Int. J. Appl. Comput. Math (2022) 8:19
https://doi.org/10.1007/s40819-021-01233-w

ORIGINAL PAPER

Shehu Transform in Quantum Calculus and Its Applications

Arvind Kumar Sinha[1] · Srikumar Panda[1]

Accepted: 24 December 2021 / Published online: 9 January 2022

Abstract
In this paper, we introduce the concept of Shehu transform in q-calculus namely q-Shehu transform and establish some properties. We also give some applications of q-Shehu transform for solving some ordinary and partial differential equations with initial and boundary values problems to show its effectiveness and performance of the proposed transform.

Keywords Quantum calculus · q-Jackson integrals · q-Derivative · Shehu transform · q-Shehu transform · Convolution · Applications of q-Shehu transform

Mathematics Subject Classification 33D05 · 33D60 · 35A22 · 44A15

Introduction

Researchers are actively involved in the overall transformation of theme development because it is suitable for describing and analyzing physical systems [1–20]. Jackson [21] introduced q-calculus. Now, the q-calculus has become very important in various fields of science and technology. The concept of q-calculus can be used in fractions and control problems [22]. Some integral transformations have different q analogs. The research is carried out on the q-calculus [23–25]. Maitama and Zhao [1] first established the Shehu transform to solve partial differential equations in the time domain as a generalization of Sumudu and Laplace transforms. So we motivate to introduce the concept q-Shehu transform to get the advantages in q-calculus.

The Shehu transform [1] of the function $f(\varpi)$ is defined by

$$S[f(\varpi)] = R(\tau, \varrho) = \int_0^\infty e^{\frac{-\tau\varpi}{\varrho}} f(\varpi)\, d\varpi.$$

✉ Srikumar Panda
srikumarpanda79@gmail.com

Arvind Kumar Sinha
aksinha.maths@nitrr.ac.in

1 Department of Mathematics, National Institute of Technology Raipur (C.G.), Raipur 492010, India

We introduce the concept of Shehu transform in q-calculus namely q-Shehu transform and establish some properties. We also give some applications of q-Shehu transform for solving some ordinary and partial differential equations with initial and boundary values problems to show its effectiveness and performance of the proposed transform.

Preliminaries

We use literature's [23,26–28] for our study.

The q-shifted factorials for $q \in (0, 1)$ and $\kappa \in \mathbb{C}$ are defined as

$$(\kappa, q) = 1, (\kappa, q)_n = \prod_{k=0}^{n-1}(1 - \kappa q^k), n = 1, 2...,$$

$$(\kappa; q)_\infty = \lim_{n\to\infty}(\kappa; q)_n = \prod_{k=0}^{\infty}(1 - \kappa q^k).$$

Also,

$$[\kappa]_q = \frac{1 - q^\kappa}{1 - q}, [\kappa]_q! = \frac{(q; q)_n}{(1 - q)^n}, \ n \in \mathbb{N}.$$

The q-derivatives $D_q f$ and D_q^+ of a function f, given by Kac and Cheung [22]

$$(D_q f)(\alpha) = \frac{f(\alpha) - f(q\alpha)}{(1 - q)\alpha}, \text{ if } (\alpha \neq 0)$$

and $(D_q f)(0) = f'(0)$ exists.

If f is differentiable, then $(D_q f)(\alpha)$ tends to $f'(\alpha)$ as q tends to 1. For $n \in \mathbb{N}$, we have

$$D_q^1 = D_q, \ \ (D_q^+)^1 = D_q^+.$$

The q-derivative of the product

$$D_q(f.g)(\alpha) = g(\alpha)D_q f(\alpha) + f(q\alpha)D_q g(\alpha).$$

The q-Jackson integral from 0 to k and from 0 to ∞ given by Jackson [21]

$$\int_0^\kappa f(\alpha)d_q\alpha = (1 - q)\alpha\sum_{n=0}^{\infty} f(\alpha q^n)q^n,$$

$$\int_0^\infty f(\alpha)d_q\alpha = (1 - q)\sum_{n=-\infty}^{\infty} f(q^n)q^n,$$

provided these sums converge absolutely.

A q-analogue of integration by parts formulae is given by the following relation:

$$\int_\kappa^\varpi g(\alpha)D_q f(\alpha)d_q\alpha = f(\varpi)g(\varpi) - f(\kappa)g(\kappa) - \int_\kappa^\varpi f(q\alpha)D_q g(\alpha)d_q\alpha.$$

Gasper and Rahamen [27], Kac and Cheung [22] have given the following relation:

$$E_q^{\rho} = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{\rho^n}{[n]_q!} = (-(1-q)z; q)_{\infty}, \tag{1}$$

$$e_q^{\rho} = \sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!} = \frac{1}{((1-q)\rho; q)_{\infty}}, \quad |z| < \frac{1}{1-q}. \tag{2}$$

The above Eqs. (1) and (2) satisfy the following equations:

$$D_q e_q^{\rho} = e_q^{\rho}, \ D_q E_q^{\rho} = E_q^{q\rho}, \ \text{and}$$
$$e_q^{\rho} E_q^{-\rho} = E_q^{-\rho} e_q^{\rho} = 1.$$

Jackson [21] has introduced the following concept and many researchers [22,26,28] have given important results on it,

$$\Gamma(\vartheta) = \int_0^{\infty} \alpha^{\vartheta-1} e^{-\alpha} d_q\alpha \text{ by}$$
$$\Gamma_q(\vartheta) = \frac{(q;q)_{\infty}}{(q^{\vartheta};q)_{\infty}}(1-q)^{\vartheta-1}, \quad \vartheta \neq 0, -1, -2...... .$$

If satisfies the following conditions

$$\Gamma_q(\vartheta+1) = [\vartheta]_q \Gamma(\vartheta), \Gamma_q(1) = 1, \ \text{and}$$
$$\lim_{q \to 1^-} \Gamma_q(\vartheta) = \Gamma(\vartheta), \quad Re(\vartheta) > 0.$$

The function Γ_q has the following q -integral representations

$$\Gamma_q(\gamma) = \int_0^{\frac{1}{1-q}} \vartheta^{\gamma-1} E_q^{-q\vartheta} d_q\vartheta = \int_0^{\frac{\infty}{1-q}} \vartheta^{\gamma-1} E_q^{-qt} d_q t.$$

