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ORIGINAL PAPER

Fast and Accurate Calculations of Fourth-Order Non-self-adjoint Sturm–Liouville Eigenvalues for Problems in Physics and Engineering

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Abstract

Finding the eigenvalues of non-self-adjoint boundary value problems is a very difficult task, especially when the problems are of higher-order or when high-index eigenvalues are required. In fact, the lack of oscillation theorems of non-self-adjoint problems as well as the distribution and scatteration of the eigenvalues in the complex plane, makes the computational process of the eigenvalues a strong and difficult challenge. In this paper, we propose a fast and accurate numerical technique based on the Chebyshev spectral collocation method for approximating the eigenvalues of fourth-order non-self-adjoint Sturm–Liouville boundary value problems. This technique transforms the non-self-adjoint problem into a generalized eigenvalue problem by employing the spectral differentiation matrices to determine the derivatives of Chebyshev polynomials at the Chebyshev–Gauss–Lobatto nodes. The excellent performance of the suggested technique is investigated by considering three numerical examples among which singular ones. The numerical results and comparison with other methods indicate that this technique is easy to implement, considerably accurate and requires less computational costs even when high-index eigenvalues are required.

Keywords Fourth-order non-self-adjoint Sturm–Liouville problems · Spectral collocation method · Chebyshev differentiation matrix · Eigenvalues

Mathematics Subject Classification 65N35 · 34B24 · 47A45 · 34L16

Introduction

Fourth-order non-self-adjoint Sturm–Liouville problems arise in many important scientific and engineering applications. For example, the problems arise in the stability of hydrodynamic and magnetohydrodynamics, lasers, electromagnetic scattering and inverse scattering problems in lossy media, and photonic crystal bres can be modelled mathematically by non-self-adjoint problems [1–4]. When studying this type of problems, the eigenvalues are

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having particular importance. In fact, the eigenvalues and corresponding eigenfunctions often give a perfect description of the problem and indicate many exceptional behaviors. To illustrate, in scattering theory, eigenvalues can be used to obtain useful information on the physical properties of the scattering targets [5–7]. In mechanical vibrations theory, eigenvalues can be interpreted physically as the transition point between an oscillatory and a monotonically decaying behavior (see Refs. [8–10] and references therein). In the wavelet theory, interesting non-classical wavelets can be obtained from eigenfunctions associated with the eigenvalues and related functions for non-self-adjoint spectral problems [11–13]. Unlike the self-adjoint Sturm-Liouville problems which there are many numerical techniques and several successful software packages available in the literatures for computing their eigenvalues (see, e.g., [14–22] and references therein), there exist much fewer numerical techniques for the non-self-adjoint problems. Fundamentally, the lack of oscillation theorems of non-self-adjoint problems and the distribution and scatteration of the eigenvalues in the complex plane, makes the computational process of the eigenvalues a strong and difficult challenge [1,2,23,24]. Some methods have been proposed and analyzed both theoretically and computationally for second-order non-self-adjoint Sturm-Liouville problems (see, e.g. [24–28] and references cited therein). There are a few numerical methods on the computation of eigenvalues of the high-order non-self-adjoint Sturm-Liouville problems (for instance, see [30–32]). Due to the increasing importance of non-self-adjoint eigenvalue problems, and their important applications in physical and engineering sciences, there is a requirement to extend and modified some numerical techniques, so as to be capable to compute eigenvalues of such problems. In particular, those techniques that allows us to compute eigenvalues of the high-order non-self-adjoint Sturm-Liouville problems, efficiently. The main objective of this paper is to provide a numerical technique that can be used to deal directly with fourth-order non-self-adjoint Sturm-Liouville problems, as well as that it has the capability to calculate the eigenvalues of singular problem and/or find the high-index eigenvalues with lower cost. Recently in [19,20], we have proposed a new efficient technique based on Chebyshev differentiation matrix for computing eigenvalues of self-adjoint Sturm-Liouville problems. It has been shown that those numerical technique has the ability to solve regular and singular self-adjoint Sturm-Liouville problems, efficiently. Moreover, the technique finds the high index eigenvalues with high accuracy and less computational costs. On the best of our knowledge there is no Chebyshev differentiation matrices solution to a general class of regular and singular fourth-order non-self-adjoint boundary eigenvalue problems. In this study, the principal objective is to extend the scope of implementations of Chebyshev differentiation matrix technique for the numerical solution of fourth-order non-self-adjoint Sturm-Liouville problem of the form:

$$(p_1(x)y''(x))'' = (s(x)y'(x))' + (\lambda w(x) - q(x))y(x), \quad x \in (a, b), \tag{1}$$

in $L^2_w(a, b)$ subject to some four point specified conditions at the boundary $x \in \{a, b\}$ on y, y', $p_1(x)y''$ and/or $(p_1(x)y'')' - s(x)y'$. The cofficients in Eq. (1) are complex-valued, satisfying $p_1(x)$, $s(x)$, $q(x)$ and $w(x)$ are continuous on $[a, b]$ and $p_1(x), w(x) \neq 0, \forall x \in [a, b]$ [31]. In this work, both the clamped (CBCs) and the hinged (HBCs) boundary conditions are considered with Eq. (1):

$$\texttt{CBCs:} \begin{cases} y(a) = 0, & y'(a) = 0, \\ y(b) = 0, & y'(b) = 0, \end{cases} \tag{2}$$

or

$$\texttt{HBCs:} \begin{cases} y(a) = 0, & y''(a) = 0, \\ y(b) = 0, & y''(b) = 0. \end{cases} \tag{3}$$

Furthermore, combinations of boundary conditions (2) and (3) are possible. In (1) the parameter λ is a spectral parameter, $L_w^2(a,b)$ is the weighted Hilbert space of all Lebesgue measurable complex-valued functions f on (a,b) satisfying $\int_a^b |f(x)|^2 w(x)dx < \infty$ with the inner product

