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ORIGINAL PAPER

Combination Projective Synchronization in Fractional-Order Chaotic System with Disturbance and Uncertainty

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Abstract
In this manuscript, we study combination projective synchronization (CPS). In CPS, matrix projective combination synchronization (MPCS) and inverse matrix projective combination synchronization (IMPCS) between non-identical fractional-order complex chaotic systems subjected to uncertainty and external disturbance is investigated. Matrix projective synchronization (MPS) and inverse matrix projective synchronization is obtained when the scaling factor is a constant matrix, which gives the assurance of high security in secure communication and image encryption. Based on the Lyapunov stability theory and appropriate active control technique, the MPCS and IMPCS between two master systems and one slave system has been achieved. Based on the MPCS synchronization, a scheme of secure communication is presented, and the message signals are transmitted using the chaotic signal masking method. Finally, numerical simulations have been provided, which shows that our theoretical results are in complete agreement will the graphical one.

Keywords Combination synchronization · Matrix projective synchronization · Inverse matrix projective synchronization · Active control · Fractional-order chaotic systems

Introduction

For the last two decades, Fractional calculus is portraying a significant role in the study of nonlinear dynamical systems. Fractional calculus is a generalization of integer order integration and differentiation. Fractional-order calculus has many advantages in the field of engineering and sciences such as secure communication [1], data encryption [2], Financial systems [3], ecological systems [4], biomedical engineering [5], electromagnetic wave [6], etc.

Chaos theory is a branch of mathematics focus on the behavior of the nonlinear dynamical system that is highly sensitive to the initial values. In recent times, the control and

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synchronization of chaotic systems become an attractive field for researchers. Despite the observation made by Poincare, Lorenz [7] in 1963 gives the first introduction of chaos in a deterministic system. Further, Pecora and Carroll [8] firstly introduced the synchronization of chaotic systems between two identical chaotic systems. After that, researchers performing synchronization in a non-identical system having different properties.

A large variation technique have been utilized to analysis the synchronization of the FO chaotic systems such as active control [9], adaptive control [10], sliding mode control (SMC) [11], optimal control [12], adaptive SMC [13,14], feedback control method [15], time-delayed feed-back control [16] and robust adaptive SMC [14] etc. In which many types of synchronization for the FO chaotic systems have been performed such as projective synchronization (PS) [17], complete synchronization [18], anti synchronization [19], hybrid synchronization [20], hybrid projective synchronization [21], function projective synchronization (FPS) [22], compound synchronization [23], dual combination synchronization [24], double compound synchronization [25] etc.

Nowadays, among all types of chaotic synchronization, PS has been primarily considered. Firstly introduced the concept of PS by Mainieri and Rehaceh in [26]. In PS, the master and slave system could be synchronized up to a scaling factor α. When a scaling function replaces the scaling factor in PS, then FPS is obtained. The unpredictability of the scaling function in FPS can additionally enhance the security of communication. The generalization of PS is FPS. FPS signifies that the master and slave systems could be synchronized up to a scaling function is discussed in [22,27]. To provide high protection in connection, scaling factor α of PS can be extended to a constant arbitrary matrix, and a different synchronization kind develops and is called MPS. It synchronizes of chaotic systems with a different dimension, and it gives the application in secure communication [28]. When the scaling function is generalized to a constant matrix, then MPS is obtained. Another method is the IMPS technique, that is when each slave system state synchronizes with a linear combination of master system states. In [29], the author investigates the MPS and IMPS between chaotic systems of identical dimensional and non-identical dimensional in discrete-time chaotic systems. In [30], the IMPS among non-identical dimensional of FO chaotic systems has been proposed. Also, in [31], the author discussed the synchronization of FO hyperchaotic systems disturbed by uncertainty and external disturbance using the MPS and IMPS scheme. Moreover, [32] presented the dynamical analysis and MPS in identical new FO Rabinovich systems. For the synchronization of two different delayed FO neural networks with disturbance, the author discussed the quasi MPS and quasi IMPS in [33].

In the real-world, system uncertainty and external disturbances are exist everywhere in reality. Besides, owing to unmodeled dynamics, structure differences of the system, and estimation and surrounding noises, the chaotic systems should be deal with uncertainties and external disturbances. Uncertainty and disturbances increase the instability of the systems, energy fluctuations, and also destroy the synchronization performance, which can not be avoided in the real application, which was discussed in [34]. Thus, it is essential to explore the synchronization of FO chaotic systems having unknown external disturbances and uncertainty. For chaos synchronization, unknown model uncertainties have a lousy effect on the chaotic dynamics system and synchronization behavior and diminish the performance of the real system. In [35], Synchronization of chaotic systems with disturbance is illustrated. For the decrease of the chattering problem, an adaptive SMC was proposed for two uncertain chaotic systems [36]. Moreover, in [37], the author discussed the application of synchronization of chaotic systems with uncertainties and external disturbances.

The important contribution of this research are summarized as follows.

- This paper proposed combination projective synchronization in fractional-order chaotic system with disturbance and uncertainty.
- It is based on Lyapunov stability theory, and an active control technique with fast convergence is designed for the matrix projective combination synchronization and inverse matrix projective combination synchronization.
- This paper proposed an application of a secure communication scheme based on matrix projective combination synchronization.
- The design of the controller is easy and simple.
- Simulation result with a comparison example shows the effectiveness of the introduced method.

Therefore, in this paper, we will be presenting the scheme of combination projective synchronizing for the FO complex chaotic systems disturbed by model uncertainties and external disturbances. The organization of the paper is as follows: In the second section, it contains preliminaries. Third section provides the problem formulation of MPCS and IMPCS in FO complex chaotic system with disturbance and uncertainty. In fourth section, includes an example of matrix and inverse matrix projective combination synchronization in FO complex chaotic system with disturbance and uncertainty. Fifth section consists of the numerical simulation. Sixth section contains the comparison of given synchronization with previously published work. In the seventh section, the application of the obtained control scheme on MPCS is investigated. Finally, concluding remarks are pointed out in the last section.

Preliminaries

Definition 1 [38] The Caputo's derivative for function $h(t)$ with fractional order α is define by:

$$cD_y^{\alpha}h(y) = \frac{1}{\Gamma(n-\alpha)}\int_c^y \frac{h^n(x)}{(y-x)^{\alpha-n+1}}dx \tag{1}$$

where $n-1 < \alpha < n$ and $\Gamma()$ is the Euler's Gamma function.

Due to wide range of applications of Caputo's fractional derivative definition, we have also used the Caputo's fractional derivative in our proposed research work.

Considering the FO non-linear dynamical system

$$D_y^{\alpha}y_i = h_i(y_1, y_2, \ldots, y_n), (0 < \alpha < 1, i = 1, 2, \ldots, n), \tag{2}$$

Its equilibrium point $E^* = (y_1^*, y_2^*, y_3^*, \ldots, y_n^*)$ are calculating by solving $h_i(y_1, y_2, \ldots, y_n) = 0$

Stability Criterion 1: [39] System (2) is asymptotically stable iff all the eigen value λ_i of the jacobi matrix $J = \frac{\partial h}{\partial y}$, where $h = [h_1, h_2, \ldots, h_n]^T$, calculated at the equilibrium point E^* fulfill the condition $|arg\lambda_i| > \frac{\alpha\pi}{2}$

Problem Formulation

Introduce the Scheme of Fractional Matrix Projective Combination Synchronization (MPCS) Method

[31] Consider two non-identical n-dimensional FO complex chaotic master systems, which are disturbed by uncertainty and disturbance are taken as

$$D^{\alpha}X_1 = A_1X_1 + H_1(X_1) + \delta\theta_1(X_1) + D_1(t) \tag{3}$$

$$D^{\alpha}X_2 = A_2X_2 + H_2(X_2) + \delta\theta_2(X_2) + D_2(t) \tag{4}$$

where $A_1 \in R^{n\times n}$, $A_2 \in R^{n\times n}$ are the coefficient constant matrix of the linear parts of the systems (3), and (4), respectively. $X_1 = [x_{11}, x_{12}, \ldots x_{1n}]^T \in R^n$ and $X_2 = [x_{21}, x_{22}, \ldots x_{2n}]^T \in R^n$ are the state vector of master systems (3), and (4), respectively; $H_1(X_1), H_2(X_2) \in R^n$ are the non-linear terms of system (3), and (4), respectively; $\delta\theta_1(X_1)$, $\delta\theta_2(X_2) \in R^n$, $(|\delta\theta_1(X_1)| \le m_1, |\delta\theta_2(X_2)| \le m_2, m_1, m_2 > 0)$ are the model uncertainties. $D_1(t), D_2(t) \in R^n$, $(|D_1(t)| \le n_1, |D_2(t)| \le n_2, n_1, n_2 > 0)$ are the external disturbance of system (3), and (4), respectively.

