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Riccati-Type Equations Associated with Higher Order Ordinary Differential Equations

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Abstract

It is shown, similarly as the solution of the Riccati equation can be given via quotients of two linear independent solutions of the second order ordinary differential equation, the solution of the Riccati-Abel differential equation is presented by three linear independent solutions of the third order ordinary differential equation. The method is extended to the class of generalized Riccati-type equations governed by the characteristic polynomials of the associated higher order ordinary differential equations. An explicit functional dependence between solutions of the higher degree Riccati equations with the linear independent solutions of the associated ordinary differential equations established.

Keywords Trigonometry · Polynomial · Euler formula · Riccati–Abel equation · Ordinary differential equation

Introduction

Consider the first order differential equation

$$\frac{du}{d\phi} = P(u), \tag{1.1}$$

where the function $P(u)$ is a polynomial. If the function $P(u)$ is restricted by a second order polynomial then the Eq. (1.1) is the Riccati equation [1]. For a given second order ordinary differential equation one may put in correspondence the Riccati equation. The relevance of the ordinary Riccati equation to the second order ordinary differential equation can be specified by the following features.

1. The corresponding Riccati equation is governed by the quadratic polynomial which is the characteristic polynomial of the associated second order ordinary differential equation.
2. The solution of the Riccati equation is formed from the linear independent solutions of the associated differential equation [2].

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The aim of the present work is to derive the Riccati-type equation possessing with the similar properties as the ordinary Riccati equation.

The higher order Riccati equation associated with the higher order ordinary differential equation has to be the first order differential equation governed with the same characteristic polynomial. Consider n- order ordinary differential equation

$$P\left(\frac{d}{d\phi}\right)\Psi(\phi) = 0, \tag{1.2}$$

with n-degree characteristic polynomial $P(X)$. The task is to derive the Ricati-type equation of the form

$$\Phi(u)\frac{du}{d\phi} = P(u), \tag{1.3}$$

induced by the polynomials $P(u)$ and $\Phi(u)$ with $deg(\Phi(u)) < deg(F(u))$. Furthermore, we shall establish a link between set of linear independent solutions of the differential equation (1.2) and the solutions of the corresponding Riccati-type equations.

The Riccati equations of the type (1.3) with polynomials $\Phi(u, \phi)$, $P(u, \phi)$ recently had been studied from various points of view (see, for example [3,4] and references therein). The works have been devoted to construct appropriate solutions of the Riccati-type equations with different dependence of the coefficients on the parameter of differentiation.

The main purpose of the present paper is to establish a link between higher order linear differential equations and the first order nonlinear differential equations. The paper is presented by the following sections. In order to give a main idea, in "Ordinary Differential Equation of Second Order and Its Associated Riccati Equation" section, we recall the interconnections between second order ordinary differential equation and its associated Riccati equation. In "Riccati–Abel Equations Associated with Third Order Ordinary Differential Equation" section, it is suggested a method to construct a link between the solution of the Riccati–Abel equation [5] and the third order ordinary linear differential equation. In "Higher Order Riccati-Type Equation Governed by n-Degree Polynomial" section, the system of Riccati-type equations are reduced to the system of algebraic equations. In "Solutions of the Riccati-Type Equations in Terms of the General Trigonometric Functions" section, the linear independent solutions of the higher order ordinary differential equations are identified with the functions of the generalized trigonometry. The solutions of the Riccati-type equations are expressed via the generalized trigonometric functions.

Ordinary Differential Equation of Second Order and Its Associated Riccati Equation

Consider the ordinary differential equation of second order

$$\frac{d^2}{d\phi^2}\Psi - a_1\frac{d}{d\phi}\Psi + a_2\Psi = 0, \tag{2.1}$$

where $a_1, a_2 \in C$. The second order differential equation can be cast in the following "two-component" form [6]

$$\frac{d}{d\phi}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -a_2 & a_1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix},$$

after identifying the components as

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \Psi \\ d\Psi/d\phi \end{pmatrix}. \tag{2.2}$$

Let $x_1, x_2 \in C$ be roots of the characteristic polynomial

$$P(x) = x^2 - a_1 x + a_2.$$

In terms of the roots $x_1.x_2$ one may easily form the set of the linear independent solutions of the differential equation (2.1):

$$\Psi_1(\phi) = C_1 \exp(x_1\phi),\ \ \Psi_2(\phi) = C_2 \exp(x_2\phi). \tag{2.3}$$

