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ORIGINAL PAPER

# On Oscillatory and Asymptotic Behavior of Higher Order Neutral Differential Equations with Impulsive Conditions

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**Abstract**
In the present article, we considered a class of $n$th order impulsive neutral differential equations. The study on the oscillatory and asymptotic behavior of solutions for the higher-order neutral differential equation is theoretical and practical. Various techniques appeared for these studies. We reduced this class into a class of non-impulsive neutral differential equations by using suitable substitutions. Through a comparison strategy involving first-order differential equations, we studied the oscillatory and asymptotic behavior of solutions. Sufficient conditions are obtained for asymptotic as well as oscillatory bounded solutions. Several examples have illustrated the effectiveness of the requirements.

**Keywords** Neutral differential equations · Higher order · Impulsive conditions · Oscillation criteria · Asymptotic behavior

**Mathematics Subject Classification** 34K11 · 34K40 · 34G10 · 34G45 · 35R12

## Introduction

The differential equations having the higher-order derivatives with and without delay are called neutral differential equations. Higher-order neutral differential equations are used to model many mathematical phenomena in natural science and technology. Initially, the existence and uniqueness of solutions for different types of neutral equations have been studied. In recent years, extensive considerations have been given to their oscillatory nature by many researchers [1,3,5,11,12,17,18,22,23,28,33,34,36] and in the last few decades the characteristics of such neutral equations with even/odd order have been studied [18,22,37]. The asymptotic and oscillation properties of higher-order neutral equations with some relaxed

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conditions on coefficients are investigated in [1,3]. Yildiz et al. [33,34] have considered neutral type nonlinear higher-order functional differential equations with oscillating coefficients. Basic definitions and results on oscillation for neutral type differential equations are given in [2].

It is a well-known fact that the motions on the earth are not always uniform as various kinds of resistance appear during the motions. If high-intensity forces act for a short duration, then the motions caused by these forces are called impulsive motions. In mathematical models, these types of motions are described by impulsive differential equations. The differential equations with impulsive effect can be used to simulate those discontinuous processes in which impulses occur. So, it becomes an important tool to handle the natural function of mathematical models and phenomena such as in optimal control, electric circuit, biotechnology, population dynamics, fractals, neural network, viscoelasticity, and chemical technology. One of the main advantages of the impulses can be seen in the paper of Sugie and Ishihara [26]. They provided the model in which the mass point might oscillate with impulsive effect; however, the mass point didn't oscillate without an impulsive effect. For more work on impulsive impact, refer to the article of Feng et al. [6] as well as Raheem and Maqbul [24].

In 1989, some researchers started to investigate the oscillatory nature of differential equations with impulses and were at the initial stage of its development. Later on, authors in papers [6,8–10,18] have extended the study of oscillation to parabolic and hyperbolic impulsive partial differential equations. The oscillatory and asymptotic nature of the solutions for a higher-order delay differential equation with impulses were examined by some researchers using comparison results with associated delay differential equations without impulses[7,19,37]. We have often seen that even non-impulsive neutral delay differential equations may have solutions of oscillatory nature due to some additional controls.

In literature, we noticed Riccati techniques are widely used to obtain Kamenev, and Philos-type oscillation criteria [6,27,31]. Oscillation theory extended to the first-order impulsive differential equations with variable delays in [11]. Several results for third-order delay differential equations were discussed by Tiryaki and Aktas [29]. For the oscillation results on second and fourth-order dynamical systems, refer to the paper [12,38]. Oscillatory and non-oscillatory solutions play a significant role in many applied problems in natural sciences and engineering. The research on oscillation and asymptotic behavior of impulsive differential equation is emerging as an important area of study and is developing rapidly [4,20,25,30,35].

After considering the above formulations, we study the oscillation and asymptotic behavior of solutions for higher-order neutral differential equations with impulsive conditions. We converted the impulsive differential equations into non-impulsive differential equations by using suitable substitutions. Moreover, we reduced the $n$th order neutral differential equation into the first-order equation using generalized Riccati transformation. It allows using the comparison theorems to establish the oscillation results. The obtained conditions are sufficient for asymptotic as well as oscillatory bounded solutions. Philos-type oscillation criteria are proved for taking $n$ as an even integer.

Necessary lemmas and fundamental assumptions are provided in "Preliminaries and Assumptions" section. Main results are obtained in "Main Results" section for the problem (1) by using generalized Riccati transformations and comparison theorems. And in "Frequency-Amplitude Formulation" section, the applicability of the main results is demonstrated by several examples.

Here, we established the oscillation results for the following model of impulsive neutral differential equation of order $n$:

$$\begin{cases} v^{(n)}(x) + k(x)u(\eta_2(x)) = 0, \quad x \neq x_p, \\ u^{(r)}(x_p) - u^{(r)}(x_p^-) = d_p u^{(r)}(x_p^-), \quad r = 0, 1, 2, \ldots, n-1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 1, 2, 3, \ldots, \end{cases} \tag{1}$$

where $v(x) = u(x) + \alpha u(\eta_1(x))$, $\eta_1(x) \leq x$, $\eta_2(x) \leq x$, $x > x_0$, $\alpha > 0$, $d_p > 0$, $v^{(r)}(x)$ denote the $r$th ($r \geq 1$) order derivatives.

