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ORIGINAL PAPER

Asymptotic Expansion of Wavelet Transform for Small Values of *a*: An Oscillatory Case

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Abstract
In paper [R S Pathak and Ashish Pathak, Asymptotic expansion of Wavelet Transform for small value a , The Wavelet Transforms, World scientific, 164–168, (2009).], we presented a simple methode for deriving asymptotic expansion of wavelet transform $(W_\psi f)(b,a) = \frac{a^{\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} e^{ibw}\hat{f}(w)\overline{\hat{\psi}(aw)}dw$ with $b \in \mathbb{R}$ for small values of a by considering the asymptotic expansion of $\overline{\hat{\psi}}(w)$ and $\hat{f}(w)$ as $w \to 0$ and $w \to \infty$ respectively. In present paper, we consider the asymptotic expansion of $\psi(t)$ as $t \to \infty$ and $f(t)$ as $t \to 0$ and then find the asymptotic expansion for small value a. Some example are given as illustration.

Keywords Asymptotic expansion · Wavelet transform · Fourier transform · Mellin transform

Mathematics Subject Classification 44A05 · 41A60

Introduction

Using Mellin transform technique of Wong and asymptotic expansions of the Fourier transform of the function and the wavelet Pathak and Pathak [1–3] have obtained the asymptotic expansions of the continuous wavelet transform for large and small values of dilation and translation parameters.

Lopez and Pagol [4] have obtained the asymptotic expansions of Mellin convolutions by means of analytic continuation and oscillatory case. Recently Pathak et al. [5] derived asymptotic expansion of continuous wavelet transform of the large value of dilation parameter by exploiting “sum and subtract method ” due to Lopez [4]. In this paper using same technique, we derive the asymptotic expansion of wavelet transform for small dilation parameter

In “Preliminaries” section we give some definition and technical results which are useful the further section. In “Main Results” section we introduce the main result of the paper. In

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"Example" section we shall compute asymptotic expansion of Mexican Hat wavelet transform and Morlet wavelet transform as a special case for small a.

Preliminaries

The general wavelet transform of f with respect to the wavelet ψ is defined by

$$(W_\psi f)(b,a) = a^{\frac{-1}{2}} \int_{-\infty}^{\infty} f(t)\, \overline{\psi\left(\frac{t-b}{a}\right)} dt,\ b \in \mathbb{R},\ a > 0, \tag{1}$$

provided the integral exists [6].

Using Fourier transform it can also be expressed as

$$(W_\psi f)(b,a) = \frac{a^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{ibw} \hat{f}(w) \overline{\hat{\psi}(aw)} dw \tag{2}$$

where,

$$\hat{f}(w) = \int_{-\infty}^{\infty} e^{-itw} f(t) dt \tag{3}$$

$$\left(W_\psi f\right)(b,a) = \frac{\sqrt{a}}{2\pi} \left\{ \int_0^\infty e^{ib\omega} \overline{\hat{\psi}(a\omega)} \hat{f}(\omega) d\omega + \int_0^\infty e^{-ib\omega} \overline{\hat{\psi}(-a\omega)} \hat{f}(-\omega) d\omega \right\}. \tag{4}$$

For deriving the asymptotic expansion of wavelet transform, we requir that $f \in \mathbb{K}$ and $\psi \in \mathbb{H}$; where

Definition 1 We denote by $\mathbb{K}$ the set of function $\psi \in L^1_{Loc}(0,\infty)$ satisfyings :

(i) $\hat{\psi} \in L^1_{Loc}(0,\infty)$.

(ii) $\psi(t)$ has an asymptotic expansion of the form:

$$\psi(t) = \sum_{j=0}^{n-1} b_j\, t^{-j-\alpha} + \psi_n(t);\ as\, t \to \infty$$

where, $0 < \alpha < 1$.

(iii) As $\psi(t) = 0(t^{\rho_1});\ as\ t \to 0+$ and $\overline{\hat{\psi}}(w) = O(w^{-\rho_1 - 1});\ as\ \rho_1 \in \mathbb{R}, w \to \infty+$.

If $\psi \in \mathbb{K}$. Then by using [7, theorem 14, 323], we have the asymptotic expansion of $\bar{\hat{\psi}}(w)$ as

Case I If $0 < \alpha < 1$, then

$$\overline{\hat{\psi}}(w) = \sum_{j=0}^{\infty} b_j^*\, w^{j+\alpha-1} + \sum_{j=0}^{\infty} d_j^* w^j;\ as\ w \to 0^+,$$

where,

$$b_j^* = e^{\frac{i\pi\alpha}{2}}\, i^{j-1}\, b_j\, \Gamma(1-j-\alpha),\ \ d_j^* = \frac{(-1)^{j+1}}{j!} M[\psi; j+1].$$

Case II If $\alpha = 1$, we have

$$\overline{\hat{\psi}}(w) = \sum_{j=0}^{\infty} c_j^* \log(w)\, w^j + \sum_{j=0}^{\infty} \gamma_j^* w^j;\ as\ w \to 0^+.$$

where,

$$c_j^* = \frac{(-1)^{j+1}\, i^j}{j!}\, b_j, \quad \gamma_j^* = \frac{(-1)^{j+1}\, i^j\, (\gamma + \frac{i\pi}{2})}{j!}\, b_j - i^j\, d_{j+1}$$

for $j = 0, 1, 2, 3, \ldots$ and $d_{j+1} = \lim_{z\to 0}\left[\psi_{j+1,\, j+1}(w) + \frac{(-1)^j}{j!}\, b_j \log(w)\right]$

Definition 2 We denote by $\mathbb{H}$ the set of function $f \in L^1_{Loc}(0, \infty)$ satisfying:
(i) $f(t)$ is n times continuously differential in $(0, \infty)$ and $\hat{f} \in L^1_{Loc}(0, \infty)$.
(ii) $f(t)$ has an asymptotic expansion of the form

$$f(t) = \sum_{k=0}^{\infty} a_k t^{k-1};\ as\ t \to 0+$$

(iii) Each of the integral

$$\int_1^{\infty} f_k(t) e^{iwt} dw\ ;\ \ k = 0, 1, 2, 3 \ldots, n$$

converges uniformly for all sufficiently large w.
(iv) As

$$f(t) = O(t^{-\rho_2});\ as\ t \to \infty+,\ \rho_2 \in \mathbb{R}$$

and

$$\hat{f}(w) = O(w^{-\rho_2 - 1});\ \ as\ w \to 0+.$$

If $f \in \mathbb{H}$. Then by using [7, theorem 1, 199],We have an asymptotic expansion of $\hat{f}(w)$ at $w \to \infty+$ is

$$\hat{f}(w) = \sum_{k=0}^{n-1} a_k^* w^{-k} + O(w^{-n})$$

where, $a_k^* = a_k\, e^{\frac{i\pi k}{2}}\, \Gamma(k)$.

