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A Two-Level Trade-Credit Approach to an Integrated Price-Sensitive Inventory Model with Shortages

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Abstract

This paper deals with an economic order quantity model with deterioration for two different demand functions under two-level of trade-credit policy. The demand functions of the proposed model are two types: (i) exponential function of the price (ii) price with the negative power of constant. Shortages are allowed and fully backlogged as deterioration arises. A nonlinear constraint optimization problem is then formulated considering cost and profit parameters. The main objective of this paper is to find out the optimal selling price, optimal deteriorating length and optimal cycle length for the optimal total profit of the chain. Some theoretical as well as numerical outcomes are studied to show the validity of the proposed model. A sensitivity analysis is carried out to study the effect of changes of key parameters of the inventory system.

Keywords Deterioration · Inventory model · Price-dependent demand · Shortages · Trade-credit

Introduction

The trade-credit policy plays a vital role in supply chain management. The existence of trade-credit financing in inventory control problems should not be overlooked in economic order quantity models. In practice, deterioration is also an important factor for cost/profit analysis of the inventory analysis. Ghare and Schrader [6], was the first who introduced a constant deterioration rate on inventory control problem. This model was extended by Covert and Philip [4] considering two-parameter Weibull distribution deterioration rate. In this continuation, we may refer some related articles on deterioration done by Chang and Dye [2], Min et al. [15], Wang and Jiang [34] and Tiwari et al. [33], among others.

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Thereafter, Dave and Patel [5] introduced an inventory model with time dependent demand, deterioration and shortages. After a few years' later, Sachan [18] extended the model of Dave and Patel [5] incorporating shortage. Giri et al. [7] discussed an inventory model for ramp-type demand rate including time-dependent deterioration rate. Skouri et al. [26] improved this model by considering general ramp type demand rate and Weibull distributed deterioration together with partial backlogging. An EOQ inventory model having time-varying deterioration and partial backlogging was developed by Sana [20]. Taleizadeh [27] developed an EOQ model for evaporating items with partial back ordering and advance payments scheme. In this line of research works, an inventory model with time dependent demand and deterioration rate through finite replenishment rate was discussed by Sarkar [21]. Moreover, Sarkar [22] discussed a production-inventory model with three types of continuously disseminated deterioration functions. In this continuation, few related research works done by by Taleizadeh and Pentico [28], Wee [35], Hariga [9], Wee and Law [36], Moon et al. [16], Chung and Wee [3], Sana [19], Shaikh et al. [25], Mashud et al. [13] and Tiwari et al. [32] are referred to the readers.

In trade-credit financing, the pioneering work in inventory literature was done by Goyal [8]. Goyal [8] assumed that the supplier only could offer the trade-credit period but the retailer did not has opportunity to offer any credit-period to his/her customer. This circumstance is identified as single-level trade-credit or one-level-trade-credit policy where each stage contains a member and shares only two stages of the chain. Perceptibly, it would be more interesting to study if the retailer had the power to offer the trade-credit to his/her customer. This singularity is known as two-levels of trade-credit policy where the chain integrates with three stages and one member belongs to each stage. [10] explored the EOQ inventory model under two-levels of trade-credit policy in this extension when the supplier provided to the retailer a permissible delay and then consequently the retailer also offered to its customer a permissible delay. Later, a new factor deterioration is introduced and extended Goyal's [8] model by Aggarwal and Jaggi [1]. Thereafter, Teng and Goyal [30] proposed an optimal ordering policies for a retailer in a supply chain with consideration of upstream and downstream trade-credits. Considering deterioration as an important factor, Liao [12] presented an inventory model with two-level trade-credit for non-instantaneous deteriorating items. Teng and Chang [29] investigated two-level trade credit facilities in production inventory model. Min et al. [14] proposed an inventory model for deteriorating items with stock-dependent demand rate considering two-levels of trade-credit policy. Wu et al. [37] added deterioration with expiration dates and developed an inventory model under two-level of trade-credit policy. Mukherjee and Mahata [17] also developed an inventory model considering trade-credit policy and deterioration. Some supplementary related researches on trade-credit policy are Sarkar et al. [23], Sarkar et al. [24], Ting [31] etc. A comparative study is given in Table 1.

This paper investigates an inventory model for a deteriorating item considering two-level of trade-credit policy, different price-dependent demand and fully backlogged shortage. Here, the demand functions of the customers are (i) exponential function of selling price and (ii) negative power of selling price. The deterioration is considered on-hand inventory of the product in the retailer's warehouse. The shortages are permitted and are fully backlogged. The average profit function is then formulated and maximized analytically as well as numerically. We consider two numerical examples and prove the concavity of each objective function graphically to describe and validate the inventory model. Sensitivity analysis is carried out to study the effect of key parameters on optimal solution of the objective function which are realistic in practice.

Table 1 Related research works with price-dependent demand, shortage and trade-credit

Authors	Demand type	Shortages	Deterioration	Trade-credit type
Goyal [8]	Constant	No shortage	No deterioration	Single-level
Min et al. [14]	Stock dependent	No shortage	Constant	Two-level
Sana [20]	Price-dependent	Partial backorder	Time varying	No
Wu et al. [37]	Trade-credit dependent	No shortage	Time varying	Two-level
Tiwari et al. [32]	Price-dependent	Partial backorder	Time varying	Two-level partial trade-credit
Mukherjee and Mahata [17]	Time and credit dependent	No shortage	No deterioration	Two-level
Mashud et al. [13]	Constant	Partial backorder	Constant	No
This paper	Price-dependent	Partial backorder	Constant	Two-level

Assumption and Notations

The proposed model in this paper is based on the subsequent assumptions.

1. The rate of replenishment is instantaneous and the lead-time is negligible.
2. In this paper, two different price-sensitive demand is considered as follows

   (i) $D(p) = ae^{\left(\frac{-p}{\alpha}\right)}$, $a > 0, \alpha > 1$
   (ii) $D(p) = ap^{-\alpha}$, $a > 0, \alpha > 1$

3. The infinite planning horizon is considered.
4. The shortage is permitted and fully backlogged.

Notations

The following notations are used throughout the paper.

Notations	Description
c_o	Replenishment cost per order
c_p	Purchasing cost per unit
c_h	Holding cost per unit per unit time
c_b	Shortage cost per unit per unit time
θ	Deterioration rate
S	Maximum stock per cycle
R	Maximum shortage level
I_e	Interest which can be earned per \$ per year by retailer
I_p	Interest charges per \$ in stocks per year by the supplier
M	The retailer's trade-credit period offered by supplier in years

Notations	Description
N	The supplier's trade-credit period offered by retailer in years
$I(t)$	Inventory level at any time t where $0 \le t \le T$
X_1	Total inventory cost for the first case
X_2	Total inventory cost for the second case
$TP_i\ (p, t_1, T)$	The total profit per unit time for i = 1,2
*	Optimal values of the parameters
Decision variables	
t_1	Time at which the stock is zero
p	Selling price
T	Total cycle length

The Model

The proposed inventory model is developed on the basis of above-mentioned assumptions. Initially, an enterprise purchases goods (S + R) units. This stock is depleted owing to meet up the customers' demands with products' deterioration. At time $t = t_1$, stock reaches at zero. Then, shortage appears which is fully backlogged. Therefore, the inventory systems are defined by the following differential equations (Fig. 1).

