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ORIGINAL PAPER

Biorthogonal Wavelets in Sobolev Space Over Local Fields of Positive Characteristic

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Abstract
In continuation of the paper (Pathak and Singh in Int J Wavelets Multiresolut Inf Process 16(4):1850027, 2018). A general construction of biorthogonal wavelets on Sobolev space over local fields of positive characteristic $H^s(\mathbb{K})$ is given. Some results are discussed. Finally, we obtain Riesz bases for $H^s(\mathbb{K})$ in form of wavelet under some assumption on the wavelets and scaling functions where scaling functions depend on levels.

Keywords Wavelets · Multiresolution analysis · Local fields · Sobolev space · Biorthogonal wavelets · Riesz basis

Introduction

The theory of wavelet on local field and related groups has been developed by Benedetto and Benedetto [1,2]. Albeverio and Kozyrev [3–5] and their collaborators gave multiresolution analysis and wavelets on the $p-adic$ field $\mathbb{Q}_p$. MRA on a local field is defined by Jiang et al. [6] and the corresponding orthonormal wavelets are constructed.

These concepts have been extended by Behra and Jahan [7]. Recently, Pathak and Singh modified the classical definition of multiresolution analysis and constructed orthonormal wavelets in Sobolev space over local fields of positive characteristic $H^s(\mathbb{K})$ see [8–12].

The theory of biorthogonal wavelets are discussed by Cohen et al. [13], Chui and Wang [14] and others. The idea of biorthogonal wavelets on local field are discussed by Behra and Jahan [15].

In [16], biorthogonal wavelets are constructed in $H^s(\mathbb{R})$. In this paper we generalize the concept of biothogonal wavelets to Sobolev space over $\mathbb{K}$.

This article is divided as follows. Section 2 contains the general notations and definitions. Also in this section, we give some basic concepts of theory of distributions over local fields and defined Sobolev space. In Sect. 3 Riesz basis in $H^s(\mathbb{K})$ is given. Section 4 contains

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multiresolution point of view in Sobolev space. In Sect. 5, it is proved that wavelets associated with dual MRAs generate Riesz bases for $H^s(\mathbb{K})$.

Notation and Definitions

In this section, we recall some notations and definitions of local fields and distribution over local fields which will be used throughout the paper. The following list of notation and definitions are given below :

- Throughout this paper $\mathbb{K}$ denotes the local field of positive characteristic.
- dx is the normalized Haar measure for $\mathbb{K}^+$.
- $|\alpha|$ is the valuation of $\alpha \in \mathbb{K}$. If $\alpha \neq 0 (\alpha \in \mathbb{K})$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. Let $|0| = 0$. The valuation or absolute value has the following properties:

  (i) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$;
  (ii) $|xy| = |x||y|$;
  (iii) $|x + y| \leq max(|x|, |y|)$.

  The condition (iii) is called the ultrametric inequality or non-Archimedean property. It follows that $|x + y| = max(|x|, |y|)$ if $|x| \neq |y|$.
- We will use following notations for the numbers, $\mathbb{Z} = set\ of\ integers$; $\mathbb{N} = set\ of\ natural\ numbers$; $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.
- Let π be a prime element in $\mathbb{K}$.
- For $k \in \mathbb{Z}$, $\mathfrak{P}^k = \{x \in \mathbb{K} : |x| \leq q^{-k}\}$ is a compact subgroup of $\mathbb{K}^+$. $\mathfrak{P}^0 = \mathbb{D}$ is called ring of integres in $\mathbb{K}$.
- $|\mathfrak{P}^k| = q^{-k}$ and $|\mathbb{D}| = 1$.
- χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathbb{D}$ but is non trivial on $\mathfrak{P}^{-1}$. For $y \in \mathbb{K}$, $\chi_y(x) = \chi(yx)$, $x \in \mathbb{K}$.
- The "natural"order on the sequence is denoted by $\{w(k) \in \mathbb{K}\}_{k=0}^{\infty}$ and is described as follows.
  $\mathbb{D}/\mathfrak{P} \cong GF(q) = \tau$, $q = p^s$, p is a prime, $s \in \mathbb{N}$ and $\Omega : \mathbb{D} \to \tau$ the canonical homomorphism of $\mathbb{D}$ onto τ. $\tau = GF(q)$ is a vector space over $GF(p) \subset \tau$. We consider a set $\{1 = \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{s-1}\} \subset \mathbb{D}^* = \mathbb{D} \backslash \mathfrak{P}$ in such a way that $\{\Omega(\varepsilon_k)\}_{k=0}^{s-1}$ is a basis of $GF(q)$ over $GF(p)$.
  For k, $0 \leq k < q$, $k = a_0 + a_1 p + \cdots + a_{s-1} p^{s-1}$, $0 \leq a_i < p$, $i = 0, 1, \ldots, s-1$, we define
  $$w(k) = (a_0 + a_1 \varepsilon_1 + \cdots + a_{s-1} \varepsilon_{c-1}) \pi^{-1} \quad (0 \leq k < q).$$
  For $k = b_0 + b_1 q + \cdots + b_r q^r$, $0 \leq b_i < q$, $k \geq 0$, we set
  $$w(k) = w(b_0) + \pi^{-1} w(b_1) + \cdots + \pi^{-r} w(b_r).$$
- Note that for $k, l \geq 0$, $w(k + l) \neq w(k) + w(l)$. However, it is true that for all $r, l \geq 0$, $w(rq^l) = \pi^{-l} w(r)$, and for $r, l \geq 0$, $0 \leq t < q^l$, $w(rq^l + t) = w(rq^l) + w(t) = \pi^{-l} w(r) + w(t)$.
- For $k \in \mathbb{N}_0$, we denote $\chi_{w(k)}$ by χ_k.
- $\mathfrak{S}(\mathbb{K})$ is the space of all finite linear combinations of characteristic function of balls of $\mathbb{K}$. Also $\mathfrak{S}(\mathbb{K})$ is dense in $L^p(\mathbb{K})$, $1 \leq p < \infty$.
- $\mathfrak{S}'(\mathbb{K})$ is the space of distributions.

