx)^2 + x\sin(t)u'(t)|dt,$$

Table 6 The exact and numerical solution of Example 6

x_i	Approximate solution	Exact solution	Absolute error
0	− 0.000787671	0	7.8767 E−4
0.1	0.0992115	0.0998334	6.2188 E−4
0.2	0.198211	0.198669	4.5814 E−4
0.3	0.295223	0.29552	2.9696 E−4
0.4	0.38928	0.389418	1.3881 E−4
0.5	0.479441	0.479426	1.5874 E−4
0.6	0.564809	0.564642	1.6664 E−4
0.7	0.644531	0.644218	3.1311 E−4
0.8	0.717811	0.717356	4.5491 E−4
0.9	0.783919	0.783327	5.9173 E−4
1	0.842194	0.841471	7.2330 E−4

and also

$$|g(x,t,u'(t)dt)| \leq \frac{2}{3}(tx)^2 + \frac{2}{3}|u'(t)|,$$

where $c_1(x) = -\frac{1}{45}x\left(2x + 3\sin^2(1)\right) - \frac{6\sin(x)}{5}$, $c_2(x,t) = \frac{2}{3}(tx)^2$, $d_1 = \frac{1}{5}$, $d_2 = \frac{2}{3}$, $b = 1$, then $2d_1b^2 + d_1d_2b^2 = \frac{2}{5} + \frac{2}{15} = \frac{8}{15} < 1$. It is clear that the Assumption 1–4 of Theorem 2 is satisfied, therefore the given NBVP has a continuous solution. Then, we use the modified decomposition method to obtain the exact solution of this example. From Eq. (41) we get

$$\begin{aligned} u(x) &\approx 0.236024x + \sin(x) - 0.262926e^{\frac{x}{\sqrt{5}}} + 0.264841e^{-\frac{x}{\sqrt{5}}} \\ &\quad + \left(-0.666667x + 0.742651e^{\frac{x}{\sqrt{5}}} - 0.748061e^{-\frac{x}{\sqrt{5}}}\right)\int_0^1 \sin(t)u'(t)dt. \end{aligned}$$

By using modified decomposition method we can get the following recurence relation

$$\begin{aligned} u_0(x) &= \sin(x), \\ u_1(x) &\approx 0.236024x - 0.262926e^{\frac{x}{\sqrt{5}}} + 0.264841e^{-\frac{x}{\sqrt{5}}} \\ &\quad + \left(-0.666667x + 0.742651e^{\frac{x}{\sqrt{5}}} - 0.748061e^{-\frac{x}{\sqrt{5}}}\right)\int_0^1 \sin(t)u_0'(t)dt \approx 0, \\ u_{l+1}(x) &= k(x,t)u_l'(t)dt, \quad l \geq 1. \end{aligned}$$

It is clear that each component of u_l, $l \geq 1$ is zero. This in turn gives the exact solution by

$$u(x) = \sin(x).$$

Now, we use the finite difference Simpson's approach to finding the numerical solution of this example.

Table 6 and Fig. 10 below give the approximate solution of this example and compare it with the exact solution to show the accuracy of the presented method.

Now, we study the continous dependence on u_0.

If we take $\int_0^1 u^*(s)ds = 1.00001 - \cos(1)$, $\quad u^{*'}(0) = 1.00001$. then the exact solution of Example 6 is given by

$$u^*(x) = \sin(x) + 4.122539 \times 10^{-17}x + 2.23607 \times 10^{-5}\sinh\frac{x}{\sqrt{5}} + 4.5086196 \times 10^{-6}\cosh\frac{x}{\sqrt{5}}.$$

