Int. J. Appl. Comput. Math (2022) 8:4
https://doi.org/10.1007/s40819-021-01154-8

ORIGINAL PAPER

Biorthogonal Wavelet Packets in $H^s(\mathbb{K})$

Guru P. Singh[1] **· Ashish Pathak**[1]

Accepted: 16 September 2021 / Published online: 7 December 2021

Abstract
The concepts of multiresolution analysis(MRA), wavelets, and biorthogonal wavelets in Sobolev space over local fields of positive characteristic ($H^s(\mathbb{K})$) are developed by Pathak and Singh [8,9]. In this paper, we constructed biorthogonal wavelet packets in Sobolev space $H^s(\mathbb{K})$ and derived their biorthogonality at each level by means of Fourier transform.

Keywords Multiresolution analysis · Local fields · Sobolev space · Biorthogonal wavelets · Fourier transforms · Wavelet packets

Introduction

In recent years, local fields have attracted attention of many mathematicians. Benedetto and Benedetto gave wavelet theory on local fields and related group(s). Their approach are not based on MRA. The definition of MRA on local fields of positive characteristic is given by Jiang et al. [5]. They developed a theory for constructing orthonormal wavelets on local fields $\mathbb{K}$. Their concepts have been extended by Behra and Jahan in different setups. Recently Pathak, Singh, and Kumar[8–10] modified the concept of MRA on Sobolev space over local fields $\mathbb{K}$ and constructed orthonormal wavelets from the MRA. They developed a theory of biorthogonal wavelets on Sobolev space over local fields [9]. Also, they constructed multilevel wavelet packets on Sobolev space over local fields $\mathbb{K}$.

In this article, we developed a theory for constructing biorthogonal wavelet packets associated with dual MRA on Sobolev space over local fields $\mathbb{K}$.

This article is divided in following sections. In Sect. 1, we discuss some properties of local fields and Sobolev space over $\mathbb{K}$. In Sect. 2, We recall dual MRA on $H^s(\mathbb{K})$ and Sect. 3 contains biorthogonal wavelet packets corresponding to these MRAs and in its subsection, we have proved their orthogonality at j^{th} level.

Throughout the paper $\mathbb{K}$ denotes the local field of positive characteristic, $\varkappa$ is a fixed character on $\mathbb{K}^+$, $\mathfrak{p}$ be a fixed prime element in $\mathbb{K}$ used for dilation, and $v(k) \in \mathbb{K},\ k \in \mathbb{N}_0 =$

✉ Ashish Pathak
ashishpathak@bhu.ac.in

Guru P. Singh
gurusinghab9092@gmail.com

1 Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi 221005, Uttar Pradesh, India

$\{0, 1, 2, 3, ...\}$ is used for translation. The Sobolev space $H^s(\mathbb{K})$, $s \in \mathbb{R}$, consists of all those $f \in \mathfrak{D}'(\mathbb{K})$ (space of continuous linear functional on $\mathfrak{D}(\mathbb{K})$ and $\mathfrak{D}(\mathbb{K})$ is the space of all finite linear combinations of characteristic functions of balls of $\mathbb{K}$) such that $\hat{\gamma}^{\frac{s}{2}} \hat{f}(\zeta) \in L^2(\mathbb{K})$, which satisfy:

$$\|f\|^2_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\gamma}^s(\zeta)|\hat{f}(\zeta)|^2 d\zeta, \quad \text{where} \quad \hat{\gamma}^s(\zeta) = (max(1, |\zeta|))^s,$$

the corresponding inner product is defined by

$$\langle f, g\rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\gamma}^s(\zeta)\hat{f}(\zeta)\overline{\hat{g}(\zeta)}d\zeta,$$

where

$$\hat{f}(\zeta) = \int_{\mathbb{K}} f(x)\overline{\varkappa_\zeta(x)}dx, \quad \zeta \in \mathbb{K}.$$

For more detail, refer to [5,8,13].

Dual Multiresolution Analysis on $H^s(\mathbb{K})$

Pathak and Singh modified the classical multiresolution analysis on $L^2(\mathbb{K})$ and defined MRA and dual MRA on $H^s(\mathbb{K})$(see [8,9]). Now, we recall the theory of dual MRA in Sobolev space over $\mathbb{K}$.