- $\hat{f}(\zeta)$ is the Fourier transform of $f \in \mathfrak{S}(\mathbb{K})$ and is defined by

$$\hat{f}(\zeta) = \int_{\mathbb{K}} f(x)\overline{\chi_\zeta(x)}dx, \quad \zeta \in \mathbb{K},$$

  and the inverse transform by

$$f(x) = \int_{\mathbb{K}} \hat{f}(\zeta)\chi_x(\zeta)d\zeta, \quad x \in \mathbb{K}.$$

- Let $s \in \mathbb{R}$, we denote Sobolev space over local fields by $H^s(\mathbb{K})$ is the space of all functions in $\mathfrak{S}'(\mathbb{K})$ such that

$$\hat{\gamma}^{\frac{s}{2}}(\zeta)\hat{f}(\zeta) \in L^2(\mathbb{K}), \quad \text{where } \hat{\gamma}^s(\zeta) = (max(1, |\zeta|))^s.$$

- The inner product in $H^s(\mathbb{K})$ is denoted by

$$\langle f, g\rangle = \langle f, g\rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\gamma}^s(\zeta)\hat{f}(\zeta)\overline{\hat{g}(\zeta)}d\zeta.$$

- The space $\mathfrak{S}(\mathbb{K})$ is also dense in $H^s(\mathbb{K})$.

For more details refer to [6,8,17,18].

Riesz Basis in $H^s(\mathbb{K})$

In this section we give definitions related to Riesz basis and deduce certain results.

Definition 1 Two families of functions $\{\psi_k : k \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_k : k \in \mathbb{N}_0\}$ in $H^s(\mathbb{K})$ are said to be biorthogonal if

$$\langle \psi_k, \tilde{\psi}_{k'}\rangle = \delta_{k,k'} \quad \text{for every } \; k, k' \in \mathbb{N}_0.$$

A collection $\{\psi_k : k \in \mathbb{N}_0\}$ of functions in $H^s(\mathbb{K})$ is said to be linearly independent if for any l^2-sequence $\{a_k : k \in \mathbb{N}_0\}$ of coefficients such that if $\sum_{k\in\mathbb{N}_0} a_k\psi_k = 0$ in $H^s(\mathbb{K})$, then, $a_k = 0$ for all $k \in \mathbb{N}_0$. It can be easily shown that biorthogonal families are linearly independent.

Lemma 1 *Let $\{\psi_k : k \in \mathbb{N}_0\}$ be a collection of functions in $H^s(\mathbb{K})$. Suppose there is a collection $\{\tilde{\psi}_k : k \in \mathbb{N}_0\}$ in $H^s(\mathbb{K})$ which is biorthogonal to $\{\psi_k : k \in \mathbb{N}_0\}$. Then $\{\psi_k : k \in \mathbb{N}_0\}$ is linearly independent.*

Proof Let $\{a_k : k \in \mathbb{N}_0\}$ be an l^2-sequence satisfying $\sum_{k\in\mathbb{N}_0} a_k\psi_k = 0$ in $H^s(\mathbb{K})$. Then for each $k' \in \mathbb{N}_0$, we have

$$0 = \langle 0, \tilde{\psi}_{k'}\rangle = \left\langle \sum_{k\in\mathbb{N}_0} a_k\psi_k, \tilde{\psi}_{k'}\right\rangle = \sum_{k\in\mathbb{N}_0} a_k\langle\psi_k, \tilde{\psi}_{k'}\rangle = a_{k'}.$$

Therefore, $\{\psi_k : k \in \mathbb{N}_0\}$ is linearly independent. □

Definition 2 A sequence of functions $\{g_k : k \in \mathbb{N}_0\}$ is called a Riesz basis of Sobolev space $(H^s(\mathbb{K}), \|.\|_{H^s(\mathbb{K})})$ if

1. $\{g_k : k \in \mathbb{N}_0\}$ is linearly independent, and

2. there exist constants A_1 and A_2 with $0 < A_1 \leq A_2 < \infty$ such that

$$A_1^2 \|h\|^2_{H^s(\mathbb{K})} \leq \sum_{k \in \mathbb{N}_0} |\langle h, g_k \rangle_{H^s(\mathbb{K})}|^2 \leq A_2^2 \|h\|^2_{H^s(\mathbb{K})} \quad \text{for every } h \in H^s(\mathbb{K}). \tag{1}$$

If above sequence satisfies the condition in item 2 of Definition 2 then it is called frame of $H^s(\mathbb{K})$ and the numbers A_1 and A_2 are called frame bounds.

Remark 1 A sequence of functions $\{g_k\}_{k \in \mathbb{N}_0}$ is called a Riesz basis of Sobolev space $(H^s(\mathbb{K}), \|.\|_{H^s(\mathbb{K})})$. If for any $h \in H^s(\mathbb{K})$, there is a sequence $\{c_k : k \in \mathbb{N}_0\}$ such that $h = \sum_{k \in \mathbb{N}_0} c_k g_k$ which converges in $H^s(\mathbb{K})$ and

$$A_1^2 \sum_{k \in \mathbb{N}_0} |c_k|^2 \leq \left\| \sum_{k \in \mathbb{N}_0} c_k g_k \right\|^2_{H^s(\mathbb{K})} \leq A_2^2 \sum_{k \in \mathbb{N}_0} |c_k|^2, \tag{2}$$

where the constants A_1 and A_2 satisfy $0 < A_1 \leq A_2 < \infty$ and independent of h. The right hand ineqaulity in (1) and (2) is known as the pre-Riesz condition for $\{g_k\}_{k \in \mathbb{N}_0}$.

It can be easly shown that the above two definitions of Riesz bases are euivalent to each other.

Lemma 2 *Let $\{\phi^{(j)}\}_{j \in \mathbb{Z}} \in H^s(\mathbb{K})$. If $\{\phi_{j,k} = q^{\frac{j}{2}} \phi^{(j)}(\pi^{-j} \cdot - w(k)) : k \in \mathbb{N}_0\}$ satisfies the Riesz condition, we have*

$$A_j^2 \sum_{l \in \mathbb{N}_0} |c_l|^2 \leq \left\| \sum_{l \in \mathbb{N}_0} c_l g_l \right\|^2_{H^s(\mathbb{K})} \leq B_j^2 \sum_{l \in \mathbb{N}_0} |c_l|^2 \tag{3}$$

where, $0 < A_j \leq B_j < \infty$ and are independent of $\{c_l\}_{l \in \mathbb{N}_0}$. Let

$$\sigma^2_{\phi^{(j)}} = \sum_{k \in \mathbb{N}_0} \hat{\gamma}^s(\pi^{-j}(\zeta + w(k))) |\hat{\phi}^{(j)}(\zeta + w(k))|^2. \tag{4}$$

Then ,

$$A_j \leq \sigma^2_{\phi^{(j)}} \leq B_j \quad \text{a.e.} \quad \zeta \in \mathbb{K}. \tag{5}$$

Moreover,

$$|\hat{\phi}^{(j)}(\pi^j \zeta)| \leq \sqrt{B_j} \hat{\gamma}^{-\frac{s}{2}}(\zeta). \tag{6}$$

Proof See [12]. □

Multiresolution Point of View in $H^s(\mathbb{K})$

Here we discuss certain results associated to multiresolution analysis in $H^s(\mathbb{K})$.

