



Divided Differences Calculus in Matrix Representation

Robert M. Yamaleev¹

Published online: 3 September 2019
© Springer Nature India Private Limited 2019

Abstract

A systematic description of actions of the divided differences operators on power and exponential functions is given. The results of actions of these operators on entire functions are presented by the matrices whose elements are functions of coefficients of a characteristic (pivot) polynomial. Effective algorithms of calculation of the matrices are constructed using the properties of the companion matrix of the pivoting polynomial. Degeneration of the roots of the pivot polynomial reduces the n -order divided differences operator to $n - 1$ order operator of differentiation. The exponential type invariant functions with respect to higher order derivatives are constructed.

Keywords Vandermonde matrix · Divided differences · Trigonometry · Pascal matrix · Polynomial · Invariant functions

Introduction

The divided differences method is used as a numerical procedure for interpolating polynomials given a set of points [1]. In the textbooks the divided differences are used as an origin of the differential and integral calculus [2].

An important feature of the divided differences operators is that the result of action of these operators on analytical functions is expressed by the complete homogeneous symmetric forms of nodes. In turn, the complete homogeneous symmetric forms are unambiguously expressed via coefficients of the characteristic polynomial. If the nodes of the divided differences operators are cumulated near the one of them, e.g., the roots of the characteristic polynomial are degenerated, then the divided differences of the power functions are presented by the Pascal triangle-form matrix. In that case one may define exponential-type invariant functions.

The most prominent examples correspond to divided differences of the power and exponential functions. Divided differences of the power functions are just the generalized Chebyshev functions [3]. An action of the divided differences operators on exponential function straightforward leads one to the system of generalized trigonometric functions [4].

In this paper we develop a matrix approach to the divided difference calculus. In contrast to other approaches, the present method mostly requires the characteristic polynomial, the

✉ Robert M. Yamaleev
yamaleev@jinr.ru

¹ Joint Institute for Nuclear Research, LIT, Dubna, Russia

Vandermonde [5] and the Pascal matrices [6]. The paper is organized as follows. In “Divided Differences and Their Pivot Polynomials” section, the definitions and main properties of the divided differences operators are done and a notion of the pivot polynomial is introduced. The results of actions of the divided differences on the polynomials are formulated in a matrix representation. In “Complete System of Divided Differences Operators” section, the complete system of divided differences operators is introduced. In “Degenerated Divided Difference Operators and Their Invariant Functions” section, a set of the functions invariant with respect to the degenerated divided differences operators is constructed. The paper ends with concluding remarks in the section “Conclusion”.

Divided Differences and Their Pivot Polynomials

Terminology and Notations

Through the text we shall use the upper index as the degree of a variable. Another types of the upper indices are included inside brackets.

Operation of *divided differences* (thereafter, DD) has different equivalent definitions. Usually, it is started with the zeroth DD of the function $f(x)$ with respect to x_i , which is value of $f(x)$ at the point. We shall use an alternative start definition of DD operation:

$$D_2(1)f(x) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}. \tag{1}$$

This formula corresponds to the second order DD- operation on the function $f(x)$.

In order to extend this definition to the case of $n > 2$ order we shall use remarkable properties of the Vandermonde matrix [7]. Consider the following set of x points $[x_1, x_2, x_3, \dots, x_n]$ and corresponding values of a function $f(x)$ at these points. On the given n data points $(x_k, y_k), k = 1, \dots, n$, where $y_k = f(x_k)$, define n -dimensional column- vectors

$$\mathbf{v}_k = [x_1^k, x_2^k, \dots, x_n^k]^T, \quad k = 0, 1, \dots, n - 1, \tag{2}$$

and

$$\mathbf{f} = [y_1, y_2, y_3, \dots, y_{n-1}, y_n]^T, \tag{3}$$

and define the Vandermonde matrix

$$VM(x; n) = \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \dots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \dots & x_2^2 & x_2 & 1 \\ x_3^{n-1} & x_3^{n-2} & \dots & x_3^2 & x_3 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_n^{n-1} & x_n^{n-2} & \dots & x_n^2 & x_n & 1 \end{pmatrix}. \tag{4}$$

In terms of the vectors \mathbf{v}_k the Vandermonde matrix (4) is defined in the following compact form

$$VM(x; n) = |\mathbf{v}_{n-1}, \mathbf{v}_{n-2}, \dots, \mathbf{v}_1, \mathbf{v}_0|, \tag{5}$$

with determinant

$$V(x; n) = Det(VM(x; n)) = \prod_{i>k} (x_i - x_k). \tag{6}$$

In a consistent manner define the matrix $WM_1(f; x; n)$ by replacing the first column of the Vandermonde matrix $\mathbf{v}^{(n-1)}$ by the vector \mathbf{f} ,

$$WM(f; x; n) = |\mathbf{f}, \mathbf{v}_{n-2}, \mathbf{v}_{n-3}, \dots, \mathbf{v}_{(1)}, \mathbf{v}_0|. \tag{7}$$