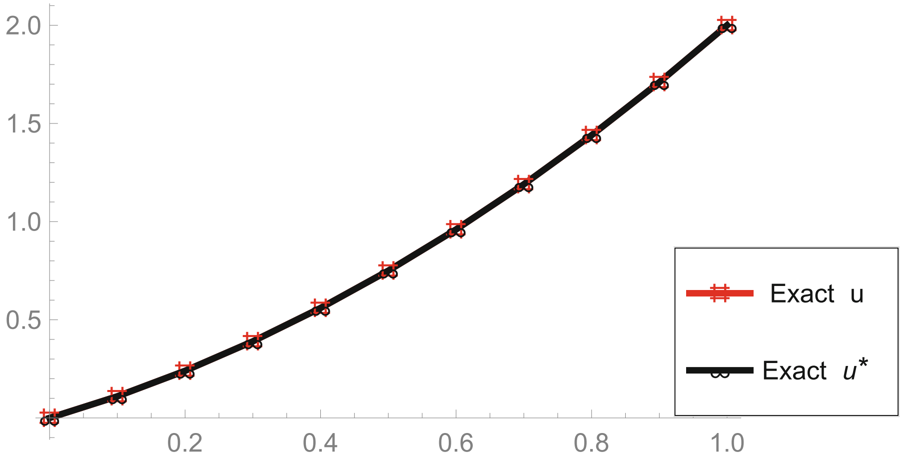

Fig. 10 Comparison between the approximate and exact solutions of Example 6

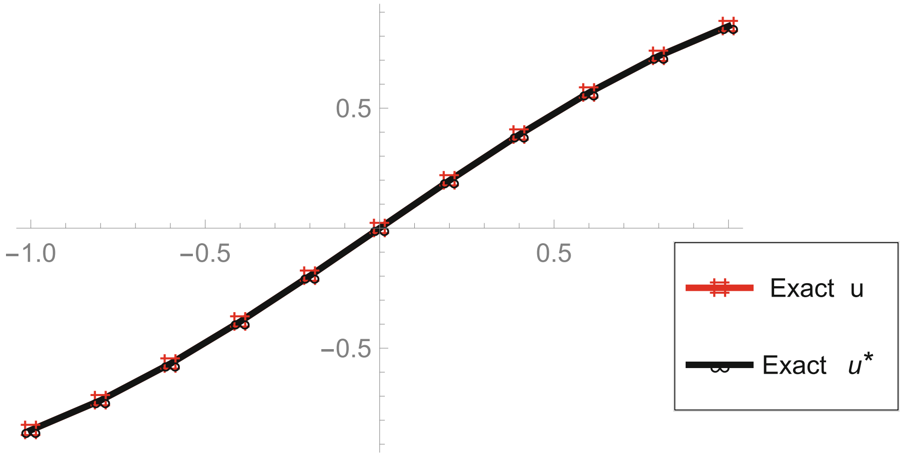

Fig. 11 Comparison between the d exact solutions of u and u^*

Since $|u_0 - u_0^*| = 0.00001 \implies \|u - u^*\| = 4.122539 \times 10^{-17}x + 2.23607 \times 10^{-5} \sinh \frac{x}{\sqrt{5}} + 4.5086196 \times 10^{-6} \cosh \frac{x}{\sqrt{5}} \leq 1.53037 \times 10^{-5}$.

Then, Example 6 is a continous dependence on u_0. It is showing that in Fig. 9.

Conclusion

In this work, the existence, uniqueness and the continuous dependence of the NBVP have been studied. Some examples are introduced to illustrate the benefits of our results, also, by using the modified decomposition method, we get the exact solution. Furthermore a numerical study of this system has been presented, by solving the proposed models numerically using the finite difference Simpson's method. Some numerical solutions are compared with exact answers to show the accuracy of our methods, and some figures are obtained that illustrate this approach. It is evident from the presented Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 that the numerical results that we obtained are entirely consistent with the analytical study that we have carried out. Thus, through the survey that we conducted on some examples, one can say that we have made a clear contribution in solving the integral differential equations in the form of the proposed system analytically and numerically, in full accordance with the analytical study that conducted.

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Declarations

Conflict of interest There is no conflict of interest between the authors or anyone else regarding this manuscript.

Ethical Statement The authors confirm that all the results they obtained are new and there is no conflict of interest with anyone.

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