The q-integral representation Γ_q is defined in [23,29] as follows:
For all $\gamma, \vartheta > 0$, we have

$$\Gamma_q(\gamma) = K_q(s) \int_0^{\frac{\infty}{1-q}} \alpha^{\gamma-1} e_q^{-\alpha} d_q\alpha, \ \text{and}$$

$$B_q(\vartheta, \gamma) = K_q(\vartheta) \int_0^{\infty} \alpha^{\vartheta-1} \frac{(-\alpha q^{\gamma+1}; q)_{\infty}}{(-\alpha; q)_{\infty}} d_q\alpha,$$

where,

$$K_q(\vartheta) = \frac{(-q,-1;q)_\infty}{(-q^\vartheta,-q^{1-\vartheta};q)_\infty}.$$

If $\frac{\log(1-q)}{\log(q)} \in \mathbb{Z}$, we obtain

$$\Gamma_q(\gamma) = K_q(\gamma)\int_0^{\frac{\infty}{1-q}} \alpha^{\gamma-1} e_q^{-\alpha} d_q\alpha = \int_0^{\frac{\infty}{1-q}} \vartheta^{\gamma-1} E_q^{-q\vartheta} d_q\vartheta.$$

Main results

Definition 3.1 Λ be a function defined by $\mathbb{R}_{q_1,+}$, we defined the q-Shehu transform of a function Λ as

$$S_q(\Lambda)(\varpi) = S_q[\Lambda(\tau,\varrho)](\varpi) = \frac{1}{(1-q)}\int_0^\infty e_q^{\frac{-\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi. \tag{3}$$

Property 3.2 (*Linearity property*) Let the functions $M\Lambda(\varpi)$ and $N\rho(\varpi)$ be in set A, then $[M\Lambda(\varpi) + N\rho(\varpi)] \in A$, where M and N are non-zero arbitrary constants, and

$$S_q[M\Lambda(\varpi) + N\rho(\varpi)] = MS_q[\Lambda(\varpi)] + NS_q[\rho(\varpi)].$$

Proof

$$S_q\,[M\Lambda(\varpi) + N\rho(\varpi)] = \frac{1}{(1-q)}\int_0^\infty e_q^{\frac{-\tau\varpi}{\varrho}} (M\Lambda(\varpi) + N\rho(\varpi))\, d_q\varpi$$
$$= \frac{1}{(1-q)}\int_0^\infty e_q^{\frac{-\tau\varpi}{\varrho}} (M\Lambda(\varpi))\, d_q\varpi$$

$$+\frac{1}{(1-q)}\int_0^\infty e_q^{\frac{-\tau\varpi}{\varrho}}(N\rho(\varpi))\,d_q\varpi$$

$$=\frac{M}{(1-q)}\int_0^\infty e_q^{\frac{-\tau\varpi}{\varrho}}\Lambda(\varpi)\,d_q\varpi$$

$$+\frac{N}{(1-q)}\int_0^\infty e_q^{\frac{-\tau\varpi}{\varrho}}\rho(\varpi)\,d_q\varpi$$

$$=MS_q[\Lambda(\varpi)]+NS_q[\rho(\varpi)].$$

Property 3.3 (*Change of the scale property*) Let the function $\Lambda(M(\varpi))$ be in a set A, where M is an arbitrary constant, then

$$S_q[\Lambda(M\varpi)]=\frac{\varrho}{M}R\Big(\frac{\tau}{M},\varrho\Big).$$

Proof We obtain

$$S_q[\Lambda(M\varpi)]=\frac{1}{(1-q)}\int_0^\infty e_q^{\frac{-\tau\varpi}{\varrho}}\Lambda(M\varpi)\,d_q\varpi.$$

Substituting $\eta=M\varpi\Rightarrow\varpi=\frac{\eta}{M}$ and $d_q\varpi=\frac{d_q\eta}{M}$.

Therefore,

$$S_q[\Lambda(M\varpi)]=\frac{1}{M(1-q)}\int_0^\infty e_q^{-\frac{\tau\eta}{\varrho M}}\Lambda(\eta)\,d_q\eta$$

$$=\frac{1}{M(1-q)}\int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho M}}\Lambda(\varpi)\,d_q\varpi$$

$$=\frac{\varrho}{M(1-q)}\int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho M}}\Lambda(\varrho\,\varpi)\,d_q\varpi$$

$$=\frac{\varrho}{M}R\Big(\frac{\tau}{M},\varrho\Big).$$

Property 3.4 q-Shehu transform shoes the following:
(1) If $\Lambda(\varpi) = 1$ be in set A, then

$$S_q(1) = \frac{\varrho}{(1-q)\tau}.$$

Proof Using the concept (3), we obtain

$$S_q(1) = \frac{1}{(1-q)} \int_0^{\infty} e_q^{-\frac{\tau\varpi}{\varrho}} d_q\varpi$$

$$= -\frac{\varrho}{(1-q)\tau}\left[e_q^{-\frac{\tau\varpi}{\varrho}}\right]_0^{\infty}$$

$$= \frac{\varrho}{(1-q)\tau}.$$

(2) If $\Lambda(\varpi) = \varpi$ in a set A, then

$$S_q(\varpi) = \frac{\varrho^2}{(1-q)\tau^2}.$$

Proof Using the concept (3), we obtain

$$S_q(\varpi) = \frac{1}{(1-q)} \int_0^{\infty} \varpi e_q^{-\frac{\tau\varpi}{\varrho}} d_q\varpi$$

$$= \frac{1}{(1-q)} \frac{\varrho}{\tau} \int_0^{\infty} e_q^{-\frac{\tau\varpi}{\varrho}} d_q\varpi$$

$$= \frac{1}{(1-q)} \frac{\varrho^2}{\tau^2}.$$

(3) If $\Lambda(\varpi) = \frac{\varpi^n}{n!}$, n= 0, 1, 2.. in a set A, then

$$S_q\left[\frac{\varpi^n}{n!}\right] = \frac{1}{(1-q)}\left(\frac{\varrho}{\tau}\right)^{n+1}.$$

