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ORIGINAL PAPER

Stream Function Solution of the Brinkman Equation in Parabolic Cylindrical Coordinates

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Abstract
The present work concerns the general stream function solution of the Brinkman equation in parabolic cylindrical coordinates, arising in the study of fluid flow through porous medium. Analytical stream function solutions of this equation are available in the coordinates (Cartesian, cylindrical polar, spherical polar and prolate spheroidal coordinates). Stream function solution of the Stokes equation in parabolic cylindrical coordinates is also investigated analytically. The parabolic cylinder functions are a class of functions which are the solution of Weber differential equation. A transformation of parabolic cylinder function into the Whittaker function is used. Method of inverse operator is applied to obtain particular integral in solving the Stokes equation. Explicit expressions of velocity components and vorticity are also reported.

Keywords Brinkman equation · Weber differential equation · Parabolic cylinder function · Whittaker function

Mathematics Subject Classification 35G05 · 35C05 · 76S05

Introduction

The study of viscous fluid flow through porous media is of interest to a wide range of researchers due to its numerous applications in many fields such as bio-mechanics, physical sciences, chemical engineering, etc.. Due to vast applications, several conceptual models have been developed for describing fluid flow through porous media [11]. Henry Darcy (1856), stated that the seepage velocity of fluid flow through porous medium is proportional to the

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driving pressure gradient commonly known as Darcy law. Mathematically,

$$\nabla p = -\frac{\mu}{k}\ \vec{v}. \tag{1}$$

Brinkman [1] proposed the modified version of the Darcys law for porous medium and provided an expression of the form

$$\nabla p = -\frac{\mu}{k}\ \vec{v} + \mu_e\ \Delta\vec{v}. \tag{2}$$

Here, $\vec{v}$ is the seepage velocity, p is the pressure, μ is the fluid viscosity, k is the permeability and μ_e is the effective viscosity of the fluid, flowing in the porous medium. Previously, several authors solved these Eqs. (1) and (2) analytically in various coordinates systems, such as, Cartesian, cylindrical polar, spherical polar and prolate spheroidal coordinates.

A Cartesian-tensor solution of the Brinkman equation was investigated by Qin and Kaloni [14] and they also evaluated the drag force on a porous sphere in an unbounded medium. Pop and Cheng [13] evaluated a particular solution of the Brinkman equation in the cylindrical polar coordinates. They also presented the streamlines and velocity profiles for the flow past a circular cylinder embedded in a constant porosity medium. By using the theory of generalised eigenfunctions, Dassios et al. [2] obtained the complete semi-separable stream function solution of the Stokes equations in prolate spheroidal coordinates. Khuri and Wazwaz [9] reported the solution of a second order partial differential equation $E^2\psi = 0$ in the toroidal coordinates which arised in the case of irrotational fluid motion. Zlatanovski [19] in his celebrated paper, investigated the general Stokes stream function solution of the Stokes equations and the Brinkman equations for axisymmetric creeping flow in the prolate spheroidal coordinates. He also reported the general stream function solutions of these equations in spherical polar coordinates as a limiting case.

Qudais et al. [15] investigated the two dimensional incompressible fluid flow past parabolic bodies with uniform stream, and solved numerically the Navier-Stokes and energy equations in the parabolic coordinates for stream function. Haddad et al. [7] reported the numerical solutions for two-dimensional fluid flow past a parabolic cylinder embedded in porous media using the Brinkman-Forchheimer model in the parabolic cylindrical coordinates. Gil et al. [6] described the various methods for the evaluation of real parabolic cylinder functions and their derivatives.