Corresponding slave system is taken as:

$$D^{\alpha}Y_1 = B_1Y_1 + H_3(Y_1) + \delta\theta_3(Y_1) + D_3(t) + U \tag{5}$$

where $B_1 \in R^{n\times n}$ is the coefficient constant matrix of the linear parts of the system (5); $Y_1 = [y_{11}, y_{12}, \ldots y_{1n}]^T \in R^n$ is state vector of slave system (5); $H_3(Y_1) \in R^n$ is the non-linear terms of system (5); $\delta\theta_3(Y_1) \in R^n$, $(|\delta\theta_3(Y_1)| \le m_3, m_3 > 0)$ is the model uncertainty; $D_3(t) \in R^n$, $(|D_3(t)| \le n_3, n_3 > 0)$ is the external disturbance of system (5), and $U = (u_{11}, u_{12}, \ldots, u_{in})$ is a vector controller to be designed.

The MPCS for the FO chaotic complex system disturbed by disturbance and uncertainty is defined as follows.

Definition 2 [30] The n-dimensional disturb master systems (3) and (4) and n-dimensional disturb slave system (5) are said to be MPCS, if there exist a controller $U = (u_{11}, u_{12}, \ldots, u_{1n})^T$ and given constant matrix $M = (M_{ij})_{n\times n}$, such that the synchronization error will be

$$e = Y_1 - M(X_1 + X_2) \tag{6}$$

satisfies $lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|Y_1 - M(X_1 + X_2)\| = 0$, where M presents the projective matrix, $\|.\|$ represents the Euclidean norm, and $e = (e_{11}, e_{12}, \ldots, e_{in})$.

Our aim is to outline the suitable controller U to achieve MPCS between two master systems (3) and (4) and slave system (5) as follows:

Theorem 1 [31,40] *The n-dimensional disturb systems (3), (4) and (5) fulfil the overall MPCS under the suitable controller.*

$$\begin{aligned} U = {} & K_1Y_1 - (B_1 + K_1)M(X_1 + X_2) + M[A_1X_1 + H_1(X_1) \\ & + \delta\theta_1(X_1) + D_1(t) + A_2X_2 + H_2(X_2) + \delta\theta_2(X_2) + D_2(t)] \\ & - H_3(Y_1) - \delta\theta_3(Y_1) - D_3(t) \end{aligned} \tag{7}$$

where $K_1 \in R^{n\times n}$ is the gain matrices. Then, the matrix projective combination synchronization will be achieved between the considered systems (3), (4), and (5). If and only if all the eigenvalue λ_i of $B_1 + K_1$ satisfy $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$, where $i = 1, 2, \ldots, n$.

Proof Apply Caputo derivative in error system, using Eqs. (3), (4), and (5), the dynamical system can be obtain.

$$\begin{aligned} D^\alpha e &= D^\alpha Y_1 - M(D^\alpha X_1 + D^\alpha X_2) \\ D^\alpha e &= B_1 Y_1 + H_3(Y_1) + \delta\theta_3(Y_1) + D_3(t) + U - M[A_1 X_1 + H_1 X_1 \\ &\quad + \delta\theta_1(X_1) + D_1(t) + A_2 X_2 + H_2 X_2 + \delta\theta_2(X_2) + D_2(t)]. \end{aligned} \tag{8}$$

using the appropriate control function given in Eq. (7) in equation system (8), the error system of the MPCS is reduced in the following form.

$$D^\alpha e = (B_1 + K_1)(Y_1 - M(X_1 + X_2)) = (B_1 + K_1)e \tag{9}$$

Clearly, $e = Y_1 - M(X_1 + X_2)$ is the only equilibrium point of the the system (9) and Jacobi matrix at this fixed point is $(B_1 + K_1)$.

The gain matrix K_1 is chosen such that the eigenvalues λ_i of the Jacobi matrix $(B_1 + K_1)$ satisfy the condition $|arg(\lambda_1)| > \frac{\alpha\pi}{2}$,where $i = 1, 2, \ldots, n$.

Therefore, $lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|Y_1 - M(X_1 + X_2)\| = 0$ indicates system are MPCS and hence this finish the proof. □

Introduce the Scheme of Inverse Matrix Projective Combination Synchronization (IMPCS) Method

For systems (3), (4) and (5), the IMPCS can be define in definition (2), which is written in below.

Definition 3 [30] The n-dimensional disturbed master systems (3), and (4) and n-dimensional disturbed slave system (5) are said to be IMPCS if there exist a controller $U = (u_{11}, u_{12}, \ldots, u_{1n})^T$ and the given invertible constant matrix $N = (N_{ij})_{n\times n}$, such that the synchronization error

$$e = (X_1 + X_2) - NY_1 \tag{10}$$

satisfies $lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|(X_1 + X_2) - NY_1\| = 0$, where N presents the projective matrix.

Our aim is to described the suitable controller U to achieve IMPCS between two master systems (3), and (4) and slave system (5) as follows:

Theorem 2 [31,40] *The n-dimensional disturbed systems* (3)*,* (4)*, and* (5) *fulfil the overall IMPCS under the suitable controller.*

$$\begin{aligned} U = N^{-1}[&(A_1 + A_2 + K_1 + K_2)NY_1 - (K_1 + K_2)(X_1 + X_2) \\ &+ H_1(X_1) + H_2(X_2) + \delta\theta_1(X_1) + \delta\theta_2(X_2) + D_1(t) + D_2(t) \\ &- A_1 X_2 - A_2 X_1] - B_1 Y_1 - H_3(Y_1) - \delta\theta_3(Y_1) - D_3(t) \end{aligned} \tag{11}$$

where $K_1 \in R^{n\times n}$, $K_2 \in R^{n\times n}$ are the gain matrices. Then, the IMPCS will be achieved between the considered systems (3)*,* (4)*, and* (5)*. If and only if all the eigenvalue λ_i of $A_1 + A_2 + K_1 + K_2$ satisfy $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$, where $i = 1, 2, \ldots, n$.*

Proof Apply Caputo derivative in error system, using Eqs. (3), (4), and (5), the dynamical system can be obtain.

$$\begin{aligned} D^{\alpha}e &= (D^{\alpha}X_1 + D^{\alpha}X_2) - N(D^{\alpha}Y_1) \\ D^{\alpha}e &= (A_1X_1 + H_1X_1 + \delta\theta_1(X_1) + D_1(t) + A_2X_2 + H_2X_2 + \delta\theta_2(X_2) \\ &\quad + D_2(t)) - M[B_1Y_1 + H_3(Y_1) + \delta\theta_3(Y_1) + D_3(t) + U] \end{aligned} \tag{12}$$

using the appropriate control function (11) in (12), the error system of the IMPCS is reduced in the following form .

$$\begin{aligned} D^{\alpha}e &= (A_1 + A_2 + K_1 + K_2)(X_1 + X_2 - NY_1) \\ &= (A_1 + A_2 + K_1 + K_2)e \end{aligned} \tag{13}$$

The gain matrix K_1,K_2 are selected such that the eigenvalues λ_i of the Jacobi matrix $(A_1 + A_2 + K_1 + K_2)$ satisfy the condition $|arg(\lambda_1)| > \dfrac{\alpha\pi}{2}$, where $i = 1, 2, \ldots, n$.

Therefore, $lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|(X_1 + X_2) - NY_1\| = 0$ indicates system are inverse matrix projective synchronized and hence this finish the proof. □

Remark

(1) Selecting matrix projective $M = N = I$, systems (3), (4), and (5) can achieve the complete combination synchronization.
(2) Selecting matrix projective $M = N = -I$, system (3), (4), and (5) can achieve the anti combination synchronization.
(3) If $M = pI$(or $N = pI$),(p=constant and $p \neq 1, -1$), they can achieve projective combination synchronization (or inverse projective combination synchronization).
(4) If $M = diag(p_1, p_2, \ldots, p_n)$ (or $N = diag(p_1, p_2, \ldots, p_n)$),($p_i = constant, i = 1, 2, \ldots, n$), then system (3), (4), and (5) achieve the modified combination projective synchronization (or modified inverse combination projective synchronization).
(5) If matrix $M = 0$ (or $N = 0$), then the MPCS or IMPCS turn into stable problem of FO complex chaotic system.
(6) If the external disturbance $D_1(t)$, $D_2(t) = 0$, and $\delta\theta_1$, $\delta\theta_2 = 0$, then it turns into MPCS and IMPCS without model external disturbance and model uncertainty.

Numerical Example of MPCS and IMPCS

In this section, fractional complex Lorenz and T system are taken as the master systems and Lu system is taken as a slave system in order to achieve MPCS and IMPCS.