Proof Using the concept (3), we obtain

$$S_q\left[\varpi^n\right] = \frac{1}{(1-q)} \int_0^{\infty} e_q^{-\frac{\tau\varpi}{\varrho}} \varpi^n d_q\varpi = \frac{\varrho}{(1-q)\tau} n \int_0^{\infty} \varpi^{n-1} e_q^{-\frac{\tau\varpi}{\varrho}} d_q\varpi$$

$$= \frac{\varrho^2}{(1-q)\tau^2} n(n-1) \int_0^\infty \varpi^{n-2} e_q^{-\frac{\tau\varpi}{\varrho}} d_q\varpi$$

$$= \frac{\varrho^3}{(1-q)\tau^3} n(n-1)(n-2) \int_0^\infty \varpi^{n-3} e_q^{-\frac{\tau\varpi}{\varrho}} d_q\varpi$$

$$= \frac{\varrho^4}{(1-q)\tau^4} n(n-1)(n-2)(n-3) \int_0^\infty \varpi^{n-4} e_q^{-\frac{\tau\varpi}{\varrho}} d_q\varpi$$

$$= \frac{\varrho^5}{(1-q)\tau^5} n(n-1)(n-2)(n-3)(n-4) \int_0^\infty \varpi^{n-5} e_q^{-\frac{\tau\varpi}{\varrho}} d_q\varpi$$

$$= - - -$$

$$= \frac{n!}{(1-q)} \left(\frac{\varrho}{\tau}\right)^{n+1}.$$

(4) If $\Lambda(\varpi) = e_q^{(M\varpi)}$ in set A, then

$$S_q\left[e_q^{(M\varpi)}\right] = \frac{1}{(1-q)} \frac{\varrho}{(\tau - M\varrho)}.$$

Proof Using the concept (3), we obtain

$$S_q[e_q^{(M\varpi)}] = \frac{1}{(1-q)} \int_0^\infty e_q^{-\left(\frac{(\tau - M\varrho)\varpi}{\varrho}\right)} d_q\varpi$$

$$= -\frac{1}{(1-q)} \frac{\varrho}{\tau - M\varrho} \left[e_q^{-\left(\frac{(\tau - M\varrho)\varpi}{\varrho}\right)}\right]_0^\infty$$

$$= \frac{1}{(1-q)} \frac{\varrho}{(\tau - M\varrho)}.$$

(5) If $\Lambda(\varpi) = \varpi e_q^{(M\varpi)}$ in set A, then

$$S_q\left[\varpi e_q^{(M\varpi)}\right] = \frac{1}{1-q} \frac{\varrho^2}{(\tau - M\varrho)^2}.$$

Proof Using the concept (3), we obtain

$$S_q\left[\varpi e_q^{M\varpi}\right] = \frac{1}{(1-q)} \int_0^\infty \varpi\ e_q^{-\left(\frac{(\tau - M\varrho)\varpi}{\varrho}\right)} d_q\varpi$$

$$= -\frac{1}{(1-q)} \frac{\varrho}{\tau - M\varrho}\left[\varpi e_q^{-\left(\frac{(\tau - M\varrho)\varpi}{\varrho}\right)}\right]_0^\infty$$

$$+\frac{1}{(1-q)} \frac{\varrho}{\tau - M\varrho} \int_0^\infty e_q^{-\left(\frac{(\tau - M\varrho)\varpi}{\varrho}\right)} d_q\varpi$$

$$= -\frac{1}{(1-q)} \frac{\varrho^2}{(\tau - M\varrho)^2}\left[e_q^{-\left(\frac{(\tau - M\varrho)\varpi}{\varrho}\right)}\right]_0^\infty$$

$$= \frac{1}{1-q} \frac{\varrho^2}{(\tau - M\varrho)^2}.$$

(6) If $\Lambda(\varpi) = \sin_q(M\varpi)$ in set A, then

$$S_q\left[sin_q(M\varpi)\right] = \frac{M\varrho^2}{\tau^2 - \tau^2 q + M^2\varrho^2 - M^2\varrho^2 q}.$$

Proof Using the concept (3), we obtain

$$S_q\left[sin_q(M\varpi)\right] = \frac{1}{1-q} \int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \sin_q(M\varpi)\, d_q\varpi$$

$$= -\frac{1}{(1-q)} \frac{\varrho}{\tau}\left[e_q^{-\frac{\tau\varpi}{\varrho}} \sin_q(M\varpi)\right]_0^\infty$$

$$+\frac{1}{(1-q)} \frac{(M\varpi)}{\tau} \int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \cos_q(M\varpi)\, d_q\varpi$$

$$= -\frac{1}{(1-q)} \frac{M\varrho^2}{\tau^2}\left[e_q^{-\frac{\tau\varpi}{\varrho}} \cos_q(M\varpi)\right]_0^\infty$$

$$-\frac{1}{(1-q)} \frac{M^2\varrho^2}{\tau^2} \int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \sin_q(M\varpi)\, d_q\varpi$$

$$= \frac{1}{(1-q)}\frac{M\varrho^2}{\tau^2} - \frac{1}{(1-q)}\frac{M^2\varrho^2}{\tau^2}\int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \sin_q(M\varpi)\, d_q\varpi.$$

$$\Rightarrow S_q\left[sin_q(M\varpi)\right] = \frac{1}{(1-q)}\frac{M\varrho^2}{\tau^2} - \frac{M^2\varrho^2}{\tau^2}S_q\left[sin_q(M\varpi)\right]$$

$$\Rightarrow S_q\left[sin_q(M\varpi)\right] + \frac{M^2\varrho^2}{\tau^2}S_q[sin_q(M\varpi)] = \frac{1}{(1-q)}\frac{M\varrho^2}{\tau^2}$$

$$\Rightarrow S_q[sin_q(M\varpi)]\left[1 + \frac{M^2\varrho^2}{\tau^2}\right] = \frac{1}{(1-q)}\frac{M\varrho^2}{\tau^2}$$

$$\Rightarrow S_q\left[sin_q(M\varpi)\right] = \frac{M\varrho^2}{\tau^2 - \tau^2 q + M^2\varrho^2 - M^2\varrho^2 q}.$$

(7) If $\Lambda(\varpi) = \cos_q(M\varpi)$ in set A, then

$$S_q[cos_q(M\varpi)] = \frac{\varrho\tau}{\tau^2 - \tau^2 q + M^2\varrho^2 - qM^2\varrho^2}.$$