Definition 1 Two families of functions $\{\varphi_k : k \in \mathbb{N}_0\}$ and $\{\tilde{\varphi}_k : k \in \mathbb{N}_0\}$ in $H^s(\mathbb{K})$ are said to be biorthogonal if

$$\langle\varphi_k, \tilde{\varphi}_{k'}\rangle = \delta_{k,k'} \text{ for every } k, k' \in \mathbb{N}_0.$$

If pair of scaling functions $\varphi^{(j)}, \tilde{\varphi}^{(j)} \in H^s(\mathbb{K})$ are biorthogonal, then

$$\langle\varphi^{(j)}(.), \tilde{\varphi}^{(j)}(. - v(k))\rangle = \delta_{0,k}, \quad k \in \mathbb{N}_0. \tag{1}$$

Theorem 1 *Let $\tilde{\varphi}^{(j)}$, $\varphi^{(j)} \in H^s(\mathbb{K})$ and $j \in \mathbb{Z}$, then the distributions $\tilde{\varphi}^{(j)}_{j,k} = q^{\frac{j}{2}}\tilde{\varphi}^{(j)}(\mathfrak{p}^{-j}x - v(k))$; $k \in \mathbb{N}_0$ and $\varphi^{(j)}_{j,k} = q^{\frac{j}{2}}\varphi^{(j)}(\mathfrak{p}^{-j}x - v(k))$ are biorthogonal in $H^s(\mathbb{K})$ if and only if*

$$\sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(k)))\hat{\varphi}^{(j)}(\zeta + v(k))\overline{\hat{\tilde{\varphi}}^{(j)}(\zeta + v(k))} = 1 \quad a.e. \tag{2}$$

Moreover, we also have

$$\lim_{j\to\infty} \hat{\varphi}^{(j)}(\mathfrak{p}^j\zeta)\overline{\hat{\tilde{\varphi}}^{(j)}(\mathfrak{p}^j\zeta)} \le \hat{\gamma}^{-s}(\zeta). \tag{3}$$

Dual multiresolution analysis of $H^s(\mathbb{K})$ are defined as follows

$$V_j \subset V_{j+1}, \quad \tilde{V}_j \subset \tilde{V}_{j+1}.$$

Correspondingly, since $\varphi^{(j)} \in V_j \subset V_{j+1}$; $\tilde{\varphi}^{(j)} \in \tilde{V}_j \subset \tilde{V}_{j+1}$, we have

$$\varphi^{(j)} = \sum_{k\in\mathbb{N}_0} h^{(j)}_k \varphi^{(j+1)}_{j+1,k}; \; \tilde{\varphi}^{(j)} = \sum_{k\in\mathbb{N}_0} \tilde{h}^{(j)}_k \tilde{\varphi}^{(j+1)}_{j+1,k}. \tag{4}$$

Taking Fourier transform of the equation (4), we get

$$\hat{\varphi}^{(j)}(\zeta)=m_0^{(j+1)}(\mathfrak{p}\zeta)\hat{\varphi}^{(j+1)}(\mathfrak{p}\zeta)\,;\ \hat{\tilde{\varphi}}^{(j)}(\zeta)=\tilde{m}_0^{(j+1)}(\mathfrak{p}\zeta)\hat{\tilde{\varphi}}^{(j+1)}(\mathfrak{p}\zeta). \tag{5}$$

Associated wavelets $\psi_r^{(j)}$ and $\tilde{\psi}_r^{(j)}$ $(0\le r\le q-1)$ are given as follows :
$\hat{\psi}_r^{(j)}(\mathfrak{p}^j\zeta)=m_r^{(j+1)}(\mathfrak{p}^{j+1}\zeta)\hat{\varphi}^{(j+1)}(\mathfrak{p}^{j+1}\zeta)$ and $\hat{\tilde{\psi}}_r^{(j)}(\mathfrak{p}^j\zeta)=\tilde{m}_r^{(j+1)}(\mathfrak{p}^{j+1}\zeta)\hat{\tilde{\varphi}}^{(j+1)}(\mathfrak{p}^{j+1}\zeta)$.
For more detail (see [9]).

Biorthogonal Wavelet Packets on Sobolev Space over Local Fields

For construction of biorthogonal wavelet packets the following splitting lemma is required.