$$(f,g) = \int_a^b f(x)\overline{g(x)}w(x)dx, \tag{4}$$

and norm

$$\|f\|_{L_w^2(a,b)} = \left(\int_a^b |f(x)|^2 w(x)dx\right)^{1/2}. \tag{5}$$

When the problem (1)–(2) and/or (3) is singular, either because (a,b) is an infinite interval or because one or more of the coefficients functions (mean one of $p_1(x)$, $s(x)$, $q(x)$ and $w(x)$ in (1)) is badly behaved near an endpoint, we must use an interval truncation procedure [33], then apply our numerical technique to compute the eigenvalues of the resulting problem. In this technique the derivative of Chebyshev polynomials at grid points can be determined by using the spectral differentiation matrices and then converts the non-self-adjoint problem (1) into a generalized eigenvalue problem. The efficiency and precision of our technique is reflected in its ability to deal with regular and singular problems, adaptability to handle general types of boundary conditions and accuracy of the estimates of eigenvalues with less computational costs even when the high index eigenvalues are computed. We will apply the technique to three different numerical examples among which singular ones, firstly, to demonstrate the accuracy and efficiency of the technique, the eigenvalues of a singular fourth-order non-self-adjoint problem which has exact solution are computed. The famous problem in the hydrodynamic stability, which is known by the Orr-Sommerfeld equation (frequently used as test example in the engineering literature), will be our second test problem. The final test problem is the square of the anharmonic oscillator problem. The remaining structure of this paper is organized as follows: The Chebyshev polynomials are briefly described in Sect. 2.1. In Sect. 2.2 the Chebyshev spectral collocation technique is introduced and explained by applying it to solve fourth-order non self-adjoint Sturm-Liouville problems. Convergence and stability of technique is summarized in Sect. 2.3. To examine the accuracy and efficiency of the technique, three numerical experiments are discussed in Sect. 3. Section 4 includes our conclusions

Chebyshev Spectral Collocation Technique

In this section, we investigate the application of the Chebyshev spectral collocation method for the calculation of the eigenvalues of the fourth-order non-self-adjoint Sturm-Liouville problems (1)–(2) or (1)–(3). We recall that, by using the following change of independent variable:

$$x = \frac{b-a}{2}(X+1), \quad x \in [a,b] \longrightarrow X = \frac{2x}{b-a} - 1, \quad X \in [-1,1], \tag{6}$$

the interval $[a,b]$ can be transformed into $[-1,1]$. If (a,b) is an infinite interval, then an interval truncation procedure must be adopted. Fourth-order non-self-adjoint problem (1) can be numerically solved using the spectral collocation method. In that case, the spectral representation of eigenfunction $y(x)$ of this problem can be expanded as:

$$y(x) = \sum_{j=0}^{n} \overline{\hat{y}_j(x)} T_j(x), \tag{7}$$

where $\{T_j(x)\}$ is a sequence of orthogonal polynomials of of degree $\leq n$ and $\hat{y}_j$ is the set of coefficients to be determined. Here, Chebyshev polynomials of the first kind are used to expand the solution of problem (1). Before getting into this topic of using Chebyshev polynomials in the spectral representation of solution, let's briefly summarize some of important properties of these type of orthogonal polynomials.

The Chebyshev Polynomials of the First Kind

Definition 1 The Chebyshev polynomials of the first kind, $T_n(x)$ of order n, are the eigenfunctions of the singular self-adjoint Sturm-Liouville problems

$$\sqrt{(1-x^2)}\left(\sqrt{(1-x^2)}T_n'(x)\right)' + n^2 T_n(x) = 0, \tag{8}$$

where $x \in (-1, 1)$. If $T_n(x)$ is normalized so that $T_n(1) = 1$, then

$$T_n(x) = cos(n\theta), \;\; \theta = cos^{-1}x. \tag{9}$$

From this representation, it is clear that the Chebyshev polynomials are cosine functions after a change of variable. This is the main reason of their widespread popularity in the numerical estimation of non-periodic boundary value problems [34–36]. The recursive relation of Chebyshev polynomials is the following:

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x). \end{aligned} \tag{10}$$

Some properties of the Chebyshev polynomials are

$$|T_n(x)| \leq 1, \;\; x \in [-1, 1], \tag{11}$$

$$|T_n'(x)| \leq n^2, \;\; x \in [-1, 1], \tag{12}$$

$$T_n(\pm 1) = (\pm 1)^n, \tag{13}$$

$$T_n'(\pm 1) = (\pm 1)^{n+1} n^2. \tag{14}$$

More properties of this sequence of polynomials can be found in Refs. [34–39] and references therein.

The Chebyshev Differentiation Matrices Technique

In this subsection, we are interested in applying the spectral differentiation matrices technique to solve the fourth-order non-self-adjoint Sturm-Liouville problems. At first, let us rewrite Eq. (1) in the following form

$$y^{(4)} + 2L_1(x)y''' + L_2(x)y'' - S(x)y' + Q(x)y = \lambda W(x)y, -1 < x < 1, \tag{15}$$

where $L_1(x) = \frac{p_1'(x)}{p_1(x)}$, $L_2(x) = \frac{p_1''(x) - s(x)}{p_1(x)}$, $S(x) = \frac{s'(x)}{p_1(x)}$, $Q(x) = \frac{q(x)}{p_1(x)}$ and $W(x) = \frac{w(x)}{p_1(x)}$ are defined on the interval of $-1 \leq x \leq 1$. Now let us consider the basis functions T_j that are

Chebyshev polynomials of degree $\leq N$ satisfying $T_j(x_k) = \delta_{j,k}$ for the Chebyshev-Gauss-Lobatto nodes x_k [19,20,34–36]:

$$x_k = cos\left(\frac{k\pi}{N}\right), \quad 0 \leq k \leq N, \tag{16}$$

which are the extrema of the Chebyshev polynomial T_j on the interval $[-1, 1]$. Therefore, the spectral differentiation matrix for the collocation points can be obtained by interpolating a polynomial $P(x)$ through the collocation points, namely the polynomial

$$P(\mathbf{x}) = \sum_{j=0}^{N} T_j(x)\overline{y_j(x)}, \tag{17}$$

which agrees with $y(\mathbf{x})$ at the $N+1$ interior collocation points x_k, $0 \leq k \leq N$ given in (16), i.e.

$$P(\mathbf{x}) = \mathbf{y}, \tag{18}$$

where $\mathbf{x} = (x_0, x_1, \cdots, x_N)^T$ and $\mathbf{y} = (y_0, y_1, \cdots, y_N)^T$. Then the d-th derivative of the interpolating polynomial $P(\mathbf{x})$ at the nodes is given by [19,20]

$$P^{(d)}(\mathbf{x}) = D^{(d)}\mathbf{y}, \tag{19}$$

where the i, j-th element of the differentiation matrices $D^{(d)}$ is $T_j^{(d)}(x_i)$, $0 \leq i, j \leq N$. For each $N \geq 1$, let the rows and columns of the $(N+1) \times (N+1)$ Chebyshev differentiation matrix $D_N^{(1)}$ be indexed from 0 to N. Then the entries of the matrix are [19,20,40]

$$\begin{aligned}
(D_N^{(1)})_{00} &= \frac{2N^2+1}{6}, \quad (D_N^{(1)})_{NN} = -\frac{2N^2+1}{6}, \\
(D_N^{(1)})_{jj} &= \frac{-x_j}{2(1-x_j^2)}, \quad j = 1, \cdots, N-1, \\
(D_N^{(1)})_{ij} &= \frac{c_i}{c_j}\frac{(-1)^{i+j}}{(x_i - xj)}, \quad i \neq j, \quad j = 1, \cdots, N-1,
\end{aligned} \tag{20}$$

where

$$c_i = \begin{cases} 2, & i = 0 \ or \ N \\ 0, & otherwise. \end{cases}$$

Evaluating Eq. (15) at each interior node x_k, $k = 0, 1, \cdots, N$, leads to

$$\begin{aligned}
&y^{(4)}(x_k) + 2L_1(x_k)y'''(x_k) + L_2(x_k)y''(x_k) - S(x_k)y'(x_k) \\
&\quad + Q(x_k)y(x_k) = \lambda W(x_k)y(x_k).
\end{aligned} \tag{21}$$