Mathematical model of FO complex Lorenz system [41]:

$$\begin{cases} \dfrac{d^{\alpha}x_{11}}{dt^{\alpha}} &= a_{11}(x_{13} - x_{11}) + sin4x_{11} - 0.5sin4t \\ \dfrac{d^{\alpha}x_{12}}{dt^{\alpha}} &= a_{11}(x_{14} - x_{12}) + 2cos4x_{12} - 0.5sin4t \\ \dfrac{d^{\alpha}x_{13}}{dt^{\alpha}} &= a_{12}x_{11} - x_{13} - x_{11}x_{15} - 0.5cos4x_{13} - 0.5cos4t \\ \dfrac{d^{\alpha}x_{14}}{dt^{\alpha}} &= a_{12}x_{12} - x_{14} - x_{12}x_{15} - 0.25sin4x_{14} - 0.25sin4t \\ \dfrac{d^{\alpha}x_{15}}{dt^{\alpha}} &= x_{11}x_{13} + x_{12}x_{14} - a_{13}x_{15} - 0.5cos4x_{15} + 2cos4t \end{cases} \tag{14}$$

where $x_{11}, x_{12}, x_{13}, x_{14}, x_{15}$ ate the state variale, a_{11}, a_{12}, a_{13} are parameters of system and parameters values $a_{11} = 10$, $a_{12} = 180$, $a_{13} = 1$ and initial value $(x_{11}(0), x_{12}(0), x_{13}(0), x_{14}(0), x_{15}(0)) = (2, 3, 5, 6, 9)$, and $\alpha = 0.95$.

Now comparing the system (14) with system (3), we get

$$A_1 = \begin{bmatrix} -a_{11} & 0 & a_{11} & 0 & 0 \\ 0 & -a_{11} & 0 & a_{11} & 0 \\ a_{12} & 0 & -1 & 0 & 0 \\ 0 & a_{12} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -a_{13} \end{bmatrix} = \begin{bmatrix} -10 & 0 & 10 & 0 & 0 \\ 0 & -10 & 0 & 10 & 0 \\ 180 & 0 & -1 & 0 & 0 \\ 0 & 180 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$H_1(X_1) = \begin{bmatrix} 0 \\ 0 \\ -x_{11}x_{15} \\ -x_{12}x_{15} \\ x_{11}x_{13} + x_{12}x_{14} \end{bmatrix}, \delta\theta_1(X_1) = \begin{bmatrix} sin4x_{11} \\ 2cos4x_{12} \\ -0.5cos4x_{13} \\ -0.25sin4x_{14} \\ -0.5cos4x_{15} \end{bmatrix}, D_1(t) = \begin{bmatrix} -0.5sin4t \\ -0.5sin4t \\ -0.5cos4t \\ -0.25sin4t \\ 2cos4t \end{bmatrix},$$

,

Mathematical model of FO complex T system [42]:

$$\begin{cases} \dfrac{d^\alpha x_{21}}{dt^\alpha} = a_{21}(x_{23} - x_{21}) + sin2x_{21} - 0.5sin2t \\ \dfrac{d^\alpha x_{22}}{dt^\alpha} = a_{21}(x_{24} - x_{22}) + cos2x_{22} - 0.5sin2t \\ \dfrac{d^\alpha x_{23}}{dt^\alpha} = (a_{22} - a_{21})x_{21} - a_{21}x_{21}x_{25} - 0.5cos2x_{23} - 0.5cos2t \\ \dfrac{d^\alpha x_{24}}{dt^\alpha} = (a_{22} - a_{21})x_{22} - a_{21}x_{22}x_{25} - 0.25sin2x_{24} - 0.5sin2t \\ \dfrac{d^\alpha x_{25}}{dt^\alpha} = x_{21}x_{23} + x_{22}x_{24} - a_{23}x_{25} - 0.5cos2x_{25} - 0.5cos2t \end{cases} \tag{15}$$

where $x_{21}, x_{22}, x_{23}, x_{24}, x_{25}$ are the state variable, a_{21}, a_{22}, a_{23} are parameters of system and parameters values $a_{21} = 2.1$, $a_{22} = 30$, $a_{23} = 0.6$ and initial value $(x_{21}(0), x_{22}(0), x_{23}(0), x_{24}(0), x_{25}(0)) = (8, 7, 6, 8, 7)$, and $\alpha = 0.95$

Now comparing the system (15) with system (4), we obtain

$$A_2 = \begin{bmatrix} -a_{21} & 0 & a_{21} & 0 & 0 \\ 0 & -a_{21} & 0 & a_{21} & 0 \\ a_{22} - a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} - a_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{23} \end{bmatrix} = \begin{bmatrix} -2.1 & 0 & 2.1 & 0 & 0 \\ 0 & -2.1 & 0 & 2.1 & 0 \\ 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & -0.6 \end{bmatrix},$$

$$H_2(X_2) = \begin{bmatrix} 0 \\ 0 \\ -a_{21}x_{21}x_{25} \\ -a_{21}x_{22}x_{25} \\ x_{21}x_{23} + x_{22}x_{24} \end{bmatrix}, \delta\theta_2(X_2) = \begin{bmatrix} sin2x_{21} \\ cos2x_{22} \\ -0.5cos2x_{23} \\ -0.25sin2x_{24} \\ -0.5cos2x_{25} \end{bmatrix}, D_2(t) = \begin{bmatrix} -0.5sin2t \\ -0.5sin2t \\ -0.5cos2t \\ -0.5sin2t \\ -0.5cos2t \end{bmatrix},$$

Mathematical model of FO complex Lu system [43]:

$$\begin{cases} \dfrac{d^\alpha y_{11}}{dt^\alpha} = b_{11}(y_{13}-y_{11}) + 6sin10y_{11} + 1.5cos10t + u_{11} \\ \dfrac{d^\alpha y_{12}}{dt^\alpha} = b_{11}(y_{14}-y_{12}) - 4cos10y_{12} - 0.5sin10t + u_{12} \\ \dfrac{d^\alpha y_{13}}{dt^\alpha} = -y_{11}y_{15} + b_{12}y_{13} - 3sin10y_{13} - 1.5sin10t + u_{13} \\ \dfrac{d^\alpha y_{14}}{dt^\alpha} = -y_{12}y_{15} + b_{12}y_{14} - 2cos10y_{14} - 0.25cos10t + u_{14} \\ \dfrac{d^\alpha y_{15}}{dt^\alpha} = y_{11}y_{13} + y_{12}y_{14} - b_{13}y_{15} - 5cos10y_{15} + 0.5cos10t + u_{15} \end{cases} \tag{16}$$

where $y_{11}, y_{12}, y_{13}, y_{14}, y_{15}$ are the state variable, b_{11},b_{12},b_{13} are parameters of system and parameters values $b_{11} = 40$, $b_{12} = 22$, $b_{13} = 5$ and initial value $(y_{11}(0), y_{12}(0), y_{13}(0), y_{14}(0), y_{15}(0)) = (1, 2, 3, 4, 5)$ and $\alpha = 0.95$. Where $u_{11}, u_{12}, u_{13}, u_{14}, u_{15}$ are control functions.

Now comparing the system (16) with system (5), we get:

$$B_1 = \begin{bmatrix} -b_{11} & 0 & b_{11} & 0 & 0 \\ 0 & -b_{11} & 0 & b_{11} & 0 \\ 0 & 0 & b_{12} & 0 & 0 \\ 0 & 0 & 0 & b_{12} & 0 \\ 0 & 0 & 0 & 0 & -b_{13} \end{bmatrix} = \begin{bmatrix} -40 & 0 & 40 & 0 & 0 \\ 0 & -40 & 0 & 40 & 0 \\ 0 & 0 & 22 & 0 & 0 \\ 0 & 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix},$$

$$H_3(Y_1) = \begin{bmatrix} 0 \\ 0 \\ -y_{11}y_{15} \\ -y_{12}y_{15} \\ y_{11}y_{13} + y_{12}y_{14} \end{bmatrix}, \delta\theta_3(Y_1) = \begin{bmatrix} 6sin10y_{11} \\ -4cos10y_{12} \\ -3sin10y_{13} \\ -2cos10y_{14} \\ -5cos10y_{15} \end{bmatrix}, D_3(t) = \begin{bmatrix} 1.5cos10t \\ -0.5sin10t \\ -1.5sin10t \\ -0.25cos10t \\ 0.5cos10t \end{bmatrix}$$

To Achieve Matrix Projective Combination Synchronization

According to Theorem 1, there exist a matrix projective $M \in R^{5\times5}$, so that the systems (14), (15), and (16) realize the MPCS.