## Preliminaries and Assumptions

Throughout the paper, we consider the following assumptions:

(C) $\eta_r : (x_0, \infty) \to \mathbb{R}$, $r = 1, 2$ are continuous functions with the following conditions:

(i) $\eta_r(x) \leq x$, $\eta_1(\eta_2(x)) = \eta_2(\eta_1(x))$
(ii) $\eta_r'(x) = 1$ and $\eta_r''(x) = 0$
(iii) $\lim_{x\to\infty} \eta_r(x) = \infty$.

**Lemma 1** $U(x) = \prod_{x_0<x_p\leq x} (1 + d_p)^{-1} u(x)$ *satisfies*

$$V^{(n)}(x) + k(x)£(x)U(\eta_2(x)) = 0, \tag{2}$$

*where*

$$V(x) = U(x) + \alpha \prod_{\eta_1(x)<x_p\leq x} (1 + d_p)^{-1} U(\eta_1(x)) \tag{3}$$

*and*

$$£(x) = \prod_{\eta_2(x)<x_p\leq x} (1 + d_p)^{-1}$$

*if and only if* $u(x)$ *satisfies* (1) *on the interval* $(x_0, \infty)$ .

***Proof*** Let $U(x) = \prod_{x_0<x_p\leq x} (1+d_p)^{-1} u(x)$ satisfies (2). Then we will show that $u(x)$ satisfies (1) on the interval $(x_0, \infty)$.

Obviously,

$$u(x) = \prod_{x_0<x_p\leq x} (1 + d_p)U(x) \text{ and } v(x) = \prod_{x_0<x_p\leq x} (1 + d_p)V(x).$$

Thus

$$v^{(n)}(x) = \prod_{x_0<x_p\leq x} (1 + d_p)V^{(n)}(x).$$

Using (2), we get

$$v^{(n)}(x) = \prod_{x_0<x_p\leq x} (1 + d_p) \left[ -k(x) \prod_{\eta_2(x)<x_p\leq x} (1 + d_p)^{-1} U(\eta_2(x)) \right]$$

$$= -k(x) \prod_{x_0 < x_p \le \eta_2(x)} (1 + d_p) U(\eta_2(x)).$$

Therefore, for $x \neq x_p$, we have

$$v^{(n)}(x) + k(x) u(\eta_2(x)) = 0.$$

Also, we obtain

$$u^{(r)}(x) = \prod_{x_0 < x_p \le x} (1 + d_p) U^{(r)}(x), \quad r = 0, 1, 2, \ldots, n - 1,$$

which implies that

$$u^{(r)}(x_p) = (1 + d_p) u^{(r)}(x_p^-), \quad r = 0, 1, 2, \ldots, n - 1.$$

This shows that $u(x)$ satisfied (1).

Conversely, we assume $u(x) = \prod_{x_0 < x_p \le x} (1 + d_p) U(x)$ satisfies (1). Then we will show that $U(x)$ satisfies (2) on $(x_0, \infty)$.

As $V(x) = \prod\limits_{x_0 < x_p \le x} (1 + d_p)^{-1} v(x)$, we have

$$V^{(n)}(x) = \prod_{x_0 < x_p \le x} (1 + d_p)^{-1} v^{(n)}(x).$$

Using (1), we obtain

$$\begin{aligned} V^{(n)}(x) &= -k(x) \prod_{x_0 < x_p \le x} (1 + d_p)^{-1} u(\eta_2(x)) \\ &= -k(x) \prod_{\eta_2(x) < x_p \le x} (1 + d_p)^{-1} U(\eta_2(x)) \\ &= -k(x) \pounds(x) U(\eta_2(x)). \end{aligned}$$

Now, we can easily show that $U^{(r)}(x_p^-) = U^{(r)}(x_p)$. This shows that $U(x)$ satisfied (2). □

**Lemma 2** [19] *A non zero solution $u(x)$ of (1) is oscillatory on $(x_0, \infty)$ if and only if the corresponding solution $U(x) = \prod_{x_0 < x_p \le x} (1 + d_p)^{-1} u(x)$ of (2) is oscillatory on $(x_0, \infty)$. Moreover,* $\lim x \to \infty u(x) = 0$ *if and only if* $\lim\limits_{x \to \infty} U(x) = 0$.