Denote determinants of this matrix by $W(f; x; n)$. Let us notice that

$$W(x^k; x; n) = V(x; n) \text{ for } k = n - 1, \tag{8}$$

and,

$$W(x^k; x; n) = 0, \text{ for } k < n - 1. \tag{9}$$

Definition 1 The n -th order DD- operator acting on the function $f(x)$ is defined by the following ratio

$$D_n f(x) = \frac{W(f; x; n)}{V(x; n)}. \tag{10}$$

From (7)–(8) immediately it follows that

$$D_n x^k = 1, \text{ for } k = n - 1, \tag{11}$$

$$D_n x^k = 0, \text{ for } k < n - 1. \tag{12}$$

Pivot Polynomial and Its Companion Matrix

Let $x_1, x_2, \dots, x_n \in R$ be a set of nodes on which the DD-operation of n -order is defined. Define a polynomial of the form

$$P(X) = X^n - a_1 X^{n-1} + a_2 X^{n-2} + \dots + (-1)^n a_n, \tag{13}$$

the roots of which constitutes the set of nodes of the DD-operator. This is the *pivot polynomial* associated with the DD-operators.

In the theory of polynomials an important part takes the notion of the companion matrix E which explicitly is presented as follows

$$E = \begin{pmatrix} 0 & 0 & \cdot & 0 & 0 & (-1)^{n-1} a_n \\ 1 & 0 & \cdot & 0 & 0 & (-1)^{n-2} a_{n-1} \\ 0 & 1 & \cdot & 0 & 0 & (-1)^{n-3} a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & 0 & -a_2 \\ 0 & 0 & \cdot & 0 & 1 & a_1 \end{pmatrix}. \tag{14}$$

It is convenient to present this matrix via the following n -dimensional column- vectors

$$(w_i)_{ij} = \delta_{i,j=i+1}, \quad i, j = 1, \dots, n;$$

and

$$\mathbf{b}^{(1)} = [(-1)^{n-1} a_n, \dots, -a_2, a_1]^T. \tag{15}$$

Then,

$$E = |w_1, w_2, \dots, w_{n-1}, \mathbf{b}^{(1)}|. \tag{16}$$

In addition, with the companion matrix E one may generate $(n - 2)$ vectors by

$$\mathbf{b}^{(k+1)} = E\mathbf{b}^{(k)}, \quad k = 1, 2, 3, \dots, n - 2. \tag{17}$$

On the basis of the vectors w_k and $\mathbf{b}^{(k)}$ the powers of the companion matrix E can be presented simply by

$$E^p = |w_p, \dots, w_{n-1}, \mathbf{b}^{(1)}, \dots, \mathbf{b}^{(p)}|, \quad p = 2, 3, \dots, n - 1. \tag{18}$$

Divided Differences of Power Functions

The result of action of the DD-operator on a power function is expressed via complete homogeneous symmetric forms of the roots of the pivot polynomial $P(X)$. According to Vieta’s theorem the complete homogeneous symmetric form can be presented by the coefficients of the polynomial. This means, the results of actions of DD-operators are unambiguously expressed via the coefficients of the pivot polynomial $P(X)$.

Example 1

$$D_2(1)x^2 = \frac{x_1^2 - x_2^2}{x_1 - x_2} = a_1, \quad D_3(1) = \frac{x_1^3 - x_2^3}{x_1 - x_2} = a_1^2 - a_2, \tag{19}$$

where a_1, a_2 are coefficients of the pivot polynomial

$$P(X) = X^2 - a_1X + a_2. \tag{20}$$

$$D_3(1)x^3 = \frac{x_1^3(x_2 - x_3) + x_2^3(x_3 - x_1) + x_3^3(x_1 - x_2)}{(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)} = a_1, \tag{21}$$

$$D_3(1)x^4 = \frac{x_1^4(x_2 - x_3) + x_2^4(x_3 - x_1) + x_3^4(x_1 - x_2)}{(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)} = a_1^2 - a_2,$$

$$D_3(1)x^5 = a_3 - 2a_1a_2 + a_1^3,$$

$$D_3(1)x^6 = 2a_1a_3 - 3a_1^2a_2 + a_2^2 + a_1^4, \tag{22}$$

where a_1, a_2, a_3 are coefficients of the cubic polynomial

$$P(X) = X^3 - a_1X^2 + a_2X + a_3. \tag{23}$$

It is now almost natural to elaborate general algorithm of calculations of DD-operations on power functions which is associated with the upper-triangular matrices.