Projective matrix can be chosen as:

$$M = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix},$$

The error function $e = Y_1 - M(X_1 + X_2)$ can be obtained as, where $e = [e_{11}, e_{12}, e_{13}, e_{14}, e_{15}]^T$

$$\begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \\ e_{15} \end{bmatrix} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} (x_{11}+x_{21}) \\ (x_{12}+x_{22}) \\ (x_{13}+x_{23}) \\ (x_{14}+x_{24}) \\ (x_{15}+x_{25}) \end{bmatrix}$$

Error function will be

$$\begin{cases} e_{11} = y_{11} - (x_{11} + x_{21}) + (x_{13} + x_{23}) \\ e_{12} = y_{12} + (x_{12} + x_{22}) - (x_{13} + x_{23}) - (x_{15} + x_{25}) \\ e_{13} = y_{13} + 2(x_{13} + x_{23}) - 2(x_{15} + x_{25}) \\ e_{14} = y_{14} - (x_{13} + x_{23}) + 2(x_{14} + x_{24}) - 2(x_{15} + x_{25}) \\ e_{15} = y_{15} + 2(x_{15} + x_{25}) \end{cases} \tag{17}$$

The error dynamics system can be described as:

$$\begin{cases} D^{\alpha} e_{11} = b_{11}(y_{13} - y_{11}) + 6sin10y_{11} + 1.5cos10t - (a_{11}(x_{13} - x_{11}) + sin4x_{11} \\ \quad - 0.5sin4t + a_{21}(x_{23} - x_{21}) + sin2x_{21}) - 0.5sin2t) + (a_{12}x_{11} - x_{13} \\ \quad - x_{11}x_{15} - 0.5cos4x_{13} - 0.5cos4t + (a_{22} - a_{21})x_{21} - a_{21}x_{21}x_{25} \\ \quad - 0.5cos2x_{23} - 0.5cos2t) + u_{11} \\ D^{\alpha} e_{12} = b_{11}(y_{14} - y_{12}) - 4cos10y_{12} - 0.5sin10t + (a_{11}(x_{14} - x_{12}) + 2cos4x_{12} \\ \quad - 0.5sin4t + a_{21}(x_{24} - x_{22}) + cos2x_{22} - 0.5sin2t) - (a_{12}x_{11} - x_{13} \\ \quad - x_{11}x_{15} - 0.5cos4x_{13} - 0.5cos4t + (a_{22} - a_{21})x_{21} - a_{21}x_{21}x_{25} \\ \quad - 0.5cos2x_{23} - 0.5cos2t) - (x_{11}x_{13} + x_{12}x_{14} - a_{13}x_{15} - 0.5cos4x_{15} \\ \quad + 2cos4t + x_{21}x_{23} + x_{22}x_{24} - a_{23}x_{25} - 0.5cos2x_{25} - 0.5cos2t) + u_{12} \\ D^{\alpha} e_{13} = -y_{11}y_{15} + b_{12}y_{13} - 3sin10y_{13} - 1.5sin10t + 2(a_{12}x_{11} - x_{13} - x_{11}x_{15} \\ \quad - 0.5cos4x_{13} - 0.5cos4t + (a_{22} - a_{21})x_{21} - a_{21}x_{21}x_{25} - 0.5cos2x_{23} \\ \quad - 0.5cos2t) - 2(x_{11}x_{13} + x_{12}x_{14} - a_{13}x_{15} - 0.5cos4x_{15} + 2cos4t \\ \quad + x_{21}x_{23} + x_{22}x_{24} - a_{23}x_{25} - 0.5cos2x_{25} - 0.5cos2t) + u_{13} \\ D^{\alpha} e_{14} = -y_{12}y_{15} + b_{12}y_{14} - 2cos10y_{14} - 0.25cos10t - (a_{12}x_{11} - x_{13} - x_{11}x_{15} \\ \quad - 0.5cos4x_{13} - 0.5cos4t + (a_{22} - a_{21})x_{21} - a_{21}x_{21}x_{25} - 0.5cos2x_{23} \\ \quad - 0.5cos2t) + 2(a_{12}x_{12} - x_{14} - x_{12}x_{15} - 0.25sin4x_{14} - 0.25sin4t \\ \quad + (a_{22} - a_{21})x_{22} - a_{21}x_{22}x_{25} - 0.25sin2x_{24} - 0.5sin2t) - 2(x_{11}x_{13} \\ \quad + x_{12}x_{14} - a_{13}x_{15} - 0.5cos4x_{15} + 2cos4t + x_{21}x_{23} + x_{22}x_{24} - a_{23}x_{25} \\ \quad - 0.5cos2x_{25} - 0.5cos2t) + u_{14} \\ D^{\alpha} e_{15} = y_{11}y_{13} + y_{12}y_{14} - b_{13}y_{15} - 5cos10y_{15} + 0.5cos10t + 2(x_{11}x_{13} \\ \quad + x_{12}x_{14} - a_{13}x_{15} - 0.5cos4x_{15} + 2cos4t + x_{21}x_{23} + x_{22}x_{24} \\ \quad - a_{23}x_{25} - 0.5cos2x_{25} - 0.5cos2t) + u_{15} \end{cases} \tag{18}$$

Choosing the satisfactory control gain matrix K_1

$$K_1 = \begin{bmatrix} 0 & 0 & -40 & 0 & 0 \\ 0 & 0 & 0 & -40 & 0 \\ 0 & 0 & -23 & 0 & 0 \\ 0 & 0 & 0 & -23 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

In view of Theorem 1 the control functions will be:

$$U_1 = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \end{bmatrix} = \begin{bmatrix} -40y_{13} + 40(x_{11} + x_{21}) - 40(x_{13} + x_{23}) - 10x_{11} + 10x_{13} + sin4x_{11} \\ -0.5sin4t - 2.1sin4t - 2.1x_{21} + 2.1x_{23} + sin2x_{21} - 0.5sin5t \\ -180x_{11} + x_{13} + x_{11}x_{15} + 0.5cos4x_{13} + 0.5cos4t - 27.9x_{21} \\ +2.1x_{21}x_{25} + 0.5cos5x_{23} + 0.5sin2t - 6sin10y_{11} - 1.5cos10t \\ \\ -40y_{14} - 40(x_{12} + x_{22}) + 40(x_{13} + x_{23}) + 40(x_{15} + x_{25}) + 10x_{12} \\ -10x_{14} - 2cos4x_{12} + 0.5sin4t + 2.1x_{22} - 2.1x_{24} - cos2x_{22} \\ +0.5sin2t + 180x_{11} - x_{13} - x_{11}x_{15} - 0.5cos4x_{13} - 0.5cos4t \\ +27.9x_{21} - 2.1x_{21}x_{25} - 0.5cos2x_{23} - 0.5cos2t - x_{15} + x_{11}x_{13} \\ +x_{12}x_{14} - 0.5cos4x_{15} + 2cos4t - 0.6x_{25} + x_{21}x_{23} \\ +x_{22}x_{24} - 0.5cos2x_{25} - 0.5cos2t + 4cos10y_{12} + 0.5sin10t \\ \\ -23y_{13} - 2(x_{13} + x_{23}) + 2(x_{15} + x_{25}) - 360x_{11} + 2x_{13} + 2x_{11}x_{15} \\ +cos4x_{13} + cos4t - 55.8x_{21} + 4.2x_{21}x_{25} + 1cos2x_{23} + 1cos2t \\ -2x_{15} + 2x_{11}x_{13} + 2x_{12}x_{14} - cos4x_{15} + 4cos4t - 1.2x_{25} + 2x_{21}x_{23} \\ +2x_{22}x_{24} - 1cos2x_{25} - cos2t + y_{11}y_{15} + 3sin10y_{13} + 1.5sin10t \\ \\ -23y_{14} + (x_{13} + x_{23}) - 2(x_{14} + x_{24}) + 2(x_{15} + x_{25}) + 180x_{11} \\ -x_{13} - x_{11}x_{15} - 0.5cos4x_{13} - 0.5cos4t + 27.9x_{21} + 2.1x_{21}x_{25} \\ -0.5cos2x_{23} - 0.5cos2t - 360x_{12} + 2x_{14} + 2x_{12}x_{15} + 0.5sin4x_{14} \\ +0.5sin4t - 55.8x_{22} + 4.2x_{22}x_{25} + 0.5sin2x_{24} + 1sin2t \\ -2x_{15} + 2x_{11}x_{13} + 2x_{12}x_{14} - cos4x_{15} + 4cos4t - 1.2x_{25} + 2x_{21}x_{23} \\ +2x_{22}x_{24} - cos2x_{25} - 1cos2t + y_{12}y_{15} + 2cos10y_{14} + 0.25cos10t \\ \\ -10(x_{15} + x_{25}) + 2x_{15} - 2x_{11}x_{13} - 2x_{12}x_{14} + cos4x_{15} - cos4t \\ +1.2x_{25} - 2x_{21}x_{23} - 2x_{22}x_{24} + 1cos2x_{25} + 1cos2t - y_{11}y_{13} \\ -y_{12}y_{14} + 5cos10y_{15} - 0.5cos10t \end{bmatrix}$$

Using the value of controllers in $u_{11}, u_{12}, u_{13}, u_{14}, u_{15}$ in error dynamics (18), now error dynamics system can be obtained as

$$\begin{cases} D^\alpha e_{11} = -40e_{11} \\ D^\alpha e_{12} = -40e_{12} \\ D^\alpha e_{13} = -e_{13} \\ D^\alpha e_{14} = -e_{14} \\ D^\alpha e_{15} = -5e_{15} \end{cases} \tag{19}$$

Hence, the MPCS between master systems (14), (15), and slave system (16) is achieved.