**Lemma 3** [5] *Let the nth order derivative of $V(x)$ has a constant sign and not identically zero on a subinterval of $[x_0, \infty)$. If $V(x)$ and its derivatives up-to order $n - 1$ are of constant sign in $[x_0, \infty)$, then there exists an integer $q > 0$ and $\tau \ge x_0$ such that $0 \le q \le n - 1$, and $(-1)^{n+q} V(x) V^{(n)}(x) > 0$,*

$$V(x) V^{(r)}(x) > 0 \quad \text{for} \quad r = 0, 1, 2, \ldots, q - 1 \quad \text{when} \quad q \ge 1$$

*and*

$$(-1)^{q+r} V(x) V^{(r)}(x) > 0 \quad \text{for} \quad r = q, q + 1, \ldots, n - 1 \quad \text{when} \quad q \le n - 1$$

*on $[\tau, \infty)$.*

**Lemma 4** [18] *Let $V$ be a function defined in Lemma 3. If $\lim_{x \to \infty} V(x) \neq 0$, then for every $\mu \in (0, 1)$, there exists $x_\mu \in [\tau, \infty)$ such that*

$$|V(x)| \ge \frac{\mu}{(n - 1)!} x^{n-1} |V^{(n-1)}(x)|$$

*on* $[\tau_\mu, \infty)$.

**Lemma 5** [18] *Let V be a function defined in Lemma 3. If* $V^{(n-1)}(x)V^n(x) \leq 0$, *then for any constant* $\nu \in (0, 1)$ *and sufficiently large* $x$, *there exists a constant* $M > 0$ *satisfying*

$$|V^{(1)}(\nu x)| \geq Mx^{n-2}|V^{(n-1)}(x)|.$$

**Lemma 6** [18] *If U is a positive solution of* (2), *then corresponding function*

$$V(x) = U(x) + \alpha \prod_{\eta_1(x)<x_p\leq x} (1 + d_p)^{-1} U(\eta_1(x)),$$

*satisfies* $V(x) > 0$, $V^{(n-1)}(x) > 0$, $V^{(n)}(x) < 0$ *eventually.*

## Main Results

**Theorem 7** *If first order neutral differential inequality:*

$$[Y(x) + \alpha Y(\eta_1(x))]' + \frac{\mu}{(n-1)!} J(x)£(x)\eta_2^{n-1}(x)Y(\eta_2(x)) \leq 0,$$

*where*

$$J(x) = \min\{k(x), k(\eta_1(x))\}$$

*has no eventually positive solution then every non zero solution of* (1) *is oscillatory.*

***Proof*** Let on contrary $U(x)$ be an eventually positive solution of (2). From (2), we have

$$V^{(n)}(x) + \alpha V^{(n)}(\eta_1(x)) + J(x)£(x) \Bigg[ U(\eta_2(x)) + \alpha \prod_{\eta_1(\eta_2(x))<x_p\leq\eta_2(x)} (1 + d_p)^{-1}$$
$$U(\eta_2(\eta_1(x)))\Bigg] \leq 0.$$

Using (3), we get

$$V^{(n)}(x) + \alpha V^{(n)}(\eta_1(x)) + J(x)£(x)V(\eta_2(x)) \leq 0. \tag{4}$$

From Lemma 4, we have

$$V(\eta_2(x)) \geq \frac{\mu}{(n-1)!}\eta_2^{n-1}(x)V^{(n-1)}(\eta_2(x)).$$

Using above inequality, we get

$$V^{(n)}(x) + \alpha V^{(n)}(\eta_1(x)) + \frac{\mu}{(n-1)!} J(x)£(x)\eta_2^{n-1}(x)V^{(n-1)}(\eta_2(x)) \leq 0. \tag{5}$$

If we assume $Y(x) = V^{(n-1)}(x)$, then first order neutral differential inequality

$$[Y(x) + \alpha Y(\eta_1(x))]' + \frac{\mu}{(n-1)!} J(x)£(x)\eta_2^{n-1}(x)Y(\eta_2(x)) \leq 0,$$

has an eventually positive solution which is a contradiction to the condition of theorem. Applying Lemma 2, result follows. □

**Corollary 8** *Let $n$ be an even integer and there exists a constant $K > 0$ such that*

$$\prod_{\eta_1(x)<x_p\le x}(1+d_p)^{-1} \le K.$$

*If first order differential inequality:*

$$Y'(x) + \frac{\mu(1-\alpha K)\eta_1^{n-1}(x)k(x)\pounds(x)}{(n-1)!}Y(x) \le 0,$$

*has no eventually positive solution, then every bounded solution of (1) is oscillatory.*

***Proof*** Let $U(x)$ be a bounded and non-oscillatory solution of (2). We may assume that $U(x)$ is eventually positive. Since $U(x) > 0$ is bounded, $V(x)$ is also bounded and $V(x) > 0$ eventually. As $n$ is even and $V(x)$ is bounded, by using Lemma 3, we have $q = 1$ i.e.