Consider the following n - degree polynomial

$$F(a; \mathbf{Z}) = I + \sum_{k=1}^{n-1} (-1)^k a_k \mathbf{Z}^k, \quad \mathbf{Z}^n = 0, \tag{24}$$

where \mathbf{Z} is the nilpotent matrix

$$\mathbf{Z} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{25}$$

Proposition 2.1 *The results of action of the DD-operators on the power function are given by the last column of the matrix inverse to the matrix $F(a; \mathbf{Z})$.*

Example 2 Explicit form of the polynomial $F(a; \mathbf{Z})$ is specified by the following upper-triangular matrix

$$F(a; \mathbf{Z}) = \begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & a_4 \\ 0 & 1 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

The matrix inverse to this matrix is

$$F^{-1}(a; \mathbf{Z}) = \begin{pmatrix} 1 & a_1 & -a_2 + a_1^2 & a_3 - 2a_1a_2 + a_1^3 & -a_4 + 2a_1a_3 - 3a_1^2a_2 + a_2^2 + a_1^4 \\ 0 & 1 & a_1 & -a_2 + a_1^2 & a_3 - 2a_1a_2 + a_1^3 \\ 0 & 0 & 1 & a_1 & -a_2 + a_1^2 \\ 0 & 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

It is seen, entries of this matrix contain the table of actions of the DD-operators on the power function. A complete information is contained on the last column, or, on the first line of the matrix. In fact, compare the entries of this matrix with following table.

Example 3

$$\begin{pmatrix} D_n x^{n-1} = b_1 & & & & 1 \\ D_n x^n = b_2 = & & & & a_1 \\ D_n x^{n+1} = b_3 = & & & & -a_2 + a_1^2 \\ D_n x^{n+2} = b_4 = & & & & a_3 - 2a_1a_2 + a_1^3 \\ D_n x^{n+3} = b_5 = & & & & -a_4 + 2a_1a_3 - 3a_1^2a_2 + a_2^2 + a_1^4 \\ \dots & & & & \dots \\ D_n x^{n+k} = b_{k+2} = & & & & \sum_{j=1}^{k+1} (-1)^{j+1} a_j b_{i-j}. \end{pmatrix}. \tag{28}$$

The matrix form of the table of results is presented as follows. If the power functions are arranged as coefficients the following polynomial

$$F(x^{n-1}; \mathbf{Z}) = x^{n-1}I + x^{(n-1)+1}\mathbf{Z} + \dots + x^{(n-1)+k}\mathbf{Z}^k + \dots + x^{2(n-1)}\mathbf{Z}^{n-1}, \tag{29}$$

then, the general table of results of action of DD-operator on power function can be cast into the following matrix equation

$$D_n F(x^{n-1}; \mathbf{Z}) = (F(a; \mathbf{Z}))^{-1}. \tag{30}$$

This representation is quite general, unique remark is that the coefficients used in the right hand side has to correspond to the pivoting polynomial.

Example 4

$$D_2 \begin{pmatrix} x & x^2 \\ 1 & x \end{pmatrix} = \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix}^{-1},$$

Here the coefficients belong the pivot polynomial $P(X) = X^2 - a_1X + a_2$;

$$D_3 \begin{pmatrix} x^2 & x^3 & x^4 \\ 0 & x^2 & x^3 \\ 0 & 0 & x^2 \end{pmatrix} = \begin{pmatrix} 1 & -a_1 & a_2 \\ 0 & 1 & -a_1 \\ 0 & 0 & 1 \end{pmatrix}^{-1},$$

with coefficients of the pivot polynomial $P(X) = X^3 - a_1X^2 + a_2X + a_3$

$$D_4 \begin{pmatrix} x^3 & x^4 & x^5 & x^6 \\ 0 & x^3 & x^4 & x^5 \\ 0 & 0 & x^3 & x^4 \\ 0 & 0 & 0 & x^3 \end{pmatrix} = \begin{pmatrix} 1 & -a_1 & a_2 & -a_3 \\ 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}, \tag{31}$$

with coefficients of the third order pivot polynomial

$$P(X) = X^4 - a_1X^3 + a_2X^2 - a_3X + a_4.$$

The elements of the inverse matrix $F^{-1}(a; \mathbf{Z})$ are calculated according to the following recursive algorithm.

Algorithm 2.1

$$b_i = \sum_{j=1}^{i-1} (-1)^{j+1} a_j b_{i-j}, \quad b_1 = 1, \quad b_2 = a_1. \tag{32}$$

This algorithm is a consequence of the following lemma.

Lemma 2.1 Consider the power function $f(x; k) = x^{n+k}$, $k = 1, 2, \dots$, at the roots x_j , $j = 1, 2, \dots, n$ of the pivot polynomial $P(X)$. The power function $f(x; k)$ at these points can be reduced to the $m = n - 1$ degree polynomial of the form

$$(x_j)^{n+k} = R_m^{(k)}(x_j) = b_m^{(k)}(x_j)^m + b_{m-1}^{(k)}(x_j)^{m-1} \dots + b_1^{(k)}(x_j) + b_0^{(k)}, \quad j = 1, 2, \dots, n, \tag{33}$$

where coefficients $b_i^{(k)}$ are calculated according to algorithm

$$b_i^{(k)} = \sum_{j=1}^{i-1} (E^{k-1})_i^j a_j, \quad k > 1. \tag{34}$$

where E is the companion matrix of the pivot polynomial $P(X)$.