Lemma 1 *Let* $\{q^{\frac{j}{2}}\varphi^{(j)}(\mathfrak{p}^{-j}.-v(m)) : m\in\mathbb{N}_0\}$, $\{q^{\frac{j}{2}}\tilde{\varphi}^{(j)}(\mathfrak{p}^{-j}.-v(m)) : m\in\mathbb{N}_0\}$ *are biorthogonal system in* $H^s(\mathbb{K})$ *and* $V_j = span\{q^{\frac{j}{2}}\varphi^{(j)}(\mathfrak{p}^{-j}.-v(m)) : m\in\mathbb{N}_0\}$, $\tilde{V}_j = span\{q^{\frac{j}{2}}\tilde{\varphi}^{(j)}(\mathfrak{p}^{-j}.-v(m)) : m\in\mathbb{N}_0\}$. *Let* $\hat{\psi}_r^{(j)}(\zeta)=m_r^{(j+1)}(\mathfrak{p}\zeta)\hat{\varphi}^{(j+1)}(\mathfrak{p}\zeta)$, $\hat{\tilde{\psi}}_r^{(j)}(\zeta)=\tilde{m}_r^{(j+1)}(\mathfrak{p}\zeta)\hat{\tilde{\varphi}}^{(j+1)}(\mathfrak{p}\zeta)$, $0\le r\le q-1$. *Then* $\{\psi_{r,j,m}^{(j)}(.) : 0\le r\le q-1, m\in\mathbb{N}_0\}$, $\{\tilde{\psi}_{r,j,m}^{(j)}(.) : 0\le r\le q-1, m\in\mathbb{N}_0\}$ *are biorthogonal if and only if*

$$M^{(j)}(\zeta)(\tilde{M}^{(j)})^*(\zeta)=I, \tag{6}$$

where $M^{(j)}(\zeta)=[m_{r_1}^{(j)}(\mathfrak{p}\zeta+\mathfrak{p}v(r_2))]_{r_1,r_2=0}^{q-1}$ *and* $\tilde{M}^{(j)}(\zeta)=[\tilde{m}_{r_1}^{(j)}(\mathfrak{p}\zeta+\mathfrak{p}v(r_2))]_{r_1,r_2=0}^{q-1}$ *for a.e.* $\zeta\in\mathfrak{D}$.

Proof Let $M^{(j)}(\zeta)(\tilde{M}^{(j)})^*(\zeta)=I$. Then, we have

$$\begin{aligned}
&\langle\psi_{r_1,j,m}^{(j)}(.),\ \tilde{\psi}_{r_2,j,n}^{(j)}(.)\rangle\\
&=\int_{\mathbb{K}}\hat{\gamma}^s(\zeta)q^{-\frac{j}{2}}\hat{\psi}_{r_1}^{(j)}(\mathfrak{p}^j\zeta)\bar{\varkappa}_m(\mathfrak{p}^j\zeta)q^{-\frac{j}{2}}\overline{\hat{\tilde{\psi}}_{r_2}^{(j)}}(\mathfrak{p}^j\zeta)\varkappa_n(\mathfrak{p}^j\zeta)d\zeta\\
&=\int_{\mathfrak{D}}\sum_{l\in\mathbb{N}_0}\hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta+v(l)))\hat{\psi}_{r_1}^{(j)}(\zeta+v(l))\overline{\hat{\tilde{\psi}}_{r_2}^{(j)}}(\zeta+v(l))\bar{\varkappa}_m(\zeta)\varkappa_n(\zeta)d\zeta\\
&=\int_{\mathfrak{D}}\sum_{l\in\mathbb{N}_0}\hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta+v(l)))m_{r_1}^{(j+1)}(\mathfrak{p}\zeta+\mathfrak{p}v(l))\\
&\qquad\times\hat{\varphi}^{j+1}(\mathfrak{p}\zeta+\mathfrak{p}v(l))\overline{\tilde{m}_{r_2}^{(j+1)}(\mathfrak{p}\zeta+\mathfrak{p}v(l))}\,\overline{\hat{\tilde{\varphi}}^{j+1}(\mathfrak{p}\zeta+\mathfrak{p}v(l))}\bar{\varkappa}_m(\zeta)\varkappa_n(\zeta)d\zeta\\
&=\int_{\mathfrak{D}}\sum_{i=0}^{q-1}\sum_{l\in\mathbb{N}_0}\hat{\gamma}^s(\mathfrak{p}^{-j-1}(\mathfrak{p}\zeta+\mathfrak{p}v(ql+i)))m_{r_1}^{(j+1)}(\mathfrak{p}\zeta+\mathfrak{p}v(ql+i)))\\
&\qquad\times\overline{\tilde{m}_{r_2}^{(j+1)}(\mathfrak{p}\zeta+\mathfrak{p}v(ql+i)))}\,\hat{\varphi}^{j+1}(\mathfrak{p}\zeta+\mathfrak{p}v(ql+i)))\hat{\tilde{\varphi}}^{j+1}(\mathfrak{p}\zeta+\mathfrak{p}v(ql+i)))\\
&\qquad\times\bar{\varkappa}_m(\zeta)\varkappa_n(\zeta)d\zeta\\
&=\int_{\mathfrak{D}}\{\sum_{i=0}^{q-1}m_{r_1}^{(j+1)}(\mathfrak{p}\zeta+\mathfrak{p}v(i)))\overline{\tilde{m}_{r_2}^{(j+1)}(\mathfrak{p}\zeta+\mathfrak{p}v(i)))}\}\bar{\varkappa}_m(\zeta)\varkappa_n(\zeta)d\zeta\\
&=\int_{\mathfrak{D}}\delta_{r_1,r_2}\bar{\varkappa}_m(\zeta)\varkappa_n(\zeta)d\zeta\\
&=\delta_{r_1,r_2}\delta_{m,n}.
\end{aligned}$$