Deo and Tiwari [3] investigated the stream function solution of irrotational flow equation $E^2\psi = 0$ in the bispherical polar coordinates and toroidal coordinates, where E^2 is the well-known differential operator as defined in the book [8]. Srivastava and Deo [16] solved the Brinkman equation for variable permeability on the presence of uniform magnetic field in a channel filled with porous medium. Expressions for velocity and acceleration of a moving body were obtained in the parabolic cylindrical coordinates by Omonile et al. [12]. Zaytoon et al. [18] investigated the Weber's inhomogeneous differential equation for both initial value problems and boundary value problems. Deo et al. [4] obtained the stream function solution of the Brinkman equation in the cylindrical polar coordinates. Recently, Deo and Maurya [5] investigated the generalized stream function solution for the Brinkman equation in cylindrical polar coordinates. Lack of analytical stream function solutions of the Brinkman/Stokes equations in parabolic cylindrical coordinates, motivate us to carry forward the present research work.

In this research work, we have obtained the stream function solution (in analytical form) of the Brinkman equation for parabolic cylindrical coordinates. Analytical stream function solution of the Stokes equation is also obtained in the same coordinates. Explicit expressions of velocity components and vorticity are reported and the expressions for pressure

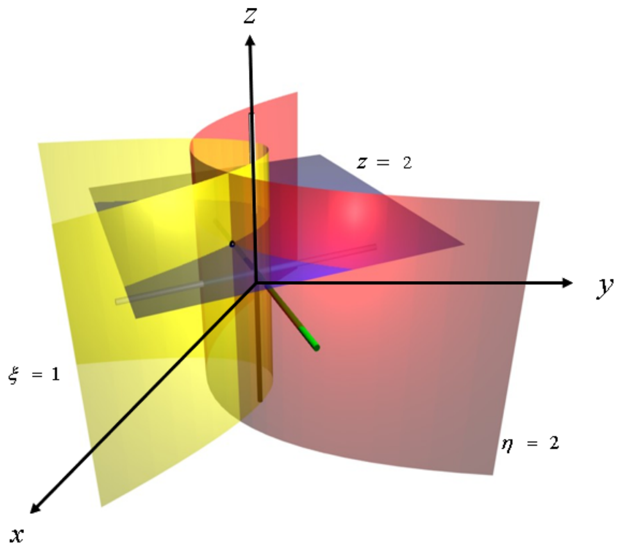

Fig. 1 Geometry of the parabolic cylindrical coordinates

and stress components are evaluated. However, the expressions for pressure and stresses are cumbersome, so they are not mentioned here.

Mathematical Formulation and Solution

Transformation equations between parabolic cylindrical coordinates (ξ, η, z) and the Cartesian coordinates (x, y, z) are:

$$x = c\,(\xi^2 - \eta^2),\ \ y = 2c\xi\eta,\ \ z = z, \tag{3}$$

where,

$$c > 0,\ -\infty < \xi < \infty,\ 0 \le \eta < \infty \text{ and } -\infty < z < \infty.$$

The scale factors (h_ξ, h_η, h_z) of the parabolic cylindrical coordinates are:

$$h_\xi = h_\eta = 2c\sqrt{\xi^2 + \eta^2},\ \ h_z = 1. \tag{4}$$

The gradient operator in the parabolic cylindrical coordinates is given by

$$\nabla \equiv \frac{\hat{\xi}}{2c\sqrt{\xi^2 + \eta^2}}\frac{\partial}{\partial \xi} + \frac{\hat{\eta}}{2c\sqrt{\xi^2 + \eta^2}}\frac{\partial}{\partial \eta} + \hat{z}\,\frac{\partial}{\partial z}, \tag{5}$$

where, $\hat{\xi},\ \hat{\eta},\ \hat{z}$ are unit base vectors along coordinates axes $\xi,\ \eta,\ z$, respectively. Also, the Laplacian operator in the parabolic cylindrical coordinates comes out as

$$\Delta \equiv \frac{1}{4c^2(\xi^2 + \eta^2)}\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right] + \frac{\partial^2}{\partial z^2}. \tag{6}$$

Assuming that fluid viscosity μ, effective viscosity μ_e and permeability of porous medium k are independent of space coordinates $(\xi,\ \eta,\ z)$. Applying the curl operator on both sides in the Eq. (2), we get

$$(\Delta - \alpha^2)\nabla \times \vec{v} = \vec{0}, \tag{7}$$

where, $\alpha^2 = \frac{\mu}{\mu_e k}$.