$$(-1)^{(1+r)}V(x)V^{(r)}(x) > 0, \quad r = 1, 2, \ldots, n-1.$$

In particular $V'(x) > 0$. From (3), we have

$$U(x) \ge (1-\alpha K)V(x).$$

Using above inequality in (2), we get

$$V^{(n)}(x) + (1-\alpha K)k(x)\pounds(x)V(\eta_2(x)) \le 0. \tag{6}$$

Using Lemma 4, we get

$$V(\eta_1(x)) \ge \frac{\mu}{(n-1)!}\eta_1^{n-1}(x)V^{(n-1)}(\eta_1(x)).$$

As $V^{(n-1)}(x)$ is decreasing, we have

$$V(\eta_1(x)) \ge \frac{\mu}{(n-1)!}\eta_1^{n-1}(x)V^{(n-1)}(x).$$

Using above inequality in (6), we get

$$V^{(n)}(x) + \frac{\mu(1-\alpha K)\eta_1^{n-1}(x)k(x)\pounds(x)}{(n-1)!}V^{(n-1)}(x) \le 0.$$

If we assume $Y(x) = V^{(n-1)}(x)$, then first order differential inequality

$$Y'(x) + \frac{\mu(1-\alpha K)\eta_1^{n-1}(x)k(x)\pounds}{(n-1)!}Y(x) \le 0,$$

has an eventually positive solution which is a contradiction to the condition of theorem. Applying Lemma 2, result follows. □

**Theorem 9** *Let $n$ be an even integer and $\frac{\eta_1(x)}{2} \le \eta_2(x)$. We assume that there exist real valued continuously differentiable functions $\Psi(x, y)$, $\phi(x, y)$ with domain $D_1 = \{(x, y) | x \ge y \ge x_0 > 0\}$, and continuously differentiable function $\rho$ with the domain $[x_0, \infty)$ satisfying the following conditions:*

(A1) $\Psi(x, x) = 0$ *for* $x \ge x_0$ *and* $\Psi(x, y) > 0$ *for* $x > y \ge x_0$;
(A2) $\frac{\partial}{\partial x}\Psi(x, y) \ge 0$, $\frac{\partial}{\partial y}\Psi(x, y) \le 0$;
(A3) $\frac{\partial \Psi(x,y)}{\partial y} + \Psi(x, y)\frac{\rho'(y)}{\rho(y)} = \phi(x, y), \quad (x, y) \in D_1.$

*Further, assume that*

$$\limsup_{x\to\infty} \frac{1}{\Psi(x,x_0)} \int_T^x \left[ \Psi(x,y)\rho(y)J(y)£(y) - \frac{(1+\alpha)\phi^2(x,y)\rho(y)}{2M\Psi(x,y)\eta_1^{n-2}(y)} \right] dy = \infty. \quad (7)$$

*Then every bounded solution of* (2) *is oscillatory.*

***Proof*** Let $U(x)$ be a bounded and non-oscillatory solution of (2). We may assume that $U(x)$ is an eventually positive. Since $U(x) > 0$ is bounded, $V(x)$ is also bounded and $V(x) > 0$ eventually. As $n$ is even and $V(x)$ is bounded, by using Lemma 3, we have $q = 1$ i.e.

$$(-1)^{(1+r)} V(x) V^{(r)}(x) > 0, \quad r = 1, 2, \ldots, n-1.$$

In particular, $V'(x) > 0$.

From (2), we have

$$V^{(n)}(x) + \alpha V^{(n)}(\eta_1(x)) + J(x)£(x) \Bigg[ U(\eta_2(x)) + \alpha \prod_{\eta_1(\eta_2(x))<x_p\le\eta_2(x)} (1+d_p)^{-1} U(\eta_2(\eta_1(x))) \Bigg] \le 0.$$

Using (3), we get

$$V^{(n)}(x) + \alpha V^{(n)}(\eta_1(x)) + J(x)£(x)V(\eta_2(x)) \le 0. \quad (8)$$

Define

$$\chi_1(x) = \rho(x) \frac{V^{(n-1)}(x)}{V\left(\frac{\eta_1(x)}{2}\right)}. \quad (9)$$

Differentiating with respect to $x$, we get

$$\chi_1^{(1)}(x) = \rho'(x) \frac{V^{(n-1)}(x)}{V\left(\frac{\eta_1(x)}{2}\right)} + \rho(x) \left[ \frac{V^{(n)}(x)}{V\left(\frac{\eta_1(x)}{2}\right)} - \frac{V^{(n-1)}(x) V^{(1)}\left(\frac{\eta_1(x)}{2}\right)}{2V^2\left(\frac{\eta_1(x)}{2}\right)} \right]. \quad (10)$$

Using Lemma 5, we obtain

$$\begin{aligned} V^{(1)}\left(\frac{\eta_1(x)}{2}\right) &\ge M\eta_1^{n-2}(x) V^{(n-1)}(\eta_1(x)) \\ &\ge M\eta_1^{n-2}(x) V^{(n-1)}(x). \end{aligned} \quad (11)$$