However, the problem is to find solutions of the Eq. (2.1) as functions of the coefficients. For that aim, consider a companion matrix of the polynomial $P(x)$ defined by the (2×2) matrix

$$E = \begin{pmatrix} 0 & -a_2 \\ 1 & a_1 \end{pmatrix}. \tag{2.4}$$

Since the matrix E obeys the quadratic equation

$$P(E) = E^2 - a_1 E + a_2 I = 0, \tag{2.5}$$

the following expansion of $\exp(E\phi)$ holds true

$$\exp(E\phi) = g_1(\phi; a_1, a_2)E + g_0(\phi; a_1, a_2). \tag{2.6}$$

In a diagonal form this matrix equation is separated into two equations

$$\begin{aligned} \exp(x_2\phi) &= x_2\ g_1(\phi; a_1, a_2) + g_0(\phi; a_1, a_2), \\ \exp(x_1\phi) &= x_1\ g_1(\phi; a_1, a_2) + g_0(\phi; a_1, a_2), \end{aligned} \tag{2.7}$$

which admits explicit expressions for the "trig"-functions [7]. Apparently, g_0 and g_1 are two linearly independent solutions of the second order differential equation (2.1). As it has been proved in [8], the functions g_0 and g_1 explicitly depend of the coefficients $a_1,\ a_2$. These functions obey the system of evolution equations generated by companion matrix E:

$$\frac{d}{d\phi}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} 0 & -a_2 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}, \tag{2.8}$$

The pair of solutions Ψ_1, Ψ_2 and the pair g_0, g_1 in [6] and [9] had been introduced within the framework of the concept "binormality". The scalar product of these vectors forms the Courant-Snyder invariant. Form a ratio of two equations from (2.7) as follows

$$\exp((x_2 - x_1)\phi) = \frac{x_2\ g_1(\phi; a_1, a_2) + g_0(\phi; a_1, a_2)}{x_1\ g_1(\phi; a_1, a_2) + g_0(\phi; a_1, a_2)}. \tag{2.9}$$

Let $g_1(\phi; a_1, a_2) \neq 0$, then,

$$\exp((x_1 - x_2)\phi) = \frac{x_1 - u}{x_2 - u}, \tag{2.10}$$

where

$$u = -\frac{g_0(\phi; a_1, a_2)}{g_1(\phi; a_1, a_2)}. \tag{2.11}$$

Differential equation for this function is derived from the system of equations (2.8),

$$u^2 - a_1 u + a_2 = \frac{du}{d\phi}. \tag{2.12}$$

This is the Riccati equation associated with differential equation (2.1) and governed by the characteristic polynomial $P(x)$:

$$\frac{du}{d\phi} = P(u). \tag{2.13}$$

Direct integration of Eq. (2.12) leads one to the inverse function $\phi(u)$ of the solution,

$$\int \frac{dx}{x^2 - a_1 x + a_2} = \int d\phi. \tag{2.14}$$

On making use of the formula

$$\frac{x_1 - x_2}{x^2 - a_1 x + a_2} = \frac{1}{x - x_1} - \frac{1}{x - x_2}, \tag{2.15}$$

the integral (2.14) is easily calculated

$$\int_w^u \frac{dx}{x^2 - a_1 x + a_2} = \frac{1}{m_{12}} \left(\log \frac{u - x_1}{u - x_2} - \log \frac{w - x_1}{w - x_2} \right) = \phi(u) - \phi(w), \tag{2.16}$$

where $m_{12} = x_1 - x_2$. By inverting the logarithm function, we recover the exponential function from (2.10),

$$\exp(m_{12}\phi) = \frac{u - x_1}{u - x_2}, \tag{2.17}$$

where the function $u(\phi)$ is defined as the quotient of trigonometric functions g_0, g_1.

If $\phi = \phi_0$ where $u(\phi_0) = 0$, then

$$\exp(m_{12}\phi_0) = \frac{x_1}{x_2}. \tag{2.18}$$

As soon as the point $\phi = \phi_0$ is determined, one may calculate the function $u(\phi)$ from the algebraic equation (2.17). Since $a_1 = x_1 + x_2$, the formula (2.18) is read as

$$a_1 = m_{12} \coth(m_{12}\phi_0/2). \tag{2.19}$$

Consequently, the function u is presented as follows

$$u(\phi, \phi_0) = \frac{1}{2} m_{12} \coth(m_{12}\phi_0/2) - \frac{1}{2} m_{12} \coth(m_{12}\phi/2). \tag{2.20}$$

The formulae (2.19) and (2.20) imply the following link between solution of the Riccati equation and first coefficient of the characteristic polynomial [10]

$$u(\phi, \phi_0) = \frac{1}{2} a_1(\phi_0) - \frac{1}{2} a_1(\phi). \tag{2.21}$$

This formula expresses one of the most important features of the associated Riccati equation: the formula provides us with a law of evolution of the first coefficient of the characteristic polynomial

$$a_1(\phi) = a_1(\phi_0) - 2u(\phi, \phi_0). \tag{2.22}$$

Under this translation the next coefficients of the polynomial are calculated on making use of the Pascal matrix [11,12].