Main Results

The following two theorem give the asymptotic expansion of $(W_\psi f)(b, a)$ at $a \to 0+$ are:

Theorem 1 *Let* $\psi \in \mathbb{K}$, $f \in \mathbb{H}$, $b \in \mathbb{R} - 0$ *and* $0 < \alpha < 1$.*Then for any* $j, k \in \mathbb{N}$ *such that*

$$(k-1) - j < (k-1) - (j + \alpha - 1) < 1 < k - (j-1) < k - (j + \alpha - 2),$$

$$(W_\psi f)(b, a) = \frac{1}{2\pi}\left\{\sum_{k=0}^{n-1} a_k^* \Big[M[\bar{\hat{\psi}}(w) e^{\frac{ibw}{a}}; 1-k] + (-1)^{-k} M[\bar{\hat{\psi}}(-w) e^{\frac{-ibw}{a}}; 1-k]\Big] a^{k-1/2}\right.$$

$$+ \sum_{j=0}^{m-1} b_j^* \Big[M[\hat{f}(w) e^{ibw}; j+\alpha] + (-1)^{j+\alpha-1} M[\hat{f}(-w) e^{-ibw}; j+\alpha]\Big] a^{j+\alpha-1/2}$$

$$+\sum_{j=0}^{m-1} d_j^*\Big[M[\hat{f}(w)e^{ibw}; j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}; j+1]\Big]a^{j+1/2}$$
$$+R_{n,m}(a)\Big\} \tag{5}$$

$K(k)$ is the index k for which $(j-k) \leq 1 < j-(k-1)$ and $J(j)$ is the index j for which $(j-1)-k < 1 \leq (j-k)$. Let $M[g; z]$ denote the Mellin transform of g.

If $j-k=1$, for some pair (j, k), then in formula (5), the corresponding sum of the terms

$$\begin{aligned}
&a_k^*\Big[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}}; 1-k] + (-1)^{-k} M[\bar{\hat{\psi}}(-w)e^{\frac{-ibw}{a}}; 1-k]\Big]a^{k-1/2}\\
&\quad+b_j^*\Big[M[\hat{f}(w)e^{ibw}; j+\alpha] + (-1)^{j+\alpha-1} M[\hat{f}(-w)e^{-ibw}; j+\alpha]\Big]a^{j+\alpha-1/2}\\
&\quad+d_j^*\Big[M[\hat{f}(w)e^{ibw}; j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}; j+1]\Big]a^{j+1/2}
\end{aligned} \tag{6}$$

must be replace by

$$\begin{aligned}
&a^{j+1/2}\Big\{d_j^*\Big[M[\hat{f}(w)e^{ibw}; j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}; j+1]\Big]\\
&\quad+b_j^*\Big[M[\hat{f}(w)e^{ibw}; j+\alpha] + (-1)^{j+\alpha-1} M[\hat{f}(-w)e^{-ibw}; j+\alpha]\Big]\\
&\quad+a_k^* \lim_{z\to 0}\Big[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}}; z+1-k] + (-1)^{-k} M[\bar{\hat{\psi}}(-w)e^{\frac{-ibw}{a}}; z+1-k]\Big]\Big\}
\end{aligned} \tag{7}$$

and the remainder term $R_{n,m}(a)$ as $a \to 0+$ is given

$$R_{n,m}(a) = \left\{ \begin{array}{ll} O(a^m + a^{n-1}), & when\ \ n \neq m+1, \\ O(a^m \log(a)), & when\ \ n = m+1. \end{array} \right\} \tag{8}$$

Proof Define $\hat{f}_o(w) = \hat{f}(w)$ and $\bar{\hat{\psi}}_o(w) + \overline{\hat{\psi}}_{\alpha-1}(w) = \overline{\hat{\psi}}(w)$. For any k, there exists j such that for $0 < \alpha < 1$, we have $(k-1)-j < (k-1)-(j+\alpha-1) < 1 < k-(j-1) < k-(j+\alpha-2)$. For given (k, j), the following integral exists:

$$\int_0^\infty e^{ibw}\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw. \tag{9}$$

In the following algorithms, we increase (k, j) step by step from $(0, 0)$ to (n, m):

(a) For $0 < \alpha < 1$ and (k, j) satisfying

$$(k-1)-j < (k-1)-(j+\alpha-1) < 1 < k-(j-1) < k-(j+\alpha-2),$$

do the following if $(k-j) < 1$, then $k-(j+\alpha-1) < 1$, go to (b); if $(k-j) > 1$, then $k-(j+\alpha-1) > 1$, go to (c) ; if $(k-j) = 1$, then $k-(j+\alpha-1) > 1$, go to (d).

(b). In this step, we take $\hat{f}_k(w) = a_k^* w^{-k} + \hat{f}_{k+1}(w)$ in (9) and (iii) of Lemma 3 of [4],then for $0 < \alpha < 1$, we have

$$\int_0^\infty e^{ibw}\,\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw = a_k^*\,a^{k-1}\Big[M[\bar{\hat{\psi}}(w)\,e^{ibw/a};1-k] - \sum_{j=0}^{K(k)-1} d_j^* B_j(1-k;a;b)$$
$$-\sum_{j=0}^{K(k)-1} b_j^* B_j(\alpha-k;a;b)\Big] + \int_0^\infty e^{ibw}\,\hat{f}_{k+1}(w)\bar{\hat{\psi}}_j(aw)dw\,,$$

where,

$$B_j(l;a;b) = \left(\frac{a\,e^{\frac{i\pi}{2}}}{b}\right)^{(l+j)} \Gamma(l+j).$$

Similarly by replacing w by $-w$, we get

$$\int_0^\infty e^{-ibw}\,\hat{f}_k(-w)\bar{\hat{\psi}}_j(-aw)dw$$
$$= (-1)^k a_k^*\,a^{k-1}\Big[M[\bar{\hat{\psi}}(-w)e^{-ibw/a};1-k] - \sum_{j=0}^{K(k)-1} d_j^*\,(-1)^j$$
$$\times \overline{B_j(1-k;a;b)} - \sum_{j=0}^{K(k)-1} b_j^*(-1)^{j+\alpha-1}\,\overline{B_j(\alpha-k;a;b)}\Big]$$
$$+\int_0^\infty e^{-ibw}\,\hat{f}_{k+1}(-w)\bar{\hat{\psi}}_j(-aw)dw,$$

go to (a) with k replaced by $k+1$.