Theorem 1 *Let $\tilde{\phi}^{(j)}, \phi^{(j)} \in H^s(\mathbb{K})$ and $j \in \mathbb{Z}$, then the distribution $\tilde{\phi}_{j,k} = q^{\frac{j}{2}} \tilde{\phi}^{(j)}(\pi^{-j} x - w(k))$; $k \in \mathbb{N}_0$ and $\phi_{j,k} = q^{\frac{j}{2}} \phi^{(j)}(\pi^{-j} x - w(k))$ are biorthogonal in $H^s(\mathbb{K})$ if and only if*

$$\sum_{k \in \mathbb{N}_0} \hat{\gamma}^s(\pi^{-j}(\zeta + w(k))) \hat{\phi}^{(j)}(\zeta + w(k)) \overline{\hat{\tilde{\phi}}^{(j)}(\zeta + w(k))} = 1 \quad \text{a.e.} \tag{7}$$

Moreover, we also have

$$\lim_{j\to\infty} \hat{\phi}^{(j)}(\pi^j\zeta)\overline{\hat{\tilde{\phi}}^{(j)}(\pi^j\zeta)} \leq \hat{\gamma}^{-s}(\zeta). \tag{8}$$

Proof For $k \in \mathbb{N}_0$ and from the biorthogonality of $\phi_{j,k}$ and $\tilde{\phi}_{j,k}$, we have

$$\begin{aligned}\delta_{k,0} &= \left\langle q^{\frac{j}{2}}\phi^{(j)}(\pi^{-j}\cdot -w(k)), q^{\frac{j}{2}}\tilde{\phi}^{(j)}(\pi^{-j}.)\right\rangle_{H^s(\mathbb{K})}\\ &= \int_{\mathbb{K}} \hat{\gamma}^s(\pi^{-j}\zeta)\hat{\phi}^{(j)}(\zeta)\bar{\hat{\tilde{\phi}}}^{(j)}(\zeta)\bar{\chi}_k(\zeta)d\zeta.\end{aligned}$$

Splitting the integral and using the fact that $\chi_k(w(l)) = 1\ \forall\, l, k \in \mathbb{N}_0$, we have

$$\delta_{k,0} = \int_{\mathbb{D}} \sum_{l=0}^{\infty} \hat{\gamma}^s(\pi^{-j}(\zeta + w(l)))\hat{\phi}^{(j)}(\zeta + w(l))\overline{\hat{\tilde{\phi}}^{(j)}(\zeta + w(l))}\bar{\chi}_k(\zeta)d\zeta. \tag{9}$$

Since $\{\chi_k(\cdot)\}_{k=0}^{\infty}$ is a complete basis over $\mathbb{D}$, then from (9) we get required result (7). □

Theorem 2 *Let $\tilde{\phi}^{(j)}, \phi^{(j)}, \in H^s(\mathbb{K})$, for every $j \in \mathbb{Z}$. Assume that two families $\phi_{j,k} = q^{\frac{j}{2}}\phi^{(j)}(\pi^{-j}x - w(k))$ and $\tilde{\phi}_{j,k} = q^{\frac{j}{2}}\tilde{\phi}^{(j)}(\pi^{-j}x - w(k))$; $k \in \mathbb{N}_0$, satisfies pre-Riesz condition. We condiser the projection map P_j*

$$P_j : H^s(\mathbb{K}) \to H^s(\mathbb{K}), \qquad P_j f = \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\phi}_{j,k}\rangle_{H^s(\mathbb{K})}\phi_{j,k}.$$

If $\lim_{j\to+\infty} \bar{\hat{\phi}}^{(j)}(\pi^j\zeta)\hat{\tilde{\phi}}^{(j)}(\pi^j\zeta) = \hat{\gamma}^{-s}(\zeta)$ a.e. then

$$\lim_{j\to+\infty} \langle P_j f, g\rangle_{H^s(\mathbb{K})} = \langle f, g\rangle_{H^s(\mathbb{K})} \quad \text{for every} \quad f, g \in H^s(\mathbb{K}). \tag{10}$$

Moreover, for every $f \in H^s(\mathbb{K})$,

$$\lim_{j\to-\infty} ||P_j f||_{H^s(\mathbb{K})} = 0. \tag{11}$$

Proof For all $j \in \mathbb{Z}$, we have

$$\begin{aligned}\langle P_j f, g\rangle_{H^s(\mathbb{K})} &= \sum_{k\in\mathbb{N}_0} q^j \int_{\mathbb{K}} \hat{\gamma}^s(\pi^{-j}\zeta)\hat{f}(\pi^{-j}\zeta)\overline{\hat{\tilde{\phi}}^{(j)}}(\zeta)\chi_k(\zeta)d\zeta\\ &\quad \int_{\mathbb{K}} \hat{\gamma}^s(\pi^{-j}\zeta)\bar{\hat{g}}(\pi^{-j}\zeta)\hat{\phi}^{(j)}(\zeta)\bar{\chi}_k(\zeta)d\zeta\\ &= \sum_{k\in\mathbb{N}_0} q^j \int_{\mathbb{K}} \Big\{ \sum_{l\in\mathbb{N}_0} \int_{\mathbb{D}} \hat{\gamma}^s(\pi^{-j}(\zeta + w(l)))\\ &\quad \hat{f}(\pi^{-j}(\zeta + w(l)))\overline{\hat{\tilde{\phi}}^{(j)}}(\zeta + w(l))\chi_k(\zeta + w(l))d\zeta\Big\}\\ &\quad \times\hat{\gamma}^s(\pi^{-j}\zeta)\bar{\hat{g}}(\pi^{-j}\zeta)\hat{\phi}^{(j)}(\zeta)\bar{\chi}_k(\zeta)d\zeta.\end{aligned}$$

Since $f, g \in \mathfrak{S}(\mathbb{K})$, so the $\sum_{l\in\mathbb{N}_0}$ contains only finite non-zero terms and $\chi_k(w(l)) = 1$ for $k, l \in \mathbb{N}_0$, then we get

$$\langle P_j f, g\rangle_{H^s(\mathbb{K})} = \sum_{k\in\mathbb{N}_0} q^j \int_{\mathbb{K}} \left(\int_{\mathbb{D}} \left\{ \sum_{l\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j}(\zeta + w(l))) \hat{f}(\pi^{-j}(\zeta + w(l))) \overline{\hat{\hat{\phi}}^{(j)}}(\zeta + w(l)) \right\} \chi_k(\zeta) d\zeta \right) \times \hat{\gamma}^s(\pi^{-j}\zeta)\bar{\hat{g}}(\pi^{-j}\zeta)\hat{\phi}^{(j)}(\zeta)\bar{\chi}_k(\zeta)d\zeta.$$