Let the velocity of an incompressible viscous fluid in the porous medium is perpendicular to the direction of z-axis, so that we may take

$$\vec{v} = (v_\xi(\xi,\eta),\ v_\eta(\xi,\eta),\ 0).$$

The equation of continuity for an incompressible viscous fluid is

$$\nabla \cdot \vec{v} = 0,$$

i.e.,

$$\frac{\partial}{\partial \xi}\left[2c\ v_\xi\sqrt{\xi^2+\eta^2}\right] + \frac{\partial}{\partial \eta}\left[2c\ v_\eta\sqrt{\xi^2+\eta^2}\right] = 0. \tag{8}$$

Introducing the stream function $\psi(\xi,\ \eta)$ in terms of velocity components satisfying the Eq. (8) by

$$v_\xi = -\frac{1}{2c\sqrt{\xi^2+\eta^2}}\frac{\partial \psi}{\partial \eta},\ v_\eta = \frac{1}{2c\sqrt{\xi^2+\eta^2}}\frac{\partial \psi}{\partial \xi}. \tag{9}$$

Now, since

$$\nabla \times \vec{v} = \hat{z}\ \Delta\psi. \tag{10}$$

So, from Eqs. (7) and (10), we obtain

$$\Delta(\Delta - \alpha^2)\psi(\xi,\eta) = 0. \tag{11}$$

Let $\psi(\xi,\eta) = \psi_1(\xi,\eta) + \psi_2(\xi,\eta)$ be the general stream function solution of the Eq. (11) such that

$$\Delta\psi_1(\xi,\eta) = 0,\ \text{and}\ (\Delta - \alpha^2)\psi_2(\xi,\eta) = 0.$$

Solving the equation $\Delta\psi_1(\xi,\eta) = 0$ by applying the method of separation of variables, we get

$$\psi_1(\xi,\eta) = \sum_{l=0}^{\infty}\left[A_l\ e^{l\xi} + B_l\ e^{-l\xi}\right]_{\sin l\eta}^{\cos l\eta}, \tag{12}$$

where, A_l and B_l are arbitrary constants. It is also mentioned that $\psi_1(\xi,\eta)$ is general stream function solution of the irrotational flow equation $\Delta\psi_1(\xi,\eta) = 0$, in the parabolic cylindrical coordinates $(\xi,\ \eta,\ z)$.

Solution for $\psi_2(\xi,\eta)$:

Since, $\psi_2(\xi,\eta)$ is satisfying the equation $(\Delta - \alpha^2)\psi_2(\xi,\eta) = 0$. Then, inserting the value of the Laplacian operator Δ by the Eq. (6) in the above equation, we obtain

$$\frac{\partial^2\psi_2}{\partial\xi^2} + \frac{\partial^2\psi_2}{\partial\eta^2} - 4\alpha^2c^2(\xi^2+\eta^2)\psi_2 = 0. \tag{13}$$

To find the general solution of the Eq. (13), we shall apply the technique of separation of variables by assuming that $\psi_2(\xi,\eta) = G(\xi)T(\eta)$. Thus, we have

$$\frac{1}{G}\left[\frac{d^2G}{d\xi^2} - \beta^2\xi^2 G\right] = -\frac{1}{T}\left[\frac{d^2T}{d\eta^2} - \beta^2\eta^2 T\right] = m, \tag{14}$$

where, $\beta^2 = 4\alpha^2 c^2$ and $m = 1, 2, 3, \cdots$.