(c) we take

$$\bar{\hat{\psi}}_j(aw) = b_j^*(aw)^{j+\alpha-1} + d_j^*(aw)^j + \overline{\hat{\psi}}_{j+1}(aw). \tag{10}$$

Now using (10) in (9) and (iii) of Lemma 2 of [4], we get

$$\int_0^\infty e^{ibw}\,\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw = b_j^* a^{j+\alpha-1}\Big[M[e^{ibw}\,\hat{f}(w);j+\alpha] - \sum_{k=0}^{J(j)-1} a_k^*\,A_k(j+\alpha;b)\Big]$$
$$+d_j^* a^j\Big[M[e^{ibw}\,\hat{f}(w);j+1] - \sum_{k=0}^{J(j)-1} a_k^*\,A_k(j+1;b)\Big]$$
$$+\int_0^\infty e^{ibw}\,\hat{f}_k(w)\overline{\hat{\psi}}_{j+1}(aw)dw;$$

where,

$$A_k(l;b) = \Gamma[l-k]\left(\frac{e^{\frac{i\pi}{2}}}{b}\right)^{l-k}$$

Similarly by replacing w by $-w$, we get

$$\int_0^\infty e^{-ibw}\hat{f}_k(-w)\bar{\hat{\psi}}_j(-aw)dw$$
$$= b_j^*(-1)^{j+\alpha-1}a^{j+\alpha-1}\Bigg[M[e^{-ibw}\hat{f}(-w);\, j+\alpha] - \sum_{k=0}^{J(j)-1} a_k^*(-1)^{-k}$$
$$\times\overline{A_k(j+\alpha;b)}\Bigg] + d_j^*(-1)^j a^j\Bigg[M[e^{-ibw}\hat{f}(-w);\, j+1]$$
$$-\sum_{k=0}^{J(j)-1} a_k^*\overline{A_k(j+1;b)}\Bigg] + \int_0^\infty e^{-ibw}\hat{f}_k(-w)\overline{\hat{\psi}}_{j+1}(-aw)dw,$$

Go to (a) with j replaced by $j+1$.

(d) In these step, we take

$$\bar{\hat{\psi}}_j(aw) = \bar{\hat{\psi}}_j^1(aw) + \bar{\hat{\psi}}_j^2(aw) \tag{11}$$

where, $\bar{\hat{\psi}}_j^1(aw) = b_j^*(aw)^{j+\alpha-1} + \bar{\hat{\psi}}_{j+1}^1(aw)$ and $\bar{\hat{\psi}}_j^2(aw) = d_j^*(aw)^j + \bar{\hat{\psi}}_{j+1}^2(aw)$. By using (11) and $\hat{f}_k(w) = a_k^* w^{-k} + \hat{f}_{k+1}(w)$ in (9) and recall proof (d) of theorem 2 of [4], we get

$$\int_0^\infty e^{ibw}\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw = b_j^* a^{j+\alpha-1}\Bigg[M[e^{ibw}\hat{f}(w);\, j+\alpha] - \sum_{k=0}^{J(j)-1} a_k^*\, A_k(j+\alpha;b)\Bigg]$$
$$+a^j\lim_{z\to 0}\Bigg\{d_j^*\Bigg[M[e^{ibw}\hat{f}(w);\, z+j+1] - \sum_{k=0}^{J(j)-1} a_k^* A_k(z+j+1;b)$$
$$-d_j^* e^{\frac{i\pi z}{2}}\frac{\Gamma(z)}{b^z}\Bigg] + a_k^* a^{-z}\Bigg[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}};\, z+1-k]$$
$$-\sum_{j=0}^{K(k)-1} d_j^* B_j(z+1-k;a;b)\Bigg]\Bigg\} + \int_0^\infty e^{ibw}\hat{f}_k(w)\,\bar{\hat{\psi}}_{j+1}^1(aw)dw$$
$$+\int_0^\infty e^{ibw}\hat{f}_{k+1}(w)\,\bar{\hat{\psi}}_{j+1}^2(aw)dw.$$

Using $a^{-z} = 1 - z\log(a) + O(z^2)$; when $z\to 0^+$ and

$$M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}};\, z+1-k] = \int_0^\infty e^{\frac{ibw}{a}}\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)w^{z-k}dw = \frac{d_j^*}{z} + O(1);\, when\, z\to 0^+.$$

We get

$$\int_0^\infty e^{ibw}\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw = b_j^* a^{j+\alpha-1}\Bigg[M[e^{ibw}\hat{f}(w);\, j+\alpha] - \sum_{k=0}^{J(j)-1} a_k^*\, A_k(j+\alpha;b)\Bigg]$$
$$+a^j\Bigg\{d_j^*\Bigg[M[e^{ibw}\hat{f}(w);\, j+1] - \sum_{k=0}^{J(j)-1} a_k^* A_k(j+1;b) - a_k^*\log(a)\Bigg]$$

$$+a_k^* \lim_{z\to 0}\left[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}}; z+1-k] - \sum_{j=0}^{K(k)-1} d_j^* B_j(z+1-k; a; b) - \frac{d_j^*}{z}\right]$$

$$-a_k^* d_j^*[\frac{i\pi}{2} - \log(b) - \gamma]\Big\} + \int_0^\infty e^{ibw}\hat{f}_k(w)\,\bar{\hat{\psi}}^1_{j+1}(aw)dw$$

$$+\int_0^\infty e^{ibw}\hat{f}_{k+1}(w)\,\bar{\hat{\psi}}^2_{j+1}(aw)dw$$

Similarly by replacing w by $-w$, we get

$$\int_0^\infty e^{-ibw}\hat{f}_k(-w)\bar{\hat{\psi}}_j(-aw)dw$$

$$= b_j^*(-1)^{j+\alpha-1}a^{j+\alpha-1}\Bigg[M[e^{-ibw}\hat{f}(-w); j+\alpha] - \sum_{k=0}^{J(j)-1}(-1)^{-k}a_k^*$$

$$\times\overline{A_k(j+\alpha; b)}\Bigg] + a^j\Bigg\{(-1)^j d_j^*\Bigg[M[e^{-ibw}\hat{f}(-w); j+1]$$

$$-\sum_{k=0}^{J(j)-1} a_k^*(-1)^{-k}\overline{A_k(j+1; b)} - (-1)^{-k}a_k^*\log(a)\Bigg]$$