Inventory Model with Respect to the Demand Function $D = ae^{-\left(P/\alpha\right)}$

$$\frac{dI_1(t)}{dt} + \theta I(t) = -ae^{-\left(P/\alpha\right)} \quad 0 < t \le t_1 \tag{1}$$

$$\frac{dI_2(t)}{dt} = -ae^{-\left(P/\alpha\right)} \quad t_1 < t \le T \tag{2}$$

Solving the above (1) and (2) differential equations with the boundary conditions $I_1(t) = S\ at\ t = 0\ and\ I_2(t) = -R\ at\ t = T\ and\ I(t)\ is\ continuous\ at\ t = t_1$, we have

$$I_1(t) = \frac{a}{\theta}e^{-\left(P/\alpha\right)}\left[e^{\theta(t_1-t)} - 1\right] \quad t_1 < t \le T \tag{3}$$
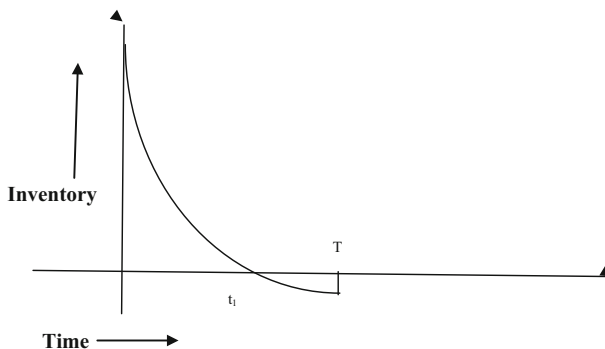

Fig. 1 Inventory versus time

Using $I_1(t) = S$ at $t = 0$ in the above, we have

$$S = \frac{a}{\theta} e^{-(P/\alpha)} \big[e^{\theta t_1} - 1 \big] \tag{4}$$

From Eq. (2) we have

$$I_2(t) = a e^{-(P/\alpha)} [t_1 - t] \tag{5}$$

Using $I_2(t) = -R$ at $t = T$, we have

$$R = a e^{-(P/\alpha)} [T - t_1] \tag{6}$$

Now, $t_1 = T - \frac{R}{a} e^{(P/\alpha)}$
Total sales revenue is

$$SR = p \left[a e^{-(P/\alpha)} t_1 + a e^{-(P/\alpha)} [T - t_1] \right]$$

$$= p a e^{-(P/\alpha)} T \tag{7}$$

The cost components per cycle are as follows:

(a) 　　　　　　　　　　Ordering cost $= C$

(b)
$$\text{Holding cost} = C_h \left[\int_0^{t_1} I_1(t) dt \right]$$

$$i.e., = \frac{C_h a}{\theta^2} e^{-(p/\alpha)} \big[e^{\theta t_1} - \theta t_1 - 1 \big] \tag{8}$$

(c) 　　　　　　The purchase cost $= c_p(S + R)$

(d)
$$\text{The shortage cost} = C_b \left[\int_{t_1}^T -I_2(t) dt \right]$$

$$i.e., = \frac{1}{2} C_b a e^{\left(\frac{-p}{\alpha} \right)} (T - t_1)^2 \tag{9}$$

Annual Interest Payable to Supplier

Here, there are three cases may arise where the retailer charges interest for keeping the items in stock per year. These three cases are studied separately as follows

Case I $T \geq M$
In this case, two sub-cases may arise, **(I-1).** $M \leq t_1 \leq T$ and **(I-2).** $t_1 \leq M \leq T$. These are discussed separately below.

Subcase (I-1) According to Fig. 2, if the trade-credit period M is at the left of t1, then Eq. (6) implies that $T \geq t_1 = T - \frac{R}{a} e^{(P/\alpha)} \geq M$.
Thus, the payable annual interest is

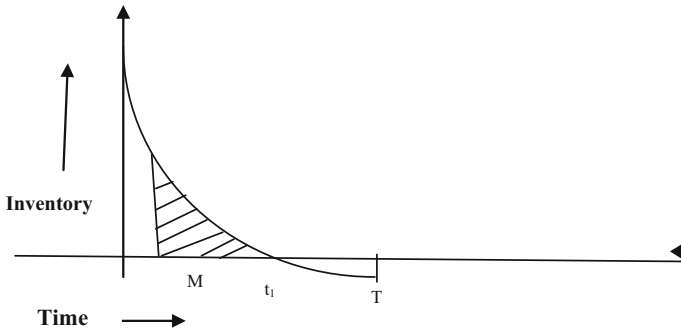$$IC_{11} = \frac{c_p I_p}{T} \left[\int_M^{t_1} I_1(t) dt \right]$$
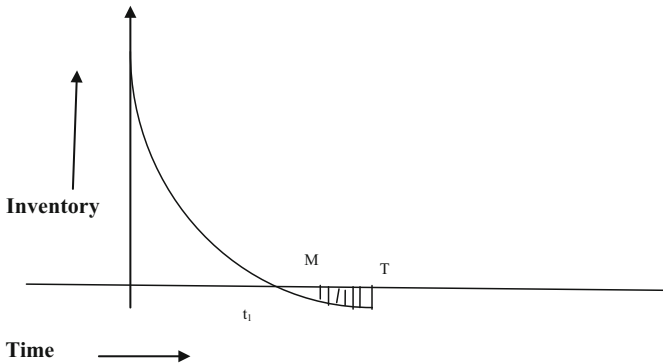
Fig. 2 Inventory versus time

Fig. 3 Inventory veruss time

$$= \frac{c_p I_p}{T} \left[\frac{D}{\theta} \int_M^{t_1} \left[e^{\theta(t_1 - t)} - 1 \right] dt \right]$$

$$= \frac{c_p I_p D}{T \theta^2} \left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1 \right]$$

(10)

Subcase (I-2): According to Fig. 3, if trade-credit period M is at the right of t1 then Eq. (6) implies that $T \geq M \geq t_1 = T - \frac{R}{a} e^{\left(P/\alpha \right)}$

Thus, the annual payable interest is zero

Case II $N \leq T \leq M$

The payable annual interest for this case is zero.

Case III $0 \leq T \leq N$

The payable annual interest is zero also in this case.

Interest Earn by Retailer Annually

Here, there are three cases arise where supplier charges interest per year. These three cases are as follows:

Case I: $T \geq M$

When $T \geq M$ occurs then there are two sub-cases which are sub-case (I-1) $M \leq t_1 \leq T$ and

Subcase (I-1): According to the Fig. 2, if M lies at the left of t1 then Eq. (6) indicates that $M \leq t_1 = T - \frac{R}{a}e^{\left(P/\alpha\right)} \leq T$.

Thus, the annual interest earned is

$$IE_{11} = \frac{pI_e}{T} \int_N^{t_1} Dt \, dt$$

$$= \frac{pI_eD}{T} \int_N^{t_1} t \, dt$$

$$= \frac{pI_eD}{2T} \left[t_1^2 - N^2\right] \tag{11}$$

Subcase (I-2): Accordingly to Fig. 3, if the trade-credit period M lies at the right side of t1 then Eq. (6) implies that $t_1 = T - \frac{R}{a}e^{\left(P/\alpha\right)} \leq M \leq T$ and we have

$$IE_{12} = \frac{pI_e}{T} \left[\int_N^{t_1} Dt \, dt + \int_N^{t_1} Dt_1 \, dt \right]$$

$$= \frac{pI_eD}{T} \left[\frac{\left(t_1^2 - N^2\right)}{2} + t_1(t_1 - N) \right] \tag{12}$$

Case II: $N \leq T \leq M$

The annual interest earn for this case is

$$IE_{13} = \frac{pI_e}{T} \left[\int_N^T DT \, dt + \int_T^M DT \, dt \right]$$

$$= \frac{pI_eD}{2T} [T(T - N) + T(M - T)]$$

$$= \frac{pI_eD}{2} [M - N] \tag{13}$$

Case III $0 < T \leq N$

The annual interest earn for this case is

$$IE_{14} = \frac{pI_e}{T} \int_N^M DT \, dt$$

$$= \frac{pI_e}{T} \int_N^M ae^{-P/\alpha} T \, dt$$

$$= pI_e ae^{-P/\alpha} [M - N] \tag{14}$$

Therefore, *Total inventory cost*, $X_1 = $ <ordering cost>+<purchase cost>+<holding cost>+<shortage cost>+<interest payable per cycle>-<interest earn per cycle>

Total profit $= Total \ sales \ revenue, \ SR - Total \ inventory \ cost, \ X_1$

Therefore, the constrained optimization problem of this proposed model is as follows:

Problem 1 Maximize

$$TP_1(p, t_1, T) = \frac{1}{T}\begin{cases} TP_{11}(p, t_1, T), & when\ T \geq t_1 \\ TP_{12}(p, t_1, T) & when\ t_1 \leq M \leq T \\ TP_{13}(p, t_1, T), & when\ N \leq T \leq M \\ TP_{14}(p, t_1, T), & when\ 0 < T \leq N \end{cases} \tag{15}$$

Subject to $o \leq t_1 \leq T$
Where,

$$TP_{11}(p, t_1, T) = \frac{1}{T}[SR - X_1] - [IC_{11} - IE_{11}]$$

$$TP_{12}(p, t_1, T) = \frac{1}{T}[SR - X_1] + [IE_{12}]$$

$$TP_{13}(p, t_1, T) = \frac{1}{T}[SR - X_1] + [IE_{13}]$$

$$TP_{14}(p, t_1, T) = \frac{1}{T}[SR - X_1] + [IE_{14}]$$

Inventory Model with Respect to the Demand Function $D = ap^{-\alpha}$

The governing differential equation are as follows:

$$\frac{dI_1(t)}{dt} + \theta I(t) = -ap^{-\alpha} \quad 0 < t \leq t_1 \tag{16}$$

$$\frac{dI_2(t)}{dt} = -ap^{-\alpha} \quad t_1 < t \leq T \tag{17}$$

From Eq. (16), we have

$$I_1(t) = \frac{a}{\theta}p^{-\alpha}\left[e^{\theta(t_1-t)} - 1\right] \quad t_1 < t \leq T \tag{18}$$

Using $I_1(t) = 0\ at\ t = t_1\ and I_1(t) = S$ at $t = 0$ conditions, one can obtain

$$S = \frac{a}{\theta}p^{-\alpha}(e^{\theta t_1} - 1) \tag{19}$$

From Eq. (17), we have

$$I_2(t) = ap^{-\alpha}(t_1 - t) \quad t_1 < t \leq T \tag{20}$$

Using $I_2(t) = 0\ at\ t = t_1$ and $I_2(t) = -R$ at $t = T$, we have

$$R = -ap^{-\alpha}(t_1 - T) \tag{21}$$

This implies

$$t_1 = T - \frac{R}{a}p^{\alpha} \tag{22}$$

Total sales revenue is

$$SR = p\big[ap^{-\alpha}t_1 + ap^{-\alpha}[T - t_1]\big]$$
$$= ap^{-\alpha+1}T \tag{23}$$

The cost factors per cycle are as follows:

(a)   Ordering cost = C

(b)

$$\text{Holding cost} = C_h \left[\int_0^{t_1} I_1(t)\,dt \right]$$

$$\text{i.e.,} = \frac{C_h a}{\theta^2} p^{-\alpha}\big[e^{\theta t_1} - \theta t_1 - 1\big] \tag{24}$$

The purchase cost $= c_p(S + R)$

(c)

(d)

$$\text{The shortage cost} = C_b \left[\int_{t_1}^{T} -I_2(t)\,dt \right]$$

$$\text{i.e.} = \frac{1}{2} C_b a p^{-\alpha}(T - t_1)^2 \tag{25}$$

Annual Interest Payable to Supplier

Here, there are three cases to occur in interest charged for the items kept in stock per year. These three cases are examined separately as follows

Case I $T \geq M$
There are two sub-cases such as (I-1) $M \leq t_1 \leq T$ and (I-2) $t_1 \leq M \leq T$. We discuss them accordingly as below.

Subcase (I-1) As per Fig. 2, if M is situated at the left side of t1 then Eq. (22) provides $T \geq t_1 = T - \frac{R}{a}p^{\alpha} \geq M$.
Therefore, the annual interest payable is

$$IC_{21} = \frac{c_p I_p}{T} \left[\int_M^{t_1} I_1(t)\,dt \right]$$

$$= \frac{c_p I_p}{T} \left[\frac{ap^{-\alpha}}{\theta} \int_M^{t_1} (e^{\theta(t_1 - t)} - 1)\,dt \right]$$

$$= \frac{c_p I_p D}{T\theta^2} \big[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1\big] \tag{26}$$

Subcase (I-2) As per Fig. 3 if M is located at the right side of t1 as shown in the figure, then Eq. (22) implies that $T \geq M \geq t_1 = T - \frac{R}{a}p^{\alpha}$
Subsequently, the annual interest payable is zero

Case II $N \leq T \leq M$
The annual interest payable is zero here.
Case III $0 < T \leq N$
The annual interest payable is zero here also.

Annual Interest Earn by the Retailer

Here, annual interest is earned by the retailer which is offered by the supplier. In this situation, there are three cases may arise for interest earned per year. These three cases are calculated independently below.

Case I $T \geq M$

When this condition holds then there are another two sub-cases which are sub-case (I-1) $M \leq t_1 \leq T$ and sub-case (I-2) $t_1 \leq M \leq T$. All are discussed separately as below

Subcase (I-1) According to the Fig. 2, if M is lies at the left side of t1 then Eq. (22) gives $T \geq t_1 = T - \frac{R}{a}p^\alpha \geq M$.

Consequently, the annual interest earn is

$$IE_{21} = \frac{pI_e}{T} \int_N^{t_1} Dt dt$$

$$= \frac{pI_e}{T} \int_N^{t_1} ap^{-\alpha} t dt$$

$$= \frac{pI_e ap^{-\alpha}}{2T}\left[t_1^2 - N^2\right] \tag{27}$$

Subcase (I-2)

Accordingly Fig. 3, if M is located at the right hand side of t1 then Eq. (22) infers that $T \geq M \geq t_1 = T - \frac{R}{a}p^\alpha$.

$$IE_{22} = \frac{pI_e}{T}\left[\int_N^{t_1} Dt dt + \int_N^{t_1} Dt_1 dt\right]$$

$$= \frac{pI_e ap^{-\alpha}}{T}\left[\frac{(t_1^2 - N^2)}{2} + t_1(t_1 - N)\right] \tag{28}$$

Case II: $N \leq T \leq M$

The annual interest earned for this case is

$$IE_{23} = \frac{pI_e}{T}\left[\int_N^T DT dt + \int_T^M DT dt\right]$$

$$= \frac{pI_e ap^{-\alpha}}{2T}[T(T - N) + T(M - T)]$$

$$= \frac{pI_e ap^{-\alpha}}{2}[M - N] \tag{29}$$

Case III $0 < T \leq N$

The annual interest earned for this case is

$$IE_{24} = \frac{pI_e}{T} \int_N^M DT\,dt$$

$$= \frac{pI_e}{T} \int_N^M ap^{-\alpha}T\,dt$$

$$= pI_eap^{-\alpha}[M - N] \tag{30}$$

Therefore, $Total\ inventory\ cost,\ X_2 = $ <ordering cost>+<purchase cost>+<holding cost>+<shortage cost>+<interest payable per cycle>-<interest earn per cycle>

Therefore, the constrained optimization problem is as follows

Problem 2 Maximize

$$TP_2(p, t_1, T) = \frac{1}{T}\begin{cases} TP_{21}(p, t_1, T), & when\ T \geq t_1 \\ TP_{22}(p, t_1, T), & when\ t_1 \leq M \leq T \\ TP_{23}(p, t_1, T), & when\ N \leq T \leq M \\ TP_{24}(p, t_1, T), & when\ 0 < T \leq N \end{cases} \tag{31}$$