Using above inequality in (10), we get

$$\chi_1^{(1)}(x) \le \rho'(x) \frac{V^{(n-1)}(x)}{V\left(\frac{\eta_1(x)}{2}\right)} + \rho(x) \frac{V^{(n)}(x)}{V\left(\frac{\eta_1(x)}{2}\right)} - \frac{M\eta_1^{n-2}(x)}{2\rho(x)} \left[ \rho(x) \frac{V^{(n-1)}(x)}{V\left(\frac{\eta_1(x)}{2}\right)} \right]^2.$$

Using (9), we get

$$\chi_1^{(1)}(x) \le \frac{\rho'(x)}{\rho(x)} \chi_1(x) + \rho(x) \frac{V^{(n)}(x)}{V\left(\frac{\eta_1(x)}{2}\right)} - \frac{M\eta_1^{n-2}(x)}{2\rho(x)} \chi_1^2(x). \quad (12)$$

Define another function

$$\chi_2(x) = \rho(x)\frac{V^{(n-1)}(\eta_1(x))}{V\left(\frac{\eta_1(x)}{2}\right)}. \tag{13}$$

Differentiating with respect to $x$, we get

$$\chi_2^{(1)}(x) = \rho'(x)\frac{V^{(n-1)}(\eta_1(x))}{V\left(\frac{\eta_1(x)}{2}\right)} + \rho(x)\left[\frac{V^{(n)}(\eta_1(x))}{V\left(\frac{\eta_1(x)}{2}\right)} - \frac{V^{(n-1)}(\eta_1(x))V^{(1)}\left(\frac{\eta_1(x)}{2}\right)}{2V^2\left(\frac{\eta_1(x)}{2}\right)}\right].$$

Using (11) and (13), we get

$$\chi_2^{(1)}(x) \le \frac{\rho'(x)}{\rho(x)}\chi_2(x) + \rho(x)\frac{V^{(n)}(\eta_1(x))}{V\left(\frac{\eta_1(x)}{2}\right)} - \frac{M\eta_1^{n-2}(x)}{2\rho(x)}\chi_2^2(x). \tag{14}$$

Using (12) and (14), we get

$$\begin{aligned}\chi_1^{(1)}(x) + \alpha\chi_2^{(1)}(x) \le \rho(x)\frac{V^{(n)}(x) + \alpha V^{(n)}(\eta_1(x))}{V\left(\frac{\eta_1(x)}{2}\right)} + \frac{\rho^{(1)}(x)}{\rho(x)}\chi_1(x) \\ -\frac{M\eta_1^{n-2}(x)}{2\rho(x)}\chi_1^2(x) + \alpha\frac{\rho'(x)}{\rho(x)}\chi_2(x) - \alpha\frac{M\eta_1^{n-2}(x)}{2\rho(x)}\chi_2^2(x).\end{aligned}$$

Using (8), we get

$$\begin{aligned}\chi_1^{(1)}(x) + \alpha\chi_2^{(1)}(x) \le -\rho(x)J(x)£(x) + \frac{\rho'(x)}{\rho(x)}\chi_1(x) - \frac{M\eta_1^{n-2}(x)}{2\rho(x)}\chi_1^2(x) \\ +\alpha\frac{\rho^{(1)}(x)}{\rho(x)}\chi_2(x) - \alpha\frac{M\eta_1^{n-2}(x)}{2\rho(x)}\chi_2^2(x).\end{aligned}$$

Multiplying by $\Psi(x, y)$ and integrating from $T$ to $x$, we get

$$\begin{aligned}&\int_T^x \Psi(x, y)\rho(y)J(y)£(y)dy \\ &\le -\int_T^x \Psi(x, y)\chi_1^{(1)}(y)dy - \alpha\int_T^x \Psi(x, y)\chi_2^{(1)}(y)dy \\ &\quad + \int_T^x \frac{\rho^{(1)}(y)}{\rho(y)}\Psi(x, y)\chi_1(y)dy - \frac{M}{2}\int_T^x \frac{\eta_1^{n-2}(y)}{\rho(y)}\Psi(x, y)\chi_1^2(y)dy \\ &\quad +\alpha\int_T^x \frac{\rho^{(1)}(y)}{\rho(y)}\Psi(x, y)\chi_2(y)dy - \alpha\frac{M}{2}\int_T^x \frac{\eta_1^{n-2}(y)}{\rho(y)}\Psi(x, y)\chi_2^2(y)dy \\ &= \Psi(x, T)\chi_1(T) + \int_T^x \left[\phi(x, y)\chi_1(y) - \frac{M\eta_1^{n-2}(y)}{2\rho(y)}\Psi(x, y)\chi_1^2(s)\right]ds \\ &\quad +\alpha\Psi(x, T)\chi_2(T) + \alpha\int_T^x \left[\phi(x, y)\chi_2(y) - \frac{M\eta_1^{n-2}(y)}{2\rho(y)}\Psi(x, y)\chi_2^2(y)\right]dy.\end{aligned}$$