Proof Using the concept (3), we obtain

$$S_q\left[\cos_q(M\varpi)\right] = \frac{1}{1-q}\int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \cos_q(M\varpi)\, d_q\varpi$$

$$= -\frac{1}{(1-q)}\frac{\varrho}{\tau}\left[e_q^{-\frac{\tau\varpi}{\varrho}} \cos_q(M\varpi)\right]_0^\infty$$

$$-\frac{1}{(1-q)}\frac{M\varrho}{\tau}\int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \sin_q(M\varpi)\, d_q\varpi$$

$$= \frac{1}{(1-q)}\frac{\varrho}{\tau} - \frac{1}{(1-q)}\frac{M\varrho^2}{\tau^2}\left[e_q^{-\frac{\tau\varpi}{\varrho}} \sin_q(M\varpi)\right]_0^\infty$$

$$-\frac{1}{(1-q)}\frac{M^2\varrho^2}{\tau^2}\int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \cos_q(M\varpi)\, d_q\varpi$$

$$= \frac{1}{(1-q)}\frac{\varrho}{\tau} - \frac{1}{(1-q)}\frac{M^2\varrho^2}{\tau^2}\int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \cos_q(M\varpi)\, d_q\varpi$$

$$= \frac{1}{(1-q)}\frac{\varrho}{\tau} - \frac{M^2\varrho^2}{\tau^2} S_q[\cos_q(M\varpi)].$$

$$\Rightarrow S_q[\cos_q(M\varpi)] = \frac{\varrho\tau}{\tau^2 - \tau^2 q + M^2\varrho^2 - qM^2\varrho^2}.$$

Theorem 3.5 *If the Shehu transform of a function $\Lambda(\varpi)$ exists, then*

$$S_q\left[\Lambda(\varpi - M)H(\varpi - M)\right] = e_q^{-\frac{\tau M}{\varrho}} S_q[\Lambda(\varpi - M)],$$

where $H(\varpi)$ is Heaviside unit step function defined by $H(\varpi - M) = 1$, when $\eta > M$ and $H(\varpi - M) = 0$ when $\eta < M$.

Proof We have by definition,

$$S_q\left[\Lambda(\varpi - M)\, H(\varpi - M)\right] = \frac{1}{(1-q)} \int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi - M) H(\varpi - M)\, d_q\varpi$$

$$= \frac{1}{(1-q)} \int_0^M e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi - M)\, d_q\varpi, \quad \varpi > M.$$

By putting $A = \varpi - M$

$$S_q[\Lambda(\varpi - M)] = \frac{1}{(1-q)} \int_0^\infty e_q^{\frac{-\tau(A+M)}{\varrho}} \Lambda(A)\, d_qA$$

$$= \frac{1}{(1-q)} e_q^{-\frac{\tau M}{\varrho}} \int_0^\infty e_q^{-\frac{\tau A}{\varrho}} \Lambda(A)\, d_qA$$

$$= e_q^{-\frac{\tau M}{\varrho}} S_q\{\Lambda(A)\}.$$

Theorem 3.6 *If the Shehu transform of the $\Lambda(\varpi)$ exists where $\Lambda(\varpi)$ is a periodic function of periods Λ (That is $\Lambda(\varpi + A) = \Lambda(\varpi), \forall\varpi$), then*

$$S_q\{\Lambda(\varpi)\} = \frac{\left[1 - e_q^{-\frac{\tau M}{\varrho}}\right]^{-1}}{(1-q)} \int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi.$$

Proof

$$S_q[\Lambda(\varpi)] = \frac{1}{(1-q)} \int_0^{\infty} e_q^{\frac{\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi$$

$$= \frac{1}{(1-q)} \int_0^{A} e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi$$

$$+\frac{1}{(1-q)} \int_A^{\infty} e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi.$$

Setting $\varpi = A + M$ in the second integral, we have

$$S_q[\Lambda(\varpi)] = \frac{1}{(1-q)} \int_0^{A} e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi$$

$$+\frac{1}{(1-q)} \int_0^{\infty} e_q^{-\frac{\tau(A+M)}{\varrho}} \Lambda(A+M) d_q A$$

$$= \frac{1}{(1-q)} \int_0^{A} e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi + \frac{e_q^{-\frac{\tau M}{\varrho}}}{(1-q)} \int_0^{\infty} e_q^{-\frac{\tau A}{\varrho}} \Lambda(A) d_q A$$

$$= \frac{1}{(1-q)} \int_0^{A} e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi + e_q^{-\frac{\tau M}{\varrho}} S_q\{\Lambda(\varpi)\}.$$

$$\Rightarrow S_q\{\Lambda(\varpi)\} = \frac{\left[1 - e_q^{-\frac{\tau M}{\varrho}}\right]^{-1}}{(1-q)} \int_0^{\infty} e_q^{-\frac{\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi.$$

Theorem 3.7 (q - Shehu Convolution product) *The Convolution of two functions $\Lambda(\varpi)$ and $\rho(\varpi)$ is denoted by $(\Lambda * \rho)(\varpi)$ and defined as*

$$(\Lambda * \rho)(\varpi) = \frac{1}{(1-q)} \int_0^{\varpi} \Lambda(\varpi - N)\, \rho(N)\, d_q N.$$

Convolution theorem

Statement. Let $\Lambda_1(\varpi)$ and $\Lambda_2(\varpi)$ be a positive scalar functions of ϖ and let $\rho_1(\varpi)$ and $\rho_2(\varpi)$ be their q- Shehu transforms, then

$$S_q\{\Lambda_1(\varpi) * \Lambda_2(\varpi)\} = \rho_1(\varpi).\rho_2(\varpi),$$

where $\Lambda_1(\varpi) * \Lambda_2(\varpi) = \frac{1}{(1-q)} \int_0^{\varpi} \Lambda(\varpi - N)\, \rho(N)\, d_q N.$

Proof We have

$$S_q\{\Lambda_1(\varpi) * \Lambda_2(\varpi)\} = S_q\Big[\frac{1}{(1-q)} \int_0^{\varpi} \Lambda(\varpi - N)\, \rho(N)\, d_q N\Big]$$

$$= \frac{1}{(1-q)^2} \int_0^{\infty} e_q^{\frac{-\tau\varpi}{\varrho}} \int_0^{\varpi} \Lambda(\varpi - N)\, \rho(N)\, d_q N\, d_q \varpi.$$

Let $\varpi - N = M \Rightarrow d_q \varpi = d_q M.$

$$\therefore S_q\{\Lambda_1(\varpi) * \Lambda_2(\varpi)\} = \frac{1}{(1-q)^2} \int_{N=0}^{\infty} \int_{N=\varpi}^{\infty} e_q^{\frac{-\tau(N+M)}{\varrho}} \Lambda(M)\rho(N) d_q N d_q M$$

$$= \rho_1(\varpi).\, \rho_2(\varpi).$$

Applications

Application 4.1 We take the first order ODE

$$\frac{d_q \Lambda(\varpi)}{d_q \varpi} + \Lambda(\varpi) = 0, \tag{4}$$

with $\Lambda(0) = 1$.