Remark 1 Eventually, this formula can be written also in the more transparent form

$$\begin{pmatrix} b_0^{(k)} \\ b_1^{(k)} \\ b_2^{(k)} \\ \vdots \\ b_{n-2}^{(k)} \\ b_{n-1}^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdot & 0 & 0 & (-1)^{n-1} a_n \\ 1 & 0 & \cdot & 0 & 0 & (-1)^{n-2} a_{n-1} \\ 0 & 1 & \cdot & 0 & 0 & (-1)^{n-3} a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & 0 & -a_2 \\ 0 & 0 & \cdot & 0 & 1 & a_1 \end{pmatrix}^k \begin{pmatrix} (-1)^{n-1} a_n \\ (-1)^{n-2} a_{n-1} \\ (-1)^{n-3} a_{n-2} \\ \vdots \\ -a_2 \\ a_1 \end{pmatrix} \tag{35}$$

In particularly, if $f(x) = x^n$, $k = 0$, then

$$b_{n-1}^{(0)} = (-1)^{l+1} a_l. \tag{36}$$

Example 5 As an example let us consider a presentation of the power function $f(x) = x^k, k > 3$ at roots of the cubic polynomial

$$X^3 = a_1X^2 - a_2X + a_3. \tag{37}$$

The algorithm of calculation of the coefficients of the polynomial

$$(x_j)^{3+k} = R_2^{(k)}(x_j) = b_2^{(k)}(x_j)^2 + b_1^{(k)}(x_j)^1 + b_0^{(k)}, \quad j = 1, 2, 3, \tag{38}$$

is read as

$$\begin{pmatrix} b_0^{(k)} \\ b_1^{(k)} \\ b_2^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & a_1 \end{pmatrix}^k \begin{pmatrix} a_3 \\ -a_2 \\ a_1 \end{pmatrix}. \tag{39}$$

This algorithm provides one with all coefficients of the polynomial R . In addition, this algorithm can be formulated in an alternative way by taking into account a peculiar structure of the matrix E^k . From formulas (18) and (35) it follows that the information on the coefficients b_k can be directly extracted from corresponding column- vectors of the companion matrix E^p .

Complete System of Divided Differences Operators

Definitions

Now, let us explore another kinds of the DD-operators. In Section 2 the matrix $WM(f; x; n)$ has been obtained from the Vandermonde matrix $VM(x; n)$ by replacing the first column with the vector \mathbf{f} . However, besides of this possibility one may build the set of matrices by replacing other columns of the Vandermonde matrix with the vector \mathbf{f} .

Example 6

$$WM_1 \begin{pmatrix} f(x_1) & x_1 & 1 \\ f(x_2) & x_2 & 1 \\ f(x_3) & x_3 & 1 \end{pmatrix}, \quad WM_2 = \begin{pmatrix} x_1^2 & f(x_1) & 1 \\ x_2^2 & f(x_2) & 1 \\ x_3^2 & f(x_3) & 1 \end{pmatrix}, \quad WM_3 = \begin{pmatrix} x_1^2 & x_1 & f(x_1) \\ x_2^2 & x_2 & f(x_2) \\ x_3^2 & x_3 & f(x_3) \end{pmatrix}. \tag{40}$$

By defining determinants of these matrices

$$W_k = Det(WM_k), \quad k = 1, 2, 3, \tag{41}$$

one may introduce three kinds of the divided differences operators,

$$D_3(k) f(x) = \frac{1}{V(x; 3)} Det(WM_k(f; x; 3)), \quad k = 1, 2, 3. \tag{42}$$

Algorithm 3.1 The general scheme can be specified as follows:

$$\begin{aligned} WM_1(f; x; n) &= |\mathbf{f}, \mathbf{v}_{n-2}, \mathbf{v}_{n-3}, \dots, \mathbf{v}_1, \mathbf{v}_0|. \\ WM_2(f; x; n) &= |\mathbf{v}_{n-1}, \mathbf{f}, \mathbf{v}_{n-3}, \dots, \mathbf{v}_1, \mathbf{v}_0|. \\ &\dots \dots \dots \\ WM_k(f; x; n) &= |\mathbf{v}_{n-1}, \mathbf{v}_{n-2}, \dots, \mathbf{v}_{n-k+1}, \mathbf{f}, \mathbf{v}_{n-k-1}, \dots, \mathbf{v}_1, \mathbf{v}_0|. \end{aligned} \tag{43}$$

Correspondingly, the complete set of DD- operators are defined as

$$D_k(p) = \frac{Det WM_p(f; : n)}{V(x; n)}, \quad k = 1, 2, \dots, n; \quad p = 1, 2, \dots, n. \tag{44}$$

In accordance with the notations introduced in Section 2, it is taken as $WM = WM_1$ and $D_k = D_k(1)$.

The power function $f(x) = x^k$ is one of basic functions of the divided difference calculus. For that function the following important relationship holds true.