As mentioned in Eq. (18), the interpolating polynomial $P(x)$ is known to be satisfy Eq. (15) at all $N+1$ collocation points. Consequently, we will get the following collocation equations

$$\begin{aligned}
&P^{(4)}(x_k) + 2L_1(x_k)P'''(x_k) + L_2(x_k)P''(x_k) - S(x_k)P'(x_k) \\
&\quad + Q(x_k)P(x_k) = \lambda W(x_k)P(x_k).
\end{aligned} \tag{22}$$

Here, we recall that the

$$P'''(x) = \sum_{j=1}^{N-1} T_j'''(x_k)\overline{y_j}, \tag{23}$$

and

$$P^{(4)}(x) = \sum_{j=1}^{N-2} T_j^{(4)}(x_k)\overline{y_j}. \tag{24}$$

In addition, for the clamped boundary conditions (2), we have

$$\begin{cases} P(-1) = P'(-1) = 0, \\ P(1) = P'(1) = 0, \end{cases} \tag{25}$$

and for the hinged boundary conditions (3), we get

$$\begin{cases} P(-1) = P''(-1) = 0, \\ P(1) = P''(1) = 0, \end{cases} \tag{26}$$

where $P''(-1) = \sum_{j=1}^{N} T_j^{''}(-1)y_j$ and $P''(1) = \sum_{j=1}^{N} T_j^{''}(1)y_j$. Rewriting Eq. (22) in the differentiation matrix form, we get

$$\left(D^{(4)} + 2\hat{L_1}D^{(3)} + \hat{L_2}D^{(2)} - \hat{S}D^{(1)} + (\hat{Q} - \lambda\hat{W})\right)\mathbf{y} = \mathbf{0}, \tag{27}$$

where $\hat{L_1} = diag(L_1)$, $\hat{L_2} = diag(L_2)$, $\hat{S} = diag(S)$, $\hat{Q} = diag(Q)$ and $\hat{W} = diag(W)$. The boundary conditions in (25) and (26) are modelled in the following forms respectively

$$\begin{cases} I_l\mathbf{y} = D_l^{(1)}\mathbf{y} = \mathbf{0}, \\ I_r\mathbf{y} = D_r^{(1)}\mathbf{y} = \mathbf{0}, \end{cases} \tag{28}$$

and

$$\begin{cases} I_l\mathbf{y} = D_l^{(2)}\mathbf{y} = \mathbf{0}, \\ I_r\mathbf{y} = D_r^{(2)}\mathbf{y} = \mathbf{0}, \end{cases} \tag{29}$$

where indices l and r are represented boundary conditions at endpoints -1 and 1, respectively. Inserting four boundary conditions (28) or (29) into (27), yields the following generalized eigenvalue problems

$$\begin{pmatrix} D^{(4)} + 2\hat{L_1}D^{(3)} + \hat{L_2}D^{(2)} - \hat{S}D^{(1)} + \hat{Q} \\ I_l \\ D_l^{(1)} \\ I_r \\ D_r^{(1)} \end{pmatrix}\mathbf{y} = \lambda \begin{pmatrix} \hat{W}I \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \end{pmatrix}\mathbf{y}, \tag{30}$$

or

$$\begin{pmatrix} D^{(4)} + 2\hat{L_1}D^{(3)} + \hat{L_2}D^{(2)} - \hat{S}D^{(1)} + \hat{Q} \\ I_l \\ D_l^{(2)} \\ I_r \\ D_r^{(2)} \end{pmatrix}\mathbf{y} = \lambda \begin{pmatrix} \hat{W}I \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \end{pmatrix}\mathbf{y}, \tag{31}$$

respectively. Now, the approximate eigenvalues of non self-adjoint problem (1)–(2) or (1)–(3) can be obtained by solving the generalized eigenvalue problem (30) or (31).

Convergence Behavior and Error Estimate

In this subsection, we briefly summarize the convergence and stability of the proposed technique for estimating the eigenvalues of fourth-order non-self-adjoint Sturm-Liouville problems that outlined in the previous subsection. First, let us denote by $\mathbb{P}_N$ be the set of Chebyshev polynomials $\{T_n(x)\}$ of degree $\leq N$. Note that, these polynomials are orthogonal over the interval $[-1, 1]$ with respect to a weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$, namely, for naturals n and m

$$\int_{-1}^{1} T_n(x)T_m(x)w(x)dx = 0, \quad if \ \ n \neq m. \tag{32}$$

The classical Weierstrass theorem and the results reported in Ref. [11,41–45] about systems, whose constituent functions are complex valued implies that such a system is complete in the space $L^2_w(-1, 1)$. This space is the space of functions $f(x)$ over the interval $(-1, 1)$ such that satisfying

$$\int_{-1}^{1} |f(x)|^2 w(x)dx < \infty. \tag{33}$$

We see that the space $L^2_w(-1, 1)$ with the inner product

$$(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)dx, \tag{34}$$

and norm

$$\|f\|_{L^2_w(-1,1)} = \left(\int_{-1}^{1} |f(x)|^2 w(x)dx\right)^{1/2}, \tag{35}$$

is a Hilbert space. Now if the $y(x) \in L^2_w(-1, 1)$ is a solution of fourth-order non self-adjoint problem (1), then the Chebyshev series expansion of $y(x)$ is

$$y(x) = \sum_{j=0}^{\infty} \overline{\hat{y}_j(x)}T_j(x), \ \ \texttt{where} \ \ \hat{y}_j = \frac{(y, T_j)}{\|T_j\|^2_{L^2_w(-1,1)}}. \tag{36}$$