Riccati–Abel Equations Associated with Third Order Ordinary Differential Equation

Let us start with Riccati–Abel differential equation of the form [13]

$$u^3 - a_1u^2 + a_2u - a_3 = \frac{du}{d\phi}, \quad a_1, a_2, a_3 \in C. \tag{3.1}$$

The integral

$$\int_w^u \frac{dx}{x^3 - a_1x^2 + a_2x - a_3} \tag{3.2}$$

is calculated by applying the expansion

$$\frac{1}{x^3 - a_1x^2 + a_2x - a_3} = \frac{(x_3 - x_2)}{V}\frac{1}{x - x_1} + \frac{(x_1 - x_3)}{V}\frac{1}{x - x_2} + \frac{(x_2 - x_1)}{V}\frac{1}{x - x_3}, \tag{3.3}$$

where

$$V = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1), \tag{3.4}$$

means the determinant of the Vandermonde matrix

$$W[x_1, x_2, x_3] := \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}.$$

The distinct constants $x_1, x_2, x_3 \in C$ are roots of the cubic polynomial

$$P(x) = x^3 - a_1x^2 + a_2x - a_3. \tag{3.5}$$

After integration the differential equation (3.1) is reduced to the following algebraic equation

$$\frac{(x_3 - x_2)}{V}\log\frac{u - x_1}{w - x_1} + \frac{(x_1 - x_3)}{V}\log\frac{u - x_2}{w - x_2} + \frac{(x_2 - x_1)}{V}\log\frac{u - x_3}{w - x_3} = \phi(u) - \phi(w). \tag{3.6}$$

By inverting the logarithms we come to the exponential function of the form

$$(u - x_1)^{m_{32}}(u - x_2)^{m_{13}}(u - x_3)^{m_{21}} = \exp(V\phi), \tag{3.7}$$

where

$$m_{ij} = (x_i - x_j), \; i, j = 1, 2, 3, \; \text{with } m_{21} + m_{32} + m_{13} = 0. \tag{3.8}$$

Thus, the problem of solution of differential equation (3.1) is reduced to the problem of solution of the algebraic equation (3.7). Notice, the values $m_{i,j}, i, j = 1, 2, 3$ are elements of the cofactor matrix corresponding to the elements on the third line of the Vandermonde's matrix $W[x_1, x_2, x_3]$. In our notations these quantities are given by

$$A_1^{(3)} = x_2 - x_3, \; A_2^{(3)} = x_3 - x_1, \; A_3^{(3)} = x_1 - x_2. \tag{3.9}$$

According to the properties of the cofactor matrix we have two identities [14]

$$x_1A_1^{(3)} + x_2A_2^{(3)} + x_3A_3^{(3)} = 0, \; x_1^2A_1^{(3)} + x_2^2A_2^{(3)} + x_3^2A_3^{(3)} = V. \tag{3.10}$$

In addition, let us recall the following useful matrix identity

$$\begin{pmatrix} x_2x_3 & x_3x_1 & x_1x_2 \\ -(x_2+x_3) & -(x_3+x_1) & -(x_1+x_2) \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_1^{(3)} & A_1^{(2)} & A_1^{(1)} \\ A_2^{(3)} & A_2^{(2)} & A_2^{(1)} \\ A_3^{(3)} & A_3^{(2)} & A_3^{(1)} \end{pmatrix} = V \begin{pmatrix} 1 & -a_1 & a_2 \\ 0 & 1 & -a_1 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

The differential form of the algebraic equation (3.7) is

$$\left(\frac{A_1^{(3)}}{u-x_1} + \frac{A_2^{(3)}}{u-x_2} + \frac{A_3^{(3)}}{u-x_3} \right) du = V d\phi. \tag{3.12}$$

The use of the identities (3.10), (3.11)leads us to the Riccati–Abel differential equation of the form

$$\frac{d}{d\phi} u(\phi) = P(u). \tag{3.13}$$

Now, let us use elements of the cofactor matrix corresponding to the elements on the second line of the Vandermonde matrix. For these elements the following identities hold true

$$x_1 A_1^{(2)} + x_2 A_2^{(2)} + x_3 A_3^{(2)} = V, \quad x_1^2 A_1^{(2)} + x_2^2 A_2^{(2)} + x_3^2 A_3^{(2)} = 0. \tag{3.14}$$

This case corresponds to the Riccati-type equation of the following form

$$\Phi(u) \frac{d}{d\phi} u(\phi) = P(u), \tag{3.15}$$

where

$$\Phi(u) = u - a_1. \tag{3.16}$$

Next, let us establish a relationship between linear independent solutions of third order ordinary differential equation

$$\frac{d^3}{d\phi^3}\Psi - a_1 \frac{d^2}{d\phi^2}\Psi + a_2 \frac{d}{d\phi}\Psi - a_3\Psi = 0, \tag{3.17}$$

and the Riccati–Abel equation (3.13) and the Riccati-type equation (3.15).