By the convergence theorem of Fourier series on $\mathbb{D}$, we obtain

$$\begin{aligned}\langle P_j f, g\rangle_{H^s(\mathbb{K})} &= \int_{\mathbb{K}} \hat{\gamma}^s(\zeta)\hat{f}(\zeta)\overline{\hat{\hat{\phi}}^{(j)}(\pi^j\zeta)}\{\sum_{l\in\mathbb{N}_0}\hat{\gamma}^s (\zeta + \pi^{-j}w(l))\bar{\hat{g}}(\zeta + \pi^{-j}w(l))\hat{\phi}^{(j)}(\pi^j\zeta + w(l))\}d\zeta \\ &= \int_{\mathbb{K}} \hat{\gamma}^{2s}(\zeta)\hat{f}(\zeta)\bar{\hat{g}}(\zeta)\overline{\hat{\hat{\phi}}^{(j)}(\pi^j\zeta)}\hat{\phi}^{(j)}(\pi^j\zeta)d\zeta \\ &\quad + \int_{\mathbb{K}}\sum_{l\in\mathbb{N}}\hat{\gamma}^s(\zeta)\gamma^s(\zeta+\pi^{-j}w(l))\hat{f}(\zeta) \overline{\hat{\hat{\phi}}^{(j)}(\pi^j\zeta)}\bar{\hat{g}}(\zeta+\pi^{-j}w(l))\hat{\phi}^{(j)}(\pi^j\zeta + w(l))\}d\zeta \qquad (12)\\ &= I_1 + I_2 \quad (say). \qquad (13)\end{aligned}$$

Now by using Lemma 2 and Cauchy–Schwarz inequality, we get

$$\begin{aligned}|I_2| &\le \int_{\mathbb{K}} \hat{\gamma}^s(\zeta)|\hat{f}(\zeta)||\overline{\hat{\hat{\phi}}^{(j)}(\pi^j\zeta)}|\sum_{k=1}^{\infty}\hat{\gamma}^s (\zeta+\pi^{-j}w(k))|\bar{\hat{g}}(\zeta+\pi^{-j}w(l))||\hat{\phi}^{(j)}(\pi^j\zeta+w(l))|d\zeta \\ &\le \sqrt{B_j\tilde{B}_j}\int_{\mathbb{K}}\hat{\gamma}^{\frac{s}{2}}(\zeta)|\hat{f}(\zeta)|\sum_{k=1}^{\infty}\hat{\gamma}^{\frac{s}{2}}(\zeta+\pi^{-j}w(k))|\bar{\hat{g}}(\zeta+\pi^{-j}w(l))|d\zeta \\ &\le \sqrt{B_j\tilde{B}_j}\sum_{k=1}^{\infty}\left\|\hat{\gamma}^{\frac{s}{2}}(\cdot)\hat{f}(\cdot)\right\|_{L^2(\mathbb{K})}\left\|\hat{\gamma}^{\frac{s}{2}}(\cdot+\pi^{-j}w(k))\hat{g}(\cdot+\pi^{-j}w(k))\right\|_{L^2(\mathbb{K})}.\end{aligned}$$

Again since $\hat{g}\in\mathfrak{S}(\mathbb{K})$ therefore $\exists\, l$ such that support of $\hat{g}(\zeta)$ is $\mathfrak{P}^{-l}$, i.e., if $j > l$ then for any such $l\in\mathbb{N}$, $\hat{g}(\zeta+\pi^{-j}w(l)) = 0$. This shows that $\lim_{j\to\infty}|I_2| = 0$.

By using the Hypothesis of the theorem, we see that

$$\lim_{j\to+\infty}\langle P_j f, g\rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}}\hat{\gamma}^s(\zeta)\hat{f}(\zeta)\bar{\hat{g}}(\zeta)d\zeta$$

Now, let $f\in H^s(\mathbb{K})$. Since we know that $\mathfrak{S}(\mathbb{K})$ is dense in $H^s(\mathbb{K})$ so there exists $\sigma(\zeta)$ such that

$$\|f-\sigma\|_{H^s(\mathbb{K})} < \varepsilon, \quad \text{where } \sigma(\zeta) = \left(\hat{\gamma}^{-\frac{s}{2}}(\zeta)\hat{h}(\zeta)\right)^{\vee}, \quad \text{and} \quad h(\zeta)\in\mathfrak{S}(\mathbb{K}). \qquad (14)$$

Therefore,

$$||P_j(f-\sigma)||_{H^s(\mathbb{K})} < \varepsilon \implies ||P_j f||_{H^s(\mathbb{K})} < \varepsilon + ||P_j\sigma||_{H^s(\mathbb{K})}. \qquad (15)$$

So, we only need to show that $\lim_{j\to-\infty} ||P_j\sigma||^2_{H^s(\mathbb{K})} = 0$. Now, by using (12) and (14), we have

$$||P_j\sigma||_{H^s(\mathbb{K})} = \sup_{\|g\|\le 1} |\langle P_j\sigma, g\rangle_{H^s(\mathbb{K})}| \le B_j \sqrt{\sum_{k\in\mathbb{N}_0} |\langle \sigma, \tilde{\phi}^{(j)}_{j,k}\rangle|^2}.$$

Therefore,

$$||P_j\sigma||^2_{H^s(\mathbb{K})} \le B_j^2 \int_{\mathbb{K}} \hat{\gamma}^{\frac{s}{2}}(\zeta)\hat{h}(\zeta)\overline{\hat{\tilde{\phi}}^{(j)}(\pi^j\zeta)} \left\{ \sum_{l\in\mathbb{N}_0} \hat{\gamma}^{\frac{s}{2}}(\zeta+\pi^{-j}w(l))\bar{\hat{h}}(\zeta+\pi^{-j}w(l))\hat{\tilde{\phi}}^{(j)}(\pi^j\zeta+w(l)) \right\} d\zeta.$$

By Cauchy–Schwarz inequality, we get

$$||P_j\sigma||^2_{H^s(\mathbb{K})} \le B_j^2 \sum_{l\in\mathbb{N}_0} \left(\int_{\mathbb{K}} \hat{\gamma}^s(\zeta)|\hat{h}(\zeta)|^2|\hat{\tilde{\phi}}^{(j)}(\pi^j\zeta)|^2 d\zeta \right)^{\frac{1}{2}} \times \left(\int_{\mathbb{K}} \hat{\gamma}^s(\zeta+\pi^{-j}w(l))|\bar{\hat{h}}(\zeta+\pi^{-j}w(l))|^2|\hat{\tilde{\phi}}^{(j)}(\pi^j\zeta+w(l))\}|^2 d\zeta \right)^{\frac{1}{2}}.$$