$$+(-1)^{-k}a_k^*\lim_{z\to 0}\Bigg[M[\bar{\hat{\psi}}(-w)e^{\frac{-ibw}{a}}; z+1-k]$$

$$-\sum_{j=0}^{K(k)-1}(-1)^j d_j^*\overline{B_j(z+1-k; a; b)} - (-1)^j\frac{d_j^*}{z}\Bigg] - (-1)^{j-k}a_k^* d_j^*$$

$$\times\left[\frac{i\pi}{2} - \log(b) - \gamma\right]\Bigg\} + \int_0^\infty e^{-ibw}\hat{f}_k(-w)\,\bar{\hat{\psi}}^1_{j+1}(-aw)dw$$

$$+\int_0^\infty e^{-ibw}\hat{f}_{k+1}(-w)\,\bar{\hat{\psi}}^2_{j+1}(-aw)dw$$

Go to (a) with j replaced by $j+1$ and k replaced by $k+1$.The error bounds for the remaninder $R_{mn}(a)$ can be easly proof with the help of theorem 3 [4]. Hence this algorithm generates the required results (5)–(8). □

Theorem 2 *Let* $\psi\in\mathbb{K}$, $f\in\mathbb{H}$, $b\in\mathbb{R}-0$ *and* $\alpha=1$.*Then for any* $j,k\in\mathbb{N}$ *such that* $(k-1)-j<1<k-(j-1)$,

$$(W_\psi f)(b,a) = \sum_{k=0}^{n-1} a_k^*\left[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}}; 1-k] + (-1)^{-k}M[\bar{\hat{\psi}}(-w)e^{\frac{-ibw}{a}}; 1-k]\right]a^{k-1/2}$$

$$+\sum_{j=0}^{m-1}\gamma_j^*\left[M[\hat{f}(w)e^{ibw}; j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}; j+1]\right]a^{j+1/2}$$

$$+\sum_{j=0}^{m-1}c_j^*\Big[M[\hat{f}(w)e^{ibw}\log(aw); j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}\log(-aw); j+1]\Big]a^{j+1/2}$$

$$+R_{mn}(a). \tag{12}$$

$K(k)$ is the index k for which $(j-k)\le 1<j-(k-1)$ and $J(j)$ is the index j for which $(j-1)-k<1\le(j-k)$. Let $M[g;z]$ denote the Mellin transform of g. If $j-k=1$ for

some pair (j, k)*, then corresponding sum of the terms*

$$
\begin{aligned}
&a_k^* \left[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}}; 1-k] + (-1)^{-k} M[\bar{\hat{\psi}}(-w)e^{\frac{-ibw}{a}}; 1-k] \right] a^{k-1/2} \\
&\quad +\gamma_j^* \left[M[\hat{f}(w)e^{ibw}; j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}; j+1] \right] a^{j+1/2} \\
&\quad +c_j^* \left[M[\hat{f}(w)e^{ibw}\log(aw); j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}\log(aw); j+1] \right] a^{j+1/2}
\end{aligned}
\tag{13}
$$

must be replace by

$$
\begin{aligned}
&a^{j+1/2} \Bigg\{ \gamma_j^* \left[M[\hat{f}(w)e^{ibw}; j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}; j+1] \right] \\
&\quad +c_j^* \left[M[\hat{f}(w)e^{ibw}\log(aw); j+1] + (-1)^j M[\hat{f}(-w)e^{-ibw}\log(aw); j+1] \right] \\
&\quad +a_k^* \lim_{z\to 0} \left[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}}; z+1-k] + (-1)^{-k} M[\bar{\hat{\psi}}(-w)e^{\frac{-ibw}{a}}; z+1-k] \right] \Bigg\}
\end{aligned}
\tag{14}
$$

and the remainder term $R_{n,m}(a)$ *as* $a \to 0+$ *is given*

$$
R_{n,m}(a) = \left\{ \begin{array}{ll} O(a^m + a^{n-1}), & \textit{when} \;\; n \neq m+1, \\ O(a^m \log(a)), & \textit{when} \;\; n = m+1. \end{array} \right\}
\tag{15}
$$

Proof Define $\hat{f}_o(w) = \hat{f}(w)$ and $\bar{\hat{\psi}}_o(w) = \overline{\hat{\psi}}(w)$. For any k, there exists j such that for $\alpha = 1$, we have $(k-1) - j < 1 < k - (j-1)$. For given (k, j), the following integral exists:

$$
\int_0^\infty e^{ibw} \hat{f}_k(w) \bar{\hat{\psi}}_j(aw) dw
\tag{16}
$$

In the following algorithms, we increase (k, j) step by step from $(0, 0)$ to (n, m):

(a). For $\alpha = 1$ and (k, j) satisfying

$$
(k-1) - j < 1 < k - (j-1),
$$

do the following if $(k-j) < 1$, go to (b); if $(k-j) > 1$, go to (c); if $(k-j) = 1$, go to (d).
(b). In this step, we take $\hat{f}_k(w) = a_k^* w^{-k} + \hat{f}_{k+1}(w)$ in (16) and (iii) of Lemma 3 of [4], we have

$$
\begin{aligned}
\int_0^\infty e^{ibw} \hat{f}_k(w) \bar{\hat{\psi}}_j(aw) dw &= a_k^* a^{k-1} \Bigg[M[\bar{\hat{\psi}}(w)e^{ibw/a}; 1-k] - \sum_{j=0}^{K(k)-1} \gamma_j^* B_j(1-k; a; b) \\
&\quad - \sum_{j=0}^{K(k)-1} c_j^* B_j(1-k; a; b) \left\{ \psi(-k+j+1) - \log(b) - \frac{i\pi}{2} \right\} \Bigg] \\
&\quad + \int_0^\infty e^{ibw} \hat{f}_{k+1}(w) \bar{\hat{\psi}}_j(aw) dw.
\end{aligned}
$$

Similarly by replacing w by $-w$, we get

$$\int_0^\infty e^{-ibw}\hat{f}_k(-w)\bar{\hat{\psi}}_j(-aw)dw$$
$$= a_k^*(-1)^k a^{k-1}\Big[M[\bar{\hat{\psi}}(-w)e^{-ibw/a};1-k] - \sum_{j=0}^{K(k)-1}\gamma_j^*(-1)^j$$
$$\times\overline{B_j(1-k;a;b)} - \sum_{j=0}^{K(k)-1}c_j^*(-1)^j\,\overline{B_j(1-k;a;b)}$$
$$\times\left\{\psi(-k+j+1)-\log(b)-\frac{i\pi}{2}\right\}\Big]$$
$$+\int_0^\infty e^{-ibw}\hat{f}_{k+1}(-w)\bar{\hat{\psi}}_j(-aw)dw.$$

go to (a) with k replaced by $k+1$.
(c). Here, we take

$$\bar{\hat{\psi}}_j(aw) = c_j^*\log(aw)(aw)^j + \gamma_j^*(aw)^j + \overline{\hat{\psi}}_{j+1}(aw). \quad (17)$$