Therefore, if the solution $y(x)$ in (36) is truncated up to the Nth terms, then the approximate solution can be written as

$$P_N y(x) = \sum_{j=0}^{N} \overline{\hat{y}_j(x)}T_j(x). \tag{37}$$

Take into account the orthogonality property (32) of Chebyshev polynomials $\{T_n(x)\}$ implies that the $P_N y$ is the orthogonal projection of y on $\mathbb{P}$, that is,

$$(P_N y, T) = (y, T), \quad \forall\, T \in \mathbb{P}_N. \tag{38}$$

Moreover, from completeness property of the polynomials $\{T_n(x)\}$ it follows that

$$\| \ y - P_N y \ \|_{L^2_w(-1,1)} \to 0, as \ \ N \to \infty, \tag{39}$$

for all $y \in L^2_w(-1, 1)$. In fact, the convergence and stability of this technique can be determined based on the convergence behavior of the truncated Chebyshev series (37). Therefore, to survey the approximation properties of the Chebyshev projection (37) in a weighted Sobolev space:

$$H_w^m(-1,1) := \{y \in L_w^2(-1,1) : y^{(k)} \in L_w^2(-1,1),\ 0 \leq k \leq m\}, \tag{40}$$

equipped with the following norm

$$\begin{aligned}\|y\|_{H_w^m(-1,1)} &= \left(\sum_{k=0}^{m} \|y^{(k)}\|^2_{L_w^2(-1,1)}\right)^{1/2} \\ &= \left(\sum_{k=0}^{m} \int_{-1}^{1} |y^{(k)}(x)|^2 w(x) dx\right)^{1/2},\end{aligned} \tag{41}$$

we can give the following definition which is a fundamental condition to the convergence of the approximate solution of the system (1) generated by using the Chebyshev spectral collocation technique.

Definition 2 Suppose that $P_N y$ is the truncated Chebyshev series given in (37) as an approximate to the exact solution $y(x)$ of the system (1) for all $x \in (a, b)$ or $X \in (-1, 1)$ generated by using current technique, then this numerical technique converges if

$$\lim_{N\to\infty} \| y - P_N y \|_{L_w^2(-1,1)} = 0.$$

Following [36,46–48]), the truncation error obtained by the use of the truncated Chebyshev series $P_N y$ (37) to approximate $y(x)$ in Hilbert space norms (35) is given by:

Theorem 1 *Let $P_N y$ is the truncated Chebyshev series of y, then the truncation error $\| y - P_N y \|$ for all $y \in L_w^2(-1,1)$, $m \geq 0$ can be estimated as follows*

$$\| y - P_N y \|_{L_w^2(-1,1)} \leq C N^{-m} \| y^{(m)} \|_{L_w^2(-1,1)}, \tag{42}$$

where the constant C independent of N and depends on m.

Moreover, we can estimate the truncation error $y - P_N y$ in Sobolev norms (41) as follows:

Theorem 2 *Let $P_N y$ is the truncated Chebyshev series of y in the weighted Sobolev space $H_w^m(-1,1)$, then*

$$\| y - P_N y \|_{H_w^m(-1,1)} \leq C N^{k-m} \| y^{(m)} \|_{L_w^2(-1,1)}, \tag{43}$$

for any $m \geq 0$ and any $0 \leq k \leq m$.

Thus, the proposed technique gives spectral accuracy; that is, the accuracy depends upon the smoothness of the functions being interpolated. If the eigenfunction $y(x)$ belongs to C^∞−class, then the produced error of approximation $\| y - P_N y \| \to 0$, as $N \to \infty$ with exponential convergence rate $O(e^{-CN})$, for some $C > 0$. On the other hand, it is well known that the proposed technique in the current work can be considered as the hp-version of the finite element method (see [19,49–53]). So, by adopting this process, we would expect an estimate of the error of the n-th eigenvalue due to truncation of the Chebyshev series up to the Nth terms of the form $O(n^p e^{-CN})$.

Numerical Results and Discussion

In this section, to demonstrate the accuracy, efficiency and reliability of the numerical technique that outlined in the previous section, three of the more interesting and difficult examples in the hydrodynamic and engineering literature, including singular ones such as the Orr-Sommerfeld equations and square of the anharmonic oscillator problem are presented and discussed. All of the computational experiments were implemented using MATLAB R2014a, running in an Intel (R) Core (TM) i3-8145U CPU @ 2.10 GHz, equipped with 4 GB of Ram.

Example 1 Consider the singular fourth-order non-self-adjoint Sturm-Liouville problem (Taken from [31])

$$\begin{cases} \left(-\frac{d^2}{dx^2}+x^2\right)\left(-\frac{d^2}{dx^2}+x^2+i\right)y(x)=\lambda y(x), \quad x\in[0,\infty), \\ y(0)=y''(0)=0. \end{cases} \tag{44}$$

We will consider the truncated problem over the interval $[0, b^*]$ for various b^* and compute the eigenvalues of

$$\begin{cases} \left(-\frac{d^2}{dx^2}+x^2\right)\left(-\frac{d^2}{dx^2}+x^2+i\right)y(x)=\lambda y(x), \quad x\in[0,b^*], \\ y(0)=y''(0)=0, \quad y(b^*)=y''(b^*)=0. \end{cases} \tag{45}$$

Now, let us rewrite (45) in the following form

$$\begin{cases} y^{(4)}(x)-\left(2x^2+i\right)y''(x)-4xy'(x)+\left(x^4-2+ix^2\right)y(x) \\ \quad =\lambda y(x), \quad x\in[0,b^*], \\ y(0)=y''(0)=0, \quad y(b^*)=y''(b^*)=0. \end{cases} \tag{46}$$

Then, by the change of variables $x=\frac{b^*}{2}(X+1), \quad X\in[-1,1]$, this problem becomes

$$\begin{cases} \frac{16}{b^{*4}}Y^{(4)}(X)-\frac{4}{b^{*2}}L_2(X)Y''(X)-\frac{2}{b^*}S(x)Y'(X)+Q(X)Y(X) \\ \quad =\lambda Y(X), \quad X\in[-1,1], \\ Y(-1)=\frac{b^{*2}}{4}Y''(-1)=0, \quad Y(1)=\frac{b^{*2}}{4}Y''(1)=0, \end{cases} \tag{47}$$

where $L_2(X)=(b^*(X+1)+i)$, $S(X)=2b^*(X+1)$ and $Q(X)=(\frac{b^*}{2}(X+1))^4-2+i(\frac{b^*}{2}(X+1))$. In this case the collocation equation is

$$\frac{16}{b^{*4}}P^{(4)}(x_k)-\frac{4}{b^{*2}}\hat{L_2}(x_k)P''(x_k)-\frac{2}{b^*}\hat{S}(x_k)P'(x_k)+\hat{Q}(x_k)P(x_k)=\lambda P(x_k). \tag{48}$$