Therefore, $\{q^{\frac{j}{2}}\psi_r^{(j)}(\mathfrak{p}^{-j}.-v(m)) : 0 \le r \le q-1, m \in \mathbb{N}_0\}$ and $\{q^{\frac{j}{2}}\tilde{\psi}_r^{(j)}(\mathfrak{p}^{-j}.-v(m)) : 0 \le r \le q-1, m \in \mathbb{N}_0\}$ are biorthogonal. The converse part can be proved easily. □

Like orthogonal wavelet packets [11], Biorthogonal wavelet packets corresponding to $\varphi^{(j)}, \ \tilde{\varphi}^{(j)}$ are given by

$$w_0^{(j)} = \varphi^{(j)}, \ \tilde{w}_0^{(j)} = \tilde{\varphi}^{(j)}, \ w_n^{(j)} = \psi_n^{(j)}, \text{ and } \tilde{w}_n^{(j)} = \tilde{\psi}_n^{(j)} \ (1 \le n \le q-1),$$

where

$$\hat{\psi}_t^{(j)}(\zeta) = m_t^{(j+1)}(\mathfrak{p}\zeta)\hat{\varphi}^{(j+1)}(\mathfrak{p}\zeta), \ \hat{\tilde{\psi}}_t^{(j)}(\zeta) = \tilde{m}_t^{(j+1)}(\mathfrak{p}\zeta)\hat{\tilde{\varphi}}^{(j+1)}(\mathfrak{p}\zeta) \ (1 \le t \le q-1).$$

In general, let $w_n^{(j)}$ and $\tilde{w}_n^{(j)}$ are defined for every integer $n \ge 0$ by

$$w_{r+qn}^{(j)}(\mathfrak{p}^{-j}x) = q^{\frac{j+1}{2}} \sum_{m\in\mathbb{N}_0} h_{m,r}^{(j+1)} w_n^{(j+1)}(\mathfrak{p}^{-j-1}x - v(m)),$$

$$\tilde{w}_{r+qn}^{(j)}(\mathfrak{p}^{-j}x) = q^{\frac{j+1}{2}} \sum_{m\in\mathbb{N}_0} h_{m,r}^{(j+1)} \tilde{w}_n^{(j+1)}(\mathfrak{p}^{-j-1}x - v(m)), \text{ for } 0 \le r \le q-1.$$

Taking the Fourier transform, we get

$$\hat{w}_{r+qn}^{(j)}(\zeta) = m_r^{(j+1)}(\mathfrak{p}\zeta)\hat{w}_n^{(j+1)}(\mathfrak{p}\zeta),$$
$$\hat{\tilde{w}}_{r+qn}^{(j)}(\zeta) = \tilde{m}_r^{(j+1)}(\mathfrak{p}\zeta)\hat{\tilde{w}}_n^{(j+1)}(\mathfrak{p}\zeta).$$

We can also define $w_n^{(j)}, \tilde{w}_n^{(j)}$ for every integer $n \ge 0$ by its Fourier transform as (here [x] denotes greatest integer less than or equal to x)

$$\hat{w}_n^{(j)}(\zeta) = m_r^{(j+1)}(\mathfrak{p}\zeta)\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta),$$

$$\hat{\tilde{w}}_n^{(j)}(\zeta) = \tilde{m}_r^{(j+1)}(\mathfrak{p}\zeta)\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta),$$

where r is given by

$$r = n - q[\frac{n}{q}]. \tag{7}$$

Definition 2 The set of functions $\{w_n^{(j)} : n \ge 0\}, \{\tilde{w}_n^{(j)} : n \ge 0\}$ are defined as above are said to be biorthogonal wavelet packets with respect to dual MRA $\{V_j\}_{j\in\mathbb{Z}}, \{\tilde{V}_j\}_{j\in\mathbb{Z}}$ of $H^s(\mathbb{K})$.