The Eq. (14) can be separated into two ordinary differential equations

$$\frac{d^2G}{d\xi^2} - (\beta^2\xi^2 + m)G = 0, \tag{15}$$

and

$$\frac{d^2T}{d\eta^2} - (\beta^2\eta^2 - m)T = 0. \tag{16}$$

Equations (15) and (16) are particular cases of the Weber differential equation [10]. The parabolic cylinder functions $D_{\frac{-m-\beta}{2\beta}}(\xi\sqrt{2\beta})$ and $D_{\frac{m-\beta}{2\beta}}(i\xi\sqrt{2\beta})$ are two linearly independent solutions of Eq. (15). To get real solutions, we shall transform the parabolic cylinder function $D_\nu(\xi)$ in the Whittaker function $W_{r,s}(\xi)$ by using the following relation

$$D_\nu(\xi) = 2^{\frac{\nu}{2}+\frac{1}{4}}\xi^{-\frac{1}{2}}\ W_{\frac{\nu}{2}+\frac{1}{4},-\frac{1}{4}}\left(\frac{\xi^2}{2}\right).$$

Thus,

$$G(\xi) = C_m^{(1)}2^{-\frac{m}{4\beta}}\xi^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{-\frac{m}{4\beta},-\frac{1}{4}}\big(\xi^2\beta\big) + D_m^{(1)}2^{\frac{m}{4\beta}}\xi^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{\frac{m}{4\beta},-\frac{1}{4}}\big(-\xi^2\beta\big). \tag{17}$$

Similarly, the general solution of the Eq. (16) can be obtained as:

$$T(\eta) = C_m^{(2)}2^{\frac{m}{4\beta}}\eta^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{\frac{m}{4\beta},-\frac{1}{4}}(\eta^2\beta) + D_m^{(2)}2^{-\frac{m}{4\beta}}\eta^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{-\frac{m}{4\beta},-\frac{1}{4}}(-\eta^2\beta). \tag{18}$$

Therefore,

$$\begin{aligned}\psi_2(\xi,\eta) = (\xi\eta\beta)^{-\frac{1}{2}}&\Big[C_m^{(1)}2^{-\frac{m}{4\beta}}\xi^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{-\frac{m}{4\beta},-\frac{1}{4}}(\xi^2\beta) + D_m^{(1)}2^{\frac{m}{4\beta}}\xi^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{\frac{m}{4\beta},-\frac{1}{4}}(-\xi^2\beta)\Big]\\ &\Big[C_m^{(2)}2^{\frac{m}{4\beta}}\eta^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{\frac{m}{4\beta},-\frac{1}{4}}(\eta^2\beta) + D_m^{(2)}2^{-\frac{m}{4\beta}}\eta^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{-\frac{m}{4\beta},-\frac{1}{4}}(-\eta^2\beta)\Big],\end{aligned} \tag{19}$$

where, $C_m^{(i)}$ and $D_m^{(i)}$, i=1, 2, are arbitrary constants.

Hence, the general stream function solution of the Brinkman equation in parabolic cylindrical coordinates is:

$$\begin{aligned}\psi(\xi,\eta) = \sum_{l=0}^{\infty}\Big[A_l\,e^{l\xi} + B_l\,e^{-l\xi}\Big]_{\sin l\eta}^{\cos l\eta} + (\xi\eta\beta)^{-\frac{1}{2}}\sum_{m=0}^{\infty}\Big[C_m^{(1)}2^{-\frac{m}{4\beta}}\xi^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{-\frac{m}{4\beta},-\frac{1}{4}}(\xi^2\beta)\\ +D_m^{(1)}2^{\frac{m}{4\beta}}\xi^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{\frac{m}{4\beta},-\frac{1}{4}}(-\xi^2\beta)\Big]\Big[C_m^{(2)}2^{\frac{m}{4\beta}}\eta^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{\frac{m}{4\beta},-\frac{1}{4}}(\eta^2\beta)\\ +D_m^{(2)}2^{-\frac{m}{4\beta}}\eta^{-\frac{1}{2}}\beta^{-\frac{1}{4}}W_{-\frac{m}{4\beta},-\frac{1}{4}}(-\eta^2\beta)\Big].\end{aligned} \tag{20}$$