Following the algorithm developed in the previous section, firstly, let us define the companion matrix of the cubic characteristic polynomial

$$E := \begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & a_1 \end{pmatrix}. \tag{3.18}$$

Due to equation

$$P(E) = 0, \tag{3.19}$$

the following expansion of the exponential function holds true

$$\exp(E\phi_1 + E^2\phi_2) = g_0(\phi_1, \phi_2) + E\, g_1(\phi_1, \phi_2) + E^2\, g_2(\phi_1, \phi_2). \tag{3.20}$$

In this expansion the function $g_0(\phi_1, \phi_2)$ is the cosine-type function whereas the functions $g_k(\phi_1, \phi_2), k = 1, 2$ are referred to the sine-type functions. It is seen, the general trigonometrical functions of third order depend on the pair of parameters. It should be emphasized, the second parameter of evolution ϕ_2 plays a crucial role in solution of the problem of connection between third order ordinary differential equation and the Riccati–Abel equations. The formulae of differentiation are defined as the evolution equations generated by degrees of the companion matrix along these parameters:

$$\frac{\partial}{\partial \phi_1}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & a_1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}, \tag{3.21}$$

$$\frac{\partial}{\partial \phi_2}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & a_3 & a_3 a_1 \\ 0 & -a_2 & a_3 - a_1 a_2 \\ 1 & a_1 & -a_2 + a_1^2 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}. \tag{3.22}$$

In the diagonal form the matrix equation (3.20) is decomposed into three equations

$$\exp(x_k\phi_1 + x_k^2\phi_2) = g_0(\phi_1, \phi_2) + x_k\, g_1(\phi_1, \phi_2) + x_k^2\, g_2(\phi_1, \phi_2), k = 1, 2, 3, \tag{3.23}$$

where the values $x_1, x_2, x_3 \in C$ are eigenvalues of the matrix E. Raise to power $A_k^{(i)}, k = 1, 2, 3; i = 2, 3$, both sides of this equation and form the following product

$$\prod_{k=1}^{3} \exp(A_k^{(i)}(x_k\phi_1 + x_k^2\phi_2)) = \prod_{k=1}^{3} \left(g_0(\phi_1, \phi_2) + x_k\, g_1(\phi_1, \phi_2) + x_k^2\, g_2(\phi_1, \phi_2)\right)^{A_k^{(i)}}. \tag{3.24}$$

Notice, in this equation the terms $A_k^{(i)}, k = 1, 2, 3; i = 2, 3$ mean the elements of the cofactor matrix corresponding to the elements of the Vandermonde's matrix $W[x_1, x_2, x_3]$ and, respectively, they obey the identities (3.10) and (3.14). The use of these identities in (3.24) allows to reduce the products into the following compact forms

$$\prod_{k=1}^{3} \exp(A_k^{(3)}(x_k\phi_1 + x_k^2\phi_2)) = \exp(V\phi_2), \tag{3.25}$$

$$\prod_{k=1}^{3} \exp(A_k^{(2)}(x_k\phi_1 + x_k^2\phi_2)) = \exp(V\phi_1). \tag{3.26}$$

From these formulas it follows that the former, e.g. (3.25), is free of the variable ϕ_1, and the latter, e.g. (3.26), does not depend of the variable ϕ_2, respectively. This observation prompts us to choose the variable ϕ_1, ϕ_2 in a such way that the following constrains will be satisfied:

$$g_2(\phi_1, \phi_2) = 0 \to \phi_1 = \phi_1(\phi_2), \tag{3.27}$$

for the former product, and

$$g_2(\phi_1, \phi_2) = 0 \to \phi_2 = \phi_2(\phi_1), \tag{3.28}$$

for the latter one, correspondingly.

Define the function $u(\phi_2)$ by

$$u(\phi_2) = -\frac{g_0(\phi_1(\phi_2), \phi_2)}{g_1(\phi_1(\phi_2), \phi_2)}. \tag{3.29}$$

In these terms the formula (3.24) for $i = 3$ is read

$$\exp(V\phi_2) = (u - x_1)^{m_{32}}(u - x_2)^{m_{13}}(u - x_3)^{m_{21}}, \tag{3.30}$$

where $m_{ij} = (x_i - x_j),\ i, j = 1, 2, 3$.

The differential equation corresponding to the algebraic equation (3.30) is the Riccati–Abel equation

$$\frac{d}{d\phi_2}u(\phi_2) = u^3 - a_1u^2 + a_2u - a_3. \tag{3.31}$$

Next, redefine the function $u(\phi_1, \phi_2)$ as a function of the variable ϕ_1, then,

$$u(\phi_1) = -\frac{g_0(\phi_1, \phi_2(\phi_1))}{g_1(\phi_1, \phi_2(\phi_1))}. \tag{3.32}$$

Correspondingly, for $i = 2$ Eq. (3.24) takes the form

$$\exp(V\phi_1) = (u - x_1)^{r_{23}}(u - x_2)^{r_{31}}(u - x_3)^{r_{12}}, \tag{3.33}$$

where $r_{ij} = (x_i^2 - x_j^2),\ i, j = 1, 2, 3$.

The differential equation corresponding to the algebraic equation (3.33), as above it has been proved, is the Riccati-type equation of the form

$$(u - a_1)\frac{d}{d\phi_1}u(\phi_1) = u^3 - a_1u^2 + a_2u - a_3. \tag{3.34}$$

Notice, namely the variable ϕ_1 is the parameter of differentiation of the differential equation (3.17), meanwhile the variable ϕ_2 is the auxiliary parameter of the method.