Applying the concept of q-Shehu transform to the Eq. (4), we obtain

$$\frac{\tau}{\varrho} S_q(\tau, \varrho) - S_q\{\Lambda(0)\} + S_q(\tau, \varrho) = 0.$$

By applying initial condition, we get

$$S_q(\tau,\varrho)=\frac{\varrho^2}{(1-q)(\tau+\varrho)\tau}. \tag{5}$$

By applying inverse on Eq. (5), we get

$$\begin{aligned}\Lambda(\varpi)&=S_q^{-1}\Big[\frac{\varrho^2}{(1-q)(\tau+\varrho)\tau}\Big].\\&=S_q^{-1}\Big[\frac{\varrho}{(1-q)\tau}\Big]-S_q^{-1}\Big[\frac{\varrho}{(1-q)(\tau+\varrho)}\Big]\\&=1-e_q^{\varpi}.\end{aligned}$$

Application 4.2 Consider the following second order ODE

$$\frac{d_q^2\Lambda(\varpi)}{d\varpi^2}+\frac{d_q\Lambda(\varpi)}{d_q\varpi}=1 \tag{6}$$

with $\Lambda(0)=0,\ \frac{d_q\Lambda(\varpi(0))}{d_q\varpi}=0.$

Applying the concept of q-Shehu transform to the Eq. (6), we obtain

$$\begin{aligned}&\frac{\tau^2}{\varrho^2}S_q(\tau,\varrho)-\frac{\tau}{\varrho}S_q\{\Lambda(0)\}-S_q\{\Lambda^{/}(0)\}+\frac{\tau}{\varrho}S_q(\tau,\varrho)-S_q\{\Lambda(0)\}=\frac{1}{(1-q)}\frac{\varrho}{\tau}\\&\Rightarrow\Big[\frac{\tau^2+\varrho\tau}{\varrho^2}\Big]S_q(\tau,\varrho)=\frac{1}{(1-q)}\frac{\varrho}{\tau}.\end{aligned}$$

By applying initial conditions, we get

$$\Rightarrow R(\tau,\varrho)=\frac{\varrho^3}{(1-q)(\tau^2+\varrho\tau)\tau}$$

$$\Rightarrow R(\tau,\varrho)=\frac{\varrho^3}{(1-q)(\tau^3+\varrho\tau^2)}.$$

Taking the inverse q-Shehu transform, we get

$$\begin{aligned}\Lambda(\varpi)&=S_q^{-1}\Big[\frac{\varrho^3}{(1-q)\tau^2(\tau+\varrho)}\Big]\\&=S_q^{-1}\left\{\frac{\varrho^2}{(1-q)\tau}\Big[\frac{1}{\tau}-\frac{1}{\tau+\varrho}\Big]\right\}\\&=S_q^{-1}\Big[\frac{\varrho^2}{\tau^2(1-q)}\Big]-S_q^{-1}\Big[\frac{\varrho^2}{\tau(\tau+\varrho)(1-q)}\Big]\\&=\varpi-S_q^{-1}\Big[\frac{\varrho}{(1-q)\tau}-\frac{\varrho}{(1-q)(\tau+\varrho)}\Big]\\&=\varpi-1-e_q^{-\varpi}.\end{aligned}$$

Application 4.3 Consider the following second non-homogeneous order ODE

$$\frac{d_q^2\Lambda(\varpi)}{d\varpi^2}-3\frac{d_q\Lambda(\varpi)}{d_q\varpi}+2\Lambda(\varpi)=e_q(3\varpi) \tag{7}$$

with $\Lambda(0) = 1, \quad \dfrac{d_q\Lambda(0)}{d_q\varpi} = 0.$

Applying the concept of q-Shehu transform to the Eq. (7), we obtain

$$
\begin{aligned}
&\frac{\tau^2}{\varrho^2}\overline{\Lambda} - \frac{\tau}{\varrho}S_q\{\Lambda(0)\} - S_q\{\Lambda^{'}(0)\} - 3\Big(\frac{\tau}{\varrho}\overline{\Lambda}(\tau,\varrho) - S_q\{\Lambda(0)\}\Big) + 2\overline{\Lambda}(\tau,\varrho) \\
&= \frac{\varrho}{(1-q)(\tau-3\varrho)}. \\
&\Rightarrow \Big[\frac{\tau^2}{\varrho^2} - 3\frac{\tau}{\varrho} + 2\Big]\overline{\Lambda}(\tau,\varrho) - \frac{1}{1-q} + \frac{3\varrho}{(1-q)\tau} = \frac{\varrho}{(1-q)(\tau-3\varrho)} \\
&\Rightarrow \Big[(\frac{\varrho}{\tau}-2)(\frac{\varrho}{\tau}-1)\Big]\overline{\Lambda}(\tau,\varrho) = \frac{\varrho}{(1-q)(\tau-3\varrho)} - \frac{3\varrho}{(1-q)\tau} + \frac{1}{1-q} \\
&\Rightarrow \overline{\Lambda}(\tau,\varrho) = \frac{\varrho^3}{(1-q)(\tau-3\varrho)(\tau-2\varrho)(\tau-\varrho)} - \frac{3\varrho^3}{(1-q)\tau(\tau-2\varrho)(\tau-\varrho)} \\
&\quad + \frac{\varrho^2}{(1-q)(\tau-2\varrho)(\tau-\varrho)} \\
&\Rightarrow \overline{\Lambda}(\tau,\varrho) = \frac{\varrho^2}{(1-q)(\tau-3\varrho)}\Big[\frac{1}{\tau-2\varrho} - \frac{1}{\tau-\varrho}\Big] - \frac{3\varrho^2}{(1-q)\tau}\Big[\frac{1}{\tau-2\varrho} - \frac{1}{\tau-\varrho}\Big] \\
&\quad + \frac{\varrho}{1-q}\Big[\frac{1}{\tau-2\varrho} - \frac{1}{\tau-\varrho}\Big] \\
&\Rightarrow \overline{\Lambda}(\tau,\varrho) = \frac{\varrho^2}{(1-q)\ (\tau-3\varrho)\ (\tau-2\varrho)} - \frac{\varrho^2}{(1-q)\ (\tau-3\varrho)\ (\tau-\varrho)} - \frac{3\varrho^2}{(1-q)\ \tau\ (\tau-2\varrho)} \\
&\quad + \frac{3\varrho^2}{(1-q)\ \tau\ (\tau-\varrho)} + \frac{\varrho}{(1-q)\ (\tau-2\varrho)} - \frac{\varrho}{(1-q)\ (\tau-\varrho)} \\
&\Rightarrow \overline{\Lambda}(\tau,\varrho) = \frac{\varrho}{(1-q)(\tau-3\varrho)} - \frac{\varrho}{(1-q)(\tau-2\varrho)} - \frac{\varrho}{2\,(1-q)(\tau-3\varrho)} + \frac{\varrho}{2\,(1-q)(\tau-\varrho)} \\
&\quad - \frac{3\varrho}{2\,(1-q)(\tau-2\varrho)} + \frac{3\varrho}{2\,(1-q)\tau} - \frac{3\varrho}{(1-q)\tau} + \frac{3\varrho}{(1-q)(\tau-\varrho)} \\
&\quad + \frac{\varrho}{(1-q)(\tau-2\varrho)} - \frac{\varrho}{(1-q)(\tau-\varrho)}.
\end{aligned}
$$