The differentiation matrix relation can be written as

$$\left(\frac{16}{b^{*4}}D^{(4)}-\frac{4}{b^{*2}}\hat{L_2}D^{(2)}-\frac{2}{b^*}\hat{S}D^{(1)}+(\hat{Q}I-\lambda I)\right)\mathbf{y}=\mathbf{0}, \tag{49}$$

wher $\hat{L_2}(x_k)=diag(L_2(x_k))$, $\hat{S}(x_k)=diag(S(x_k))$ and $\hat{Q}(x_k)=diag(Q(x_k))$. The boundary conditions are modeled in the following form

$$I_l\mathbf{y}=\frac{4}{b^{*2}}D_l^{(2)}\mathbf{y}=\mathbf{0}, \quad I_r\mathbf{y}=\frac{4}{b^{*2}}D_r^{(2)}\mathbf{y}=\mathbf{0}. \tag{50}$$

Combining Eqs. (49) and (50), we get the following generalized eigenvalue problem

$$\begin{pmatrix} \frac{16}{b^{*4}}D^{(4)} - \frac{4}{b^{*2}}\hat{L_2}D^{(2)} - \frac{2}{b^{*}}\hat{S}D^{(1)} + \hat{Q}I \\ I_l \\ \frac{4}{b^{*2}}D_l^{(2)} \\ I_r \\ \frac{4}{b^{*2}}D_r^{(2)} \end{pmatrix} \mathbf{y} = \lambda \begin{pmatrix} I \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \end{pmatrix} \mathbf{y}. \tag{51}$$

Solving generalized eigenvalue problem (51), gives the eigenvalues of the problem (44). Table 1 lists the first twenty computed eigenvalues of the problem (44) together with the related absolute error and those results reported in Ref. [31]. The exact eigenvalues of problem (44) are given by

$$\lambda_n = (4n+3)(4n+3+i), \quad n = 0, 1, 2, \cdots . \tag{52}$$

From Table 1, one can see that there are variations in of the number of exact calculated eigenvalues based on the number of grid points N and the truncated interval $[0, b^*]$ used in each time. It is observed that, we can calculate the exact values of the first ten and eighteen eigenvalues by using $N = 60, 70$ respectively, over the truncated interval $[0, 10]$, whereas, calculating the exact values of the first twenty eigenvalues requires the use of $N = 115$ and a truncated interval $[0, 20]$ (see column fourth, Table 1). Here, it must be noted that by following this procedure, the approximate value of λ_{11} in the second column (i.e. when $N = 60$ over $[0, 10]$) is $2208.99331791665 + 47.000003747952i$, while the approximate value of λ_{18} with $N = 70$ over $[0, 10]$ and λ_{20} with $N = 115$ over $[0, 20]$ in the third and fourth columns are $5625.01804856698 + 75.0000917969447i$ and $6888.96815509742 + 83.0000085573244i$ respectively. So, the accuracy can be improved by increasing the number of Chebyshev collocation nodes N and/or choosing the appropriate truncated interval $[0, b^*]$. In Table 2, we introduce the results of numerical experiments for high index eigenvalues on this problem and the related absolute error. As documented in Table 2, in order to get the exact values of the high-index eigenvalues, we must increase N and choosing the appropriate truncated interval $[0, b^*]$. Numerical results in Tables 1 and 2 show the high performance of the current technique. Figure 1 displays the eigenvalues computed by truncating $[0, \infty)$ to $[0, 20]$ and using $N = 150$.

Example 2 (Orr-Sommerfeld problem) The Orr-Sommerfeld problem, in hydrodynamic stability, is an eigenvalue problem describing the linear modes of perturbation to a viscous parallel flow. This problem may be written as:

$$\begin{cases} \left(-\frac{d^2}{dx^2}+\alpha^2\right)^2 y(x) + i\alpha Re\, U(x)\left\{\left(-\frac{d^2}{dx^2}+\alpha^2\right)y(x) + U''(x)y(x)\right\} \\ \quad = \alpha Re\,\lambda\left(-\frac{d^2}{dx^2}+\alpha^2\right)y(x), \quad x \in [-1, 1], \\ y(-1) = y'(-1) = 0, \quad y(1) = y'(1) = 0, \end{cases} \tag{53}$$

where $\alpha \in \mathbb{R}$ is a wave number, Re is the Reynolds number, $U(x)$ is a real-valued function representing the undisturbed stream velocity, and the wave speed λ is the spectral parameter [30,40,54–56].

Here, we consider that $U(x) = 1 - x^2$ and $\alpha = 1$ which gives

$$\begin{cases} \left(y^{(4)}(x) - 2y''(x) + y(x)\right) + i\,Re\,\left(1 - x^2\right)\left(-y''(x) + y(x)\right) \\ -i\,Re\,y(x) = Re\,\lambda\left(-y''(x) + y(x)\right), \\ y(-1) = y'(-1) = 0, \quad y(1) = y'(1) = 0. \end{cases} \tag{54}$$

Table 1 The first twenty eigenvalues of the singular fourth-order non-self-adjoint problem (44)

n	λ_n			Abs. Err.	
	$N = 60$, [0, 10]	$N = 70$, [0, 10]	$N = 115$, [0, 20]	Current work	Ref. [31], [0, 30]
0	9 + 3i	9 + 3i	9 + 3i	0	7.9e–06
1	49 + 7i	49 + 7i	49 + 7i	0	
2	121 + 11i	121 + 11i	121 + 11i	0	
3	225 + 15i	225 + 15i	225 + 15i	0	
4	361 + 19i	361 + 19i	361 + 19i	0	
5	529 + 23i	529 + 23i	529 + 23i	0	
6	729 + 27i	729 + 27i	729 + 27i	0	
7	961 + 31i	961 + 31i	961 + 31i	0	
8	1225 + 35i	1225 + 35i	1225 + 35i	0	
9	1521 + 39i	1521 + 39i	1521 + 39i	0	0.03
10	1849 + 43i	1849 + 43i	1849 + 43i	0	
11		2209 + 47i	2209 + 47i	0	
12		2601 + 51i	2601 + 51i	0	
13		3025 + 55i	3025 + 55i	0	
14		3481 + 59i	3481 + 59i	0	
15		3969 + 63i	3969 + 63i	0	
16		4489 + 67i	4489 + 67i	0	
17		5041 + 71i	5041 + 71i	0	
18			5625 + 75i	0	
19			6241 + 79i	0	1.85
CPU time	0.037 seconds	0.041 seconds	0.055 seconds		