Theorem 3.1 *The link of determinants of the matrices $WM_j(x^k; x; n)$, $j = 1, \dots, n$ with the determinant of the Vandermonde matrix $VM(x; n)$ is given by the formula*

$$Det WM_m(x^{n+k}; x; n) = V(x; n) b_{n-m}^{(k)}, \quad m = 1, 2, \dots, n \quad k = 1, 2, \dots, \tag{45}$$

where $b_p^{(j)}$, $p = 0, 1, \dots, n - 1$ are the coefficients of the polynomials $R^{(k)}(x_j)$, $j = 1, \dots, n$.

Proof The power function $f(x) = x^{n+k}$, $k = 1, 2, 3, \dots$, is included into the matrices $WM_p(x^{n+k}; x; n)$, $p = 1, 2, \dots, n$ only at the roots of the polynomial $P(X)$, hence, in accordance with Lemma 2.1, the functions $f(x_j) = (x_j)^{n+k}$ can be replaced by the polynomial $R_{n-1}^{(k)}(x_j)$. Define a column-vector $\mathbf{r}_{n-1}^{(k)}$ with n components as follows

$$\mathbf{r}_{n-1}^{(k)} = [R_{n-1}^{(k)}(x_1), R_{n-1}^{(k)}(x_2), R_{n-1}^{(k)}(x_n)]^T, \tag{46}$$

and introduce this vector into the matrix $WM_p(x^{n+k}; x; n)$ instead of the vector \mathbf{f} . In this we get the matrix of the form

$$WR_k(f; x; n) = ||\mathbf{v}^{(n-2)}, \dots, \mathbf{v}^{(k+1)}, \mathbf{r}_{n-1}^{(m)}, \mathbf{v}^{(k)}, \dots, \mathbf{v}^{(1)}, \mathbf{v}^{(0)}||. \tag{47}$$

Expand the determinant of the matrix $WR_k(f; x; n)$ to the sum of determinants according to the well-known rule [7]:

$$Det WR(x^{n+m}; x; n) = \sum_{m=0}^{n-1} b_{n-m} Det WM_m(x^{n+k}; x; n). \tag{48}$$

Any term of the sum, apart of the term containing factor $b_{n-1}^{(j)}$ vanishes (the case of two equal columns).

The rest non-trivial determinant is equal to the determinant of the n -order Vandermonde matrix multiplied by the factor $b_{n-1}^{(j)}$, e.g.,

$$Det WM_k(x^k; x; n) = V(x; n) b_{n-1}^{(k)}. \tag{49}$$

□

Example 7 Consider action of the operators $D_3(k)$, $k = 1, 2, 3$ on function $f(x)$

$$D_3(1) = \frac{1}{V(x; 3)} \begin{pmatrix} f(x_1) & x_1 & 1 \\ f(x_2) & x_2 & 1 \\ f(x_3) & x_3 & 1 \end{pmatrix}, \quad D_3(2) = \frac{1}{V(x; 3)} \begin{pmatrix} x_1^2 & f(x_1) & 1 \\ x_2^2 & f(x_2) & 1 \\ x_3^2 & f(x_3) & 1 \end{pmatrix},$$

$$D_3(3) = \frac{1}{V(x; 3)} \begin{pmatrix} x_1^2 & x_1 & f(x_1) \\ x_2^2 & x_2 & f(x_2) \\ x_3^2 & x_3 & f(x_3) \end{pmatrix}. \tag{50}$$

It is seen, this system of equations are substantially equivalent to those defining according the Cramer’s rule for the system of algebraic equations

$$\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} D_3(1)f(x) \\ D_3(2)f(x) \\ D_3(3)f(x) \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{pmatrix}. \tag{51}$$

In the general case, one gets a system of n linear equations for n unknowns $D_n(k), k = 1, \dots, n$. Then, by applying the Cramer’s rule, one will obtain exactly the definition of the DD-operators acting on the function $f(x)$. Furthermore, for the entire function $f(x)$ and the companion matrix E of n -degree polynomial $P(X)$ the following expansion holds true

$$f(E) = I F_n + E F_{n-1} + E^2 F_{n-2} + \dots + E^{n-1} F_1, \tag{52}$$

where the components $F_k, k = 0, \dots, n-1$ are solutions of n linear algebraic system of equation with Vandermonde matrix. The explicit form of the solution of such a system allows one to conclude that

$$F_k = D_n(k)f(x), k = 1, \dots, n. \tag{53}$$

The manipulation ends up with the identity

$$\begin{aligned} f(E) &= I D_n(n)f(x) + E D_n(n-1)f(x) + E^2 D_n(n-2)f(x) \\ &+ \dots + E^{n-1} D_n(1)f(x). \end{aligned} \tag{54}$$

By taking advantage from this identity the following table of formulas can be constructed for the divided differences of the power functions