Taking the inverse q-Shehu transform, then

$$
\begin{aligned}
&\Rightarrow \overline{\Lambda}(\varpi) = S_q^{-1}\Big[\frac{\varrho}{2(1-q)(\tau-3\varrho)}\Big] - S_q^{-1}\Big[\frac{3\varrho}{2(1-q)(\tau-2\varrho)}\Big] + S_q^{-1}\Big[\frac{5\varrho}{2(1-q)(\tau-\varrho)}\Big] \\
&\quad - S_q^{-1}\Big[\frac{5\varrho}{2(1-q)(\tau-\varrho)}\Big] - S_q^{-1}\Big[\frac{3\varrho}{2\,(1-q)\ \tau}\Big] \\
&\Rightarrow \overline{\Lambda}(\varpi) = \frac{1}{2}e_q^{\varpi} - \frac{3}{2}e_q^{2\varpi} + \frac{5}{2}e_q^{\varpi} - \frac{3}{2}.
\end{aligned}
$$

Application 4.4 A semi-infinite solid $\eta > 0$ is initially at temperature zero. At time $\varpi > 0$, a constant temperature $R_0 > 0$ is applied and maintained at the face $\eta = 0$. Find the temperature at any point of the solid at any time $\varpi > 0$.

Here the temperature $\varrho(\eta,\varpi)$ at any point of the solid at any time $\varpi > 0$ is governed by one dimensional heat equation

$$\frac{\partial_q\varrho}{\partial_q\varpi} = C^2\frac{\partial_q^2\varrho}{\partial_q\eta^2}, (\eta > 0, \varpi > 0), \tag{8}$$

with the initial and boundary conditions

$$\varrho(0,\varpi) = R_0,\ \varrho(\eta,0) = 0.$$

Applying the concept of q-Shehu transform to the Eq. (8), we obtain

$$\frac{\tau}{\varrho}\overline{\varrho}(\eta,\varpi) - S_q\{\varrho(\eta,0)\} = C^2\frac{d_q^2\overline{\varrho}}{d_q\eta^2}$$
$$\Rightarrow \frac{d_q^2\overline{\varrho}}{d_q\eta^2} - \frac{\tau}{\varrho C^2}\overline{\varrho} = 0.$$

The solution is

$$\overline{\varrho} = A\ e_q^{\sqrt{\frac{\tau}{\varrho C^2}}\eta} + B\ e_q^{-\sqrt{\frac{\tau}{\varrho C^2}}\eta}. \tag{9}$$

Since ϱ is finite, when $\eta \to \infty$.

$$\therefore \overline{\varrho} \text{ is also finite, when } \eta \to \infty.$$
$$\therefore \text{ A = 0, otherwise } \overline{\varrho} \to \infty \text{ as } \eta \to \infty.$$

Taking the q-Shehu transform of the both sides, then we get

$$\varrho(0,\varpi) = R_0.$$

Therefore, $\overline{\varrho} = R_0\frac{\varrho}{(1-q)\tau}$.

$\therefore$ From (9), we have $\overline{\varrho} = B = R_0\frac{\varrho}{(1-q)\tau}$.

Hence $\overline{\varrho} = R_0\frac{\varrho}{(1-q)\tau}\ e_q^{-\sqrt{\frac{\tau}{\varrho C^2}}\eta}$.

Taking the inverse q-Shehu transform, then

$$\begin{aligned}\overline{\varrho}(\eta,\varpi) &= S_q^{-1}\Big[R_0\frac{\varrho}{(1-q)\tau}\ e_q^{-\sqrt{\frac{\tau}{\varrho C^2}}\eta}\Big]\\ &= S_q^{-1}\Big[\frac{R_0}{(1-q)\frac{\tau}{\varrho}\tau}\ e_q^{-\sqrt{\frac{\tau/\varrho}{C^2}}\eta}\Big]\\ &= R_0\ erfc_q\Big(\frac{\eta}{2C\sqrt{\Omega}}\Big).\end{aligned}$$

Application 4.5 Find the solution of the equation

$$\frac{\partial_q\varrho}{\partial_q\varpi} = K\frac{\partial_q^2\varrho}{\partial_q^2\eta^2} \tag{10}$$

which tends to zero as $\eta \to \infty$ and which satisfies the conditions $\varrho = f(\varpi)$ when $\eta = 0$, $\varpi > 0$ and $\varrho = 0$ when $\eta > 0$, $\varpi = 0$.

Applying the concept of q-Shehu transform to the Eq. (10), we obtain

$$\frac{\tau}{\varrho}\overline{\varrho}(\eta,\varpi) - S_q\{\varrho(\eta,0)\} = K\frac{d_q^2\overline{\varrho}}{d_q\eta^2} \Rightarrow \frac{d_q^2\overline{\varrho}}{d_q\eta^2} - \frac{\tau}{K\varrho}\overline{\varrho} = 0.$$

The solution is

$$\overline{\varrho} = A\, e_q^{\sqrt{\frac{\tau}{K\varrho}}\eta} + B\, e_q^{-\sqrt{\frac{\tau}{K\varrho}}\eta}.$$

Since $\varrho \to 0$ as $\eta \to \infty$.

$$\therefore \overline{\varrho} \to 0\; as\; \eta \to \infty.$$

From which it follows that $A = 0$.