Table 2 Computation of some high eigenvalues of the problem in Example 1 over [0, b^*]

n	λ_n	Abs. Err.	N	b^*	n	λ_n	Abs. Err.	N	b^*
0	9 + 3i	0	150	20	170	466489 + 683i	0	500	30
20	6889 + 83i	0	150	20	180	522729 + 723i	0	500	30
30	15129 + 123i	0	150	20	200	644809 + 803i	0	500	30
35	20449 + 143i	0	150	20	210	710649 + 843i	0	500	30
37	22801 + 151i	0	150	20	212	724201 + 851i	0	500	30
	CPU TIME	0.060670 seconds				CPU TIME	0.940698 seconds		
40	26569 + 163i	0	200	20	213	731025 + 855i	0	600	40
50	41209 + 203i	0	200	20	215	744769 + 863i	0	600	40
60	59049 + 243i	0	200	20	218	765625 + 875i	0	600	40
70	80089 + 283i	0	200	20	220	779689 + 883i	0	600	40
71	82369 + 287i	0	200	20	225	815409 + 903i	0	600	40
72	84681 + 291i	0	200	20	229	844561 + 919i	0	600	40
	CPU TIME	0.098919 seconds				CPU TIME	1.681647 seconds		
80	104329 + 323i	0	300	20	230	851929 + 923i	0	700	40
85	117649 + 343i	0	300	20	240	927369 + 963i	0	700	40
89	128881 + 359i	0	300	20	250	1006009 + 1003i	0	700	40
	CPU TIME	0.208779 seconds			260	1087849 + 1043i	0	700	40
90	131769 + 363i	0	400	30	270	1172889 + 1083i	0	700	40
100	162409 + 403i	0	400	30	280	1261129 + 1123i	0	700	40
110	196249 + 443i	0	400	30	290	1352569 + 1163i	0	700	40
120	233289 + 483i	0	400	30	299	1437601 + 1199i	0	700	40
130	273529 + 523i	0	400	30	300	1447209 + 1203i	0	700	40
140	316969 + 563i	0	400	30	301	1456849 + 1207i	0	700	40
150	363609 + 603i	0	400	30	302	1466521 + 1211i	0	700	40
160	413449 + 643i	0	400	30	303	1476225 + 1215i	0	700	40
161	418609 + 647i	0	400	30	304	1485961 + 1219i	0	700	40
	CPU TIME	0.458471 seconds				CPU TIME	2.783854 seconds		

Rewriting (54) into (15) form, leads to

$$\begin{cases} y^{(4)}(x) - L_2(x)y'' + Q(x)y(x) = Re\,\lambda(-y''(x) + y(x)), \\ y(-1) = y'(-1) = 0, \quad y(1) = y'(1) = 0, \end{cases} \tag{55}$$

where $Q(x) = \big(1 + iRe(1 - x^2) - 2iRe\big)$ and $L_2(x) = \big(2 + iRe\,(1 - x^2)\big)$. Now we transform the eigenvalue problem (55) to a generalized eigenvalue problem. To this end, we use the same procedure as we did in Example 1, except the boundary conditions which are modeled by the equations

$$I_l\mathbf{y} = D_l^{(1)}\mathbf{y} = \mathbf{0}, \quad I_r\mathbf{y} = D_r^{(1)}\mathbf{y} = \mathbf{0}. \tag{56}$$

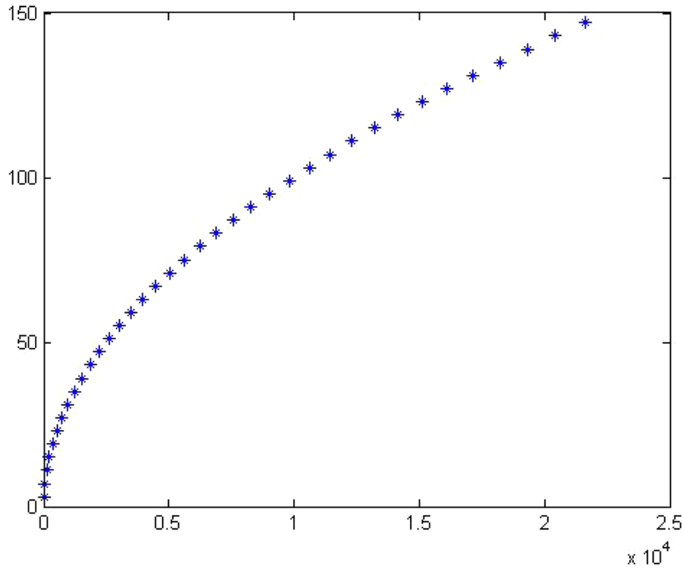

Fig. 1 The eigenvalues of the problem in Example 1 over truncated interval [0, 20] with $N = 150$

Then problem (55) is equivalent to the following generalized eigenvalue problem

$$\begin{pmatrix} D^{(4)} - \hat{L_2} D^{(2)} + \hat{Q} I \\ I_r \\ D_r^{(1)} \\ I_l \\ D_l^{(1)} \end{pmatrix} \mathbf{y} = Re\ \lambda \begin{pmatrix} -D^{(2)} + I \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \end{pmatrix} \mathbf{y}, \tag{57}$$

where $\hat{L_2}(x_k) = diag\,(L_2(x_k))$ and $\hat{Q}(x_k) = diag\,(Q(x_k))$. Table 3 shows the first twenty-five computed eigenvalues of the problem (53). For comparison, the author solved this problem using the method described by Trefethen in [40] and those described by Weideman and Reddy in [55]. As it is shown in the Table 3, there is an excellent agreement between the results of the current work and the results of Refs. [40,55], but from the last row we can see that the CPU time for the proposed method in this work has less cost than those resulting from the use of the program 40 [40] and the MATLAB program was given in Table XVII [55]. The graph of the calculated eigenvalues of the problem (53) are shown in Fig. 2.