Subject to $o \leq t_1 \leq T$

Where,

$$TP_{21}(p, t_1, T) = \frac{1}{T}[SR - X_2] - [IC_{21} - IE_{21}]$$

$$TP_{22}(p, t_1, T) = \frac{1}{T}[SR - X_2] + [IE_{22}]$$

$$TP_{23}(p, t_1, T) = \frac{1}{T}[SR - X_2] + [IE_{23}]$$

$$TP_{24}(p, t_1, T) = \frac{1}{T}[SR - X_2] + [IE_{24}]$$

Solution Procedure

In order to obtain the profit function, we need to calculate sales revenue, the ordering cost, the holding cost, the purchasing cost, the interest earned and the interest payable for the whole chain. The retailer's average profit per unit time can be written as

$$TP(p, t_1, T) = \frac{1}{T}\left[Sales\ revenue - \left\{\begin{array}{l} ordering\ cost + Holding\ cost + Purchase\ cost \\ + \ Shortage\ cost + interest\ payable \\ - \ interest\ earn \end{array}\right\}\right]$$

The necessary and sufficient conditions to find out the optimal values are

$$\frac{\partial TP_1(p, t_1, T)}{\partial p} = 0,\ \frac{\partial TP_2(p, t_1, T)}{\partial p} = 0,\ \frac{\partial TP_1(p, t_1, T)}{\partial t_1} = 0,\ \frac{\partial TP_2(p, t_1, T)}{\partial t_1} = 0,\ and$$

$$\frac{\partial^2 TP_1(p, t_1, T)}{\partial p^2} < 0,\ \frac{\partial^2 TP_2(p, t_1, T)}{\partial p^2} < 0,\ \frac{\partial^2 TP_1(p, t_1, T)}{\partial t_1^2} < 0,\ \frac{\partial^2 TP_2(p, t_1, T)}{\partial t_1^2} < 0$$

The hessian matrix H_i of TP_i at p, t_1, and T should have all negative eigenvalues for concavity of TP_i.

For Case 1: The average profit function $TP_1(p, t_1, T)$ of the whole system is

$$TP_1(p, t_1, T) = \frac{1}{T} \begin{cases} TP_{11}(p, t_1, T), & when\ T \geq t_1 \\ TP_{12}(p, t_1, T), & when\ t_1 \leq M \leq T \\ TP_{13}(p, t_1, T), & when\ N \leq T \leq M \\ TP_{14}(p, t_1, T), & when\ 0 < T \leq N \end{cases}$$

Total profit function $TP_1(p, t_1, T)$ has four branch functions. More accurately the maximum values of those four branch functions will be the required solution.

From overhead expression some continuity relation will be rise such as:

$$TP_{11}(p, t_1, T) = TP_{12}(p, t_1, T)\ is\ continuous\ at\ t_1\ point.$$

$$TP_{12}(p, t_1, T) = TP_{13}(p, t_1, T)\ is\ continuous\ at\ M\ point.$$

$$TP_{13}(p, t_1, T) = TP_{14}(p, t_1, T)\ is\ continuous\ at\ N\ point.$$

The extreme value of $TP_1(p, t_1, T)$ is determined by the local extreme points or the boundary points of (p, t_1, T), exactly when one need to search for (p, t_1, T) directly. One could easily checked whether (p, t_1, T) is originate within the allowed range, i.e. $T \geq t_1, t_1 \leq M \leq T, N \leq T \leq M, 0 \leq T \leq N$. One can select the optimal point (p^*, t_1^*, T^*) such that $TP_1^*(p, t_1, T) = \max\{TP_{11}^*(p, t_1, T),\ TP_{12}^*(p, t_1, T),\ TP_{13}^*(p, t_1, T),\ TP_{14}^*(p, t_1, T)\}$

To show the concavity of the profit function $TP_1(p, t_1, T)$ some theoretical results are develop as follows.

Theorem 1 The total profit function $TP_{11}(p, t_1, T)$ attains its maximum value with respect to p, when other decision variables t_1, T are fixed.

*See "*Appendix B*"*

Theorem 2 *The total profit function $TP_{11}(p, t_1, T)$ attains its maximum value with respect to t_1, when other decision variables p, T are fixed.*

*See "*Appendix B*".*

Theorem 3 *The total profit function $TP_{11}(p, t_1, T)$ attains its maximum value with respect to T, when other decision variables p, t_1 are fixed.*

*See "*Appendix B*".*

For Case 2:

$$TP_2(p, t_1, T) = \frac{1}{T} \begin{cases} TP_{21}(p, t_1, T), & when\ T \geq t_1 \\ TP_{22}(p, t_1, T), & when\ t_1 \leq M \leq T \\ TP_{23}(p, t_1, T), & when\ N \leq T \leq M \\ TP_{24}(p, t_1, T), & when\ 0 < T \leq N \end{cases}$$

Total profit function $TP_2(p, t_1, T)$ has four branch functions. More precisely the maximum values of those four branch functions will be the required solution.

$$TP_{21}(p, t_1, T) = TP_{22}(p, t_1, T)\ is\ continuous\ at\ t_1\ point.$$

$$TP_{22}(p, t_1, T) = TP_{23}(p, t_1, T)\ is\ continuous\ at\ M\ point.$$

$$TP_{23}(p, t_1, T) = TP_{24}(p, t_1, T) \text{ is continuous at } N \text{ point.}$$

The extreme value of $TP_2(p, t_1, T)$ is determined by the local extreme points or the boundary points of (p, t_1, T), exactly when one need to search for (p, t_1, T) directly. (p, t_1, T). One could easily check whether (p, t_1, T) is initiate inside the legal range, i.e. $T \geq t_1, t_1 \leq M \leq T, N \leq T \leq M, 0 \leq T \leq N$. One can select the optimal (p^*, t_1^*, T^*) points such that $TP_2^*(p, t_1, T) = \max\{TP_{21}^*(p, t_1, T), TP_{22}^*(p, t_1, T), TP_{23}^*(p, t_1, T), TP_{24}^*(p, t_1, T)\}$

To show the concavity of the profit function $TP_2(p, t_1, T)$, some theorem are develop which are as follows.

Theorem 4 *The total profit function* $TP_{21}(p, t_1, T)$ *attains its maximum value with respect to* p, *when other decision variables* t_1, T *are fixed.*
    See "Appendix C".

Theorem 5 *The total profit function* $TP_{21}(p, t_1, T)$ *attains its maximum value with respect to* t_1, *when other decision variables* p, T *are fixed.*
    See "Appendix C".

Theorem 6 *The total profit function* $TP_{21}(p, t_1, T)$ *attains its maximum value with respect to* T, *when other decision variables* p, t_1 *are fixed.*
    See "Appendix C".

Numerical Illustrations

Here, the solution of the projected inventory model is discussed. This model is more relevant for the deteriorating product like vegetables, fruits, sweets etc. During any season the demand of this item increases linearly or exponentially with the dependencies on selling price. One of the popular example of proposed model is ice-cream. This is the practical example of the anticipated model.

To demonstrate the inventory model, some numerical examples solved. The values of the input parameters for examples considers for the model are as follows:
    For Case 1:

Example 1 $(T \geq t_1)$We consider the numerical values of the parameters in appropriate units as follows: Ordering costs $C_0 = \$ 150$ per order, $a = 600$, $\alpha = 7$, holding cost rate $C_h = \$ 2$, backorder cost rate for inventory $C_b = \$ 3$, purchase cost of different item $C_p = \$ 6$, deterioration rate $\theta = 0.2$, interest charged rate $I_e = \$ 0.09$, interest earn rate $I_p = \$ 0.06$, delay period of retailer $M = 3.5$ $years$, the supplier's trade-credit period offered by retailer in years $N = 0.5$.

Example 2 $(t_1 \leq M \leq T)$We consider the numerical values of the parameters in appropriate units as follows: Ordering costs is $C_0 = \$ 150$ per order, $a = 600$, $\alpha = 7$, holding cost rate is $C_h = \$ 2$, backorder cost rate for inventory is $C_b = \$ 3$, purchase cost of different item is $C_p = \$ 6$, deterioration rate is $\theta = 0.2$, interest charged rate $I_e = \$ 0.09$, delay period of retailer is $M = 1.5$ years, the supplier's trade-credit period offered by retailer in years $N = 0.85$.