Expressions for Velocity Components, Vorticity, Pressure and Stress Components

Velocity components from Eq. (9) comes out to be

$$v_\xi = \frac{1}{2c\sqrt{\xi^2+\eta^2}}\Big[\sum_{l=0}^{\infty}\big[l^2\big(A_l\,e^{l\xi} + B_l\,e^{-l\xi}\big)\big]_{\sin l\eta}^{\cos l\eta}$$

$$+\left(\frac{\eta}{\beta\xi}\right)^{1/2} \sum_{m=0}^{\infty} 2^{-\frac{m}{2\beta}} \Big[C_m^{(1)}\ W_{-\frac{m}{4\beta},-\frac{1}{4}}(\xi^2\beta) + 2^{\frac{m}{2\beta}} D_m^{(1)} W_{\frac{m}{4\beta},-\frac{1}{4}}(-\xi^2\beta)\Big]$$
$$\Big[2^{\frac{m}{2\beta}} C_m^{(2)} \Big(m W_{\frac{m}{4\beta},-\frac{1}{4}}(\eta^2\beta) + (\beta - m) W_{\frac{m}{4\beta},\frac{3}{4}}(\eta^2\beta)\Big)$$
$$+D_m^{(2)}\Big[m W_{-\frac{m}{4\beta},-\frac{1}{4}}(-\eta^2\beta) - (\beta + m) W_{-\frac{m}{4\beta},\frac{3}{4}}(-\eta^2\beta)\Big]\Big]\Big]. \tag{21}$$

$$v_\eta = \frac{1}{2c\sqrt{\xi^2+\eta^2}}\Big[\sum_{l=0}^{\infty}[l^2(A_l\, e^{l\xi} + B_l\, e^{-l\xi})]_{\sin l\eta}^{\cos l\eta}$$
$$+\left(\frac{\xi}{\beta\eta}\right)^{1/2} \sum_{m=0}^{\infty} 2^{-\frac{m}{2\beta}} \Big[D_m^{(2)} W_{-\frac{m}{4\beta},-\frac{1}{4}}(-\eta^2\beta) + 2^{\frac{m}{2\beta}} C_m^{(2)}\ W_{\frac{m}{4\beta},-\frac{1}{4}}(\eta^2\beta)\Big]$$
$$\Big[C_m^{(1)}\Big(m W_{-\frac{m}{4\beta},-\frac{1}{4}}(\xi^2\beta) - (\beta + m) W_{-\frac{m}{4\beta},\frac{3}{4}}(\xi^2\beta)\Big)$$
$$+2^{\frac{m}{2\beta}} D_m^{(1)}(m W_{\frac{m}{4\beta},-\frac{1}{4}}(-\xi^2\beta) + (\beta - m) W_{\frac{m}{4\beta},\frac{3}{4}}(-\xi^2\beta))\Big]\Big]. \tag{22}$$

Now, since the vorticity vector is defined as the curl of the velocity vector, i.e.