Example 8 ($n = 2$)

$$\begin{aligned} \begin{pmatrix} D_2(2) \\ D_2(1) \end{pmatrix} \otimes (x \ x^2) &= \begin{pmatrix} 0 & -a_2 \\ 1 & a_1 \end{pmatrix} = E. \\ \begin{pmatrix} D_2(2) \\ D_2(1) \end{pmatrix} \otimes (x^2 \ x^3) &= \begin{pmatrix} -a_2 & a_2 a_1 \\ a_1 & (a_1^2 - a_2) \end{pmatrix} = E^2. \\ \begin{pmatrix} D_2(2) \\ D_2(1) \end{pmatrix} \otimes (x^k \ x^{k+1}) &= E_2^k. \end{aligned}$$

$n = 3$

$$\begin{aligned} \begin{pmatrix} D_3(3) \\ D_3(2) \\ D_3(1) \end{pmatrix} \otimes (x \ x^2 \ x^3) &= \begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & a_1 \end{pmatrix} = E_3 \\ \begin{pmatrix} D_3(3) \\ D_3(2) \\ D_3(1) \end{pmatrix} \otimes (x^2 \ x^3 \ x^4) &= \begin{pmatrix} 0 & a_3 & (a_1 a_3 - a_4) \\ 0 & -a_2 & (a_3 - a_1 a_2) \\ 1 & a_1 & (a_1^2 - a_2) \end{pmatrix} = E_3^2 \\ \begin{pmatrix} D_3(3) \\ D_3(2) \\ D_3(1) \end{pmatrix} \otimes (x^k \ x^{k+1} \ x^{k+2}) &= E_3^k \end{aligned}$$

Generalization to n -th order is done simply by use the induction method, this yields

$$[D_n(n), D_n(n-1), \dots, D_n(1)]^T [x^k, x^{k+1}, \dots, x^{k+n-1}] = E_n^k. \tag{55}$$

Representation of the Generalized Trigonometric Functions Via DD-Operators

Consider n -order ordinary differential equation with characteristic polynomial $P(X)$ defined as

$$P\left(\frac{d}{d\phi}\right)\Phi(\phi) = 0. \tag{56}$$

This equation possesses with n fundamental solutions $g_k(\phi)$, $k = 0, 1, 2, \dots, n - 1$ which let us define as the set of functions with initial data $g_0(0) = 1$, $g_k(0) = 0$, $k = 1, \dots, n - 1$.

As it has been proved in [8] this set of functions satisfies to the system of first order differential equations

$$\sum_{j=1}^n E_{kj}g_{j-1} = \frac{d}{d\phi}g_{k-1}, \tag{57}$$

where E is the companion matrix of the polynomial $P(X)$. The set of functions $g_k(\phi)$, $k = 0, 1, 2, \dots, n - 1$ constitutes the basis of the generalized trigonometry investigated in various works (see, for example, [13] and references therein.)

Since $P(E) = 0$, one may define the matrix functions of the exponentials

$$\exp(E\phi) = \sum_{k=0}^{n-1} E^k g_k, \quad E^0 = I. \tag{58}$$

By inverting this matrix equation we get explicit formulae for the g -functions. For instance,

$$\begin{aligned} g_1(\phi) &= \frac{\exp(x_1\phi) - \exp(x_2\phi)}{x_1 - x_2}, \\ g_0(\phi) &= \frac{x_2 \exp(x_1\phi) - x_1 \exp(x_2\phi)}{x_1 - x_2}, \end{aligned} \tag{59}$$

where x_1, x_2 are roots of the second degree polynomial

$$P(X) = X^2 - a_1X + a_2.$$

It is seen, on making use of the complete set of DD-operators these formulas can be written in a more compact form

$$D_2(1) \exp(x\phi) = g_1(\phi), \quad D_2(2) \exp(x\phi) = g_0(\phi).$$

In general, by inverting the system of equations (58) and by taking into account the definitions of the complete set of DD-operators, we get

$$D_2(k) \exp(x\phi) = g_{n-k}(\phi), \quad k = 1, 2, 3, \dots, n. \tag{60}$$

Degenerated Divided Difference Operators and Their Invariant Functions

At the limit, when distances between nodes are tending to zero, the DD-operators transform to the derivatives of corresponding order [10]. It is evident that the second order DD-operator will transformed to the first order derivative. Let us denote this operator by D_{x_k} .

Table 1 Table of actions of DD-operators on power function x^k

	1	x	x ²	x ³	x ⁴	x ⁵	x ⁶	x ⁷	x ⁸
Dx_2	C_1^0	C_2^1x	$C_3^2x^2$	$C_4^3x^3$	$C_5^4x^4$	$C_6^5x^5$	$C_7^6x^6$	$C_8^7x^7$	
Dx_3	0	C_2^0	C_3^1x	$C_4^2x^2$	$C_5^3x^3$	$C_6^4x^4$	$C_7^5x^5$	$C_8^6x^6$	
Dx_4	0	0	C_3^0	C_4^1x	$C_5^2x^2$	$C_6^3x^3$	$C_7^4x^4$	$C_8^5x^5$	
Dx_5	0	0	0	C_4^0	C_5^1x	$C_6^2x^2$	$C_7^3x^3$	$C_8^4x^4$	
Dx_6	0	0	0	0	C_5^0	C_6^1x	$C_7^2x^2$	$C_8^3x^3$	