Since $\hat{h} \in \mathfrak{S}(\mathbb{K})$, so there exists a characteristic funtion $\varphi_r(\zeta-\zeta_0)$ of the set $\zeta_0+\mathfrak{P}^r$, where r is some integers. Now h can be written as $\hat{h}(\zeta) = q^{\frac{r}{2}}\varphi_r(\zeta-\zeta_0)$. If $\zeta+\pi^{-j}w(k) \in \zeta_0+\mathfrak{P}^r$, then $|\pi^{-j}w(k)| \le q^{-r}$, hence $|w(k)| \le q^{-r-j}$. Then summation index l is bounded by q^{-r-j}. So using this, we get

$$\begin{aligned} ||P_j\sigma||^2_{H^s(\mathbb{K})} &\le B_j^2 q^{-r-j} \left(\int_{\mathbb{K}} \hat{\gamma}^s(\zeta)|\hat{h}(\zeta)|^2|\hat{\tilde{\phi}}^{(j)}(\pi^j\zeta)|^2 d\zeta \right)^{\frac{1}{2}} \\ &\le B_j^2 q^{-r-j} \int_{\zeta_0+\mathfrak{P}^r} \hat{\gamma}^s(\zeta)|\hat{\tilde{\phi}}^{(j)}(\pi^j\zeta)|^2 d\zeta \\ &= B_j^2 q^{-r} \int_{\pi^{-j}\zeta_0+\mathfrak{P}^{-j+r}} \hat{\gamma}^s(\pi^{-j}\zeta)|\hat{\tilde{\phi}}^{(j)}(\zeta)|^2 d\zeta. \end{aligned}$$

Therefore there exists j such that

$$||P_j\sigma||_{H^s(\mathbb{K})} < \varepsilon.$$

Hence,

$$\lim_{j\to-\infty} ||P_j f||_{H^s(\mathbb{K})} = 0 \quad \text{a.e.}$$

□

Now, for every $j \in \mathbb{Z}$, we consider that $\{\phi_{j,k}\}_{k\in\mathbb{N}_0}$ and $\{\tilde{\phi}_{j,k}\}_{k\in\mathbb{N}_0}$ are Riesz bases of its closed linear span V_j and $\tilde{V}_j$. For wavelets, we will have $V_j \subset V_{j+1}$ and $\tilde{V}_j \subset \tilde{V}_{j+1}$. Suppose $\{\phi_{j,k}\}_{k\in\mathbb{N}_0}$ and $\{\tilde{\phi}_{j,k}\}_{k\in\mathbb{N}_0}$ are biorthogonal. Then, Theorem 2 follows that the maps $P_{j+1}-P_j$ are projections onto $W_j(W_j = V_{j+1}\cap\tilde{V}_j^{\perp})$. This leads to a dual multiresolution analyses of $H^s(\mathbb{K})$,

$$V_j \subset V_{j+1}, \quad \tilde{V}_j \subset \tilde{V}_{j+1}.$$

Correspondingly, since $\phi^{(j)} \in V_j \subset V_{j+1}; \tilde{\phi}^{(j)} \in \tilde{V}_j \subset \tilde{V}_{j+1}$, we have

$$\phi^{(j)} = \sum_{k\in\mathbb{N}_0} h_k^{(j)} \phi_{j+1,k}^{(j+1)}; \qquad \tilde{\phi}^{(j)} = \sum_{k\in\mathbb{N}_0} \tilde{h}_k^{(j)} \tilde{\phi}_{j+1,k}^{(j+1)}. \tag{16}$$

Taking Fourier transform of the Eq. (16), we get

$$\hat{\phi}^{(j)}(\zeta) = m_0^{(j+1)}(\pi\zeta)\hat{\phi}^{(j+1)}(\pi\zeta); \quad \hat{\tilde{\phi}}^{(j)}(\zeta) = \tilde{m}_0^{(j+1)}(\pi\zeta)\hat{\tilde{\phi}}^{(j+1)}(\pi\zeta). \tag{17}$$

Theorem 3 *Let $m_0^{(j)}(\zeta)$ and $\tilde{m}_0^{(j)}(\zeta)$ given by (17) satisfy*

$$\sum_{r=0}^{q-1} m_0^{(j)}(\zeta + \pi w(r))\bar{\tilde{m}}_0^{(j)}(\zeta + \pi w(r)) = 1 \quad a.e.$$

Proof Proof is simple, hence omitted. □

The functions $\phi^{(j)}, \tilde{\phi}^{(j)} \in H^s(\mathbb{K})$ are biorthogonal if they satisfy

$$\delta_{k,0} = \langle \phi^{(j)}(\pi^{-j} \cdot - w(k)), \tilde{\phi}^{(j)}(\pi^{-j} \cdot) \rangle_{H^s(\mathbb{K})}. \tag{18}$$

Above equation in terms of Fourier transform is equivalent to

$$\sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j}(\zeta + w(k)))\hat{\phi}^{(j)}(\zeta + w(k))\overline{\hat{\tilde{\phi}}^{(j)}(\zeta + w(k))} = 1. \tag{19}$$

We solve it for $\tilde{\phi}^{(j)} \in V_0$, that is,

$$\tilde{\phi}^{(j)}(x) = \sum_{k\in\mathbb{N}_0} a_k^{(j)} \phi^{(j)}(x - w(k))$$

so

$$\begin{aligned} \hat{\tilde{\phi}}^{(j)}(\zeta) &= \sum_{k\in\mathbb{N}_0} a_k^{(j)} \bar{\chi}_k(\zeta)\hat{\phi}^{(j)}(\zeta) \\ &= a^{(j)}(\zeta)\hat{\phi}^{(j)}(\zeta), \quad \text{where } a^{(j)}(\zeta) = \sum_{k\in\mathbb{N}_0} a_k^{(j)} \bar{\chi}_k(\zeta). \end{aligned}$$

Substituting these values in (19), we get

$$\overline{a^{(j)}(\zeta)} = \left(\sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j}(\zeta + w(k)))|\hat{\phi}^{(j)}(\zeta + w(k))|^2 \right)^{-1}. \tag{20}$$

There are many ways to choose $\phi^{(j)}$ and $\tilde{\phi}^{(j)}$ in order to obtain such a result.