$$\vec{\omega} = \nabla \times \vec{v}.$$

So, vorticity will be

$$\vec{\omega} = \omega_z \hat{z},$$

where,

$$\omega_z = \frac{1}{4c^2(\xi^2+\eta^2)^2(\beta\xi\eta)^{5/2}}\Big[\sum_{l=0}^{\infty} 2l^2(l-\eta)(\beta\xi\eta)^{5/2}\big[A_l\, e^{l\xi} + B_l\, e^{-l\xi}\big]_{\sin l\eta}^{\cos l\eta}$$
$$+\sum_{l=0}^{\infty}\Big[(-2^{-\frac{m}{2\beta}}\beta\eta^2\xi^2)\Big[D_m^{(2)}\Big(4\beta W_{-\frac{m}{4\beta}+1,-\frac{1}{4}}(-\beta\eta^2) + (-m+\beta$$
$$+2\beta^2\eta^2) W_{-\frac{m}{4\beta},-\frac{1}{4}}(-\beta\eta^2)\Big) + 2^{\frac{m}{4\beta}} C_m^{(2)}\Big(4\beta W_{\frac{m}{4\beta}+1,-\frac{1}{4}}(\beta\eta^2)$$
$$+(m+\beta-2\beta^2\eta^2) W_{\frac{m}{4\beta},-\frac{1}{4}}(\beta\eta^2)\Big)\Big]$$
$$-2^{-2-\frac{m}{2\beta}}\xi^2(\xi^2+\eta^2)\Big[D_m^{(2)}\Big[-8\beta(m-2\beta(2+\beta\eta^2)) W_{1-\frac{m}{4\beta},-\frac{1}{4}}(-\beta\eta^2)$$
$$+16\beta^2 W_{2-\frac{m}{4\beta},-\frac{1}{4}}(-\beta\eta^2) + \big[m^2 - 4m\beta(1+\beta\eta^2) + \beta^2(3+4\beta^2\eta^4)\big] W_{-\frac{m}{4\beta},-\frac{1}{4}}(-\beta\eta^2)\Big]$$
$$+2^{\frac{m}{2\beta}} C_m^{(2)}\Big[8\beta(m-2\beta(-2+\beta\eta^2)) W_{1+\frac{m}{4\beta},-\frac{1}{4}}(\beta\eta^2) + 16\beta^2 W_{2+\frac{m}{4\beta},-\frac{1}{4}}(\beta\eta^2)$$
$$+\big(m^2 - 4m\beta(-1+\beta\eta^2) + \beta^2(3+4\beta^2\eta^4)\big) W_{\frac{m}{4\beta},-\frac{1}{4}}(\beta\eta^2)\Big]\Big]\Big]$$
$$\Big[C_m^{(1)} W_{-\frac{m}{4\beta},-\frac{1}{4}}(\beta\xi^2) + 2^{\frac{m}{2\beta}} D_m^{(1)} W_{\frac{m}{4\beta},-\frac{1}{4}}(-\beta\xi^2)\Big]$$
$$+2^{-2-\frac{m}{2\beta}}\eta^2(\xi^2+\eta^2)[D_m^{(2)} W_{-\frac{m}{4\beta},-\frac{1}{4}}(-\beta\eta^2) + 2^{\frac{m}{2\beta}} C_m^{(2)} W_{\frac{m}{4\beta},-\frac{1}{4}}(\beta\eta^2)]$$
$$\Big[C_m^{(1)}[-8\beta(m+2\beta(-2+\beta\xi^2)) W_{-\frac{m}{4\beta},-\frac{1}{4}}(\beta\xi^2) + 16\beta^2 W_{2-\frac{m}{4\beta},-\frac{1}{4}}(\beta\xi^2)$$
$$+\big(m^2 + 4m\beta(-1+\beta\xi^2) + \beta^2(3+4\beta^2\xi^4)\big) W_{-\frac{m}{4\beta},-\frac{1}{4}}(\beta\xi^2)] + 2^{\frac{m}{2\beta}}$$

$$D_m^{(1)}\Big[8\beta((m+2\beta(2+\beta\xi^2)))W_{1+\frac{m}{4\beta},-\frac{1}{4}}(-\beta\xi^2)+16\beta^2 W_{2+\frac{m}{4\beta},-\frac{1}{4}}(-\beta\xi^2)$$
$$+\big(m^2+4m\beta(1+\beta\xi^2)+\beta^2(3+4\beta^2\xi^4)\big)W_{\frac{m}{4\beta},-\frac{1}{4}}(-\beta\xi^2)\Big]\Big]\Big].$$