To Achieve Inverse Matrix Projective Combination Synchronization

According to Theorem 2, there exist a invertible $N \in R^{5\times5}$, so that the systems (14), (15), and (16) realize the IMPCS

The invertible matrix can be chosen as:

$$M = N = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, N^{-1} = \begin{bmatrix} 1 & 0 & -0.5 & 0 & 0.5 \\ 0 & -1 & -0.5 & 0 & 1 \\ 0 & 0 & -0.5 & 0 & 0.5 \\ 0 & 0 & -0.25 & -0.5 & -0.75 \\ 0 & 0 & 0 & 0 & -0.5 \end{bmatrix},$$

The error function $e = (X_1 + X_2) - NY_1$ can be obtained as

$$\begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \\ e_{15} \end{bmatrix} = \begin{bmatrix} x_{11} + x_{21} \\ x_{12} + x_{22} \\ x_{13} + x_{23} \\ x_{14} + x_{24} \\ x_{15} + x_{25} \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \end{bmatrix}$$

Error function will be obtained as:

$$\begin{cases} e_{11} = x_{11} + x_{21} - (y_{11} - y_{13}) \\ e_{12} = x_{12} + x_{22} - (-y_{12} + y_{13} + y_{15}) \\ e_{13} = x_{13} + x_{23} - 2(y_{13} - y_{15}) \\ e_{14} = x_{14} + x_{24} - (y_{13} - 2y_{14} + 2y_{15}) \\ e_{15} = x_{15} + x_{25} - (-2y_{15}) \end{cases} \tag{20}$$

The error dynamics system can be given as:

$$\begin{cases} D^{\alpha}e_{11} = (a_{11}(x_{13}-x_{11}) + sin4x_{11} - 0.5sin4t + a_{21}(x_{23}-x_{21}) \\ \qquad + sin2x_{21} - 0.5sin2t) - (b_{11}(y_{13}-y_{11}) + 6sin10y_{11} \\ \qquad + 1.5cos10t + y_{11}y_{15} - b_{12}y_{13} + 3sin10y_{13} + 1.5sin10t) \\ \qquad - u_{11} + u_{13} \\ \\ D^{\alpha}e_{12} = (-a_{11}(x_{14}-x_{12}) + 2cos4x_{12} - 0.5sin4t + a_{21}(x_{24}-x_{22}) \\ \qquad + cos2x_{22} - 0.5sin2t) - (-b_{11}(y_{14}-y_{12}) + 4cos10y_{12} \\ \qquad + 0.5sin10t - y_{11}y_{15} + b_{12}y_{13} - 3sin10y_{13} - 1.5sin10t \\ \qquad + y_{11}y_{13} + y_{12}y_{14} - b_{13}y_{15} - 5cos10y_{15} + 0.5cos10t) \\ \qquad + u_{12} - u_{13} - u_{15} \\ \\ D^{\alpha}e_{13} = (a_{12}x_{11} - x_{13} - x_{11}x_{15} - 0.5cos4x_{13} - 0.5cos4t \\ \qquad + (a_{22}-a_{21})x_{21} - a_{21}x_{21}x_{25} - 0.5cos2x_{23} - 0.5cos2t) \\ \qquad - 2(-y_{11}y_{15} + b_{12}y_{13} - 3sin10y_{13} - 1.5sin10t - y_{11}y_{13} \\ \qquad - y_{12}y_{14} + b_{13}y_{15} + 5cos10y_{15} - 0.5cos10t) \\ \qquad - 2u_{13} + 2u_{15} \\ \\ D^{\alpha}e_{14} = (a_{12}x_{11} - x_{13} - x_{11}x_{15} - 0.5cos4x_{13} - 0.5cos4t \\ \qquad + (a_{22}-a_{21})x_{21} - a_{21}x_{21}x_{25} - 0.25sin2x_{24} - 0.5sin2t) \\ \qquad - (-y_{11}y_{15} + b_{12}y_{13} - 3sin10y_{13} - 1.5sin10t - 2(-y_{12}y_{15} \\ \qquad + b_{12}y_{14} - 2cos10y_{14} - 0.25cos10t) + 2(y_{11}y_{13} + y_{12}y_{14} \\ \qquad - b_{13}y_{15} - 5cos10y_{15} + 0.5cos10t) - u_{13} + 2u_{14} - 2u_{15} \\ \\ D^{\alpha}e_{15} = (x_{11}x_{13} + x_{12}x_{14} - a_{13}x_{15} - 0.5cos4x_{15} + 2cos4t + x_{21}x_{23} \\ \qquad + x_{22}x_{24} - a_{23}x_{25} - 0.5cos2x_{25} - 0.5cos2t) + 2(y_{11}y_{13} \\ \qquad + y_{12}y_{14} - b_{13}y_{15} - 5cos10y_{15} + 0.5cos10t) + 2u_{15} \end{cases} \tag{21}$$

Choosing the satisfactory control gain matrix K_1 and K_2.

$$K_1 = \begin{bmatrix} 0 & 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & -10 & 0 \\ -180 & 0 & 0 & 0 & 0 \\ 0 & -180 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 & -2.1 & 0 & 0 \\ 0 & 0 & 0 & -2.1 & 0 \\ -27.9 & 0 & 0 & 0 & 0 \\ 0 & -27.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In view of Theorem 2 the control functions will be:

$$U_1=\begin{bmatrix}u_{11}\\u_{12}\\u_{13}\\u_{14}\\u_{15}\end{bmatrix}$$

$$=\begin{bmatrix}
-12.1(y_{11}-y_{13})+(-y_{13}+y_{15})-1.6y_{15}+12.1(x_{13}+x_{23})\\
+sin4x_{11}+sin2x_{21}-0.5sin4t-0.5sin2t+10x_{21}-10x_{23}\\
+2.1x_{11}-2.1x_{13}-103.95(x_{11}+x_{21})+0.5x_{11}x_{15}\\
+1.05x_{21}x_{25}+0.25cos4x_{13}+0.25cos2x_{23}+0.25cos4t+0.25cos2t\\
+90x_{21}-0.5x_{23}+13.95x_{11}-0.5x_{11}x_{13}-0.5x_{12}x_{14}-0.5x_{21}x_{23}\\
-0.5x_{22}x_{24}+0.25cos4x_{15}+0.25cos2x_{25}-1cos4t+0.25cos2t\\
-0.5x_{25}-0.3x_{15}+40y_{11}-40y_{13}-6sin10y_{11}-1.5cos10t\\
\\
12.1(-y_{12}+y_{13}+y_{15})+(-y_{13}+y_{15})-3.2y_{15}-12.1(x_{14}+x_{24})\\
-2cos4x_{12}-cos2x_{22}+0.5sin4t+0.5sin2t-10x_{22}+10x_{24}-2.1x_{12}\\
+2.1x_{14}-103.95(x_{11}+x_{21})+0.5x_{11}x_{15}+1.05x_{21}x_{25}\\
+0.25cos4x_{13}+0.25cos2x_{23}+0.25cos4t+0.25cos2t+90x_{21}\\
-0.5x_{23}+13.95x_{11}-x_{11}x_{13}-x_{21}x_{23}-x_{22}x_{24}+0.5cos4x_{15}\\
+0.5cos2x_{25}-2cos4t+0.5cos2t-x_{25}-0.6x_{15}+40y_{12}\\
-40y_{14}+4cos10y_{12}+0.5sin10t\\
\\
-y_{13}+y_{15}-1.6y_{15}-103.95(x_{11}+x_{21})+0.5x_{11}x_{15}\\
+1.05x_{21}x_{25}+0.25cos4x_{13}+0.25cos2x_{23}+0.25cos4t\\
+0.25cos2t+90x_{21}-0.5x_{23}+13.95x_{11}-0.5x_{11}x_{13}-0.5x_{12}x_{14}\\
-0.5x_{21}x_{23}-0.5x_{22}x_{24}+0.25cos4x_{15}+0.25cos2x_{25}-1cos4t\\
\\
0.5(-y_{13}+y_{15})+0.5(y_{13}-2y_{14}+2y_{15})-2.4y_{15}-51.975(x_{11}+x_{21})\\
+0.25x_{11}x_{15}+0.525x_{21}x_{25}+0.125cos4x_{13}+0.125cos2x_{23}\\
+0.125cos4t+0.125cos2t+45x_{21}-0.25x_{23}+6.975x_{11}\\
-103.95(x_{12}+x_{22})+0.5x_{12}x_{15}+1.05x_{22}x_{25}+0.125sin4x_{14}\\
+0.125sin2x_{24}+0.125sin4t+0.25sin2t+90x_{22}-0.5x_{24}\\
+13.95x_{12}-0.75x_{11}x_{13}-0.75x_{12}x_{14}-0.75x_{21}x_{23}-0.75x_{22}x_{24}\\
+0.375cos4x_{15}+0.375cos2x_{25}-1.5cos4t+0.375cos2t-0.75x_{25}\\
-0.45x_{15}-22y_{14}+y_{12}y_{15}+2cos10y_{14}+0.25cos10t\\
\\
-1.6y_{15}-0.5x_{11}x_{13}-0.5x_{12}x_{14}-0.5x_{21}x_{23}-0.5x_{22}x_{24}+0.25cos4x_{15}\\
+0.25cos2x_{25}-1cos4t+0.25cos2t-0.5x_{25}-0.3x_{15}+5y_{15}\\
-y_{11}y_{13}-y_{12}y_{14}+5cos10y_{15}-0.5cos10t
\end{bmatrix}$$