Now using (17) in (16) and (iii) of Lemma 2 of [4], we get

$$\int_0^\infty e^{ibw}\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw$$
$$= c_j^* a^j\Big[M[e^{ibw}\hat{f}(w)\log(aw);j+1] - \sum_{k=0}^{J(j)-1}a_k^*A_k(j+1;b)$$
$$\left\{\psi(j+1-k)-\log(b)-\frac{i\pi}{2}\right\}\Big] + \gamma_j^*a^j\Big[M[e^{ibw}\hat{f}(w);j+1]$$
$$-\sum_{k=0}^{J(j)-1}a_k^*A_k(j+1;b)\Big] + \int_0^\infty e^{ibw}\hat{f}_k(w)\overline{\hat{\psi}}_{j+1}(aw)dw.$$

Similarly by replacing w by $-w$, we get

$$\int_0^\infty e^{-ibw}\hat{f}_k(-w)\bar{\hat{\psi}}_j(-aw)dw$$
$$= c_j^*(-1)^j a^j\Big[M[e^{-ibw}\hat{f}(-w)\log(-aw);j+1] - \sum_{k=0}^{J(j)-1}a_k^*(-1)^{-k}$$
$$\times A_k(j+1;b)\left\{\psi(j+1-k)-\log(b)-\frac{i\pi}{2}\right\}\Big]$$
$$+\gamma_j^*(-1)^j a^j\Big[M[e^{-ibw}\hat{f}(-w);j+1] - \sum_{k=0}^{J(j)-1}a_k^*(-1)^{-k}$$
$$\times A_k(j+1;b)\Big] + \int_0^\infty e^{-ibw}\hat{f}_k(-w)\overline{\hat{\psi}}_{j+1}(-aw)dw.$$

Go to (a) with j replaced by $j+1$.

(d) In these step we take first $\hat{f}_k(w) = a_k^* w^{-k} + \hat{f}_{k+1}(w)$; as $w \to \infty$ and

$$\bar{\hat{\psi}}_j(aw) = \left[c_j^* \frac{d}{dj} + \gamma_j^*\right](aw)^j + \bar{\hat{\psi}}_{j+1}(aw) \tag{18}$$

Using (18) in (16) and recall proof (d) of Theorem 2 of [4], we get

$$\begin{aligned}&\int_0^\infty e^{ibw}\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw\\&=\int_0^\infty e^{ibw}\left[a_k^* w^{-k}+\bar{\hat{\psi}}_j(aw)+\left(c_j^*\frac{d}{dj}+\gamma_j^*\right)\hat{f}_{k+1}(w)a^j w^j\right]dw\\&+\int_0^\infty e^{ibw}\hat{f}_{k+1}(w)\bar{\hat{\psi}}_{j+1}(aw)dw.\end{aligned}$$

Define the function

$$M_{k,j}[z,w] = w^z\left[a_k^* w^{-k}\bar{\hat{\psi}}_j(aw)+\left(c_j^*\frac{d}{dj}+\gamma_j^*\right)\hat{f}_{k+1}(w)a^j w^j\right]; z\in\mathbb{C}$$

Therefore,

$$\int_0^\infty e^{ibw}\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw = \int_0^\infty e^{ibw}M_{k,j}(0,w)dw + \int_0^\infty e^{ibw}\hat{f}_{k+1}(w)\,\bar{\hat{\psi}}_{j+1}(aw)dw.$$

On the one hand $\bar{\hat{\psi}}_j(w) = (c_j^*\log(w)+\gamma_j^*)w^j + O(w^{j+1})$; when $w\to 0^+$ and on the other hand $\hat{f}_{k+1}(w) = -a_k^* w^{-k} + O(w^{-k-1})$; when $w\to 0^+$. Proceeding similar way [4], we get

$$\begin{aligned}&\int_0^\infty e^{ibw}M_{k,j}(0,w)dw\\&=a^j\lim_{z\to 0}\left\{\int_0^\infty \left[a_k^* e^{\frac{ibw}{a}}w^{z-k}a^{-z}\bar{\hat{\psi}}_j(w)+e^{ibw}\left(c_j^*\frac{d}{dj}+\gamma_j^*\right)w^{z+j}\hat{f}_{k+1}(w)\right]dw\right\}\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^\infty e^{ibw}\hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw\\&=a^j\lim_{z\to 0}\left\{a_k^* a^{-z}\left[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}};z+1-k]-\sum_{j=0}^{K(k)-1}\gamma_j^* B_j(z+1-k;a;b)\right.\right.\\&\left.-\sum_{j=0}^{K(k)-1}c_j^* B_j(z+1-k;a;b)\left\{\psi(z+1+j-k)-\log(b)-\frac{i\pi}{2}\right\}\right]\\&+c_j^*\left[\left[M[e^{ibw}\hat{f}(w)\log(aw);z+1+j-k]-\sum_{j=0}^{K(k)-1}a_k^*\right.\right.\\&\left.\times A_k(z+1+j;b)\left\{\psi(z+1+j)-\log(b)-\frac{i\pi}{2}\right\}\right]\\&-\frac{d}{dz}\left(a_k^* e^{\frac{i\pi z}{2}}\Gamma(z)b^{-z}\right)\Big]+\gamma_j^*\Big[M[e^{ibw}\hat{f}(w);z+1+j-k]\end{aligned}$$

$$- \sum_{k=0}^{J(j)-1} a_k^* A_k(z+1+j;b)\left\{\psi(z+1+j-k) - \log(b) - \frac{i\pi}{2}\right\} - a_k^* e^{\frac{i\pi z}{2}}$$
$$\times \Gamma(z) b^{-z}\Bigg]\Bigg\} + \int_0^\infty e^{ibw} \hat{f}_{k+1}(w)\bar{\hat{\psi}}_{j+1}(aw)dw.$$

Using $a^{-z} = 1 - z\log(a) + O(z^2)$; when $z \to 0^+$ and

$$M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}}; z+1-k] = \int_0^\infty e^{\frac{ibw}{a}} \hat{f}_k(w)\bar{\hat{\psi}}_j(aw)w^{z-k}dw$$
$$= \frac{c_j^*}{z}\left(\frac{1}{z} + \log\left(\frac{b}{a}\right) - \frac{i\pi}{z}\right) + \frac{\gamma_j^*}{z} + O(1); \, when\, z \to 0^+.$$