The fluid pressure $p(\xi,\eta)$ can be evaluated from the expression

$$dp = \frac{1}{2c\sqrt{\xi^2+\eta^2}}\left[-\frac{\mu}{k}\,v_\xi+\mu_e\left(\Delta v_\xi-\frac{v_\xi}{4c^2(\xi^2+\eta^2)^2}\right)\right]d\xi$$
$$+\frac{1}{2c\sqrt{\xi^2+\eta^2}}\left[-\frac{\mu}{k}\,v_\eta+\mu_e\left(\Delta v_\xi-\frac{v_\eta}{4c^2(\xi^2+\eta^2)^2}\right)\right]d\eta, \tag{23}$$

by using MATHEMATICA software. Due to cumbersome expression, fluid pressure is not mentioned here. Stress components $T_{\xi\xi}$, $T_{\eta\eta}$, $T_{\eta\xi}$ and $T_{\xi\eta}$ [17] can be computed by using following expressions:

$$T_{\xi\xi} = -p+\frac{\mu}{c\sqrt{\xi^2+\eta^2}}\left[\frac{\partial v_\xi}{\partial\xi}+\frac{\eta v_\eta}{\xi^2+\eta^2}\right], \tag{24}$$

$$T_{\eta\eta} = -p+\frac{\mu}{c\sqrt{\xi^2+\eta^2}}\left[\frac{\partial v_\eta}{\partial\eta}+\frac{\xi v_\xi}{\xi^2+\eta^2}\right], \tag{25}$$

and

$$T_{\xi\eta}=T_{\eta\xi}=\frac{\mu}{2c\sqrt{\xi^2+\eta^2}}\left[\frac{\partial v_\eta}{\partial\xi}+\frac{\partial v_\xi}{\partial\eta}+\frac{\eta v_\xi+\xi v_\eta}{\xi^2+\eta^2}\right]. \tag{26}$$

General Stream Function Solution of the Stokes Equation in Parabolic Cylindrical Coordinates

Consider the steady flow for an incompressible, two dimensional and highly viscid fluid is governed by the Stokes equation

$$\nabla p=\mu\Delta\vec{v},\qquad \nabla\cdot\vec{v}=0, \tag{27}$$

where, p, $\vec{v}$, μ are the pressure, velocity and dynamic viscosity of the fluid, respectively.

Operating curl on the Eq. (27), we obtain

$$\Delta\left[\Delta\psi(\xi,\eta)\right]=0. \tag{28}$$

Now, we have to find the general solution of the equation (28). For this, we may take

$$\Delta\psi=\Phi(\xi,\eta). \tag{29}$$

Then,

$$\Phi(\xi,\eta)=\sum_{m=0}^{\infty}\left[(A_m\,e^{m\xi}+B_m\,e^{-m\xi})\right]_{\sin m\eta}^{\cos m\eta}, \tag{30}$$

where, A_m and B_m are arbitrary parameters.

Therefore, from Eqs. (29) and (30), we may write

$$\Delta\psi=\left[A_m\,e^{m\xi}+B_m\,e^{-m\xi}\right]_{\sin m\eta}^{\cos m\eta},$$

i.e.,

$$\left[D_\xi^2+D_\eta^2\right]\psi=4c^2(\xi^2+\eta^2)[(A_m\,e^{m\xi}+B_m\,e^{-m\xi})\Big]_{\sin m\eta}^{\cos m\eta}, \tag{31}$$

where, $D^2_\xi \equiv \frac{\partial^2}{\partial \xi^2}$ and $D^2_\eta \equiv \frac{\partial^2}{\partial \eta^2}$.