An action of second order degenerated DD-operator on the power function gives the expected result

$$D_2x^k|_{x_1=x_2=x} = kx^{k-1}. \tag{61}$$

Since the result of action on the power function of n-order DD-operator is equal to the symmetric polynomial [9], an action of the degenerate DD-operation is obtained simply by equating the roots of the characteristic polynomial to the argument of the function. For example,

$$\begin{aligned}
 Dx_3x^k &= \frac{u^k(v-w) + v^k(w-u) + w^k(u-v)}{(u-v)(v-w)(w-u)} \\
 &= \sum_{i+l+j=k-2} (v^i w^j u^l)|_{u=v=w=x} = C_k^{k-2} x^{k-2}.
 \end{aligned}
 \tag{62}$$

General formula is expressed as

$$Dx_n x^k = C_k^{k-n+1} x^{k-n+1}. \tag{63}$$

It is instructive to display this formula in the tabular form.

The upper-triangle matrix inside the Table 1 can be easily recognized as the Pascal matrix [11,12]. On making use of the exponential representation of the Pascal matrix the Table 1 can be cast in the form

$$\begin{pmatrix} I \\ Dx_2 \\ Dx_3 \\ Dx_4 \\ Dx_5 \\ Dx_6 \end{pmatrix} \otimes (1 \ x \ x^2 \ x^3 \ x^5 \ x^6) = MF = \exp \left(x \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \right). \tag{64}$$

The exponential function $f(x) = \exp(x)$ is an invariant with respect to operation of differentiation, e.g.,

$$Dx_2 \exp(x) = \exp(x). \tag{65}$$

In a similar way, it can be found functions invariant under action of degenerated DD.

Example 9 For third order we get pair of exponential-like functions invariant under third order degenerated DD operation. That are

$$\begin{aligned}
 e_1^{(3)}(x) &= 1 + \frac{x^2}{C_2^0} + \frac{x^4}{C_2^0 C_4^2} + \frac{x^6}{C_2^0 C_4^2 C_6^4} + \dots, \\
 e_2^{(3)}(x) &= x + \frac{x^3}{C_3^1} + \frac{x^5}{C_3^1 C_5^3} + \frac{x^7}{C_3^1 C_5^3 C_7^5} + \dots.
 \end{aligned}
 \tag{66}$$

Correspondingly, for the four order operators there exist three invariant functions

$$\begin{aligned}
 e_1^{(4)}(x) &= 1 + \frac{x^3}{C_3^0} + \frac{x^6}{C_3^0 C_6^3} + \frac{x^9}{C_3^0 C_6^3 C_9^6} + \dots, \\
 e_2^{(4)}(x) &= x + \frac{x^4}{C_4^1} + \frac{x^7}{C_4^1 C_7^4} + \frac{x^{10}}{C_4^1 C_7^4 C_{10}^7} + \dots, \\
 e_3^{(4)}(x) &= x^2 + \frac{x^5}{C_5^2} + \frac{x^8}{C_5^2 C_8^5} + \frac{x^{11}}{C_5^2 C_8^5 C_{11}^8} + \dots.
 \end{aligned}
 \tag{67}$$

For n -order operators can be constructed $(n - 1)$ this kind of invariant functions. The general formula of invariant functions is

$$e_m^{(n)}(x) = x^{m-1} + \sum_{k=1}^{\infty} x^{m-1+k(n-1)} \prod_{r=1}^{r=k} \{C_{m-1+rn}^{m-1+(r-1)(n-1)}\}^{-1}, \quad m = 1, 2, \dots, n - 1.
 \tag{68}$$

These functions are closely related with n -order hyperbolic functions. By higher order hyperbolic functions we understand a fundamental set of solutions of the differential equation

$$\frac{d^n}{dx^n} e_k^{(n)} = n! e_k^{(n)}, \quad k = 1, \dots, n - 1.$$

Let us start with the case $n = 3$. Under action of the differentiation the functions e_k recursively are transformed one into the other

$$\frac{d}{dx} e_2^{(3)} = e_1^{(3)}, \quad \frac{d}{dx} e_1^{(3)} = e_2^{(3)}.$$

Thus, if the first of these functions is defined, then the others can be obtained by successive differentiation of the first one. Another example supports this important rule. Let $n = 5$, then

$$\frac{d}{dx} e_2^{(4)} = e_1^{(4)}.
 \tag{69}$$

These relations serve as a clue to describe all possible properties of the set of invariant functions. In fact, first of all we establish the following formulae of differentiation

$$\frac{d}{dx} \begin{pmatrix} e_1 \\ e_4 \\ e_3 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_4 \\ e_3 \\ e_2 \end{pmatrix}.
 \tag{70}$$