Biorthogonality of Wavelets

Let $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{V}\}_{j\in\mathbb{Z}}$ be dual MRAs with scaling function $\phi^{(j)}$ and $\tilde{\phi}^{(j)}$ respectively. Following [8], there exist integral periodic functions $m_0^{(j+1)}$ and $\tilde{m}_0^{(j+1)}$ such that $\hat{\phi}^{(j)}(\zeta) = m_0^{(j+1)}(\pi\zeta)\hat{\phi}^{(j+1)}(\pi\zeta)$ and $\hat{\tilde{\phi}}^{(j)}(\zeta) = m_0^{(j+1)}(\pi\zeta)\hat{\tilde{\phi}}^{(j+1)}(\pi\zeta)$. Assume that there exist integral periodic functions m_r and $\tilde{m}_r$, $1 \le r \le q-1$, such that

$$M^{(j)}(\zeta)(\tilde{M}^{(j)})^*(\zeta) = I, \tag{21}$$

where $M^{(j)}(\zeta) = [m_{r_1}^{(j)}(\pi\zeta + \pi w(r_2))]_{r_1,r_2=0}^{q-1}$ and $\tilde{M}^{(j)}(\zeta) = [\tilde{m}_{r_1}^{(j)}(\pi\zeta + \pi w(r_2))]_{r_1,r_2=0}^{q-1}$, $j \in \mathbb{Z}$. Now for $1 \le r \le q-1$, we define the associated wavelets $\psi_r^{(j)}$ and $\tilde{\psi}_r^{(j)}$ as follows:

$$\hat{\psi}_r^{(j)}(\pi^j\zeta) = m_r^{(j+1)}(\pi^{j+1}\zeta)\hat{\phi}^{(j+1)}(\pi^{j+1}\zeta) \quad \text{and}$$
$$\hat{\tilde{\psi}}_r^{(j)}(\pi^j\zeta) = \tilde{m}_r^{(j+1)}(\pi^{j+1}\zeta)\hat{\tilde{\phi}}^{(j+1)}(\pi^{j+1}\zeta).$$

Assume that there is $M > 0$ such that

$$\sup_{j\in\mathbb{Z}}\sup_{\zeta\in\mathbb{R}} |m_r^{(j)}(\zeta)| \le M, \qquad \sup_{j\in\mathbb{Z}}\sup_{\zeta\in\mathbb{R}} |\tilde{m}_r^{(j)}(\zeta)| \le M; \quad r \in \{1, 2, 3, \ldots, q-1\}. \tag{22}$$

In this section, our main aim is to show that the wavelets associated with dual MRAs are biorthogonal and they form Riesz bases for $H^s(\mathbb{K})$.

For every j and $1 \le r \le q-1$, we define linear and continuous operators P_j and Q_j from $H^s(\mathbb{K})$ into itself as

$$P_j f = \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\phi}_{j,k}\rangle_{H^s(\mathbb{K})}\phi_{j,k}$$

and

$$Q_j f = \sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}_{r,j,k}\rangle_{H^s(\mathbb{K})}\psi_{r,j,k}.$$

The same can be defined for $\tilde{P}_j$ and $\tilde{Q}_j$.

It can be easly shown that

$$P_{j+1} - P_j = Q_j, \tag{23}$$

and

$$f = \sum_{r=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}_{r,j,k}^{(j)}\rangle_{H^s(\mathbb{K})}\psi_{r,j,k}^{(j)} = \sum_{r=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} \langle f, \psi_{r,j,k}^{(j)}\rangle_{H^s(\mathbb{K})}\tilde{\psi}_{r,j,k}^{(j)}, \tag{24}$$

in Sobolev space.

Theorem 4 (Main Theorem) *Let $\phi^{(j)}$ and $\tilde{\phi}^{(j)}$ be scaling functions for dual MRAs and $\psi_r^{(j)}$ and $\tilde{\psi}_r^{(j)}$, $1 \le r \le q-1$, be associated wavelets satisfying the matrix condition 21. Then the collections $\{\psi_{r,j,k} : 1 \le r \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_{r,j,k} : 1 \le r \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ are biorthogonal. In addition, if*

$$\hat{\phi}^{(j)}(\zeta) \le \frac{M}{\hat{\gamma}^{\frac{s}{2}}(\pi^{-j}\zeta)(1+|\zeta|)^{\frac{1}{2}+\varepsilon}}, \quad \hat{\tilde{\phi}}^{(j)}(\zeta) \le \frac{M}{\hat{\gamma}^{\frac{s}{2}}(\pi^{-j}\zeta)(1+|\zeta|)^{\frac{1}{2}+\varepsilon}}, \tag{25}$$

$$\sup_{j\in\mathbb{Z}}\sup_{\zeta\in\mathbb{K}} |m_r^{(j)}(\zeta)| \le M|\zeta|, \quad \text{and} \quad \sup_{j\in\mathbb{Z}}\sup_{\zeta\in\mathbb{K}} |m_r^{(j)}(\zeta)| \le M|\zeta|, \tag{26}$$

for some constant $M > 0$, $\varepsilon > 0$ and for a.e. $\zeta \in \mathbb{K}$, then $\{\psi_{r,j,k}^{(j)} : 1 \le r \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_{r,j,k}^{(j)} : 1 \le r \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ form Riesz bases for $H^s(\mathbb{K})$.

Proof We start by proving $\{\psi_{r,j,k} : k \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_{r,j,k} : k \in \mathbb{N}_0\}$ are biorthogonal to each other. For this, we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j}(\zeta+w(k)))\hat{\psi}_r^{(j)}(\zeta+w(k))\overline{\hat{\tilde{\psi}}_r^{(j)}(\zeta+w(k))}\\
&\quad=\sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j}(\zeta+w(k)))m_r^{(j+1)}(\pi(\zeta+w(k)))\hat{\phi}^{j+1}(\pi(\zeta+w(k)))\\
&\quad\times\overline{\tilde{m}_r^{(j+1)}}(\pi(\zeta+w(k)))\overline{\hat{\tilde{\phi}}^{(j+1)}}(\pi(\zeta+w(k)))\\
&\quad=\sum_{r=0}^{q-1}\sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j}(\zeta+w(qk+r)))m_r^{(j+1)}\\
&\quad(\pi(\zeta+w(qk+r)))\hat{\phi}^{j+1}(\pi(\zeta+w(qk+r)))\\
&\quad\times\overline{\tilde{m}_r^{(j+1)}}(\pi(\zeta+w(qk+r)))\overline{\hat{\tilde{\phi}}^{(j+1)}}(\pi(\zeta+w(qk+r)))\\
&\quad=\sum_{r=0}^{q-1}\sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j-1}(\pi\zeta+\pi w(r)+w(k))\hat{\phi}^{j+1}(\pi\zeta+\pi w(r)\\
&\quad+w(k))\overline{\hat{\tilde{\phi}}^{(j+1)}}(\pi\zeta+\pi w(r)+w(k))\\
&\quad\times m_r^{(j+1)}(\pi\zeta+\pi w(r))\overline{\tilde{m}_r^{(j+1)}}(\pi\zeta+\pi w(r))\\
&\quad=\sum_{r=0}^{q-1} m_r^{(j+1)}(\pi\zeta+\pi w(r))\overline{\tilde{m}_r^{(j+1)}}(\pi\zeta+\pi w(r))\\
&\quad=1.
\end{aligned}
$$

Therefore, by Theorem 1, $\{\psi_{r,j,k}^{(j)} : k \in \mathbb{N}_0\}$ is biorthogonal to $\{\tilde{\psi}_{r,j,k}^{(j)} : k \in \mathbb{N}_0\}$.