Using inequality $P\psi - Q\psi^2 \le \frac{P^2}{4Q}$, we get

$$\int_T^x \Psi(x, y)\rho(s)J(y)£(y)dy$$

$$\leq \Psi(x,T)\chi_1(T) + \alpha\Psi(x,T)\chi_2(T) + \int_T^x \frac{(1+\alpha)\phi^2(x,y)\rho(y)}{2M\Psi(x,y)\eta_1^{n-2}(y)}dy,$$

$\Rightarrow$

$$\begin{aligned}
&\int_T^x \left[\Psi(x,y)\rho(y)J(y)\pounds(y) - \frac{(1+\alpha)\phi^2(x,y)\rho(y)}{2M\Psi(x,y)\eta_1^{n-2}(y)}\right]dy \\
&\leq \Psi(x,T)\chi_1(T) + \alpha\Psi(x,T)\chi_2(T) \\
&\leq \Psi(x,x_0)\chi_1(T) + \alpha\Psi(x,x_0)\chi_2(T),
\end{aligned}$$

which gives

$$\frac{1}{\Psi(x,x_0)}\int_T^x \left[\Psi(x,y)\rho(y)J(y)\pounds(y) - \frac{(1+\alpha)\phi^2(x,y)\rho(y)}{2M\Psi(x,y)\eta_1^{n-2}(y)}\right]dy < \infty,$$

which contradicts the condition (7). Applying Lemma 2, result follows. □

**Theorem 10** *Let $n$ be an odd integer and $u(x)$ be an eventually positive bounded solution of (1). Further, if there exists a constant $K_1 > 0$ such that*

$$\max\left\{\prod_{\eta_1(x)<x_p\leq x}(1+d_p)^{-1}, \prod_{\eta_2(x)<x_p\leq x}(1+d_p)^{-1}\right\} \leq K_1$$

*and*

$$\int_{x_0}^{\infty} k(x)dx = \infty, \tag{15}$$

*then* $\lim_{x\to\infty} u(x) = 0$.

***Proof*** Since $u(x)$ is eventually positive, consequently $U(x)$ and $V(x)$ are also eventually positive, therefore there exists $\varrho \geq x_0$ such that $V(x) > 0$, for $x \geq \varrho$. Let $\lim_{x\to\infty} V(x) = L$. Then $L \geq 0$. Claim $L = 0$, otherwise $L > 0$. As $n$ is an odd integer and $V(x)$ is bounded, by using Lemma 3, we have $q = 0$, (otherwise $V(x)$ is unbounded) i.e.

$$(-1)^r V(x)V^{(r)}(x) > 0, \quad \text{for} \quad r = 0, 1, 2, \ldots n-1.$$

In particular, $V'(x) < 0$ for $x \geq \varrho$. Since $V(x)$ is decreasing for $x \geq \varrho$, we have $L + \epsilon > V(x) > L$ for all $\epsilon > 0$. Choose $0 < \epsilon < \frac{L(1-\alpha K_1)}{\alpha K_1}$. It is easy to see that

$$\begin{aligned}
U(x) &= V(x) - \alpha\prod_{\eta_1(x)<x_p\leq x}(1+d_p)^{-1}U(\eta_1(x)) \\
&\geq L - \alpha\prod_{\eta_1(x)<x_p\leq x}(1+d_p)^{-1}V(\eta_1(x)) \\
&\geq L - \alpha\prod_{\eta_1(x)<x_p\leq x}(1+d_p)^{-1}(L+\epsilon) \\
&\geq L - \alpha K_1(L+\epsilon) = P(L+\epsilon) > PV(x),
\end{aligned} \tag{16}$$

where $P = \frac{L-\alpha K(L+\epsilon)}{L+\epsilon} > 0$.

Therefore, from (2), we have

$$V^{(n)}(t) = -k(x)\pounds(x)U(\eta_2(x))$$

$$\leq -Pk(x)\pounds(x)V(\eta_2(x))$$
$$\leq -PK_1k(x)L. \tag{17}$$

Integrating from $\varrho$ to $x$, we get

$$V^{(n-1)}(x) - V^{(n-1)}(\varrho) \leq -PK_1L\int_{\varrho}^{x} k(y)dy,$$

$\Rightarrow$

$$\int_{\varrho}^{x} k(y)dy \leq \frac{V^{(n-1)}(\varrho)}{PK_1L}.$$

Taking limit as $x \to \infty$, (15) contradicted. Hence $\lim_{x\to\infty} V(x) = 0$. Since $U(x) \leq V(x)$, we have

$$\lim_{x\to\infty} U(x) = 0.$$

Therefore, by applying Lemma 2, we get

$$\lim_{x\to\infty} u(x) = 0.$$

This completes the proof. □

**Corollary 11** *Condition (15) of Theorem 10 can be replaced by the following condition*

$$\int_{\varrho}^{\infty} y^{n-1}k(y)dy = \infty. \tag{18}$$

***Proof*** Multiplying (17) by $y^{n-1}$ and integrating from $\varrho$ to $x$, we get

$$\int_{\varrho}^{x} y^{n-1}V^{(n)}(y)dy \leq -PK_1L\int_{\varrho}^{x} y^{n-1}k(y)dy,$$