Example 3 $(N \leq T \leq M)$ We consider the numerical values of the parameters in appropriate units as follows: Ordering costs is $C_0 = \$ 150$ per order, $a = 600$, $\alpha = 7$, holding cost rate is $C_h = \$ 2$, backorder cost rate for inventory is $C_b = \$ 3$, purchase cost of different item

Table 2 Results for each inventory model

Examples	Condition	Results					
		S^*	R^*	t_1^*	P^*	T^*	$TP_{ij}^*(t_1, p, T)$
Case 1							
Example 1	$T \geq t_1$	115.564	63.999	1.4353	15.0883	2.3561	439.7191
Example 2	$t_1 \leq M \leq T$	140.403	78.732	1.5000	14.0813	2.4809	589.2986
Example 3	$N \leq T \leq M$	64.5597	68.863	0.6710	13.2881	1.4371	609.8228
Example 4	$0 < T \leq N$	65.7450	70.128	0.6594	13.0308	1.4114	**664.0494**
Case 2							
Example 1	$T \geq t_1$	1755.31	1064.3	1.1017	9.53668	1.8490	5396.707
Example 2	$t_1 \leq M \leq T$	4318.75	145.16	2.5000	9.79697	2.609	6241.845
Example 3	$N \leq T \leq M$	1715.02	2572.5	0.9431	8.98632	2.5000	6255.225
Example 4	$0 < T \leq N$	518.165	777.24	0.1976	7.53040	0.5000	**8326.688**

is $C_p = \$6$, deterioration rate is $\theta = 0.2$, interest charged rate $I_e = \$0.09$, delay period of retailer is $M = 3.5$ years, the supplier's trade-credit period offered by retailer in years $N = 0.5$.

Example 4 $(0 < T \leq N)$ Now, Ordering costs is $C_0 = \$150$ per order, a $= 600$, $\alpha = 7$, holding cost rate is $C_h = \$2$, backorder cost rate for inventory is $C_b = \$3$, purchase cost of different item is $C_p = \$6$, deterioration rate is $\theta = 0.2$, interest charged rate $I_e = \$0.09$, delay period of retailer is $M = 4.5$ years, the supplier's trade-credit period offered by retailer in years $N = 2.5$.

For Case 2:

Example 5 $(T \geq t_1)$ Here, Ordering costs is $C_0 = \$250$ per order, a $= 400,000$, $\alpha = 2.5$, holding cost rate is $C_h = \$0.5$, backorder cost rate for inventory is $C_b = \$1$, purchase cost of different item is $C_p = \$5$, deterioration rate is $\theta = 0.2$, interest charged rate $I_e = \$0.07$, interest earn rate $I_p = \$0.06$, delay period of retailer is $M = 2.5$ years, the supplier's trade-credit period offered by retailer in years $N = 0.5$.

Example 6 $(t_1 \leq M \leq T)$ In this cases, Ordering costs is $C_0 = \$250$ per order, a $= 400,000$, $\alpha = 2.5$, holding cost rate is $C_h = \$0.5$, backorder cost rate for inventory is $C_b = \$1$, purchase cost of different item is $C_p = \$5$, deterioration rate is $\theta = 0.2$, interest charged rate $I_e = \$0.07$, delay period of retailer is $M = 2.5$ years, retailer offered delay period to suppliers is in years $N = 0.5$ (Table 2).

Example 7 $(N \leq T \leq M)$ In this case, Ordering costs is $C_0 = \$250$ per order, a $= 400,000$, $\alpha = 2.5$, holding cost rate is $C_h = \$0.5$, backorder cost rate for inventory is $C_b = \$1$, purchase cost of different item is $C_p = \$5$, deterioration rate is $\theta = 0.2$, interest charged rate $I_e = \$0.07$, delay period of retailer is $M = 4.5$ years, retailer offered delay period to suppliers is in years $N = 2.5$.

Example 8 $(0 < T \leq N)$ Here, Ordering costs is $C_0 = \$250$ per order, a $= 400,000$, $\alpha = 2.5$, holding cost rate is $C_h = \$0.5$, backorder cost rate for inventory is $C_b = \$1$, purchase cost of different item is $C_p = \$5$, deterioration rate is $\theta = 0.2$, interest charged rate is $I_e = \$0.07$, delay period of retailer is $M = 2.5$ years, retailer offered delay period to suppliers is in years $N = 0.5$.

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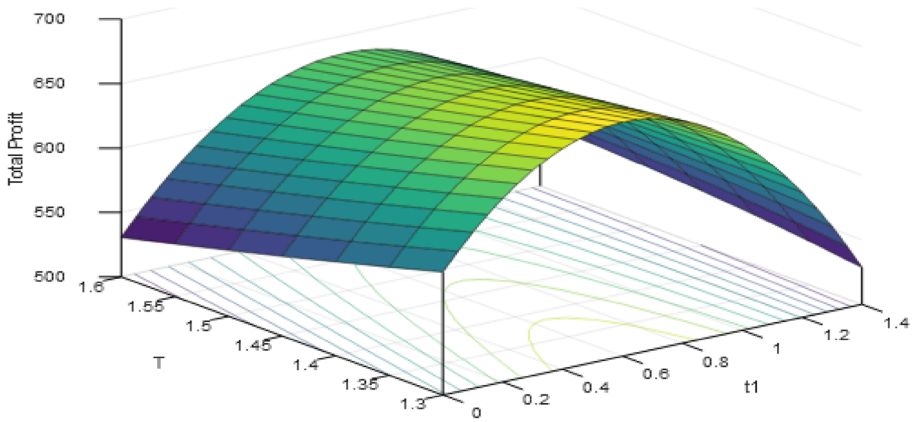## The concavity of profit function TP$_1$

Figure 1
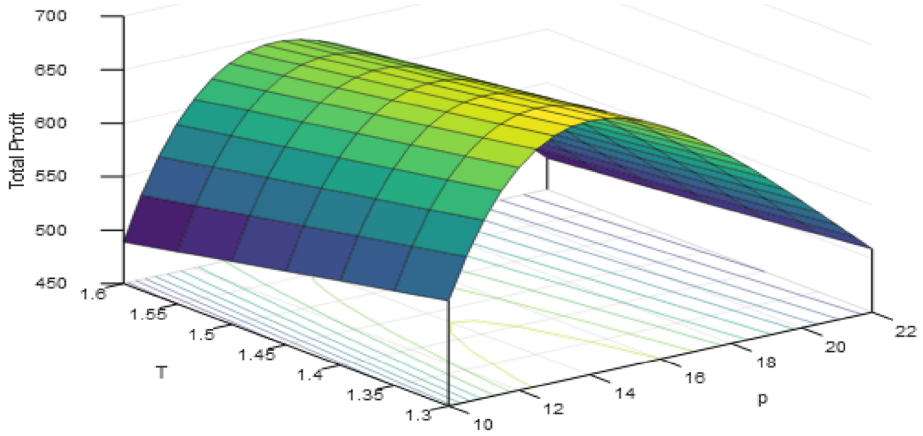
Figure 2

Figure 3
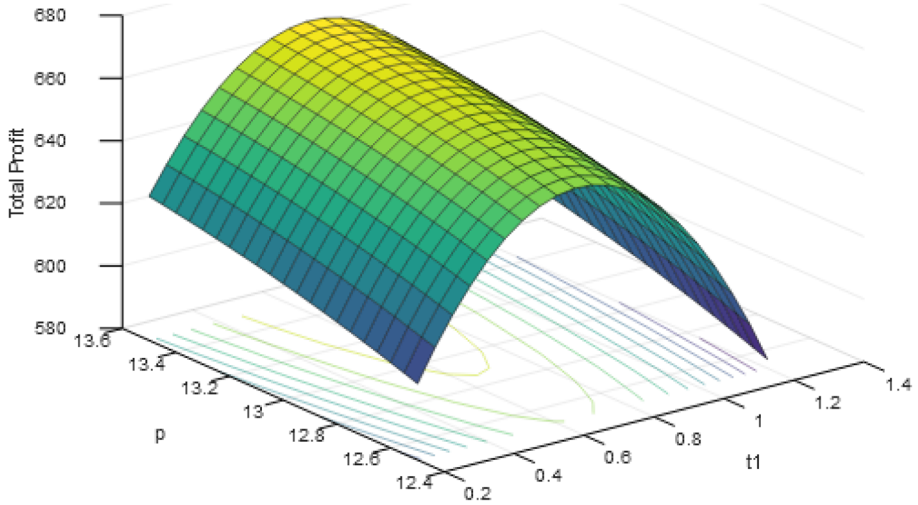
The concavity of profit function TP$_2$