Definition 3 For every $n \in \mathbb{N}_0$ and $0 \le r \le q-1$, the biorthogonal wavelet packet spaces at j^{th} level are given by

$$W_j^{[\frac{n}{q}],r} = span\{q^{\frac{j}{2}} w_n^{(j)}(\mathfrak{p}^{-j}. - v(k)) : k \in \mathbb{Z}\} \cap H^s(\mathbb{K}),$$
$$\tilde{W}_j^{[\frac{n}{q}],r} = span\{q^{\frac{j}{2}} \tilde{w}_n^{(j)}(\mathfrak{p}^{-j}. - v(k)) : k \in \mathbb{Z}\} \cap H^s(\mathbb{K}),$$

where r is given by (7).

Definition 4 Suppose $w_n^{(j)}(x), \ \tilde{w}_n^{(j)}(x)$ be biorthogonal wavelet packets corresponding to the scaling functions $\varphi^{(j)}(x), \ \tilde{\varphi}^{(j)}(x)$. Then the translates and dilates form of biorthogonal wavelet packet functions for integer j and $k \in \mathbb{N}_0$ are defined as

$$w_{j,k,n}^{(j)}(x) = q^{\frac{j}{2}} w_n^{(j)}(\mathfrak{p}^{-j}x - v(k)); \tag{8}$$

$$\tilde{w}_{j,k,n}^{(j)}(x) = q^{\frac{j}{2}} \tilde{w}_n^{(j)}(\mathfrak{p}^{-j}x - v(k)). \tag{9}$$

Lemma 2 *For $j \in \mathbb{Z}$, let $w_n^{(j+1)}, \tilde{w}_n^{(j+1)} \in H^s(\mathbb{K})$, then the distributions $\{q^{\frac{j+1}{2}} w_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}x - v(k)) : k \in \mathbb{N}_0\}$, $\{q^{\frac{j+1}{2}} \tilde{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}x - v(k)) : k \in \mathbb{N}_0\}$ are biorthogonal in $H^s(\mathbb{K})$ if and only if*

$$\sum_{k \in \mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\zeta + v(k)))\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\zeta + v(k))\overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\zeta + v(k))} = 1.$$

Proof Let

$$A^{(j+1)}(\zeta) = \sum_{k \in \mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\zeta + v(k)))\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\zeta + v(k))\overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\zeta + v(k))}.$$

Since $w_n^{(j+1)}, \tilde{w}_n^{(j+1)} \in H^s(\mathbb{K})$, then the above series converges almost everywhere and belongs to $L^1_{Loc}(\mathfrak{D})$.

Moreover, for every $l \in \mathbb{N}_0$, we have

$$\begin{aligned}
\int_{\mathfrak{D}} A^{(j+1)}\bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta &= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\zeta))\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\zeta)\overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\zeta)}\bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\zeta)q^{-\frac{j+1}{2}}\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}\zeta)\bar{\varkappa}_k(\mathfrak{p}^{j+1}\zeta)q^{-\frac{j+1}{2}} \\
&\quad \times \overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}\zeta)}\varkappa_l(\mathfrak{p}^{j+1}\zeta)d\zeta \\
&= \langle q^{\frac{j+1}{2}} w_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}x - v(k)), q^{\frac{j+1}{2}}\tilde{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}x - v(l))\rangle \\
&= 1 \text{ if } l = k \text{ and } A^{(j+1)}(\zeta) = 1.
\end{aligned}$$

□

Now, we will derive the biorthogonality of these wavelet packets.