This system of differential equations totally help us to present the other properties of the invariant functions. From this seminal equation it follows general solution which clarifies

the nature of the invariant functions. An operator form of the general solution is given by the exponential matrix of the form [13]

$$\exp(Ex) = Ie_1 + Ee_1 + \frac{1}{2!}E^2e_3 + \frac{1}{3!}E^3e_4. \tag{71}$$

where E is the companion matrix of the polynomial

$$X^4 = 4!.$$

In this polynomial equation the first coefficient is trivial, that means

$$\text{Det}(\exp(Ex)) = 1, \tag{72}$$

e.g.,

$$\text{Det} \left(Ie_1 + Ee_1 + \frac{1}{2!}E^2e_3 + \frac{1}{3!}E^3e_4 \right) = 1. \tag{73}$$

This identity is the analogy of trigonometric identity for cosine-sine functions. For third order invariant functions we get

$$(e_1^{(2)})^2 - 2(e_2^{(2)})^2 = 1. \tag{74}$$

Summation formula for the invariant function are obtained from the identity

$$\exp(Ex) \exp(Ey) = \exp(E(x + y)). \tag{75}$$

From formulae of differentiation it is seen, that the operator $\frac{d}{dx}$ is worked out a cyclic permutation.

Thus, in general the invariant functions form a fundamental set of solutions of the differential equation

$$\frac{1}{k!} \frac{d^k}{dx^k} e_m^{(k+1)} = e_m^{(k+1)}. \tag{76}$$

These functions belong to the class of higher order hyperbolic functions. There exist an extensive literature on the subject, and bibliographies have been given by H.Kaufman [14]. the most recent investigations on the subject one may meet in [8].

Conclusion

The points we have touched in this paper show that the method of expression of the results of actions of divided differences operators via coefficients of the polynomials allow to elaborate quite general formulism appropriate for all orders of the calculus. According to this formalism any order of the differential calculus can be considered as a ground level, including, also, the calculus of the definite integrals. In addition, we have shown that the definitions of the divided differences operators intimately are related with the concept of the *difference between $n \geq 2$ variables* [15]. Indeed, the statement that the generalized difference between n power functions defined at the points $x_k, k = 1, \dots, n$ is proportional to the generalized difference between the very n variables. The factor of proportionality is a multi-dimensional polynomial of the coefficients of the pivoting polynomial what provides regularity of the divided differences operators under the process of cumulating the roots of the pivoting polynomial near the one of them.

References

1. Dikoussar, N.D.: Method of basic elements. *Math. Models Comput. Simulations* **3**(4), 492–507 (2011)
2. Carl, De Boor: Divided Differences. *Surv. Approx. Theory* **1**, 46–69 (2005)
3. Dattoli, G., Licciardi, S., Sabia, E.: Generalized trigonometric functions and matrix parameterization. *Int. J. Appl. Comput. Math.* **3**(Suppl 1), 115–128 (2017)
4. Yamaleev, R.: Complex algebras on N-order polynomials and generalizations of trigonometry, oscillator model and Hamilton dynamics. *Adv. Appl. Cliff. Alg.* **15**(1), 123 (2005)
5. Klinger, A.: The Vandermonde matrix. *Amer. Math. Monthly* **74**, 571–574 (1967)
6. Aceto, L., Trigiante, D.: The matrices of Pascal and other greats. *Amer. Math. Monthly* **108**, 232–245 (2001)
7. Vein, R., Dale, P.: Determinants and their applications in mathematical physics. Springer-Verlag, New York, Inc. (1999). ISBN 0-387-98558-1
8. Yamaleev, R.M.: Geometrical and physical interpretation of evolution governed by general complex algebra. *J. Math. Anal. Appl.* **340**, 1046–1057 (2008)
9. Cornelius Jr., E.F.: Identities for complete homogeneous symmetric polynomials. *J. Algebr. Numb. Theor. Appl.* **21**(1), 109–116 (2011)
10. Blanchard, P., Devaney, R.L., Hall, G.R.: *Differential Equations*. Thompson. Coddington, E.A., Levinson, N. (1955). *Theory of Ordinary Differential Equations*. McGraw-Hill (2006)
11. Aceto, L., Trigiante, D.: The matrices of Pascal and classical polynomials. *Rendicoti del Circol Matematico in Palermo. Serie II, Suppl.* **68**, 219–228 (2002)
12. Yamaleev, R.M.: Pascal matrix representation of evolution of polynomials. *Int. J. Appl. Comput. Math.* **1**(4), 513–525 (2015)
13. Babusci, D., Dattoli, G., Di Palma, E., Sabia, E.: *Adv. Appl. Cliff. Alg.* **22**(2), 271 (2012)
14. Kaufman, H.: A bibliographical note on the higher order sine functions. *Scripta Math.* **28**, 28–36 (1967)
15. Yamaleev, R.M.: Difference between three quantities (2012). [arXiv:1209.5012](https://arxiv.org/abs/1209.5012) [math.HO]

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.