$\Rightarrow$

$$G(t) - G(\varrho) \leq -PK_1L\int_{\varrho}^{x} y^{n-1}k(y)dy, \tag{19}$$

where

$$G(x) = x^{n-1}V^{(n-1)}(x) - (n-1)x^{n-2}V^{(n-2)}(x) + (n-1)(n-2)x^{n-3}V^{(n-3)}(x)$$
$$- \cdots - (n-1)(n-2)\ldots 3.2.xV'(x) + (n-1)(n-2)(n-3)\ldots 3.2V(x).$$

Since $(-1)^rV^{(r)}(x) > 0$, $r = 0, 1, 2, \ldots, (n-1)$ for $x \geq \varrho$, $G(x) > 0$ for $x \geq \varrho$. From (19), we have

$$-G(\varrho) \leq -PK_1L\int_{\varrho}^{x} y^{n-1}k(y)dy.$$

Taking limit as $x \to \infty$, we get

$$\int_{\varrho}^{\infty} y^{n-1}k(y)dy \leq \frac{G(\varrho)}{PK_1L},$$

which contradicts (18). Thus the proof is completed. □

In the next section, following the idea used in papers [13–16], we elucidate the frequency-amplitude relationship.

## Frequency-Amplitude Formulation

According to the Lemma 1, problem (1) is equivalent to the following problem:

$$V^{(n)}(x) + k(x)\pounds(x)U(\eta_2(x)) = 0, \tag{20}$$

where $V$, £ and $U$ are defined in Lemma 1.
Equation (20) is of the following form:

$$V^{(n)} + f(U) = 0, \tag{21}$$

where $f(U) = k(x)\pounds(x)U(\eta_2(x))$.

We assume that all the conditions of Theorem 7 hold. Therefore every non zero solution of (1) is oscillatory. To find frequency- amplitude relationship, we consider the following conditions:

$$V(0) = A,\ V'(0) = V''(0) = \cdots = V^{(n-1)}(0) = 0,$$

where $A$ is its initial amplitude.

According to He's frequency-amplitude formulation, we use the following trial functions:

$$V_1(x) = A\cos w_1 x,\ \ V_2(x) = A\cos w_2 x, \ldots, V_n(x) = A\cos w_n x,$$

where $w_1$, $w_2$, ..., $w_n$ are trial frequencies.

If $n$ is even i.e. $n = 2m$, $m = 1, 2, 3, \ldots,$ then the residuals are given by:

$$\begin{aligned} R_1(x) &= f(U_1) - (-1)^{m-1} w_1^n \cos w_1 x \\ R_2(x) &= f(U_2) - (-1)^{m-1} w_2^n \cos w_2 x \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R_n(x) &= f(U_n) - (-1)^{m-1} w_n^n \cos w_n x. \end{aligned}$$

If $n$ is odd i.e.$n = 2m - 1$, $m = 1, 2, 3, \ldots,$ then the residuals are given by:

$$\begin{aligned} R_1(x) &= f(U_1) - (-1)^{m-1} w_1^n \sin w_1 x \\ R_2(x) &= f(U_2) - (-1)^{m-1} w_2^n \sin w_2 x \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R_n(x) &= f(U_n) - (-1)^{m-1} w_n^n \sin w_n x. \end{aligned}$$

According to classic procedure of He's frequency-amplitude, the approximate frequency of non linear oscillator (21) can be calculated by using above residuals.

## Application

***Example 12*** Consider the following system

$$\begin{cases} [u(x) + \alpha u(x - 2\pi)]^{(6)} + \frac{3}{x} u(x - \pi) = 0, & x \neq x_p, \\ u^{(r)}(x_p) - u^{(r)}(x_p^-) = \frac{1}{p} u^{(r)}(x_p^-), & r = 0, 1, 2, 3, 4, 5, \\ & p = 1, 2, 3, \ldots. \end{cases} \tag{22}$$

Here $n = 6$, $\eta_1(x) = x - 2\pi$, $\eta_2(x) = x - \pi$, $x > x_0$, $\alpha > 0$, $k(x) = \frac{3}{x}$, $J(x) = \frac{3}{x}$, $d_p = \frac{1}{p}$, $x_p = p\pi$.