Lemma 3 *Suppose that $w_s^{(j)}, \tilde{w}_s^{(j)} \in H^s(\mathbb{K})$ be biorthogonal wavelets corresponding to a pair of biorthogonal scaling functions $\varphi^{(j)}, \tilde{\varphi}^{(j)}$. Then we have*

$$\sum_{i=0}^{q-1} m_{r_1}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i))\overline{\tilde{m}_{r_2}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i))} = \delta_{r_1,r_2}, \quad 0 \le r_1, r_2 \le q-1. \tag{10}$$

Proof For $0 \le r_1, r_2 \le q-1$, we have

$$\begin{aligned}
\delta_{r_1,r_2} &= \sum_{k \in \mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(k)))\hat{w}_{r_1}^{(j)}(\zeta + v(k))\hat{\tilde{w}}_{r_2}(\zeta + v(k)) \\
&= \sum_{k \in \mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(k)))m_{r_1}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(k))\hat{\varphi}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(k)) \\
&\quad \times \overline{\tilde{m}_{r_2}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(k))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\zeta + \mathfrak{p}v(k))} \\
&= \sum_{i=0}^{q-1}\sum_{k \in \mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(qk+i)))m_{r_1}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(qk+i)) \\
&\quad \times \hat{\varphi}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(qk+i))\overline{\tilde{m}_{r_2}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(qk+i))}\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\zeta + \mathfrak{p}v(qk+i))}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{q-1} m_{r_1}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i)) \overline{\tilde{m}_{r_2}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i))} \\
&\quad \times \sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(qk+i)))\hat{\varphi}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(qk+l))\overline{\hat{\tilde{\varphi}}(\mathfrak{p}\zeta + \mathfrak{p}v(qk+i))} \\
&= \sum_{i=0}^{q-1} m_{r_1}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i)) \overline{\tilde{m}_{r_2}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i))}.
\end{aligned}
$$

□

Biorthogonality of Wavelet Packets at j^{th} level

In the following theorems , we obtain the biorthogonality at j^{th} level.

Theorem 2 *Let $j \in \mathbb{Z}$ and $k, l, n \in \mathbb{N}_0$. Then*

$$
\langle w_{j,k,n}^{(j)}, \tilde{w}_{j,l,n}^{(j)} \rangle = \delta_{k,l}.
$$

Proof We have

$$
\begin{aligned}
&\langle w_{j,k,n}^{(j)}, \tilde{w}_{j,l,n}^{(j)} \rangle \\
&= \langle q^{\frac{j}{2}} w_n^{(j)}(\mathfrak{p}^{-j}. - v(k)), q^{\frac{j}{2}} \tilde{w}_n^{(j)}(\mathfrak{p}^{-j}. - v(l)) \rangle \\
&= q^{-j} \int_{\mathbb{K}} \hat{\gamma}^s(\zeta)\hat{w}_n^{(j)}(\mathfrak{p}^j\zeta)\bar{\varkappa}_k(\mathfrak{p}^j\zeta)\overline{\hat{\tilde{w}}_n^{(j)}(\mathfrak{p}^j\zeta)}\varkappa_l(\mathfrak{p}^j\zeta)d\zeta \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\zeta)\hat{w}_n^{(j)}(\zeta)\overline{\hat{\tilde{w}}_n^{(j)}(\zeta)}\bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\zeta)m_r^{(j+1)}(\mathfrak{p}\zeta)\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta)\overline{\tilde{m}_r^{(j+1)}(\mathfrak{p}\zeta)}\,\overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta)} \\
&\quad \times \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathfrak{D}} \sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(n))m_r^{(j+1)}(\mathfrak{p}(\zeta + v(n))\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\zeta + v(n)) \\
&\quad \times \overline{\tilde{m}_r^{(j+1)}(\mathfrak{p}(\zeta + v(n))}\,\overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\zeta + v(n))}\bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1}\sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(qn+i))m_r^{(j+1)}(\mathfrak{p}(\zeta + v(qn+i)) \\
&\quad \times \hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\zeta + v(qn+i))\overline{\tilde{m}_r^{(j+1)}(\mathfrak{p}(\zeta + v(qn+i))} \\
&\quad \times \overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\zeta + v(qn+i))}\bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1}\sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\mathfrak{p}\zeta + \mathfrak{p}v(i) + v(n)))\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i) + v(n)) \\
&\quad \times \overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i) + v(n))}m_r^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i)) \\
&\quad \times \overline{\tilde{m}_r^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i))}\bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta
\end{aligned}
$$

$$= \int_{\mathfrak{D}} \sum_{i=0}^{q-1} m_r^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i)) \overline{\tilde{m}_r^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i))} \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta$$

$$= \int_{\mathfrak{D}} \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta = \delta_{k,l}.$$