Now let $\psi_{r,j,k}^{(j)} \in V_j$, therefore $\psi_{r,j,k}^{(j)} \in V_{j+1} \subset V_{j'}$ for $j' > j$. Hence, it will be enough to show that $\tilde{\psi}_{r',j',k'}^{(j')}$ is orthogonal to every element of $V_{j'}$. Let $f \in V_{j'}$. Hence, there exists an l^2-sequence $\{c_k^{(j')}\}$ such that $f = \sum_{k\in\mathbb{N}_0} c_k^{(j')}\phi_{j',k}$ in $H^s(\mathbb{K})$. Therefore

$$
\begin{aligned}
&\langle \tilde{\psi}_{r',j',k'}^{(j')}, \phi_{j',k}\rangle_{H^s(\mathbb{K})}\\
&\quad=q^{-j}\int_{\mathbb{K}} \hat{\gamma}^s(\zeta)\hat{\tilde{\psi}}_{r'}^{(j')}(\pi^{j'}\zeta)\overline{\chi}_{k'}(\pi^{j'}\zeta)\overline{\hat{\phi}^{(j')}}(\pi^{j'}\zeta)\chi_k(\pi^{j'}\zeta)d\zeta\\
&\quad=q^{-j}\int_{\mathbb{K}} \hat{\gamma}^s(\zeta)\tilde{m}_{r'}^{(j'+1)}(\pi^{j'+1}\zeta)\hat{\tilde{\phi}}^{(j'+1)}\\
&\quad(\pi^{j'+1}\zeta)\overline{\chi}_{k'}(\pi^{j'}\zeta)\overline{m_0^{(j'+1)}}(\pi^{j'+1}\zeta)\overline{\hat{\phi}^{(j'+1)}}(\pi^{j'+1}\zeta)\chi_k(\pi^{j'}\zeta)d\zeta\\
&\quad=\int_{\mathbb{K}} \hat{\gamma}^s(\pi^{-j'}\zeta)\hat{\tilde{\phi}}^{(j'+1)}(\pi\zeta)\overline{\hat{\phi}^{(j'+1)}}(\pi\zeta)\tilde{m}_{r'}^{(j'+1)}(\pi\zeta)\overline{m_0^{(j'+1)}}\\
&\quad(\pi\zeta)\overline{\chi}_{k'}\chi_k(\zeta)d\zeta\\
&\quad=\int_{\mathbb{D}}\sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j'}(\zeta+w(n)))\hat{\tilde{\phi}}^{(j'+1)}(\pi(\zeta+w(n)))\overline{\hat{\phi}^{(j'+1)}}(\pi(\zeta+w(n)))\tilde{m}_{r'}^{(j'+1)}\\
&\quad(\pi(\zeta+w(n)))\times\overline{m_0^{(j'+1)}}(\pi(\zeta+w(n)))\overline{\chi}_{k'}\chi_k(\zeta)d\zeta\\
&\quad=\int_{\mathbb{D}}\sum_{t=0}^{q-1}\sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j'}(\zeta+w(qn+t)))\hat{\tilde{\phi}}^{(j'+1)}(\pi(\zeta+w(qn+t)))
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\hat{\phi}^{(j'+1)}}(\pi(\zeta + w(qn+t))) \times \tilde{m}_{r'}^{(j'+1)}(\pi(\zeta + w(qn+t)))\\
&\overline{m_0^{(j'+1)}}(\pi(\zeta + w(qn+t)))\overline{\chi}_{k'}(\zeta)\chi_k(\zeta)d\zeta\\
&= \int_{\mathbb{D}} \sum_{t=0}^{q-1}\sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\pi^{-j'-1}(\pi\zeta + w(n) + \pi w(t))\hat{\tilde{\phi}}^{(j'+1)}\\
&(\pi\zeta + w(n) + \pi w(t))\overline{\hat{\phi}^{(j'+1)}}(\pi\zeta + w(n) + \pi w(t))\\
&\times \tilde{m}_{r'}^{(j'+1)}(\pi\zeta + \pi w(t))\overline{m_0^{(j'+1)}}(\pi\zeta + \pi w(t))\overline{\chi}_{k'}(\zeta)\chi_k(\zeta)d\zeta\\
&= \int_{\mathbb{D}} \left\{\sum_{t=0}^{q-1} \tilde{m}_{r'}^{(j'+1)}(\pi\zeta + \pi w(t))\overline{m_0^{(j'+1)}}(\pi\zeta + \pi w(t))\right\}\overline{\chi}_{k'}(\zeta)\chi_k(\zeta)d\zeta\\
&= 0.
\end{aligned}
$$

Hence,

$$
\langle \tilde{\psi}_{r',j',k'}^{(j')}, f\rangle_{H^s(\mathbb{K})} = \left\langle \tilde{\psi}_{r',j',k'}^{(j')}, \sum_{k\in\mathbb{N}_0} c_k^{(j')}\phi_{j',k}\right\rangle_{H^s(\mathbb{K})} = \sum_{k\in\mathbb{N}_0} c_k^{(j')}\langle \tilde{\psi}_{r',j',k'}^{(j')}, \phi_{j',k}\rangle_{H^s(\mathbb{K})} = 0.
$$

Since $\{\psi_{r,j,k} : 1 \le r \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_{r,j,k} : 1 \le r \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ are biorthogonal to each other, therefore both the collections are linearly independent by Lemma 1. We only need to verify the frame condition for these two collections to form Riesz bases for $H^s(\mathbb{K})$.