Let $\Psi(x, y) = (x - y)^2$, $\rho(y) = y$.
Clearly, $\Psi(x, x) = (x - x)^2 = 0$ for $x \geq x_0$ and $\Psi(x, y) = (x - y)^2 > 0$ for $x > y \geq x_0$.
Therefore (A1) holds.
$\frac{\partial}{\partial x}\Psi(x, y) = 2(x - y) \geq 0$, $\frac{\partial}{\partial y}\Psi(x, y) = -2(x - y) \leq 0$, for $x > y$.
Therefore (A2) holds.

$$(A3) \qquad \frac{\partial \Psi(x, y)}{\partial y} + \Psi(x, y)\frac{\rho'(y)}{\rho(y)} = -2(x - y) + (x - y)^2 . \frac{1}{y} = \frac{(x - y)(-2y + x - y)}{y} = \frac{(x - y)(x - 3y)}{y} = \phi(x, y).$$

We see that

$$\int_{x_0}^{x} \Psi(x, y)\rho(y)J(y)£(y)dy = \int_{x_0}^{x} (x - y)^2 \prod_{y-\pi < x_p \leq y} \left(\frac{p}{1 + p}\right) dy = \frac{1}{2}\int_{x_0}^{x_1} (x - y)^2 dy + \frac{1}{3}\int_{x_1}^{x_2} (x - y)^2 dy + \frac{1}{4}\int_{x_2}^{x_3} (x - y)^2 dy + \ldots = \frac{(x - x_0)^3}{6} - \frac{(x - x_1)^3}{18} - \frac{(x - x_2)^3}{36} - \frac{(x - x_3)^3}{60} - \ldots .$$

We can easily show that

$$\limsup_{x\to\infty} \frac{1}{\Psi(x, x_0)} \int_{x_0}^{x} \left[\Psi(x, y)\rho(y)J(y)£(y) - \frac{(1 + \alpha)\phi^2(x, y)\rho(y)}{2M\Psi(x, y)\eta_1^{n-2}(y)}\right] dy = \infty.$$

Thus all the conditions of Theorem 9 are fulfilled, therefore every bounded solution of (12) is oscillatory.

***Example 13*** Consider the following system

$$\begin{cases} [u(x) + \alpha u(x - 4\pi)]^{(n)} + e^{2x}u(x - 2\pi) = 0, \quad x \neq x_p, \\ u^{(r)}(x_p) - u^{(r)}(x_p^-) = \frac{1}{p}u^{(r)}(x_p^-), \quad r = 0, 1, 2, 3, ..., n - 1, \\ \quad p = 1, 2, 3, \ldots, \end{cases} \tag{23}$$

where $n$ is any positive even integer.
Here $\eta_1(x) = x - 4\pi$, $\eta_2(x) = x - 2\pi$, $x > x_0$, $\alpha > 0$, $k(x) = e^{2x}$, $d_p = \frac{1}{p}$, $x_p = p\pi$.
Let $\Psi(x, y) = (x - y)^2$, $\rho(y) = e^{-2y}$.
Clearly (A1), (A2) hold.

$$(A3) \qquad \frac{\partial \Psi(x, y)}{\partial y} + \Psi(x, y)\frac{\rho'(y)}{\rho(y)} = -2(x - y) - (x - y)^2 . \frac{2e^{-2y}}{e^{-2y}} = -2(x - y)(x - y + 1) = \phi(x, y).$$

Now, we can easily show that

$$\limsup_{x\to\infty}\frac{1}{\Psi(x,x_0)}\int_{x_0}^{x}\left[\Psi(x,y)\rho(y)J(y)\pounds(y)-\frac{(1+\alpha)\phi^2(x,y)\rho(y)}{2M\Psi(x,y)\eta_1^{n-2}(y)}\right]dy=\infty.$$

Thus all the conditions of Theorem 9 are fulfilled, therefore every bounded solution of (13) is oscillatory.

***Example 14*** Consider the following system

$$\begin{cases}[u(x)+\alpha u(x-2\pi)]^{(5)}+x^2u(x-\pi)=0, & x\neq x_p,\\ u^{(r)}(x_p)-u^{(r)}(x_p^-)=\frac{1}{p}u^{(r)}(x_p^-), & r=0,1,2,3,4,\\ \quad p=1,2,3,\ldots. \end{cases} \tag{24}$$

Here $n=5$, $k(x)=x^2$, $d_p=\frac{1}{p}$, $\eta_1(x)=x-2\pi$, $\eta_2(x)=x-\pi$. Clearly, there exits a constant $K_1>0$ such that

$$\max\left\{\prod_{x-2\pi<x_p\leq x}\left(\frac{p}{1+p}\right),\prod_{x-\pi<x_p\leq x}\left(\frac{p}{1+p}\right)\right\}\leq K_1$$

and

$$\int_{x_0}^{\infty}k(y)dy=\int_{x_0}^{\infty}y^2dy=\infty.$$

Thus all the conditions of Theorem 10 are fulfilled, therefore by using Theorem 10, we get

$$\lim_{x\to\infty}u(x)=0.$$

## Conclusion

Our main emphasis is analyzing of higher-order neutral impulsive differential equations using Riccati transformations and comparison theorems in the present work. Using these strategies, every solution of the studied equation [i.e., problem (1.1)] oscillates under certain assumptions. In addition to that, some sufficient conditions are obtained for bounded oscillatory solutions using the corresponding non-impulsive differential equation. Moreover, the asymptotic behavior of the oscillatory solutions is also discussed.

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