Figure 4
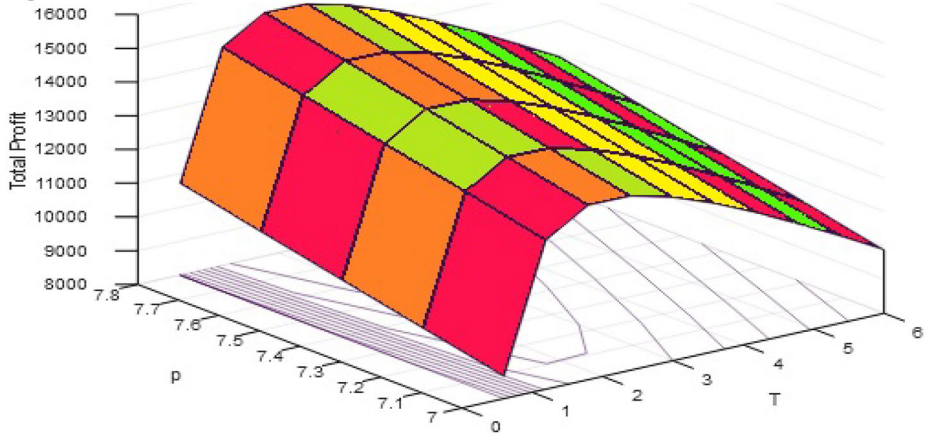
Figure 5
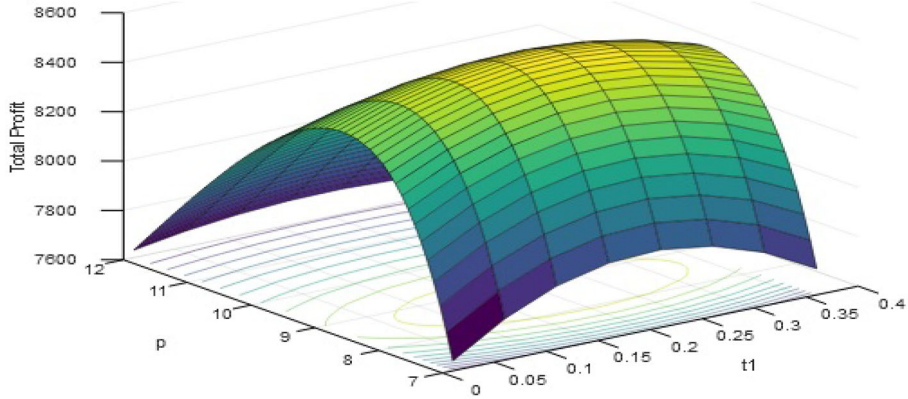
Figure 6
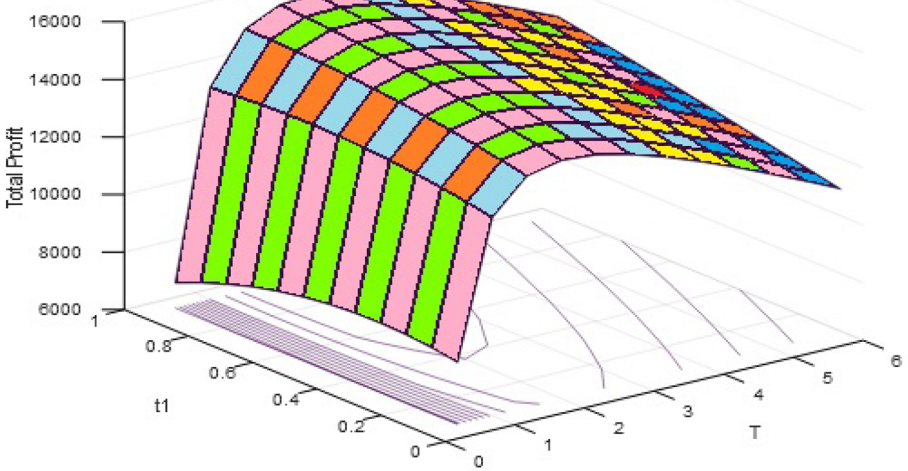
Example 9 (For different types of Demand function)

Sub Case I: Considering another price-dependent demand $D = a - bp$.
Taking all the values same as of Example 4 except the value $b = 15$.
The optimal solution is:

$$T^* = 0.8560930, \ p^* = 22.82692, \ t_1^* = 0.4055169, \quad TP^*(T^*, p^*, t_1^*) = 5044.774.$$

Similarly we can prove it for Example 8.

Sub Case II: Considering selling price and the advertisement reliant demand $D = A^\gamma (a - bp)$.
Taking all the values same as of Example 4 except the value $A_c = 2; \gamma = 0.4;$.
The optimal solution is:

$$T^* = 0.7456812, \ p^* = 22.78981, \ t_1^* = 0.3541833, \quad TP^*(T^*, p^*, t_1^*) = 6716.459.$$

Similarly we can prove it for Example 8.

Example 10 For different types of deterioration.

Sub Case I: The Weibull distribution deterioration rate is $\theta(t) = jk(t - \gamma)^{k-1}$. Here in this paper $\gamma = t_1 = 0 \,(deterioration\ free\ time)$. If $j = 0.4; k = 2; \gamma = 0$. The optimal solution is:

$$T^* = 1.100044, \ p^* = 13.10023, \ t_1^* = 0.2920156, \ TP^*(T^*, p^*, t_1^*) = 626.3627.$$

Similarly we can prove it for Example 8.

Sub Case II: Considering variable deterioration rate $\theta(t) = 0.2 + 0.1t$ and taking all the values similar as stated in Example 4, the optimal solution is

$$T^* = 1.347440, \ p^* = 13.03511, \ t_1^* = 0.5754929, \ TP^*(T^*, p^*, t_1^*) = 658.5282.$$

Similarly, we can prove it for Example 8.

Example 11 For different purposes.

 (i) When $t_1 = T$ (i.e. no shortages) then the projected model (both Case-1 and Case-2) will converted an inventory model with no shortages.
 (ii) If $M = 0 \ and \ N = 0$ then this model reduces to traditional instantaneous inventory model (both Case-1 and Case-2) with fully backlogged shortages.

Sensitivity Analysis

The above numerical example is used to study the effect of inventory system parameters on the optimal values of the initial stock level, maximum shortage level, cycle length, selling price and the optimum total profit of the system. The percentage changes in the above stated optimal values are taken as processes of sensitivity. By changing the parameters (increasing and decreasing) from -20 to $+20\%$ the analysis is carried out. The outcomes are found by changing one parameter at a time and keeping the other parameters at their original values. The outcomes of these investigation are given in Tables 3 and 4.

Table 3, gives following observations.

- The total profit is extremely delicate with respect to the purchase cost c_p, a, trade-credit period M, α and is ascetically delicate with respect to all other parameters. When the purchase cost decrease then total profit *(TP)* and maximum shortage *(R)* increases, together with selling price is also increase which is analogous to Tiwari et al. [32].
- The utmost on-hand stock-level S is exceedingly subtle with respect to the purchase cost c_p, a, α, C_0 and is temperately sensitive with respect to all other parameters.
- The peak shortage level R is vastly complex with respect to the shortage cost c_b, a, α, C_0 and is discreetly delicate with respect to all other parameters. When the value of holding cost increases then the amount of shortages increases which is analogous to Li et al. [11].
- The selling price is extremely delicate with respect to the purchase cost $c_p\ and\ \alpha$. It is abstemiously subtle with respect to other parameters. The control of c_p on p^*, AP^* is quite spontaneous and actually quite weighty. When the purchase cost c_p is increased then it leads to a significant increase in optimal selling price p^* and therefore an important decrease in demand which is parallel to Li et al. [11].
- When the holding cost is increased then total profit of the chain is decreased and also observed that with the increase in deterioration rate lessen the profit which is analogous to Tiwari et al. [33].