To show the frame condition, we have to show that there exist constants $C_1, C_2, \tilde{C}_1$ and $\tilde{C}_2$ such that for every $f \in H^s(\mathbb{K})$,

$$
C_1\|f\|^2_{H^s(\mathbb{K})} \le \sum_{r=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{r,j,k}^{(j)}\rangle_{H^s(\mathbb{K})}|^2 \le C_2\|f\|^2_{H^s(\mathbb{K})}, \tag{27}
$$

and

$$
\tilde{C}_1\|f\|^2_{H^s(\mathbb{K})} \le \sum_{r=1}^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \tilde{\psi}_{r,j,k}^{(j)}\rangle_{H^s(\mathbb{K})}|^2 \le \tilde{C}_2\|f\|^2_{H^s(\mathbb{K})}. \tag{28}
$$

To show the existence of upper bounds in (27) and (28), we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{r,j,k}^{(j)}\rangle_{H^s(\mathbb{K})}|^2 = \sum_{k\in\mathbb{N}_0} |\int_{\mathbb{K}} \hat{\gamma}^s(\zeta)\hat{f}(\zeta)q^{-\frac{j}{2}}\overline{\hat{\psi}_r^{(j)}}(\pi^j\zeta)\chi_k(\pi^j\zeta)d\zeta|^2\\
&= q^{-j}\sum_{k\in\mathbb{N}_0}\left|\int_{\mathfrak{P}^{-j}} \sum_{l\in\mathbb{N}_0} \hat{\gamma}^s(\zeta + \pi^{-j}w(l))\hat{f}(\zeta + \pi^{-j}w(l))\right.\\
&\left.\times q^{-\frac{j}{2}}\overline{\hat{\psi}_r^{(j)}}(\pi^j\zeta + w(l))\chi_k(\pi^j\zeta)d\zeta\right|^2\\
&= \int_{\mathfrak{P}^{-j}} \left|\sum_{l\in\mathbb{N}_0} \hat{\gamma}^s(\zeta + \pi^{-j}w(l))\hat{f}(\zeta + \pi^{-j}w(l))q^{-\frac{j}{2}}\overline{\hat{\psi}_r^{(j)}}(\pi^j\zeta + w(l))\right|^2 d\zeta
\end{aligned}
$$

$$\leq \int_{\mathfrak{P}^{-j}} \left(\sum_{l\in\mathbb{N}_0} \hat{\gamma}^{s(1+\delta)}(\zeta+\pi^{-j}w(l))|\hat{f}(\zeta+\pi^{-j}w(l))|^2|\overline{\hat{\psi}_r^{(j)}}(\pi^j\zeta+w(l))|^{2\delta} \right)$$
$$\times \left(\sum_{m\in\mathbb{N}_0} \hat{\gamma}^{s(1-\delta)}(\zeta+\pi^{-j}w(m))\overline{\hat{\psi}_r^{(j)}}(\pi^j\zeta+w(m))|^{2(1-\delta)} \right) d\zeta.$$

We have assumed that $\hat{\phi}^{(j)}(\zeta) \leq \frac{M}{\hat{\gamma}^{\frac{s}{2}}(\pi^{-j}\zeta)(1+|\zeta|)^{\frac{1}{2}+\varepsilon}}$, hence we have, $\hat{\psi}_r^{(j)}(\zeta) \leq \frac{M|\pi\zeta|}{\hat{\gamma}^{\frac{s}{2}}(\pi^{-j}\zeta)(1+|\zeta|)^{\frac{1}{2}+\varepsilon}}$. So, for all $\delta \in (0,1)$ such that $(1-\delta) > \frac{1}{1+2\varepsilon}$ and for all $j \in \mathbb{Z}$, the series $\sum_{m\in\mathbb{N}_0} \hat{\gamma}^{s(1-\delta)}(\zeta+\pi^{-j}w(m))\overline{\hat{\psi}_r^{(j)}}(\pi^j\zeta+w(m))|^{2(1-\delta)}$ is uniformly bounded. Hence there exists $C > 0$ such that

$$\begin{aligned} \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{r,j,k}^{(j)}\rangle_{H^s(\mathbb{K})}|^2 &\leq C\int_{\mathbb{K}} \hat{\gamma}^{s(1+\delta)}(\zeta)|\hat{f}(\zeta)|^2 \sum_{j\in\mathbb{Z}} |\hat{\psi}_r^{(j)}(\pi^j\zeta)|^{2\delta} d\zeta \\ &\leq C\|f\|^2_{H^s(\mathbb{K})} \left(\sup_{j\in\mathbb{Z}} \sup_{1<|\zeta|\leq q} \sum_{k\in\mathbb{Z}} \hat{\gamma}^{s\delta}(\pi^{-j}\zeta)|\hat{\psi}_r^{(k+j)}(\pi^k\zeta)|^{2\delta} \right) \\ &\leq C_2\|f\|^2_{H^s(\mathbb{K})}. \end{aligned}$$

For the above inequality, notice that

$$\begin{aligned} &\sup_{j\in\mathbb{Z}} \sup_{1<|\zeta|\leq q} \sum_{k=-\infty}^{0} \hat{\gamma}^{s\delta}(\pi^{-j}\zeta)|\hat{\psi}_r^{(k+j)}(\pi^k\zeta)|^{2\delta} \\ &= \sup_{j\in\mathbb{Z}} \sup_{1<|\zeta|\leq q} \sum_{k=0}^{\infty} \hat{\gamma}^{s\delta}(\pi^{-j}\zeta)|\hat{\psi}_r^{(-k+j)}(\pi^{-k}\zeta)|^{2\delta} \\ &\leq \sup_{j\in\mathbb{Z}} \sup_{1<|\zeta|\leq q} \sum_{k=0}^{\infty} \hat{\gamma}^{s\delta}(\pi^{-j}\zeta) \left[\frac{M}{\hat{\gamma}^{\frac{s}{2}}(\pi^{-j}\zeta)(1+|\pi^{-k+1}\zeta|)^{\frac{1}{2}+\varepsilon}} \right]^{2\delta} \\ &= M^{2\delta} \sup_{1<|\zeta|\leq q} \sum_{k=0}^{\infty} \frac{1}{(1+q^{k-1}|\zeta|)^{\delta(1+2\varepsilon)}} \\ &= M^{2\delta} \frac{q^{\delta(1+2\varepsilon)}}{1-q^{-\delta(1+2\varepsilon)}} < \infty. \end{aligned}$$

Also,

$$\begin{aligned} &\sup_{j\in\mathbb{Z}} \sup_{1<|\zeta|\leq q} \sum_{k=1}^{\infty} \hat{\gamma}^{s\delta}(\pi^{-j}\zeta)|\hat{\psi}_r^{(k+j)}(\pi^k\zeta)|^{2\delta} \\ &\leq \sup_{j\in\mathbb{Z}} \sup_{1<|\zeta|\leq q} \sum_{k=1}^{\infty} \hat{\gamma}^{s\delta}(\pi^{-j}\zeta) \left[\frac{M|\pi^{k+1}\zeta|}{\hat{\gamma}^{\frac{s}{2}}(\pi^{-j}\zeta)(1+|\pi^{k+1}\zeta|)^{\frac{1}{2}+\varepsilon}} \right]^{2\delta} \\ &\leq M^{2\delta} \sum_{k=1}^{\infty} q^{-k} \\ &= M^{2\delta}(q-1)^{-1} < \infty. \end{aligned}$$

Similarly, we can show that the upper bound in (28) exists.

Since upper bounds in (27) and (28) exist, we can easily show that the lower bounds in (27) and (28) also exist. □

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