□

Theorem 3 *Let $n \in \mathbb{N}_0$ and $1 \le t \le q-1$. Then, we have*

$$\langle w_{j,k,qn}^{(j)}, \tilde{w}_{j,l,t+qn}^{(j)} \rangle = 0.$$

Proof With the help of change of variable trick, we have

$$\begin{aligned}
&\langle w_{j,k,qn}^{(j)}, \tilde{w}_{j,l,t+qn}^{(j)} \rangle \\
&= \langle q^{\frac{j}{2}} w_{qn}^{(j)}(\mathfrak{p}^{-j}. - v(k)), q^{\frac{j}{2}} \tilde{w}_{t+qn}^{(j)}(\mathfrak{p}^{-j}. - v(l)) \rangle \\
&= q^{-j} \int_{\mathbb{K}} \hat{\gamma}^s(\zeta) \hat{w}_{qn}^{(j)}(\mathfrak{p}^j\zeta) \overline{\hat{\tilde{w}}_{t+qn}^{(j)}(\mathfrak{p}^j\zeta)} \bar{\varkappa}_k(\mathfrak{p}^j\zeta)\varkappa_l(\mathfrak{p}^j\zeta)d\zeta \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\zeta) \hat{w}_{qn}^{(j)}(\zeta) \overline{\hat{\tilde{w}}_{t+qn}^{(j)}(\zeta)} \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\zeta) m_0^{(j+1)}(\mathfrak{p}\zeta) \hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta) \overline{\tilde{m}_t^{(j+1)}}(\mathfrak{p}\zeta) \overline{\hat{\tilde{w}}_{[\frac{n}{q}]}}(\mathfrak{p}\zeta) \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\zeta) \hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta) \overline{\hat{\tilde{w}}_{[\frac{n}{q}]}(\mathfrak{p}\zeta)} m_0^{(j+1)}(\mathfrak{p}\zeta) \overline{\tilde{m}_t^{(j+1)}}(\mathfrak{p}\zeta) \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathfrak{D}} \sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(n))) \hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\zeta + v(n))) \\
&\qquad \times \overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\zeta + v(n)))} m_0^{(j+1)}(\mathfrak{p}(\zeta + v(n))) \overline{\tilde{m}_t^{(j+1)}}(\mathfrak{p}(\zeta + v(n))) \\
&\qquad \times \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1} \sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\zeta + v(qn+i))) \hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\zeta + v(qn+i))) \\
&\qquad \times \overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\zeta + v(qn+i)))} m_0^{(j+1)}(\mathfrak{p}(\zeta + v(qn+i))) \\
&\qquad \times \overline{\tilde{m}_t^{(j+1)}}(\mathfrak{p}(\zeta + v(qn+i))) \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1} \sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\mathfrak{p}\zeta + v(n) + \mathfrak{p}v(i)) \hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta + v(n) + \mathfrak{p}v(i))) \\
&\qquad \times \overline{\hat{\tilde{w}}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\zeta + v(n) + \mathfrak{p}v(i)))} m_0^{(j+1)}(\mathfrak{p}\zeta + v(n) + \mathfrak{p}v(i)) \\
&\qquad \times \overline{\tilde{m}_t^{(j+1)}}(\mathfrak{p}\zeta + v(n) + \mathfrak{p}v(i)) \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1} m_0^{(j+1)}(\mathfrak{p}\zeta + \mathfrak{p}v(i)) \overline{\tilde{m}_t^{(j+1)}}(\mathfrak{p}\zeta + \mathfrak{p}v(i)) \bar{\varkappa}_k(\zeta)\varkappa_l(\zeta)d\zeta \\
&= 0.
\end{aligned}$$

□

Construction of Biorthogonal Wavelet Packets

Using the theory of convolution of Fourier transform, we construct biorthogonal wavelet packets in $H^s(\mathbb{K})$ at j^{th} level in the other form.

Proposition 1 *Consider the functions* $\{w_n : n \geq 0\}$ *and* $\{\tilde{w}_n : n \geq 0\}$ *are biorthogonal wavelet packets corresponding to the dual MRA* $\{V_j : j \in \mathbb{Z}\}$ *and* $\{\tilde{V}_j : j \in \mathbb{Z}\}$ *in* $L^2(\mathbb{K})$. *Then*

$$\langle w_{j,k,m}, \tilde{w}_{j,l,n} \rangle_{L^2(\mathbb{K})} = \delta_{m,n}\delta_{k,l}, \tag{11}$$

where $w_{j,k,n}(.) = q^{\frac{j}{2}} w_n(\mathfrak{p}^{-j}. - v(k)),\ \tilde{w}_{j,k,n}(.) = q^{\frac{j}{2}} \tilde{w}_n(\mathfrak{p}^{-j}. - v(k))$, $k \in \mathbb{N}_0$ *and* $j \in \mathbb{Z}$.