Example 3 (Square of the anharmonic oscillator) Our final test problem is the square of the anharmonic oscillator [32], namely the following singular fourth-order non-self-adjoint problem:

$$l^2 y(x) = \lambda y(x), \quad x \in [0, \infty), \tag{58}$$

in which

$$ly(x) = -\frac{d^2 y}{dx^2} + (1 + 3i)x^2 y, \tag{59}$$

with the boundary conditions

$$y(0) = y'(0) = 0. \tag{60}$$

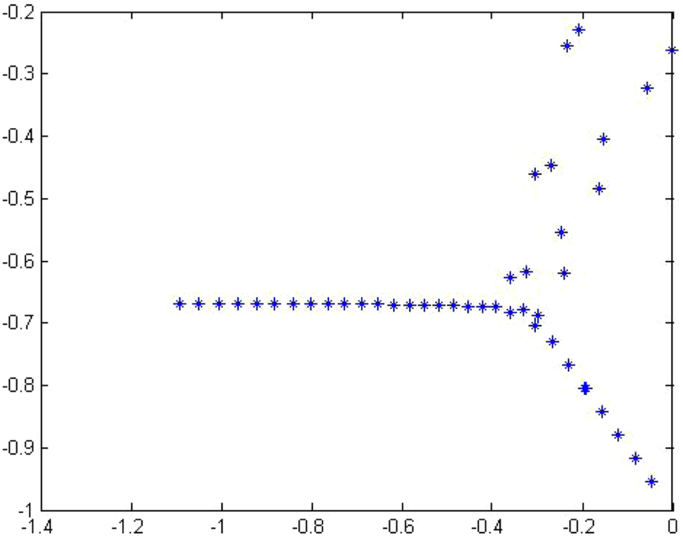

Fig. 2 The eigenvalues of the Orr-Sommerfeld problem with Reynolds number $Re = 5772$, wave number $\alpha = 1$ and $N = 151$

Here we shall use interval truncation and compute the eigenvalues of the following problem over the truncated interval $[0, b^*]$

$$\begin{cases} y^{(4)}(x) - \big(L_1(x) + L_2(x) - L_3(x)\big)y''(x) - \big(S_1(x) + S_2(x)\big)y'(x) + \\ \quad \big(Q_1(x) - 2 - 6i\big)y(x) = \lambda y(x), \;\; x \in [0, b^*], \\ y(0) = y'(0) = 0, \quad y(b^*) = y'(b^*) = 0, \end{cases} \tag{61}$$

where $L_1(x) = x^2$, $L_2(x) = 3ix^2$, $L_3(x) = (1 + 3i)x^2$, $S_1(x) = 4x$, $S_2(x) = 12ix$ and $Q(x) = (1 + 3i)x^2$. As in [32] we shall take $b^* = 30$.
The square of the anharmonic oscillator problem (58), under the change of variable $x = \frac{30}{2}(X + 1)$, becomes

$$\begin{cases} \frac{16}{30^4}Y^{(4)}(X) - \frac{4}{30^2}\big(L_1(X) + L_2(X) - L_3(X)\big)Y''(X) - \frac{2}{30}\big(S_1(X) \\ \quad +S_2(X)\big)Y'(X) + \big(Q_1(X) - 2 - 6i\big)Y(X) = \lambda Y(X), \quad X \in [-1, 1], \\ Y(-1) = \frac{2}{30}Y'(-1) = 0, \quad Y(1) = \frac{2}{30}Y'(1) = 0. \end{cases} \tag{62}$$

Using the same procedure as we did in the previous two examples, leads to the following generalized eigenvalue problem

$$\begin{pmatrix} \frac{16}{30^4}D^{(4)} - \frac{4}{30^2}\big(\hat{L_1} + \hat{L_2} - \hat{L_3}\big)D'' \; - \frac{2}{30}\big(\hat{S_1} + \hat{S_2}\big)D' \\ +\big(\hat{Q} - 2 - 6i\big)I \\ I_l \\ \frac{2}{30}D_l^{(1)} \\ I_r \\ \frac{2}{30}D_r^{(1)} \end{pmatrix} \mathbf{y} = \lambda \begin{pmatrix} I \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \end{pmatrix} \mathbf{y}, \tag{63}$$

where $\hat{L_1}(x_k) = diag(L_1(x_k))$, $\hat{L_2}(x_k) = diag(L_2(x_k))$, $\hat{L_3}(x_k) = diag(L_3(x_k))$, $\hat{S_1}(x_k) = diag(S_1(x_k))$, $\hat{S_2}(x_k) = diag(S_2(x_k))$ and $\hat{Q}(x_k) = diag(Q(x_k))$. By truncating interval $[0, \infty)$ to $[0, 30]$ and by using $N = 164, 192, 224$ and 256, we obtained the results

Table 3 The eigenvalues of the Orr-Sommerfeld eigenvalue problem with $Re = 5772$ by using $N = 151$

n	Current technique λ_n	Trefethen [40] λ_n	Weideman and Reddy [55] λ_n
1	–7.82047736239e–005 - 0.261567659267996i	–7.82025756948e–005 - 0.261567659321331i	–7.82053584006e–005 - 0.261567641825655i
2	–0.208038995689203 - 0.228413137287295i	–0.208039011775745 - 0.228413128698699i	–0.208039018672297 - 0.228413134788114i
3	–0.057671735089710 - 0.322963717466229i	–0.0576717192710218 - 0.322963727301445i	–0.0576717236172579 - 0.322963725427235i
4	–0.233198648445474 - 0.256094614950658i	–0.233198654083072 - 0.256094609698044i	–0.233198653295878 - 0.256094606984705i
5	–0.154832502776164 - 0.404415140103673i	–0.154832484455373 - 0.404415156642178i	–0.154832487458372 - 0.404415120669152i
6	–0.162183650050590 - 0.483740842765814i	–0.162183620190557 - 0.48374081848865i	–0.162183641629637 - 0.483740884983029i
7	–0.270769074066714 - 0.44581956282512i	–0.270769077358348 - 0.445819553328388i	–0.270769127682344 - 0.445819557126184i
8	–0.305993953308319 - 0.461844799659802i	–0.305993999985725 - 0.461844780062401i	–0.305993975410785 - 0.461844789000814i
9	–0.246536337068296 - 0.554941721238439i	–0.24653633905809 - 0.554941588449621i	–0.246536097728476 - 0.554941711477204i
10	–0.240182772320467 - 0.620568909774144i	–0.240182661705616 - 0.620568971649222i	–0.240182952485346 - 0.620568821831565i
11	–0.322595018913137 - 0.617625660887473i	–0.322595535780653 - 0.617625237458286i	–0.32259480108242 - 0.617625079316132i
12	–0.358080629492133 - 0.627198337295907i	–0.358080494265963 - 0.627198401431652i	–0.358081227020699 - 0.627198770158202i
13	–0.297951014190518 - 0.688845290309811i	–0.297950376862889 - 0.688844084872114i	–0.29794979203409 - 0.688846265716206i
14	–0.331467762193808 - 0.679351199666076i	–0.331467390879667 - 0.679352915618439i	–0.331469747642212 - 0.679351236958669i
15	–0.303887852636085 - 0.704538136806874i	–0.303888179846202 - 0.704537995965697i	–0.30388785317918 - 0.704538086194986i
16	–0.358294901210071 - 0.683550178852179i	–0.35829481596820 - 0.683550081764715i	–0.358294335875104 - 0.683549602733568i
17	–0.264545103782564 - 0.730319812153150i	–0.264545084183037 - 0.730320087425455i	–0.264545104398123 - 0.730319684579829i
18	–0.267444420498256 - 0.729352252979878i	–0.267444562155592 - 0.729352392632832i	–0.267444349910305 - 0.729352154215243i
19	–0.392378423336937 - 0.674728612800833i	–0.392378951241087 - 0.674728446051064i	–0.392378072112281 - 0.674728235759373i
20	–0.422236165817305 - 0.674283203798553i	–0.422236335974735 - 0.674283206999839i	–0.422235765319636 - 0.67428360905131i
21	–0.229702256500492 - 0.766236745345284i	–0.229702185062054 - 0.766236694098494i	–0.229702273462431 - 0.766236757697826i
22	–0.230442260156392 - 0.767102768682719i	–0.230442238871865 - 0.767102759041956i	–0.23044230207655 - 0.767102672915488i
23	–0.453513557693502 - 0.672735330167094i	–0.453513408481219 - 0.672735372540022i	0.453513635828121 - 0.67273547855113i
24	–0.193203807344294 - 0.803878653509797i	–0.193203821242124 - 0.80387864212388i	–0.19320381135322 - 0.803878650599485i
25	–0.193585929712041 - 0.804409275751814i	–0.193585940471334 - 0.804409280547825i	–0.193585967667824 - 0.80440929890068i
CPU TIME	0.189652 seconds	0.340769 seconds	7.885000 seconds