Table 3 Sensitivity analysis for the demand function $D(p) = ae^{\left(\frac{-p}{\alpha}\right)}$ for $0 < T \leq N$

Parameter	% Changes of parameters	% Changes in TP^*	% Changes				
			S^*	R^*	p^*	t_1^*	T^*
c_o	− 20	3.388	− 9.98	− 9.98	− 0.82	− 10.7	− 11.04
	− 10	1.64	− 4.83	− 4.83	− 0.4	− 5.2	− 5.38
	10	− 1.56	4.57	4.57	0.38	4.96	5.15
	20	− 3.047	8.9	8.9	0.75	9.71	− 11.04
a	− 20	− 23.01	− 11.21	− 11.21	0.93	12.01	12.49
	− 10	− 11.56	− 5.45	− 5.45	0.42	5.5	5.71
	10	11.64	5.18	5.18	− 0.36	− 4.72	− 4.88
	20	23.35	10.13	10.13	− 0.67	− 8.83	− 9.12
c_p	− 20	18.85	14.79	6.18	− 8.48	− 1.86	− 5.84
	− 10	9.05	7.09	3.08	− 4.25	− 0.98	− 3.00
	10	− 8.36	− 6.55	− 3.05	4.26	1.09	3.15
	20	− 16.08	− 12.61	− 6.06	8.54	2.29	6.47
c_h	− 20	0.98	11	− 2.87	− 0.25	9.75	2.78
	− 10	0.47	5.2	− 1.38	− 0.12	4.63	1.31
	10	− 0.44	− 4.69	1.27	0.11	− 4.22	− 1.18
	20	− 0.84	− 8.94	2.44	0.22	− 8.07	− 2.25
c_b	− 20	1.92	− 5.66	17.93	− 0.46	− 6.07	6.18
	− 10	0.9	− 2.64	8.18	− 0.21	− 2.84	2.8
	10	− 0.8	2.34	− 6.97	0.19	2.52	− 2.36
	20	− 1.51	4.42	− 12.99	0.36	4.78	− 4.37
θ	− 20	0.64	6.07	− 1.89	− 0.15	6.81	2.03
	− 10	0.31	2.94	− 0.92	− 0.07	3.29	0.98
	10	− 0.3	− 2.77	0.87	0.07	− 3.08	− 0.91
	20	− 0.58	− 5.4	1.7	0.14	− 5.97	− 1.75
α	− 20	− 41	− 11.04	− 11.04	− 9.83	11.81	12.29
	− 10	− 21.09	− 5	− 5	− 4.99	5.02	5.21
	10	22.02	4.21	4.21	5.08	− 3.87	− 4
	20	44.79	7.81	7.81	10.21	− 6.95	− 7.18
I_e	− 20	− 6.54	− 1.43	− 1.43	1.57	1.39	1.44
	− 10	− 3.28	− 0.71	− 0.71	0.77	0.68	0.71
	10	3.3	0.69	0.69	− 0.74	− 0.65	− 0.68
	20	6.63	1.36	1.36	− 1.47	− 1.28	− 1.33
M	− 20	− 14.594	− 3.34	− 3.34	3.68	3.29	3.417
	− 10	− 7.356	− 1.62	− 1.62	1.77	1.57	1.628
	10	7.465	1.52	1.52	− 1.64	− 1.44	− 1.488
	20	15.03	2.96	2.96	− 3.17	− 2.76	− 2.853
N	− 20	8.3	1.69	1.69	− 1.82	− 1.59	− 1.65
	− 10	4.13	0.86	0.86	− 0.93	− 0.81	− 0.84
	10	− 4.1	− 0.89	− 0.89	0.97	0.85	0.89
	20	− 8.17	− 1.8	− 1.8	1.98	1.75	1.82

Table 4 Sensitivity analysis for the demand function $D(p) = ap^{-\alpha}$ for $0 < T \le N$

Parameter	% Changes of parameters	% Changes in TP^*	% Changes				
			S^*	R^*	p^*	t_1^*	T^*
c_h	− 20	0.13	4.45	− 2.52	− 0.08	4.15	0
	− 10	0.06	2.17	− 1.23	− 0.04	2.03	0
	10	− 0.06	− 2.08	1.18	0.04	− 1.95	0
	20	− 0.12	− 4.08	2.31	0.07	− 3.83	0
c_p	− 20	41.09	67.44	45.12	− 19.9	− 3.79	− 11.58
	− 10	17.71	34.85	25.86	− 9.79	4.15	0
	10	− 13.85	− 23.9	− 18.83	9.78	− 3.83	0
	20	− 24.92	− 40.82	− 32.93	19.56	− 7.38	0
c_o	− 20	1.201	0	0	0	0	0
	− 10	0.6	0	0	0	0	0
	10	− 0.6	0	0	0	0	0
	20	− 1.201	0	0	0	0	0
θ	− 20	0.27	9.12	− 5.43	− 0.17	8.93	0
	− 10	0.13	4.36	− 2.6	− 0.08	4.28	0
	10	− 0.12	− 4.02	2.38	0.08	− 3.94	0
	20	–	–	–	–	–	–
c_b	− 20	0.62	− 12.37	9.54	− 0.39	− 12.99	0
	− 10	0.29	− 5.91	4.55	− 0.18	− 6.22	0
	10	− 0.27	5.42	− 4.17	0.17	5.75	0
	20	–	–	–	–	–	–
a	− 20	–	–	–	–	–	–
	− 10	− 10.6	− 10	− 10	0	0	0
	10	10.6	10	10	0	0	0
	20	21.2	20	20	0	0	0
α	− 20	197.3	57.15	57.15	19.35	− 18.13	− 18.3
	− 10	71.22	33.88	33.88	7.87	− 4.09	− 4.14
	10	–	–	–	–	–	–
	20	− 66.26	− 50.01	− 50.01	− 10	0	0
I_c	− 20	− 6.39	− 6.03	− 6.03	2.52	0	0
	− 10	− 3.22	− 3.04	− 3.04	1.24	0	0
	10	3.28	3.1	3.1	− 1.21	0	0
	20	6.63	6.25	6.25	− 2.4	0	0
M	− 20	− 7.95	− 7.5	− 7.5	3.17	0	0
	− 10	− 4.021	− 3.79	− 3.79	1.56	0	0
	10	4.115	3.88	3.88	− 1.51	0	0
	20	8.325	7.85	7.85	− 2.98	0	0
N	− 20	–	–	–	–	–	–
	− 10	0.62	− 8.71	− 8.71	− 0.6	− 9.89	− 10
	10	− 0.73	8.44	8.44	0.6	9.87	10
	20	− 1.52	11.85	11.85	1.05	14.49	14.69

N.B. (−) indicates non-optimal solution

- When the constant demand rate (a) increases then the total profit increase, while as the price-dependent demand rate α increases, the selling price and optimal total profit per unit time decrease which is analogous to Tiwari et al. [32].
- The period of deterioration time t_1 is discreetly subtle with respect to all other parameters although decidedly sensitive with respect to the holding cost C_h, ordering cost C_0, a, α
- Total cycle time T is exceedingly complex with respect to ordering cost C_0, $a\ and\ \alpha$ and soberly delicate with respect to all other parameters.

   Some observations made from Table 4.