Theorem 4 *Let* $\rho(.) = \gamma^{-\frac{s}{2}}(.)$ *and* $w_{j,k,n}(.),\ \tilde{w}_{j,k,n}(.)$ *as in above proposition. Then*

$$\langle w^{(j)}_{j,k,n}, \tilde{w}^{(j)}_{j,l,n} \rangle_{H^s(\mathbb{K})} = \delta_{k,l},$$

where $w^{(j)}_{j,k,n}(.) = \rho(.) * w_{j,k,n}(.),\ \tilde{w}^{(j)}_{j,k,n}(.) = \rho(.) * \tilde{w}_{j,k,n}(.)$ *and* $*$ *denotes convolution of two functions.*

Proof By using convolution theorem and (11), we have

$$\begin{aligned}
\langle w^{(j)}_{j,k,n}, \tilde{w}^{(j)}_{j,l,n} \rangle_{H^s(\mathbb{K})} &= \int_{\mathbb{K}} \hat{\gamma}^s(\zeta)\hat{\gamma}^{-\frac{s}{2}}(\zeta)\hat{w}_{j,k,n}(\zeta)\hat{\gamma}^{-\frac{s}{2}}(\zeta)\overline{\hat{\tilde{w}}_{j,l,n}(\zeta)}d\zeta \\
&= \int_{\mathbb{K}} \hat{w}_{j,k,n}(\zeta)\overline{\hat{\tilde{w}}_{j,l,n}(\zeta)}d\zeta \\
&= \int_{\mathbb{K}} w_{j,k,n}(x)\overline{\tilde{w}_{j,l,n}(x)}dx \\
&= \delta_{k,l}.
\end{aligned}$$

□

Acknowledgements The work of first author is supported by the CSIR Grant No: 09/013(0647)/2016 - EMR - 1, New Delhi.

Author Contributions All authors read and approved the final manuscript.

Funding The work of second author is supported by the CSIR Grant No: 09/013(0647)/2016 - EMR - 1, New Delhi.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Albeverio, S., Kozyrev, S.: Multidimensional basis of p-adic wavelets and representation theory, P-Adic Numbers Ultrametric. Anal. Appl. **1**, 181–189 (2009)
2. Benedetto, J.J., Benedetto, R.L.: A wavelet theory for local fields and related groups. J. Geom. Anal. **14**, 423–456 (2004)
3. Benedetto, R.L.: Examples of wavelets for local fields. In: Wavelets, frames, theory, operator (eds.) contemporary mathematics 345, pp. 27–47. R I, American Mathematical Society, Providence (2004)

4. Behra, B., Jahan, Q.: Wavelet packets and wavelet frame packets on local fields of posotive characteristic. J. Math. Anal. Appl. **395**, 1–14 (2012)
5. Jiang, H., Li, D., Jin, N.: Multiresolution analysis on local fields. J. Math. Anal. Appl. **294**, 523–532 (2004)
6. Khrennikov, A.Y., Shelkovich, V.M., Skopina, M.: p-adic refinable functions and MRA-based wavelets. J. Approx. Theory **161**, 226–238 (2009)
7. Kozyrev, S., Wavelet theory as p-adic spectral analysis (Russian), Izv. Ross. Akad. Nauk Ser. Mat. 66, : 149–158; translation in Izv. Math. **66**(2002), 367–376 (2002)
8. Pathak, A., Singh, G.P.: Wavelets in Sobolev space over loacl fields of positive characteristic. Int. J. Wavelets Multuresolut. Inf. Process **16**(04), 1850027 (2018)
9. Pathak, A., Singh, G.P.: Biorthogonal Wavelets in Sobolev Space Over Local Fields of Positive Characteristic. Int. J. Appl. Comput. Math. **6**(2), 1–13 (2020)
10. Pathak, A., Kumar, D., Singh, G.P.: The necessary and sufficient conditions for wavelet frames in Sobolev space over local fields. Bol. Soc. Paran. Mat. **39**(3), 81–92 (2021)
11. Pathak, A., Singh, G. P.: Wavelet Packet on Sobolev space over Local field of Positive Characteristic, (2020) (Pre-Print)
12. Ramakrishnan, D., Valenza, R.J.: Fourier analysis on number fields, graduate texts in mathematics 186. Springer-Verlag, New York (1999)
13. Taibleson, M.H.: Fourier analysis on local fields, mathematical notes 15. Princeton University Press, Princeton, NJ (1975)