Higher Order Riccati-Type Equation Governed by *n*-Degree Polynomial

Let $P(x)$ be a n degree polynomial over field C,

$$P(x) = x^n + \sum_{k=1}^{n}(-1)^k a_k x^{n-k}, \tag{4.1}$$

where the coefficients $a_k \in C$, and the polynomial possesses with n distinct roots $x_k \in C,\ k = 1, \ldots, n$. If C is a field and $x_1, \ldots, x_n$ are algebraically independent over C, the polynomial

$$P(x) = \prod_{k=1}^{n}(x - x_k), \tag{4.2}$$

is referred to as *generic polynomial* over C of degree n. The polynomial $P(x)$ defines structure of the higher order Riccati equation of the type

$$\frac{du}{d\phi} = P(u). \tag{4.3}$$

Integral form of this equation is

$$\int_v^u \frac{dx}{P(x)} = \phi_u - \phi_v. \tag{4.4}$$

The integral is calculated on making use of the expansion

$$\sum_{m=1}^{n} A_m^{(n-1)}(u - x_m)^{-1} = \frac{V}{P(u)}, \tag{4.5}$$

where $A_k^{(n-1)}$ is the element of the cofactor matrix corresponding to the element x_k^{n-1} of the Vandermonde matrix

$$W[x_1, x_2, \dots, x_{n-1}, x_n] := \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1}, \end{pmatrix}, \tag{4.6}$$

and V is the determinant, $V = Det(W)$.

The result of the integration is given by the algebraic equation with respect to solution $u(\phi)$,

$$\exp(V\phi) = \prod_{k=1}^{n}(u - x_k)^{A_k^{(n-1)}}. \tag{4.7}$$

Inversely, by differentiating both sides of this equation one may recover the n-order Riccati equation (4.3). In fact,

$$V \exp(V\phi)\, d\phi = du \sum_{m=1}^{n} A_m^{(n-1)}(u - x_m)^{A_m^{(n-1)}-1} \prod_{k=1,k\neq m}^{n-1} (u - x_k)^{A_k^{(n-1)}} \tag{4.8}$$

$$= du \sum_{m=1}^{n} A_m^{(n-1)}(u - x_m)^{-1} \prod_{k=1}^{n}(u - x_k)^{A_k^{(n-1)}}. \tag{4.9}$$

Owing the identities

$$\sum_{l=1}^{n}(x_l)^k A_l^{(n-1)} = 0, k = 1, 2, \dots, n-2; \quad \sum_{l=1}^{n}(x_l)^{n-1} A_l^{(n-1)} = V, \tag{4.10}$$

formula (4.9) is reduced to the following differential form [14]

$$V\, d\phi = du \sum_{m=1}^{n} A_m^{(n-1)}(u - x_m)^{-1}. \tag{4.11}$$

Transform the sum in the right-hand side of this equation by using (4.8), then,

$$\sum_{m=1}^{n} A_m^{(n-1)}(u - x_m)^{-1} = \frac{V}{\prod_{k=1}^{n}(u - x_k)}. \tag{4.12}$$

Substituting this formula into (4.11) we arrive at the n-order Riccati equation (4.3).

Following to prescriptions of the “Riccati–Abel Equations Associated with Third Order Ordinary Differential Equation” section we will extend the algebraic formula (4.8) and its differential form given by the generalized Riccati equation (4.3) by including into consideration all cofactors of the Vandemonde’s matrix. For the aim, firstly, let us extend the identity given by formula (3.11). On the set of roots x_k, $k = 1, \dots, n$ of the polynomial $P(x)$ define the *trimmed* polynomials as follows.

Define n *trimmed* polynomials by

$$T_k = \prod_{i\neq k}^{n}(x_k - x_i) = x_k^{n-1} + b_1^{(k)} x^{n-2} + \dots + b_{n-1}^{(k)},\ k = 1, 2, \cdots, n. \tag{4.13}$$

Then, the identity (3.11) is extended as follows [8]:

$$\begin{pmatrix} b_{n-1}^{(1)} & b_{n-1}^{(2)} & \dots & b_n^{(n-1)} \\ b_{n-2}^{(1)} & b_{n-2}^{(2)} & \dots & b_n^{(n-2)} \\ \dots & \dots & \dots & \dots \\ b_1^{(1)} & b_1^{(2)} & \dots & b_n^{(1)} \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} A_1^{(n-1)} & A_1^{(n-2)} & \dots & A_1^{(0)} \\ A_2^{(n-1)} & A_2^{(n-2)} & \dots & A_2^{(0)} \\ \dots & \dots & \dots & \dots \\ A_{n-1}^{(n-1)} & A_{n-1}^{(n-2)} & \dots & A_{n-1}^{(0)} \\ A_n^{(n-1)} & A_n^{(n-2)} & \dots & A_n^{(0)} \end{pmatrix}$$

$$= V \begin{pmatrix} 1 & -a_1 & \dots & a_n(-1)^n \\ 0 & 1 & \dots & a_{n-1}(-1)^{n-1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -a_1 \\ 0 & 0 & \dots & 1 \end{pmatrix}. \tag{4.14}$$