We find that the above expression can be rewritten as

$$\int_0^\infty e^{ibw} \hat{f}_k(w)\bar{\hat{\psi}}_j(aw)dw = a^j\Bigg\{ c_j^*\Bigg[\frac{d}{dj}\Bigg[M[e^{ibw}\hat{f}(w)\log(aw); 1+j] - \sum_{k=0}^{J(j)-1} a_k^*$$
$$\times A_k(1+j;b)\left\{\psi(1+j-k) - \log(b) - \frac{i\pi}{2}\right\}\Bigg]$$
$$-a_k^* \log(a)\left(\log\left(\frac{b}{a}\right) - \frac{i\pi}{2}\right)\Bigg] + \gamma_j^*\Bigg[M[e^{ibw}\hat{f}(w); 1+j-k]$$
$$- \sum_{k=0}^{J(j)-1} a_k^* A_k(1+j;b)\left\{\psi(1+j-k) - \log(b) - \frac{i\pi}{2}\right\} - a_k^*\log(a)\Bigg]$$
$$+a_k^* \lim_{z\to 0}\Bigg\{\Bigg[M[\bar{\hat{\psi}}(w)e^{\frac{ibw}{a}}; z+1-k] - \sum_{j=0}^{K(k)-1} \gamma_j^* B_j(z+1-k;a;b)$$
$$- \sum_{j=0}^{K(k)-1} c_j^* B_j(z+1-k;a;b)\left\{\psi(z+1+j-k) - \log(b) - \frac{i\pi}{2}\right\}$$
$$+\frac{c_j^*}{z}\left(\frac{1}{z} - \log(a)\right) - \frac{\gamma_j^*}{z}\Bigg] + a_k^*\Bigg[c_j^*\Bigg(\gamma^2 + \gamma\log(b) - \frac{i\pi}{2}\log(b)$$
$$+(\log(b))^2 - \frac{i\pi\gamma}{2} - \frac{\pi^2}{4}\Bigg) + \gamma_j^*\left(\frac{i\pi}{2} - \gamma - \log(b)\right)\Bigg\}\Bigg]$$
$$+ \int_0^\infty e^{ibw} \hat{f}_{k+1}(w)\,\bar{\hat{\psi}}_{j+1}(aw)dw$$

Similarly by replacing w by $-w$,we get

$$\int_0^\infty e^{-ibw} \hat{f}_k(-w)\bar{\hat{\psi}}_j(-aw)dw$$
$$= a^j\Bigg\{ (-1)^j c_j^*\Bigg[\Bigg[M[e^{-ibw}\hat{f}(-w)\log(aw); 1+j] - \sum_{k=0}^{J(j)-1} a_k^*$$
$$\times A_k(1+j;b)\left\{\psi(1+j-k) - \log(b) - \frac{i\pi}{2}\right\}\Bigg] - a_k^*\log(a)$$
$$\times\left(\log\left(\frac{b}{a}\right) - \frac{i\pi}{2}\right)\Bigg] + (-1)^j\gamma_j^*\Bigg[M[e^{-ibw}\hat{f}(-w); 1+j-k]$$
$$- \sum_{k=0}^{J(j)-1} (-1)^{-k} a_k^* A_k(1+j;b)\left\{\psi(1+j-k) - \log(b) - \frac{i\pi}{2}\right\}$$

$$-(-1)^{-k} a_k^* \log(a)\Bigg] + (-1)^{-k} a_k^* \lim_{z\to 0} \Bigg\{\Bigg[M[\bar{\hat{\psi}}(-w) e^{\frac{-ibw}{a}}; z+1-k]$$
$$- \sum_{j=0}^{K(k)-1} (-1)^j \gamma_j^* B_j(z+1-k; a; b) - \sum_{j=0}^{K(k)-1} (-1)^j c_j^*$$
$$\times B_j(z+1-k; a; b)\left\{\psi(z+1+j-k) - \log(b) - \frac{i\pi}{2}\right\}$$
$$+(-1)^j \frac{c_j^*}{z}\left(\frac{1}{z} - \log(a)\right) - (-1)^j \frac{\gamma_j^*}{z}\Bigg]$$
$$+(-1)^{-k} a_k^* \Bigg[(-1)^j c_j^* \Bigg(\gamma^2 + \gamma \log(b) - \frac{i\pi}{2}\log(b)$$
$$+(\log(b))^2 - \frac{i\pi\gamma}{2} - \frac{\pi^2}{4}\Bigg) + (-1)^j \gamma_j^* \left(\frac{i\pi}{2} - \gamma - \log(b)\right)\Bigg\}\Bigg]$$
$$+ \int_0^\infty e^{-ibw} \hat{f}_{k+1}(-w)\, \bar{\hat{\psi}}_{j+1}(-aw) dw$$

Go to (a) with j replaced by $j+1$ and k replaced by $k+1$.The error bounds for the remaninder R_{nm} can be easly proof with the help of theorem 3 [4]. Hence this algorithm generates the required results (12)–(15). □

Example

Using the aforesaid technique, we find the asymptotic expansions of Mexican Hat wavelet and Morlet wavelet transform.

Asymptotic Expansion of Mexican Hat Wavelet Transform

We consider $\psi(t) = (1-t^2)\, e^{\frac{-t^2}{2}}$ to be a Mexican Hat wavelet. Since Fourier transform of Mexican Hat $\hat{\psi}(\omega) = \sqrt{2\pi}\, \omega^2\, e^{\frac{-\omega^2}{2}}$ is locally integrable in $(-\infty, \infty)$ and has an asymptotic expansion of $\bar{\hat{\psi}}(\omega)$ as $\omega \to 0^+$ [6]

$$\bar{\hat{\psi}}(\omega) = \sum_{r=0}^{\infty} c_r^* \log(w)\omega^{2r-3} + \sum_{r=0}^{\infty} \gamma_r^* \omega^{2r-3};$$

where,

$$c_r^* = \frac{(-1)^{2r-2}\; i^{(2r-3)}}{(2r-3)!}\, \eta_r;$$
$$\gamma_r^* = \frac{(-1)^{2r-2}\; i^{2r-3}\; (\gamma + \frac{i\pi}{2})}{(2r-3)!}\, \eta_r - i^{(2r-3)}\, d_{2r-2};$$
$$\eta_r = \frac{(-1)^{r+1}(2r-1)}{2^{r-1}(r-1)!};$$
$$d_{2r-2} = \lim_{z\to 0}\left[\psi_{2r-2,\,2r-2}(w) + \frac{(-1)^{(2r-3)}}{(2r-3)!}\, \eta_r \log(w)\right];$$