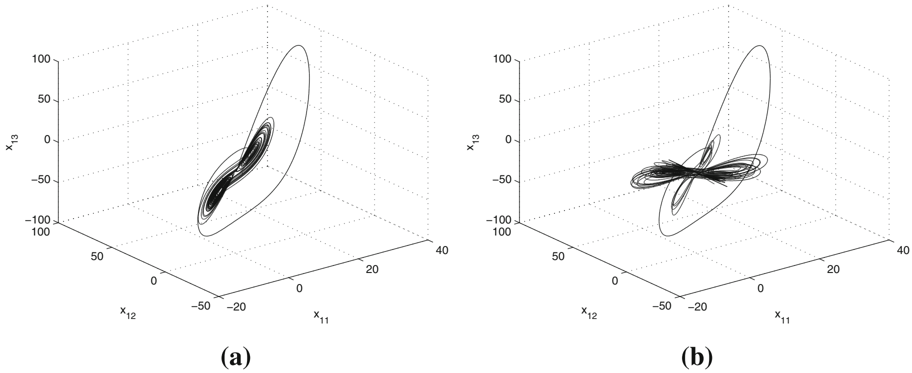

Fig. 1 Phase portrait of FO complex Lorenz chaotic system for $\alpha = 0.95$: **a** $x_{11} - x_{12} - x_{13}$ space; **b** with disturbance and uncertainty, $x_{21} - x_{22} - x_{23}$

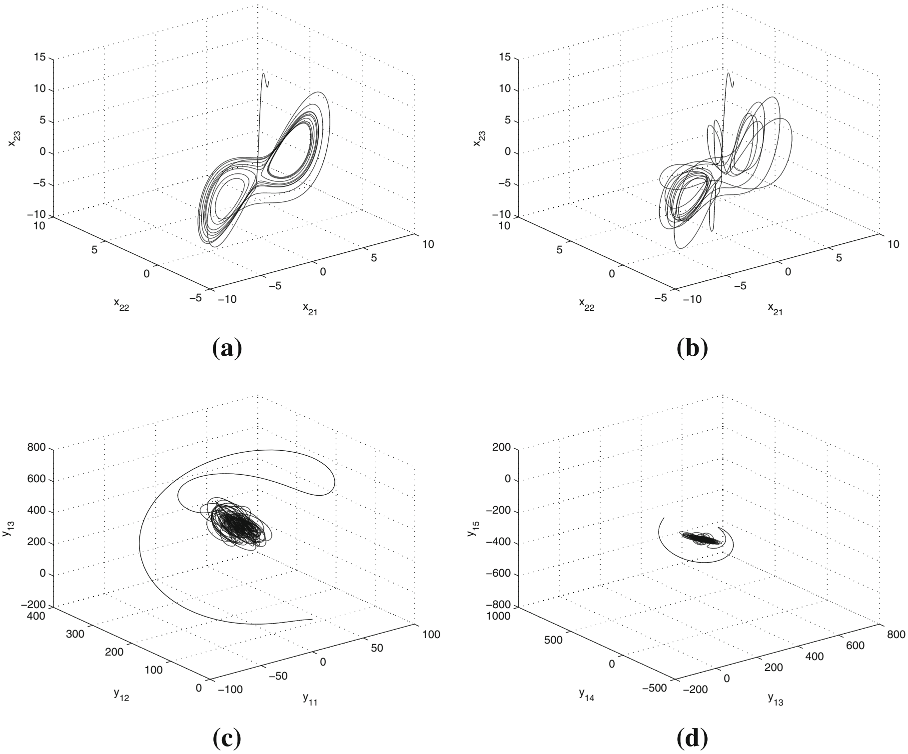

Fig. 2 Phase portrait of FO complex T chaotic system for $\alpha = 0.95$: **a** $x_{11} - x_{12} - x_{13}$ space; **b** with disturbance and uncertainty, $x_{21} - x_{22} - x_{23}$ space; Phase portrait of FO complex Lu chaotic system for $\alpha = 0.95$: **c** with disturbance and uncertainty, $y_{11} - y_{12} - y_{13}$ space ; **d** with disturbance and uncertainty, $y_{13} - y_{14} - y_{15}$ space

Using the value of controllers in $u_{11}, u_{12}, u_{13}, u_{14}, u_{15}$ in error dynamics (21), now error dynamics can be obtained as

$$\begin{cases} D^{\alpha}e_{11} = -12.1e_{11} \\ D^{\alpha}e_{12} = -12.1e_{12} \\ D^{\alpha}e_{13} = -e_{13} \\ D^{\alpha}e_{14} = -e_{14} \\ D^{\alpha}e_{15} = -1.6e_{15} \end{cases} \quad (22)$$

Hence, the IMPCS between master systems (14), (15), and slave system (16) is achieved

Numerical Simulation

In this section, during numerical simulation of MPCS and IMPCS of FO complex chaotic systems. The initial values for master systems (14), (15), and slave system (16) are $(x_{11}(0), x_{12}(0), x_{13}(0), x_{14}(0), x_{15}(0)) = (2, 3, 5, 6, 9)$, $(x_{21}(0), x_{22}(0), x_{23}(0), x_{24}(0), x_{25}(0)) = (8, 7, 6, 8, 7)$, $(y_{11}(0), y_{12}(0), y_{13}(0), y_{14}(0), y_{15}(0)) = (1, 2, 3, 4, 5)$ respectively. Hence, according to the definition of MPS error function, the initial value of error system will be $(e_{11}(0), e_{12}(0), e_{13}(0), e_{14}(0), e_{15}(0)) = (2, -15, -7, -11, 37)$, for $\alpha = 0.95$. Phase portrait of FO complex Lorenz chaotic systems for $\alpha = 0.95$ without disturbance and uncertainty and with disturbance and uncertainty illustrated in Fig. 1a, b respectively. Phase portrait of FO complex T chaotic systems for $\alpha = 0.95$ without disturbance and uncertainty and with disturbance and uncertainty shown in Fig. 2a, b respectively and Fig. 2c, d illustrates the phase portrait in 3D of FO complex Lu system. Figure 3a–e shows the state trajectories of master systems and slave system are synchronized in MPCS technique. Figure 3f describes that error of MPCS $(e_{11}(t), e_{12}(t), e_{13}(t), e_{14}(t), e_{15}(t))$ are converging to zero when times becomes large. According to the definition of IMPS error function system, the initial value of the error system for $\alpha = 0.95$ will be $(e_{11}(0), e_{12}(0), e_{13}(0), e_{14}(0), e_{15}(0)) = (2.75, 3.9, 1.8, 15.8, -7.5)$. Figure 4a–e depicts the state trajectories of master systems and slave system are synchronized using IMPCS technique. Figure 4f illustrates the error of IMPCS $(e_{11}(t), e_{12}(t), e_{13}(t), e_{14}(t), e_{15}(t))$ are converging to zero.