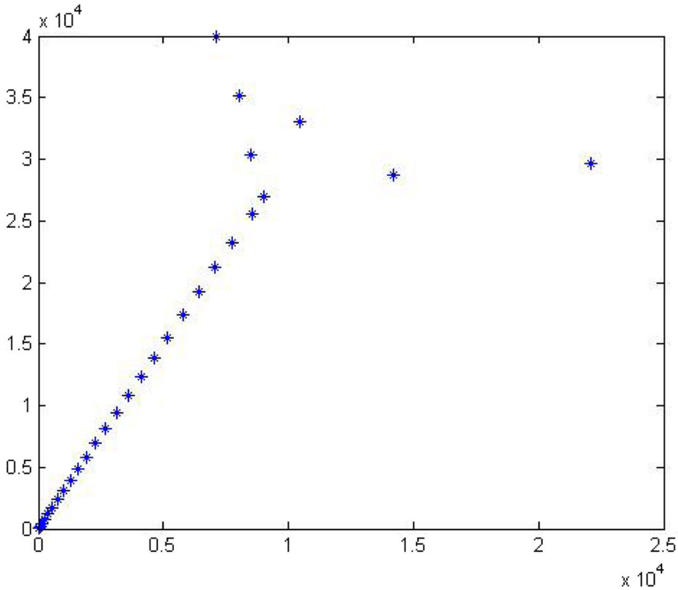

Fig. 3 The eigenvalues of the square of the anharmonic oscillator problem obtained by truncating $[0, \infty)$ to $[0, 30]$ and imposing Dirichlet boundary conditions at both ends

summarized in the second to fifth columns of Table 4. The final column of Table 4 contains some numerical results reported in Ref. [32]. It should be noted here that the results in Table 4 does not include all approximate value of eigenvalues obtained by truncating $[0, \infty)$ to $[0, 30]$ and using $N = 164, 192, 224, 256$, only high-accuracy results are included. For illustration, by using $N = 164$, we get $\lambda_6 = 576.001102491999 + 1728.00174592563i$, when $N = 194$, we get $\lambda_{10} = 1599.99326435074 + 4799.98661567798i$, but when $N = 224$, we get $\lambda_{14} = 3136.00685088521 + 9408.00461319592i$, with $N = 256$, then $\lambda_{17} = 4624.00706180033 + 13871.9956404672i$. In Figure 3 we introduce the results of numerical experiments on this problem attained by truncating $[0, \infty)$ to $[0, 30]$ and $N = 256$ with Dirichlet boundary conditions $y(0) = y'(0) = 0$ and $y(30) = y'(30) = 0$. Numerical results further demonstrate the superior performance of the current technique.

Conclusion

In this paper, an effective technique based on the Chebyshev spectral collocation approach has been applied to numerically approximate the eigenvalues of the fourth-order non-self-adjoint Sturm-Liouville problems. This approach converts the non-self-adjoint problem into a generalized eigenvalue problem based on utilizing the spectral differentiation matrices to compute derivatives of Chebyshev interpolating polynomials at interior nodes. The most salient features of this approach is not only has the ability to solve regular and singular non-self-adjoint problems efficiently, but also has ability to deal with different types of the boundary conditions. Another advantage is the fact that this approach is considerably accurate, computationally convenient and requires less computational costs even when high index eigenvalues are needed.

Table 4 The first sixteen eigenvalues of the square of anharmonic oscillator problem (58) over [0, 30]

n	$N = 164$ λ_n	$N = 192$ λ_n	$N = 224$ λ_n	$N = 256$ λ_n	Numerical results of [32] over [0, 30]
1	16 + 48i	16 + 48i	16 + 48i	16 + 48i	16.0000277 + 47.9997990ii
2	64 + 192i	64 + 192i	64 + 192i	64 + 192i	64.0000148 + 191.999366i
3	144 + 432i	144 + 432i	144 + 432i	144 + 432i	143.999846 + 431.998433i
4	256 + 768i	256 + 768i	256 + 768i	256 + 768i	255.999494 + 767.997295i
5	400 + 1200i	400 + 1200i	400 + 1200i	400 + 1200i	399.998580 + 1199.99512i
6		576 + 1728i	576 + 1728i	576 + 1728i	576.004740 + 1727.99105i
7		784 + 2352i	784 + 2352i	784 + 2352i	783.978537 + 2351.99079i
8		1024 + 3072i	1024 + 3072i	1024 + 3072i	1024.07578 + 3071.95895i
9		1296 + 3888i	1296 + 3888i	1296 + 3888i	1295.59 84 + 3887.88651i
10			1600 + 4800i	1600 + 4800i	1601.69754 + 4800.77556i
11			1936 + 5808i	1936 + 5808i	1929.22345 + 5803.26324i
12			2304 + 6912i	2304 + 6912i	2334.07204 + 6937.35200i
13			2704 + 8112i	2704 + 8112i	2603.25320 + 8007.24113i
14				3136 + 9408i	–
15				3600 + 10800i	3718.33333 + 9 428.47681i
16				4096 + 12288i	–
CPU time	0.084630 sec.	0.096242 sec.	0.119976 sec.	0.219058 sec.	

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Declarations

Conflict of interest Conflicts of interest The author declare that he have no conflicts of interest.

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