$$\overline{\hat{\psi}}(\omega) = O(1) \; as \; \omega \to \infty + .$$

Now, assume $f \in \mathbb{H}$. Then by Theorem 2, and by means of formula [8, (10, 30), pp. 318, 320], we get the asymptotic expansion of Mexican Hat wavelet transform at $a \to 0^+$ is

$$\begin{aligned}(W_\psi f)(b,a) &= \sum_{k=0}^{n-1} a_k \, e^{\frac{i\pi k}{2}} \; \Gamma[k] \, \sqrt{2\pi}\left(\frac{1}{2}\right)^{1-\frac{(3-k)}{2}} \; e^{\frac{b^2 i^2}{4a}} \Bigg[\Gamma[\frac{(3-k)}{2}] 1F1\left(\frac{(-2+k)}{2}; \frac{1}{2}; \frac{b^2}{4a}\right) \\ &\times \left(1+(-1)^{-k}\right) + i\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{-1/2} \Gamma[\frac{(4-k)}{2}] 1F1\left(\frac{(-3+k)}{2}; \frac{3}{2}; \frac{b^2}{4a}\right) \\ &\times \left(1+(-1)^{-k}\right)\Bigg] a^{k-1/2} + \sum_{r=0}^{m-1}\left(\frac{(-1)^{2r-2} \; i^{2r-3} \; (\gamma + \frac{i\pi}{2})}{(2r-3)!} \, \eta_r - i^{(2r-3)} \, d_{2r-2}\right) \\ &\times \Bigg[M[\hat{f}(w) e^{ibw}; 2r-2] + (-1)^{2r-3} M[\hat{f}(-w) e^{-ibw}; 2r-2] \Bigg] a^{2r-5/2} \\ &+ \sum_{r=0}^{m-1}\left(\frac{(-1)^{2r-2} \; i^{(2r-3)}}{(2r-3)!} \, \eta_r\right) \Bigg[M[\hat{f}(w)\log(aw) e^{ibw}; 2r-2] \\ &+ (-1)^{2r-3} M[\hat{f}(-w)\log(aw) e^{-ibw}; 2r-2] \Bigg] a^{2r-5/2} + R_{n,m}(a).\end{aligned}$$

$K(k)$ is the index k for which $(2r-3)-k \le 1 \le (2r-3)-(k-1)$. $R(r)$ is the index r for which $(2r-2)-k \le 1 \le (2r-3)-k$. If $(2r-3)-k = 1$, for some pair $(2r-3, k)$, then corresponding sum of the terms

$$\begin{aligned}& a_k \, e^{\frac{i\pi k}{2}} \; \Gamma[k] \, \sqrt{2\pi}\left(\frac{1}{2}\right)^{1-\frac{(3-k)}{2}} \; e^{\frac{b^2 i^2}{4a}} \Bigg[\Gamma[\frac{(3-k)}{2}] 1F1\left(\frac{(-2+k)}{2}; \frac{1}{2}; \frac{b^2}{4a}\right) \\ &\times \left(1+(-1)^{-k}\right) + i\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{-1/2} \Gamma[\frac{(4-k)}{2}] 1F1\left(\frac{(-3+k)}{2}; \frac{3}{2}; \frac{b^2}{4a}\right) \\ &\times \left(1+(-1)^{-k}\right)\Bigg] a^{k-1/2} + \left(\frac{(-1)^{2r-2} \; i^{2r-3} \; (\gamma + \frac{i\pi}{2})}{(2r-3)!} \, \eta_r - i^{(2r-3)} \, d_{2r-2}\right) \\ &\times \Bigg[M[\hat{f}(w) e^{ibw}; 2r-2] + (-1)^{2r-3} M[\hat{f}(-w) e^{-ibw}; 2r-2] \Bigg] a^{2r-5/2} \\ &+ \left(\frac{(-1)^{2r-2} \; i^{(2r-3)}}{(2r-3)!} \, \eta_r\right) \Bigg[M[\hat{f}(w)\log(aw) e^{ibw}; 2r-2] \\ &+ (-1)^{2r-3} M[\hat{f}(-w)\log(aw) e^{-ibw}; 2r-2] \Bigg] a^{2r-5/2}\end{aligned}$$

must be replace by

$$\begin{aligned}& a^{2r-5/2} \Bigg\{ \left(\frac{(-1)^{2r-2} \; i^{2r-3} \; (\gamma + \frac{i\pi}{2})}{(2r-3)!} \, \eta_r - i^{(2r-3)} \, d_{2r-2}\right) \\ &\times \Bigg[M[\hat{f}(w) e^{ibw}; 2r-2] + (-1)^{2r-3} M[\hat{f}(-w) e^{-ibw}; 2r-2] \Bigg] \\ &+ \left(\frac{(-1)^{2r-2} \; i^{(2r-3)}}{(2r-3)!} \, \eta_r\right) \Bigg[M[\hat{f}(w)\log(aw) e^{ibw}; 2r-2] + (-1)^{2r-3} \\ &\times M[\hat{f}(-w)\log(aw) e^{-ibw}; 2r-2] \Bigg] + \lim_{z \to 0} \Bigg[a_k \, e^{\frac{i\pi k}{2}} \; \Gamma[k+z] \sqrt{2\pi}\left(\frac{1}{2}\right)^{1-\frac{(3-k+z)}{2}} \; e^{\frac{b^2 i^2}{4a}}\end{aligned}$$

$$\times\left(\Gamma[\frac{(3-k+z)}{2}]1F1\left(\frac{(-2+k+z)}{2};\frac{1}{2};\frac{b^2}{4a}\right)\left(1+(-1)^{-k}\right)\right.$$
$$+i\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{-1/2}\Gamma[\frac{(4-k+z)}{2}]\,1F1\left(\frac{(-3+k+z)}{2};\frac{3}{2};\frac{b^2}{4a}\right)$$
$$\left.\times\left(1+(-1)^{z-k}\right)\right)\Big]\Big\}$$