Comparison of Synchronization Results with Previous Published Work

In [31] author studies the matrix projective synchronization (MPS) and inverse matrix projective synchronization (IMPS) technique for the FO hyper-chaotic system disturbed by uncertainty and disturbance using active control. They attain synchronization error approx at time t = 4 sec and t = 3.75 sec, as shown in Fig. 5a, b, respectively. Whereas in the present scheme, in which we achieved matrix projective combination synchronization (MPCS) and inverse matrix projective combination synchronization (IMPCS) disturbed by uncertainty and disturbance using the same technique. We had considered three non-identical FO complex, chaotic systems. The MPCS error and IMPCS error have been synchronized approx at t = 2.75, and t = 3.25, respectively, as shown in Fig. 5c, d. The present technique takes less time to synchronize error trajectories. This shows that our examined MPCS and IMPCS scheme using the active control technique is convenient over earlier published work.

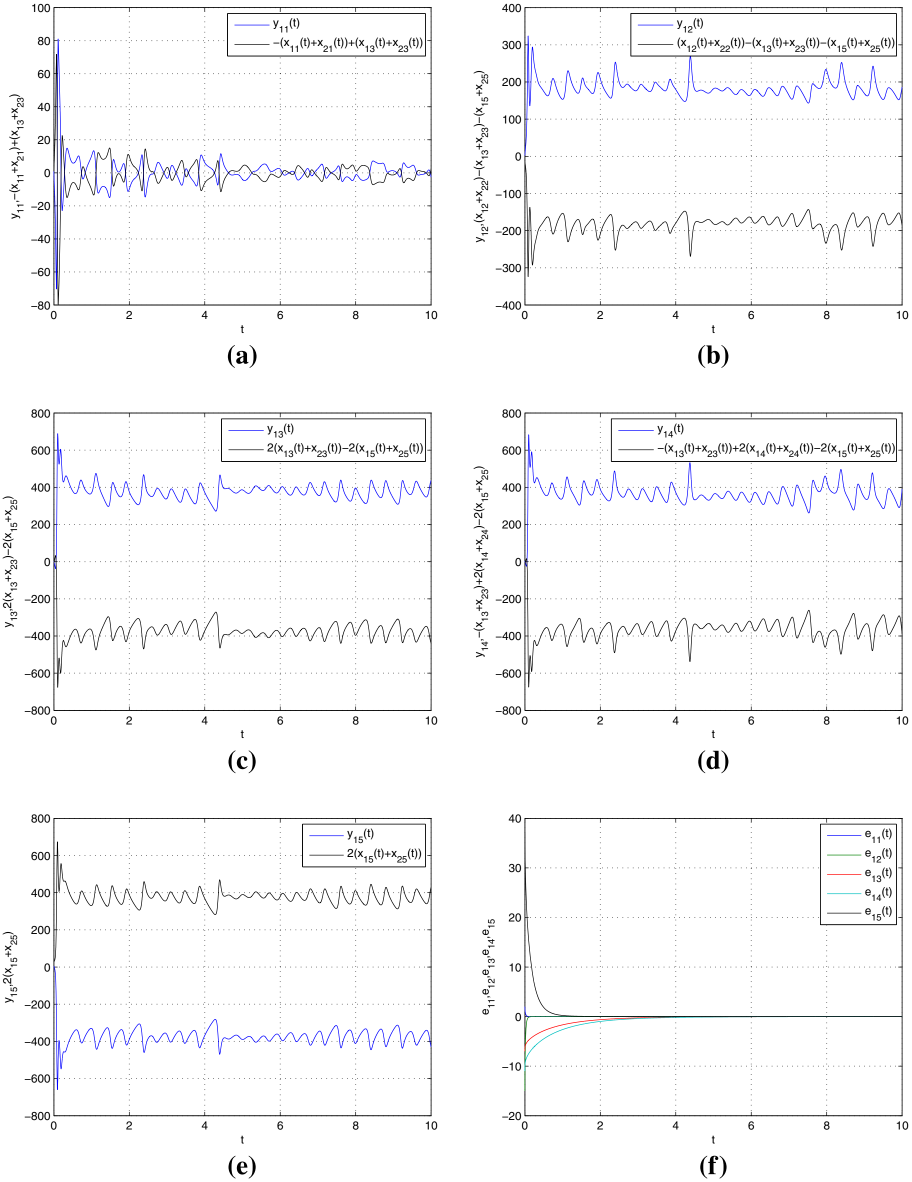

Fig. 3 The MPCS between systems (14), (15), and (16) at $\alpha = 0.95$: **a** Between $y_{11}(t)$ and $-(x_{11}(t) + x_{21}(t)) + (x_{13}(t) + x_{23}(t))$; **b** Between $y_{12}(t)$ and $(x_{12}(t) + x_{22}(t)) - (x_{13}(t) + x_{23}(t)) - (x_{15}(t) + x_{25}(t))$; **c** Between $y_{13}(t)$ and $2(x_{13}(t) + x_{23}(t)) - 2(x_{15}(t) + x_{25}(t))$; **d** Between $y_{11}(t)$ and $-(x_{13}(t) + x_{23}(t)) + 2(x_{14}(t) + x_{24}(t)) - 2(x_{15}(t) + x_{25}(t))$; **e** Between $y_{15}(t)$ and $2(x_{15}(t) + x_{25}(t))$; **f** Synchronization error system

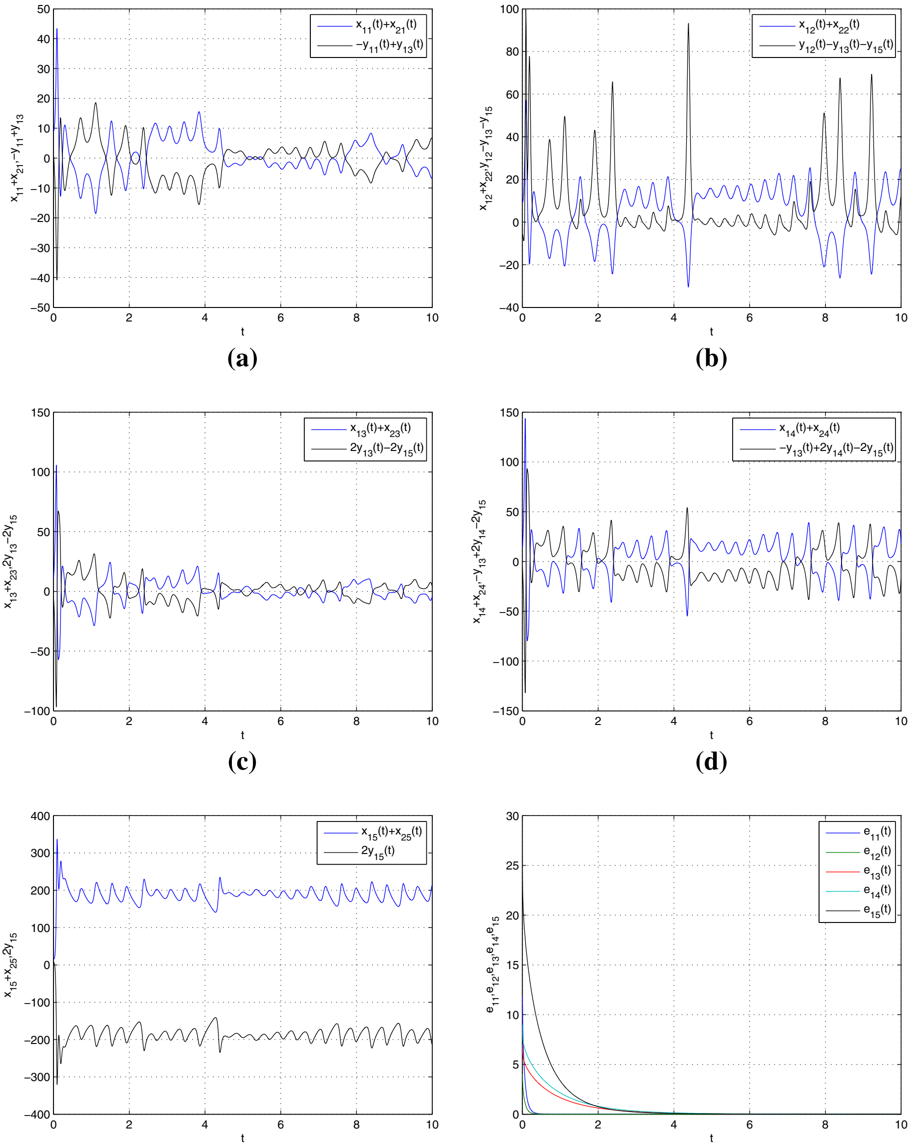

Fig. 4 The IMPCS between systems (14), (15), and (16) at $\alpha = 0.95$: **a** Between $x_{11}(t) + x_{21}(t)$ and $-y_{11}(t) + y_{13}(t)$; **b** Between $x_{12}(t) + x_{22}(t)$ and $y_{12}(t) - y_{13}(t) - y_{15}(t)$; **c** Between $x_{13}(t) + x_{23}(t)$ and $2y_{13}(t) - 2y_{15}(t)$; **d** Between $x_{14}(t) + x_{24}(t)$ and $-y_{13}(t) + 2y_{14}(t)) - 2y_{15}(t)$; **e** Between $x_{15}(t) + x_{25}(t)$ and $2y_{15}(t)$; **f** Synchronization error system