Therefore,

$$\overline{\varrho} = B\, e_q^{-\sqrt{\frac{\tau}{K\varrho}}\eta}. \tag{11}$$

Again when $\eta = 0$, $\overline{\varrho} = \dfrac{1}{1-q}\displaystyle\int_0^\infty e_q^{\frac{-\tau\varpi}{\varrho}} \Lambda(\varpi)\, d_q\varpi = \overline{\Lambda}(\rho)$.

$\therefore$ From (11), we have

$$\overline{\Lambda}(\rho) = B.$$

Hence, $\overline{\varrho} = \overline{\Lambda}(\rho)\, e_q^{-\sqrt{\frac{\tau}{K\varrho}}\eta}$.

If we take $\Lambda(\varpi) = M$ (constant),

where, constant may be real or complex.

then $\overline{\Lambda}(\rho) = \dfrac{M}{(1-q)}\dfrac{\varrho}{\tau}$.

Taking the inverse q-Shehu transform, we get

$$\begin{aligned}\varrho(\eta,\varpi) &= S_q^{-1}\Big[\frac{M}{(1-q)}.\frac{\varrho}{\tau}\, e_q^{-\sqrt{\frac{\tau/\varrho}{K^2}}\eta}\Big]\\ &= \mathrm{M}\, erfc_q\Big(\frac{\eta}{2\sqrt{K\varpi}}\Big).\end{aligned}$$

Application 4.6 An infinite long string having one end $\eta = 0$ is initially at rest on the η-axis. The end $\eta = 0$ undergoes a periodic transverse displacement given by $A_0 \sin\rho\varpi,\ \varpi > 0$, find the displacement of any point of string at $\varpi > 0$.

Here the displacement of any point of any point of the string is governed by the equation

$$\frac{\partial_q^2\varrho}{\partial_q\varpi^2} = K^2\frac{\partial_q^2\varrho}{\partial_q\eta^2}, \tag{12}$$

with the boundary and initial conditions

$$\begin{aligned}\varrho(0,\varpi) &= A_0 \sin\rho\varpi,\ \varpi > 0,\\ \varrho(\eta,0) &= 0,\ \varrho_\varpi(\eta,0) = 0,\ \eta > 0\end{aligned}$$

and the displacement is finite.

Applying the concept of q-Shehu transform to the Eq. (12), we obtain

$$\frac{\tau^2}{\varrho^2}\overline{\varrho}(\eta,\varpi) - \tau S_q\{\varrho(\eta,0)\} - S_q\{\varrho_\varpi(\eta,0)\} = K^2\frac{d_q^2\overline{\varrho}}{d_q\eta^2}. \Rightarrow \frac{d_q^2\overline{\varrho}}{d_q\eta^2} - \frac{\tau^2}{\varrho^2K^2}\overline{\varrho} = 0. \tag{13}$$

Also $\overline{\varrho}(0,\varpi) = A_0 S_q\{\sin\rho\varpi\} = A_0 \dfrac{\rho\varrho^2}{\tau^2 - \tau^2 q + \rho^2\varrho^2 - \rho^2\varrho^2 q}$ and $\overline{\varrho}(\eta,\varpi)$ is finite for $\eta > 0$.

Now solution of (13) is given by

$$\overline{\varrho} = A\, e_q^{\frac{\tau}{K\varrho}\eta} + B\, e_q^{-\frac{\tau}{K\varrho}\eta}.$$

Since $\overline{\varrho}(\eta,\varpi)$ is finite ∴ A= 0, otherwise $\overline{\varrho}(\eta,\varpi)$ becomes infinite when $\eta \to \infty$.

$$\therefore \overline{\varrho}(\eta,\varpi) = B\, e_q^{-\frac{\tau}{K\varrho}\eta}.$$

Now $\overline{\varrho}(0,\varpi) = B = A_0 \dfrac{\rho\varrho^2}{\tau^2 - \tau^2 q + \rho^2\varrho^2 - \rho^2\varrho^2 q}$.

Hence,

$$\begin{aligned}
\overline{\varrho}(\eta,\varpi) &= A_0 \frac{\rho\varrho^2}{\tau^2 - \tau^2 q + \rho^2\varrho^2 - \rho^2\varrho^2 q}\, e_q^{-\frac{\tau}{K\varrho}\eta}.\\
&= A_0\, e_q^{-\frac{\tau\left(\eta/K\right)}{\varrho}} \frac{1}{(1-q)} \int_0^\infty e_q^{-\frac{\tau\varpi}{\varrho}} \sin_q(\rho\varpi)\, d_q\varpi\\
&= A_0 \frac{1}{(1-q)} \int_0^\infty e_q^{-\frac{\tau(\varpi + \eta/K)}{\varrho}} sin_q(\rho\varpi)\, d_q\varpi\\
&= S_q\Big(sin_q(\rho\varpi - \eta/K)\, H(\rho\varpi - \eta/K)\Big).
\end{aligned}$$

Taking the inverse q-Shehu transform, then

$$\begin{aligned}
\varrho(\eta,\varpi) &= sin_q(\rho\varpi - \eta/K) \text{ when } \rho\,\varpi > \eta/K\\
&= 0 \text{ when } \rho\,\varpi < \eta/K.
\end{aligned}$$

Discussion

Maitama and Zhao [1] have introduced a new integral transform named Shehu transform to generalize Sumudu and Laplace transform for solving differential equations in the time domain. Quantum calculus is a calculus without limits. So we have applied quantum calculus in Shehu transform to explore the quantum concept in the application of Shehu transform. This is the nobility of the proposed transform.

The q-Shehu transform may be applied to solve heat and transport equations, Volterra integral equations of the first kind, Bessel's functions, and cryptography in the quantum calculus.

The proposed method is an analytic method that gives the solution of ordinary and partial differential equation with initial and boundary conditions when we compare with the model introduced by Hadid et al. [13], which optimizes the bounded interval using the fraction entropy while Zhang et al. [30] established the existence of solution type solutions for a class of fractional Choquard equations. In Zhang et al. [30], the technique was based on constrained minimization of arguments, whereas in our proposed method as Knill [31] has given advantages of quantum calculus that in the calculus, the differential form and geometric objects are treated in the same way and also it allows to do calculus on continuous functions which do not need to be smooth. So by the features mentioned above, the proposed method is more effective and high performs for solving real-life problems in differential equations.