Asymptotic Expansion of Morlet Wavelet Transform

we consider $\psi(t) = e^{iw_o t-\frac{t^2}{2}}$ to be a Morlet wavelet. Since,Fourier transform of Morlet wavelet $\hat{\psi}(w) = \sqrt{2\pi}w^2 e^{-w^2/2}$ is locally integrable in $(-\infty,\infty)$ and has an asymptotic expansion of $\bar{\hat{\psi}}(w)$ as $w \to 0^+$ is

$$\bar{\hat{\psi}}(w) = \sum_{r=0}^{\infty} c_r^* \log(w) w^r + \sum_{r=0}^{\infty} \gamma^* w^r;$$

where,

$$c_r^* = \frac{(-1)^{2r} i^{(2r-1)}}{(2r-1)!}\ \eta_{(2r-1)};$$
$$\gamma_r^* = \frac{(-1)^{2r} i^{(2r-1)}(\gamma+\frac{i\pi}{2})}{(2r-1)!}\ \eta_{(2r-1)} - i^{(2r-1)}\ d_{2r};$$
$$d_{2r} = \lim_{w\to 0}[\psi_{2r,2r}(w) + \frac{(-1)^{(2r-1)}}{(2r-1)!}\eta_{(2r-1)}\log(w)]$$
$$\eta_r = \sum_{n=0}^{r} \frac{(-1)^{r-n} i^n\ w_o^n}{2^{r-n}\ n!\,(r-n)!} t^n$$

Now, assume $f \in \mathbb{H}$. Then by Theorem 2, and by means of formula [8, (11, 31), pp. 318, 320], we get the asymptotic expansion of Morlet wavelet transform at $a \to 0^+$ is

$$(W_\psi f)(b,a) = \sum_{k=0}^{n-1} a_k e^{\frac{i\pi k}{2}}\Gamma[k]\left[\sqrt{2\pi}\ \Gamma[1-k]\, e^{-w_o^2/2} e^{\frac{(w_o^2-\frac{b^2}{a^2})}{4}} e^{\frac{ibw_o}{a}} D_{-(1-k)}\left(-w_o-\frac{ib}{a}\right)\right.$$
$$\left.\times\left(1+(-1)^{-k}\right)\right]a^{k-1/2} + \sum_{r=0}^{m-1}\left(\frac{(-1)^{2r} i^{(2r-1)}(\gamma+\frac{i\pi}{2})}{(2r-1)!}\ \eta_{(2r-1)}\right.$$
$$\left.-i^{(2r-1)}\ d_{2r}\right)\left[M[\hat{f}(w)e^{ibw};2r] + (-1)^{2r-1}M[\hat{f}(-w)e^{-ibw};2r]\right]a^{2r-1/2}$$
$$+\sum_{r=0}^{m-1}\left(\frac{(-1)^{2r} i^{(2r-1)}}{(2r-1)!}\ \eta_{(2r-1)}\right)\left[M[\hat{f}(w)e^{ibw}\log(aw);2r]\right.$$
$$\left.+(-1)^{2r-1}M[\hat{f}(-w)e^{-ibw}\log(aw);2r]\right]a^{2r-1/2} + R_{n,m}(a)$$

$K(k)$ is the index k for which $(2r-1)-k \le 1 \le (2r-1)-(k-1)$. $R(r)$ is the index r for which $(2r-2)-k \le 1 \le (2r-1)-k$. If $(2r-1)-k = 1$, for some pair $(2r-1,k)$, then corresponding sum of the terms

$$a_k e^{\frac{i\pi k}{2}}\Gamma[k]\Bigg[\sqrt{2\pi}\ \Gamma[1-k]\ e^{-w_o^2/2}e^{\frac{(w_o^2-\frac{b^2}{a^2})}{4}}e^{\frac{ibw_o}{a}}D_{-(1-k)}\Bigg(-w_o-\frac{ib}{a}\Bigg)$$
$$\times\Bigg(1+(-1)^{-k}\Bigg)\Bigg]a^{k-1/2}+\Bigg(\frac{(-1)^{2r}i^{(2r-1)}(\gamma+\frac{i\pi}{2})}{(2r-1)!}\eta_{(2r-1)}$$
$$-i^{(2r-1)}d_{2r}\Bigg)\Bigg[M[\hat{f}(w)e^{ibw};2r]+(-1)^{2r-1}M[\hat{f}(-w)e^{-ibw};2r]\Bigg]a^{2r-1/2}$$
$$+\Bigg(\frac{(-1)^{2r}i^{(2r-1)}}{(2r-1)!}\eta_{(2r-1)}\Bigg)\Bigg[M[\hat{f}(w)e^{ibw}\log(aw);2r]$$
$$+(-1)^{2r-1}M[\hat{f}(-w)e^{-ibw}\log(aw);2r]\Bigg]a^{2r-1/2}$$

must be replace by

$$a^{2r-1/2}\Bigg\{\Bigg(\frac{(-1)^{2r}i^{(2r-1)}(\gamma+\frac{i\pi}{2})}{(2r-1)!}\eta_{(2r-1)}-i^{(2r-1)}d_{2r}\Bigg)\Bigg[M[\hat{f}(w)e^{ibw};2r]$$
$$+(-1)^{2r-1}M[\hat{f}(-w)e^{-ibw};2r]\Bigg]+\Bigg(\frac{(-1)^{2r}i^{(2r-1)}}{(2r-1)!}\eta_{(2r-1)}\Bigg)\Bigg[M[\hat{f}(w)e^{ibw}\log(aw);2r]$$
$$+(-1)^{2r-1}M[\hat{f}(-w)e^{-ibw}\log(aw);2r]+\lim_{z\to 0}\Bigg[a_k e^{\frac{i\pi(k+z)}{2}}\Gamma[k+z]\Bigg[\sqrt{2\pi}\ \Gamma[1-k+z]$$
$$\times e^{-w_o^2/2}e^{\frac{(w_o^2-\frac{b^2}{a^2})}{4}}e^{\frac{ibw_o}{a}}D_{-(1-k+z)}\Bigg(-w_o-\frac{ib}{a}\Bigg)\Bigg(1+(-1)^{-k}\Bigg)\Bigg]\Bigg]\Bigg]\Bigg\}.$$

Conclusion

This paper focuses on the asymptotic expansion of continuous wavelet transform for small dilation parameter. The main advantage of this expansion is that the error terms can be easily and precisely calculated and these results are used to approximate analytical solution of different types of linear , non-linear , singular differential and integral equations which are related to the different types of wavelet transform. Apart from this direction, future work may involve seeking to apply the homotopy perturbation method, the variation iteration method or we can formed the suitable algorithm for solving the linear, non-linear , singular differential and integral equations. Another direction which can be further explored is that how to adjust the dilation parameter to priorities of different type of differential and integral equations. It is thus worth studying the properties of asymptotic expansion of different types of wavelet transform.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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