We claim that the differential form of algebraic equation

$$\exp(V\phi) = \prod_{k=1}^{n} (u - x_k)^{A_k^{(n-p)}} \tag{4.15}$$

is given by the higher order Riccati equation of the type (compare with [16])

$$\Phi(u) \frac{du}{d\phi} = P(u), \tag{4.16}$$

where $\Phi(u)$ is $(n - p - 1)$ degree polynomial of the form

$$\Phi(u) = u^{n-p-1} + \sum_{k=1}^{n-p-1} a_k u^{n-p-1-k} (-1)^k. \tag{4.17}$$

This claim it follows from observation that the differential form of Eq. (4.15) is

$$V d\phi = \sum_{k=1}^{n} \frac{A_k^{(n-p)}}{u - x_k} \, du. \tag{4.18}$$

Then, on making use of each line of the matrix formula (4.14) the following set of equations are obtained

$$\left(u^{n-p} + \sum_{k=1}^{n-p} (-1)^k a_k u^{n-p-k} \right) \frac{d}{d\phi} u = P(u), \;\; p = 2, \dots, n-1. \tag{4.19}$$

Solutions of the Riccati-Type Equations in Terms of the General Trigonometric Functions

Consider n-order ordinary differential equation with characteristic polynomial $P(x)$ defined in (4.1),

$$P\left(\frac{d}{d\phi}\right) \Psi(\phi) = 0. \tag{5.1}$$

Seeking the set of fundamental solutions explicitly depending of the coefficients of the polynomial $P(x)$ we will arrive at the set of the functions of the generalized trigonometry [13]. Following the prescriptions of "Ordinary Differential Equation of Second Order and its Associated Riccati Equation", "Riccati–Abel Equations Associated with Third Order Ordinary Differential Equation" sections, firstly, let us define the companion matrix of the characteristic polynomial $P(x)$ by $(n \times n)$ matrix with entries consisting of coefficients of the polynomial

$$E := \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^{n+1}a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & a_3 \\ 0 & 1 & \dots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix} \tag{5.2}$$

Next, consider an expansion of the exponential function with respect to matrix E as

$$\exp\left(\sum_{k=1}^{n-1} E^k \phi_k\right) = \sum_{i=0}^{n-1} E^i g_i(\phi), \quad \text{with } \phi := (\phi_1, \phi_2, \dots, \phi_{n-1}). \tag{5.3}$$

The formulae of differentiation for g-functions are obtained by using the following set of equations [15]

$$\frac{d}{d\phi_l} \exp\left(\sum_{k=1}^{n-1} E^k \phi_k\right) = E^l \exp\left(\sum_{k=1}^{n-1} E^k \phi_k\right) = \sum_{k=0}^{n-1} E^k \frac{dg_k}{d\phi_l}, \ l = 1, 2, \dots, n-1.$$

In these equations on making equal expressions at $(1, E, E^2, \dots, E^{n-1})$, we get

$$\frac{d}{d\phi_k} g_{i-1} = \left[E^k\right]_i^j g_{j-1}, \ \ k = 1, \dots, n-1, \ i = 1, 2, \dots, n. \tag{5.4}$$

Let $x_k, \ k = 1, \dots, n$ be a set of eigenvalues of the matrix E. Let Λ be a diagonal matrix of the eigenvalues of E. In diagonal form the Eq. (5.3) is given by n equations

$$\exp\left(\sum_{k=1}^{n-1} \Lambda^k \phi_k\right) = \sum_{i=0}^{n-1} \Lambda^i g_i(\phi). \tag{5.5}$$

This linear system of equations can be exploited to define the trigonometric function through the exponentials $\exp(x_k^l \phi_l), \ k = 1, \dots, n, \ l = 1, 2, \dots, n-1$.

Notice, the general trigonometric functions with n-degree characteristic polynomial depend of $(n-1)$ parameter. If we restrict ourselves only with the parameter $\phi = \phi_1$, then from the formulae of differentiation

$$\frac{d}{d\phi} g_{i-1} = [E]_i^j g_{j-1}, \ \ i = 1, 2, \dots, n;$$

it follows that the functions $g_{i-1}(\phi), \ i = 1, 2, \dots, n$, are linear independent solutions of the differential equation (5.1).

We claim that the extension of the set of parameters is indispensable in order to obtain an interconnection between solutions of higher order ordinary differential equation with its associated Riccati-type equation.

In order to recover the system of Riccati-type equations on making use of the generalized trigonometry we shall repeat the same procedure as it has been done in "Riccati–Abel Equations Associated with Third Order Ordinary Differential Equation" section. Firstly, let us construct algebraic equations.