Conclusion

We introduced the concept of Shehu transform in q-calculus namely q-Shehu transform and established some properties. We applied q-Shehu transform for solving some ordinary and partial differential equations with initial and boundary values problems to show its effectiveness and performance of the proposed transform.

Open Problems

(1) As Alfaqeih and Misirli [3] have introduced the concept of double Shehu transform and its properties with applications. Is this concept is applicable in q-calculus?

(2) Does the above study also applicable for the other transform or fractional operator?

Acknowledgements The authors are extremely thankful to Department of Mathematics, National Institute of Technology Raipur (C.G.)-492010, India, for providing facilities, space and an opportunity for the work.

Author Contributions A.K.S. and S.P. contributed equally to this work.

Funding No funding.

Data availability Not applicable.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

References

1. Maitama, S., Zhao, W.: New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations. Int. J. Anal. Appl. **17**(2), 167–190 (2019)
2. Baskonus, H.M., Bulut, H., Pandir, Y.: The natural decomposition method for linear and non-linear partial differential equations. MESA **5**(1), 111–126 (2014)
3. Alfaqeih, S., Misirli, E.: On double Shehu transform and its properties with applications. Int. J. Anal. Appl. **18**(3), 381–395 (2020)
4. Al-Omri, S.K.Q.: On the application of natural transforms. Int. J. Pure. Appl. Math. **85**(4), 729–744 (2013)
5. Belgacem, R., Baleanu, D., Bokhari, A.: Shehu transform and applications to caputo-fractional differential equations. Int. J. Anal. Appl. **17**(6), 917–927 (2019)
6. Bulut, H., Baskonus, H.M., Belgacem, F.B.M.: The analytical solution of some fractional ordinary differential equations by the Sumudu transform method. Abst. Appl. Anal. **203875**, 1–6 (2013)
7. Debnath, L., Bhatta, D.: Integral Transform and Their Applications. CRC Press, London (2015)

8. Dinc, F., Baleanu, D.: A zero-crossing technique for the multidetermination of thiamine HCl and pyridoxine HCl in their mixture by using one-dimensional wavelet transform. J. Pharm. Biol. Anal. **31**(5), 969–978 (2003)
9. Maitama, S., Zhao, W.: New homotopy analysis transform method for solving multidimensional fractional diffusion equations. Arab. J. Basic. Appl. Sci. **27**(1), 27–44 (2020)
10. Ghadle, K.P., Magar, S.K., Dole, P.V.: A new Sumudu type integral transform and its applications. Proc. Fract. Differ. Appl. **7**(3), 145–152 (2021)
11. Mtawal, A.A.H., Alkallli, S.R.: A new modified homotopy method for fractional partial differential equations with proportional delay. J. Adv. Math. **19**, 58–73 (2020)
12. Qureshi, S., Kumar, P.: Using Shehu integral transform to solve fractional order caputo type initial problems. J. Appl. Math. Comp. Mech. **18**(2), 75–83 (2019)
13. Hadid, S.B., Ibrahim, R.W., Mamoni, S.: On the en-tropic order quantity model based on the conformable calculus. Proc. Fract. Differ. Appl. **7**(4), 263–273 (2021)
14. Rashid, S., Praveen, S., Ahmed, H., Chu, Y.M.: New quantum integral inequality for some new class of generalized ψ-convex functions and their scope in physical systems. Open Phys. **19**, 35–50 (2021)
15. Rashid, S., Baleanu, D., Chu, Y.M.: Some new extension for fractional integral operator having exponential in the kernel and their applications in physical systems. Open Phys. **18**, 478–491 (2020)
16. Rashid, S., Iscan, I., Baleanu, D., Chu, Y.M.: Generation of new fractional inequalities via n-polynomials S-type convexity with applications. Adv. Differ. Equ. **264**, 1–20 (2020)
17. Rashid, S., Noor, M.A., Nisar, K.S., Baleanu, D., Rahamen, G.: A new dynamics scheme via fractional operators on time scale. Front. Phys. **8**, 1–10 (2020)
18. Rashid, S., Hammouch, Z., Baleanu, D., Chu, Y.M.: New generalizations in the sense of the weighted non-singular fractional integral operators. Fractals (2020). https://doi.org/10.1142/S0218348X20400034
19. Rashid, S., Kalsoom, H., Hammouch, Z., Ashraf, R., Baleanu, D., Chu, Y.M.: New multi-parametrized estimates having pth-order differentiability in fractional Calculus for predominating $\hbar-$ convex functions in Hilbert space. Symmetry **12**, 1–24 (2020)
20. Yang, X.J., Baleanu, D., Srivastava, H.M.: Local Fractional Integral Transform and Their Applications. Academic Press, Amsterdam (2015)
21. Jackson, F.H.: On a q-definite integral. Q. J. Appl. Math. **41**, 193–203 (1910)
22. Kac, V.G., Cheung, P.: Quantum Calculus. Springer, New York (2002)
23. Ganie, J.A., Jain, R.: On a system of q-Laplace transform of two variables with applications. J. Comp. Appl. Math. **366**, 112407 (2020)
24. Piejko, K., Sokol, J., Wieclaw, K.T.: On q-calculus and strike functions. Iran. Sci. Technol. Trans. Sci. **43**, 2879–2883 (2019)
25. Xu, L., Chu, Y.M., Rashid, S., Deeb, A.A., Nisar, K.S.: On new unified bounds for a family of functions via fractional q-calculus theory. J. Funct. Spat. **4984612**, 1–9 (2020)
26. Sole, A.D., Kac, V.: On integral representation of q-gamma and q-beta functions. Rend. Math. Linecei. **9**, 11–29 (2005)
27. Gasper, G., Rahmen, M.: Basic hypergeometric series, 2nd ed. In: Eencyclopedia of Mathematis and Its Applications. Cambridge University Press, Cambridge (2004)
28. Riyasat, M., Khan, S.: Some results on q-Hermite based hybrid polynomials. Glasnik Matematicki **53**(73), 9–31 (2018)
29. Yang, Z.M., Chu, Y.M.: Asymptotic formulas for gamma function with applications. Appl. Math. Comput. **270**, 665–680 (2015)
30. Zhang, J., Wang, W., Nie, J.: Soliton solutions for fractional Choquard equations. Proc. Fract. Differ. Appl. (2021). https://doi.org/10.18576/pfda/070103
31. Knill, O.: Lecture Notes on Quantum Multivariable Calculus, pp. 1–31 (2007). https://people.math.harvard.edu/~knill/various/quantumcalc/c.pdf

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.