The particular solution of Eq. (31) can be determined by using the method of inverse operator:

$$\begin{aligned}&\frac{1}{D^2_\xi + D^2_\eta}\Big[4c^2(\xi^2+\eta^2)\big[\big(A_m\, e^{m\xi} + B_m\, e^{-m\xi}\big)\big]^{\cos m\eta}_{\sin m\eta}\Big]\\ &\equiv \text{Real Part of}\Bigg[\frac{1}{D^2_\xi + D^2_\eta}\Big[4c^2(\xi^2+\eta^2)\Big[(A^{(1)}_m\, e^{m\xi} + B^{(1)}_m\, e^{-m\xi})e^{im\eta}\Big]\Big]\Bigg]\\ &\quad+\text{Imaginary Part of}\Bigg[\frac{1}{D^2_\xi + D^2_\eta}\Big[4c^2(\xi^2+\eta^2)\Big[(A^{(2)}_m\, e^{m\xi} + B^{(2)}_m\, e^{-m\xi})e^{im\eta}\Big]\Big]\Bigg],\end{aligned} \tag{32}$$

where, $A^{(i)}_m$ and $B^{(i)}_m$ for $i = 1, 2$, are arbitrary constants.

Therefore, the stream function solution of the Stokes equation is:

$$\begin{aligned}\psi(\xi,\eta) = \sum_{m=0}^{\infty}\Big[\Big[&A^{(m)} + \Big(m\xi(-3+m\xi) + m^2\eta^2(-1+2m\xi)\Big)B^{(m)}\\ &+\eta\big(3+2m\xi(-2+m\xi)\big)C^{(m)}\Big]e^{m\xi} + \Big[\eta^2\xi\, D^{(m)}\\ &+\eta\xi(1+m\xi)E^{(m)}\Big]e^{-m\xi}\Big]^{\cos m\eta}_{sin m\eta},\end{aligned} \tag{33}$$

where, $A^{(m)}$, $B^{(m)}$, $C^{(m)}$, $D^{(m)}$ and $E^{(m)}$ are arbitrary parameters.

Now, we are able to evaluate the velocity components v_ξ and v_η using Eq. (9), fluid pressure by Eq. (27) and stresses by Eqs. (24)–(26).

Discussion of Results

Analytical expressions of general stream function solution of an incompressible and irrotational fluid flow equation ($\Delta\psi_1(\xi,\eta) = 0$), the Stokes equation ($\Delta^2\psi(\xi,\eta) = 0$) and the Brinkman equation ($\Delta(\Delta - \alpha^2)\psi = 0$), are investigated for parabolic cylindrical coordinates. Mathematical expressions of stream function, fluid velocity, fluid pressure, vorticity of fluid particles and fluid stresses are obtained in the combinations of transcendental functions. To compute the particular solution of the Stokes equation, method of inverse operator is used. Analytical solutions are also suitable for heat and mass transfer calculations of flow problems. Further development and utilization of these solutions are possible to obtain results for engineering applications, such as drag force and permeability.

Conclusion

The analytical stream function solution of irrotational fluid flow equation $\Delta\psi_1(\xi,\eta)$=0, and the Brinkman equation $\Delta(\Delta - \alpha^2)\psi = 0$, are investigated in the parabolic cylindrical coordinates. Explicit expressions of velocity components and vorticity are reported and the expressions for pressure and stress components are also evaluated by using Mathematica software. This analytical solution is applicable to the uniform flow, creeping flow as well as three dimensional vortex flow. One can use this stream function solution of the Brinkman equation in various fluid flow problems past bodies whose geometry is based on parabolic cylindrical coordinates. Stream function solution of the Stokes equation $\Delta^2\psi(\xi,\eta) = 0$, is investigated analytically for the same coordinates. Expressions of stream function $\psi(\xi,\eta)$, velocity, pressure, vorticity and stresses are combinations of trigonometric, exponential and Whittaker functions. Since, the parabolic cylindrical coordinate system is the generalized version of cylindrical coordinate system, so using suitable substitutions, this solution can be reduced to a stream function solution that is available in the cylindrical polar coordinates. The reported analytical solutions of the Brinkman/Stokes equation can be used to investigate real life problems based on parabolic cylindrical coordinates, i.e. fluid flow through a swarm of fibrous parabolic cylindrical particles, in the process of extraction of oils/minerals through the porous parabolic cylindrical pipes, Stokes flow past a parabolic cylinder, etc..

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