Raise to power $A_m^{(n-p)}$, $p = 1, \dots, n-2$ both sides of the equations

$$\exp\left(\sum_{k=1}^{n-1}(x_m)^k\phi_k\right) = \sum_{i=0}^{n-1}(x_m)^i g_i(\phi), \quad m = 1, 2, \dots, n-1; \; \phi := (\phi_1, \phi_2, \dots, \phi_{n-1}),$$

and form the following product

$$\prod_{m=1}^{n}\exp\left(A_m^{(n-p)}\sum_{k=1}^{n-1}(x_m)^k\phi_k\right) = \prod_{m=1}^{n}\left(\sum_{i=0}^{n-1}(x_m)^i g_i(\phi)\right)^{A_m^{(n-p)}}. \tag{5.6}$$

Due to the identities

$$\sum_{k=1} x_k^i A_k^{(j)} = V\delta_{i,j}, \tag{5.7}$$

the summation in the exponential function in (5.6) is reduced to the desired item

$$\exp(V\phi_{n-p}).$$

Consequently, the formula (5.6) is simplified as follows

$$\exp(V\phi_{n-p}) = \prod_{m=1}^{n}\left(\sum_{i=0}^{n-1}(x_m)^i g_i(\phi)\right)^{A_m^{(n-p)}}. \tag{5.8}$$

This freedom allows us to choose the following set of constraints

$$g_k(\phi_1, \phi_2, \dots, \phi_{n-1}) = 0, \; k = 2, 3, \dots, n-1. \tag{5.9}$$

By setting these conditions into the right- hand side of (5.8) we get

$$\exp(V\phi_{n-p}) = \prod_{m=1}^{n}(\, g_0(\phi_{n-p}, \phi) + x_m \; g_1(\phi_{n-p}, \phi)\,)^{A_m^{(n-p)}}, \tag{5.10}$$

where $\phi(\phi_{n-p})$ means the set of the other $(n-2)$ parameters - the functions of the unique parameter ϕ_{n-p}. These functions implicitly are defined by the system of constraints (5.9).

Let $g_1 \neq 0$. Then, by introducing the function

$$u(\phi_{n-p}) = -\frac{g_0(\phi)}{g_1(\phi)}, \tag{5.11}$$

and by taking into account the identity $\sum_{m=1} A_m^{(n-p)} = 0$, we get

$$\exp(V\phi_{n-p}) = \prod_{m=1}^{n}(u(\phi) - x_m)^{A_m^{(n-p)}}, \tag{5.12}$$

which is nothing else then the algebraic equation (4.15) corresponding to the Riccati-type equation (4.16). Thus, the function $u(\phi)$ is the solution of the algebraic equation (5.12).

Under the constraints $g_k = 0,\ k = 2, \dots, n-1$ the system of differential equations (5.4) is reduced to the following set of equations [13]

$$\frac{\partial}{\partial \phi_{n-1}} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \cdots \\ g_{n-2} \\ g_{n-1} \end{pmatrix} = \begin{pmatrix} (-1)^{n+1} a_n g_1 \\ (-1)^n a_{n-1} g_1 \\ (-1)^{n-1} a_{n-2} g_1 \\ \cdots \\ -a_2 g_1 \\ g_0 + a_1 g_1 \end{pmatrix}. \tag{5.13}$$

These equations we separate into two parts:

$$\frac{\partial}{\partial \phi_{n-1}} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} (-1)^{n+1} a_n g_1 \\ (-1)^n a_{n-1} g_1 \end{pmatrix}. \tag{5.14}$$

and

$$\frac{\partial}{\partial \phi_{n-1}} \begin{pmatrix} g_2 \\ \cdots \\ g_{n-2} \\ g_{n-1} \end{pmatrix} = \begin{pmatrix} (-1)^{n-1} a_{n-2} g_1 \\ (-1)^{n-2} a_{n-3} g_1 \\ (-1)^{n-3} a_{n-4} g_1 \\ \cdots \\ -a_2 g_1 \\ g_0 + a_1 g_1 \end{pmatrix}. \tag{5.15}$$

The parameters $\phi_k, k = 1, \dots, n-1$ are not independent variables, they are connected by the constraints (5.9). These equations determine the parameters $\phi_1, \dots, \phi_{n-2}$ as functions of unique variable ϕ_{n-1} in a implicit way. These conditions admit to define the derivatives of $\phi_k, k = 1, \dots, n-2$ with respect to ϕ_{n-1} on making use of derivatives of the g-functions. Differentiating constraints (5.9) with respect to ϕ_{n-1} we arrive at the system of equations which determines the desired derivatives:

$$\frac{dg_k}{d\phi_{n-1}} = \frac{\partial g_k}{\partial \phi_{n-1}} + \sum_{j=1}^{n-2} \frac{\partial g_k}{\partial \phi_j} \frac{d\phi_j}{d\phi_{n-1}} = 0,\ k = 2, 3, \dots, n-1 \tag{5.16}$$