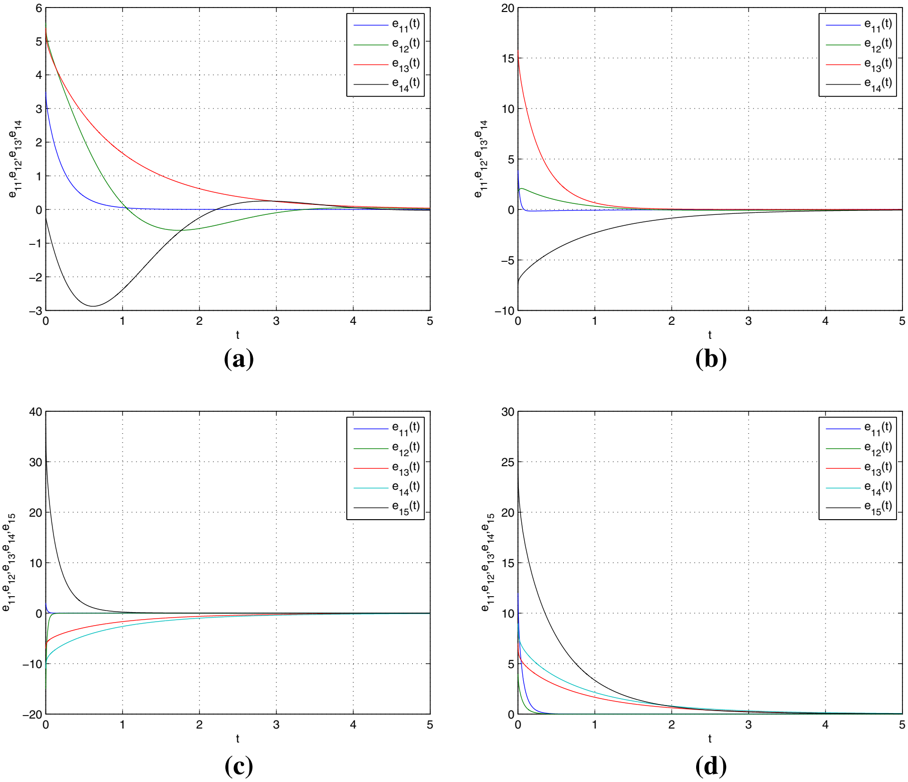

Fig. 5 Synchronization errors **a** matrix projective synchronization errors, **b** inverse matrix projective synchronization errors, **c** matrix projective combination synchronization errors, **d** inverse matrix projective combination synchronization errors

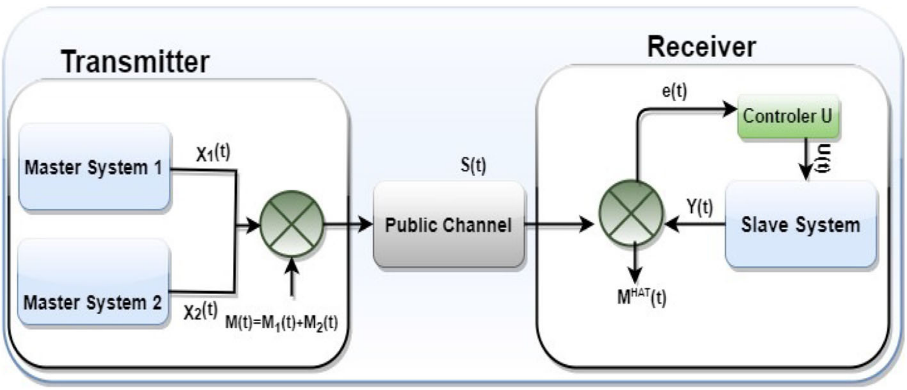

Fig. 6 Chao based secure communication system by chaotic signal masking technique

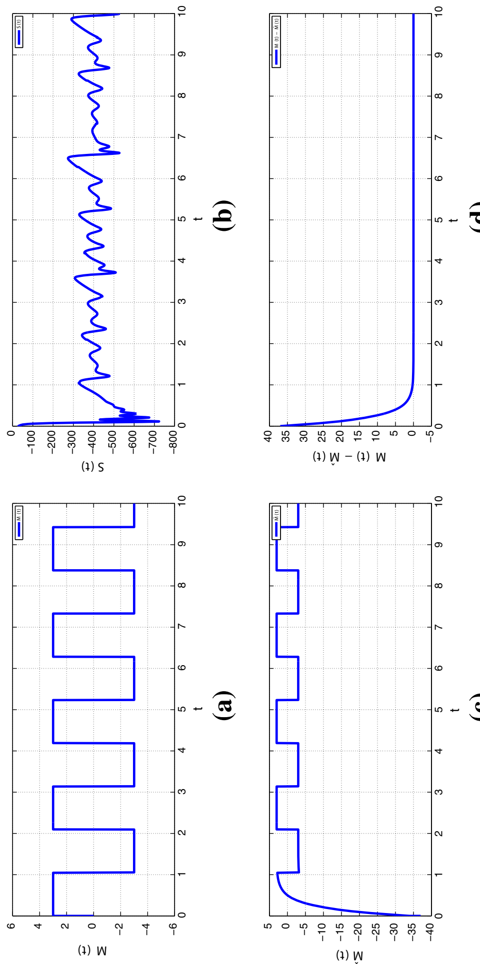

Fig. 7 **a** Information message signal $M(t)$, **b** the encrypted signal $S(t)$, **c** decrypted signal $\hat{M}(t)$, **d** error between $M(t) - \hat{M}(t)$

Secure Communication Technique

[28,44–46] The secure communication is one of the most powerful applications of chaos synchronization. The concept of a secure communication system involves the construction of a signal includes some hidden message that is to continue unpredictable by the intercepters with the transmitter signals. In this section, the secure communication application of matrix projective combination synchronization is performed, which is based on the chaotic signal masking technique. In secure communication method, the system consisting of a transmitter (or master) and receiver (or slave). The secure communication scheme is sketched as Fig. 6. We will use Eqs. (14), (15) as master systems, and Eq. (16) as a slave system. The information message signal is select to be a periodic function $M(t) = M_1(t) + M_2(t) = 3 * sign(sin3t)$, which is attached to the master signal. The encrypted information is given by $S(t) = M(t) - 2(x_{15}(t) + x_{25}(t))$ is attached to the slave signal. The decrypted message signal is given by $\hat{M}(t) = S(t) - y_{15}(t)$. We choose the message signals are in the form of $M_1(t) = sign(sin3t)$, $M_2(t) = 2 * sign(sin3t)$. The information signal $M(t) = 3 * sign(sin3t)$ and the encrypted signal $S(t)$ are shown in Fig. 7a, b, respectively. Figure 7c illustrates the decrypted signal $\hat{M}(t)$ and Fig. 7d displays the error signal $M(t) - \hat{M}(t)$. Figure 7a–d portrayed that the $M(t) = 3 * sign(sin3t)$ is recovered favourably at the receiver end.

Conclusion

Combination projective synchronization (CPS) achieved in three non-identical F0 complex chaotic systems witch are disturbed by disturbance and uncertainties. In combination projective synchronization, MPCS, and IMPCS have been presented. Initially, to obtain MPCS between non-identical FO complex chaotic systems, the control technique was introduced by controlling the linear part of the slave system. Further, to get IMPCS, the control method was introduced by controlling the linear part of the master systems. It is based on the stability analysis of the fractional derivative of the linear system, since when time becomes large, then the error system goes to zero by using a suitable controller input parameter. Due to the complexity of the introduced scheme, the MPCS and IMPCS may improve security in communication. Therefore, with the increasing demand for protection of transmission, we design an actual application in the field of secure communication. Further, in the future direction, we can study matrix hybrid complex projective compound combination synchronization interrupted by model uncertainties and mismatched disturbance in FO complex chaotic systems using the adaptive control. Finally, we have compared our results with the earlier published results. Our results display the novelty over the compared outcomes.

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