In the matrix form this system is written as

$$- \begin{pmatrix} (-1)^{n-1} a_{n-2} g_1 \\ (-1)^{n-2} a_{n-3} g_1 \\ (-1)^{n-3} a_{n-4} g_1 \\ \cdots \\ -a_2 g_1 \\ g_0 + a_1 g_1 \end{pmatrix} = \begin{pmatrix} \partial_1 g_2 & \partial_2 g_2 & \partial_3 g_2 & \dots & \partial_{n-3} g_2 & \partial_{n-2} g_2 \\ \partial_1 g_3 & \partial_2 g_3 & \partial_3 g_3 & \dots & \partial_{n-3} g_3 & \partial_{n-2} g_3 \\ \partial_1 g_4 & \partial_2 g_4 & \partial_3 g_4 & \dots & \partial_{n-3} g_4 & \partial_{n-2} g_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial_1 g_{n-3} & \partial_2 g_{n-3} & \partial_3 g_{n-3} & \dots & \partial_{n-3} g_{n-3} & \partial_{n-2} g_{n-3} \\ \partial_1 g_{n-1} & \partial_2 g_{n-1} & \partial_3 g_{n-1} & \dots & \partial_{n-3} g_{n-1} & \partial_{n-2} g_{n-1} \end{pmatrix} \begin{pmatrix} \phi'_1 \\ \phi'_2 \\ \phi'_3 \\ \cdots \\ \phi'_{n-3} \\ \phi'_{n-2} \end{pmatrix}, \tag{5.17}$$

where ϕ'_k means $d\phi_k/\phi_{n-1}$. By setting

$$\partial_k g_k = g_0,\ \ \partial_k g_{k+1} = g_1,\ k = 2, \dots, n-2,$$

the system is read as

$$-\begin{pmatrix} (-1)^{n-1}a_{n-2}g_1 \\ (-1)^{n-2}a_{n-3}g_1 \\ (-1)^{n-3}a_{n-4}g_1 \\ \dots \\ -a_2g_1 \\ g_0+a_1g_1 \end{pmatrix} = \begin{pmatrix} g_1 & g_0 & 0 & \dots & 0 & 0 \\ 0 & g_1 & g_0 & \dots & 0 & 0 \\ 0 & 0 & g_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & g_1 & g_0 \\ 0 & 0 & 0 & \dots & 0 & g_1 \end{pmatrix} \begin{pmatrix} \phi'_1 \\ \phi'_2 \\ \phi'_3 \\ \dots \\ \phi'_{n-3} \\ \phi'_{n-2} \end{pmatrix}. \tag{5.18}$$

By inverting the triangle matrix we get

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \\ \phi'_3 \\ \dots \\ \phi'_{n-3} \\ \phi'_{n-2} \end{pmatrix} = -\begin{pmatrix} 1 & u & u^2 & \dots & (u)^{n-4} & (u)^{n-3} \\ 0 & 1 & u & u^2 & \dots & (u)^{n-4} \\ 0 & 0 & 1 & u & u^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & u \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n a_{n-1} \\ (-1)^{n-1}a_{n-2} \\ \dots \\ \dots \\ -a_2 \\ -u+a_1 \end{pmatrix}, \tag{5.19}$$

where

$$u = -\frac{g_0}{g_1}. \tag{5.20}$$

From the matrix equation (5.19) the following polynomials for the derivatives are obtained

$$\begin{aligned} \frac{d\phi_1}{d\phi_{n-1}} &= u^{n-2} - a_1u^{n-3} + a_2u^{n-4} + \dots + a_{n-2}, \\ \frac{d\phi_2}{d\phi_{n-1}} &= u^{n-3} - a_1u^{n-4} + a_2u^{n-5} + \dots + a_{n-3}, \\ &\dots \\ \frac{d\phi_{n-3}}{d\phi_{n-1}} &= u^2 - a_1u + a_2, \\ \frac{d\phi_{n-2}}{d\phi_{n-1}} &= u - a_1. \end{aligned} \tag{5.21}$$

On making use of these formulas we are able to recover all equations presented in (4.19):

$$\begin{aligned} (u^{n-2} - a_1u^{n-3} + a_2u^{n-4} + \dots + a_{n-2})\frac{d}{d\phi_1}\, u &= P(u), \\ (u^{n-3} - a_1u^{n-4} + a_2u^{n-5} + \dots + a_{n-3})\frac{d}{d\phi_2}\, u &= P(u), \\ \dots& \\ (u^2 - a_1u + a_2)\frac{d}{d\phi_{n-3}}\, u &= P(u), \\ (u - a_1)\frac{d}{d\phi_{n-2}}\, u &= P(u), \\ \frac{d}{d\phi_{n-1}}\, u &= P(u). \end{aligned} \tag{5.22}$$

Conclusions

We have established that the Riccati-type equation associated with n-order ordinary differential equation

$$P\left(\frac{d}{d\phi}\right)\ \Psi = 0,$$

with characteristic polynomial

$$P(x) = x^n + \sum_{k=1}^{n}(-1)^k a_k x^{n-k},$$

is the first order non-linear differential equation

$$\Phi(u)\frac{du}{d\phi} = P(u),$$

where $\Phi(x)$ is $(n-2)$- degree polynomial of the form

$$\Phi(x) = x^{n-2} + \sum_{k=1}^{n-2}(-1)^k a_k x^{n-2-k}.$$

Seeking solutions of n-th order Riccati equation the certain relationship of these solutions with generalized trigonometric functions has been found. It was shown, in order to obtain the desired relationship one has to extend the set of variables of differentiation up till $(n-1)$ parameters. The auxiliary parameters are defined implicitly by the system of constraints for the trigonometric functions. The Riccati-type equations are formulated with respect to all set of auxiliary parameters resulting the system of equations of the triangular form.

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