- The total profit is exceedingly sensitive with respect to the purchase cost c_p, a, trade-credit period M, α and is moderately sensitive with respect to all other parameters. When purchase cost decrease then total profit *(TP)* and shortage *(R)* increases which is analogous to Tiwari et al. [32].
- When the constant demand rate (a) increases then the total profit increase, however as the price-dependent demand rate α increases as a consequence the price and the optimal total profit per unit time will decrease which is alike to Tiwari et al. [32].
- The uppermost on-hand stock-level S is extremely sensitive with respect to the purchase cost c_p, a, α, C_0 and is discreetly sensitive with respect to all other parameters.
- The maximum shortages level R is exceedingly delicate with respect to the shortage cost c_b, a, α, C_0 and is ascetically sensitive with respect to all other parameters. It is noted that when the holding cost is increased then the amount of shortages also increased which is analogous to Li et al. [11].
- The selling price is extremely delicate with respect to the purchase cost $c_p\ and\ \alpha$ and discreetly sensitive with respect to the other parameters. The influence of c_p on p^*, AP^* is quite instinctive and actually quite substantial. An increase in the purchase cost c_p pointers to a imperative increase in p^* and hence an imperative decreases in the demand which is analogous to Li et al. [11].
- When holding cost is increased then total profit of the chain is decreased and also observed that with the increase in deterioration rate lessen the profit which is analogous to Tiwari et al. [33].
- The time t_1 is reasonably exquisite with respect to all other parameters and is highly sensitive with respect to holding cost C_h, ordering cost C_0, a, α
- When the interest earned rate (I_e) is higher, it results more revenue with less quantity ordered and thus the total profit drops. When a higher interest paid rate is added to the total costs, it leads to a lesser amount of profit.
- Total cycle time T is discreetly sensitive with respect to all other parameters while highly sensitive to ordering cost C_0, $a\ and\ \alpha$.

Concluding Remarks

The demand function is considered exponentially with the price for the first problem in this paper. In second problem, demand function is used as a negative power of price. All these demand are considered under the trade-credit policy which has a great impact on total profit. The key decision factors/variables of the proposed model are total cycle length, optimal selling price, optimal profit and different scenarios for a trade-credit period where a retailer can invest safely for optimal profit in different types of demand. We have used Taylor series expansion to obtain the closed form of the cycle length, selling price and total profit. Finally, it is shown graphically the concavity of profit function with the help of MATLAB software

and also analyzed the sensitivity of different parameters by changing one and keeping the others same, which helps to the business organization to make better managerial decisions.

For further research, one can easily extend the proposed model in numerous techniques. It can be expanded for different types of variable demand which are dependent on displayed stock-level, time and others. On the other way, this model also could be generalized by considering three level of trade-credit policy. In addition, it can be also extended in fuzzy environments.

Appendix A

For demand function $D = ae^{-\frac{p}{\alpha}}$

$S = \frac{D\{e^{\theta t_1} - 1\}}{\theta}$ expanding the $e^{\theta t_1}$ in Taylor series expansion and neglect the higher order term.

$$S = D\left[t_1 + \frac{\theta}{2}t_1^2\right]$$

$C_{hol} = \frac{C_h D}{\theta^2}\left[e^{\theta t_1} - \theta t_1 - 1\right]$ expanding the $e^{\theta t_1}$ in Taylor series expansion and neglect the higher order term.

$$C_{hol} = \frac{C_h D t_1^2}{2}$$

$$R = ae^{-\left(p/\alpha\right)}[T - t_1]$$

$$C_{sho} = \frac{1}{2}C_b ae^{\left(\frac{-p}{\alpha}\right)}(T - t_1)^2$$

$$TP_{11}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} \left[c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 + \frac{C_h D t_1^2}{2}\right] \\ + \frac{c_p I_p D}{\theta^2}\left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1\right] - \frac{pI_e D}{2}\left[t_1^2 - N^2\right] \end{array}\right]\right]$$

$$TP_{11}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} + \frac{c_p I_p D}{\theta^2}\left[\theta^2(t_1 - M)^2\right] - \frac{pI_e D}{2}\left[t_1^2 - N^2\right] \end{array}\right]\right]$$

$$TP_{12}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} - pI_e D\left[\frac{(t_1^2 - N^2)}{2} + t_1(t_1 - N)\right] \end{array}\right]\right]$$

$$TP_{13}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} - \frac{pI_e DT}{2}[M - N] \end{array}\right]\right]$$

$$TP_{14}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} - pTI_e a e^{-p/\alpha}[M - N] \end{array}\right]\right]$$

For demand function $D = ap^{-\alpha}$

$S = \frac{D\{e^{\theta t_1} - 1\}}{\theta}$ expanding the $e^{\theta t_1}$ in Taylor series expansion and neglect the higher order term.

$$S = D\left[t_1 + \frac{\theta}{2}t_1^2\right]$$

$C_{hol} = \frac{C_h D}{\theta^2}\left[e^{\theta t_1} - \theta t_1 - 1\right]$ expanding the $e^{\theta t_1}$ in Taylor series expansion and neglect the higher order term.

$$C_{hol} = \frac{C_h D t_1^2}{2}$$

$$R = ae^{-(p/\alpha)}[T - t_1]$$

$$C_{sho} = \frac{1}{2}C_b a e^{\left(\frac{-p}{\alpha}\right)}(T - t_1)^2$$

$$TP_{21}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} + \frac{c_p I_p D}{\theta^2}\left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1\right] - \frac{pI_e D}{2}[t_1^2 - N^2] \end{array}\right]\right]$$

$$TP_{21}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} + \frac{c_p I_p D}{\theta^2}\left[\theta^2(t_1 - M)^2\right] - \frac{pI_e D}{2}[t_1^2 - N^2] \end{array}\right]\right]$$

$$TP_{22}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} - pI_e D\left[\frac{(t_1^2 - N^2)}{2} + t_1(t_1 - N)\right] \end{array}\right]\right]$$

$$TP_{23}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} - \frac{pI_e DT}{2}[M - N] \end{array}\right]\right]$$

$$TP_{24}(p, t_1, T) = \frac{1}{T}\left[pDT - \left[\begin{array}{l} c_o + c_p D\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b D(T - t_1)^2 \\ + \frac{C_h D t_1^2}{2} - pTI_e D[M - N] \end{array}\right]\right]$$

Appendix B

Proof of Theorem **1** By taking the second order derivative of $TP_{11}(p, t_1, T)$ with respect to p which is strictly negative, concavity of total profit function will be proved. Considering $D = ae^{-\frac{p}{\alpha}}$.

$$\frac{\partial^2 TP_{11}(p,t_1,T)}{\partial p^2} = \frac{1}{T}\left[\frac{DT}{\alpha}\left(\frac{p}{\alpha}-2\right) - \left[\begin{array}{c} c_p\frac{D}{\alpha^2}\left[T+\frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b\frac{D}{\alpha^2}(T-t_1)^2 \\ +\frac{C_h t_1^2}{2}\frac{D}{\alpha^2} + \frac{c_p I_p}{\theta^2}\frac{D}{\alpha^2}\left[\theta^2(t_1-M)^2\right] \\ -\frac{I_e}{2}[t_1^2-N^2]\frac{D}{\alpha}\left(\frac{p}{\alpha}-2\right) \end{array}\right]\right] < 0$$

For *necessary condition*, $\dfrac{\partial TP_{11}}{\partial p} = 0$

$$\frac{\partial TP_{11}}{\partial p} = \frac{1}{T}\left[DT - p\frac{D}{\alpha}T + \left[\begin{array}{c} c_p\frac{D}{\alpha}\left[T+\frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b\frac{D}{\alpha}(T-t_1)^2 \\ +\frac{C_h t_1^2}{2}\frac{D}{\alpha} + \frac{c_p I_p}{\theta^2}\frac{D}{\alpha}\left[\theta^2(t_1-M)^2\right] \\ +\frac{I_e}{2}D\left(1-\frac{p}{\alpha}\right)[t_1^2-N^2] \end{array}\right]\right] = 0$$

i.e. $\Rightarrow p\dfrac{D}{\alpha}T - DT = c_p\dfrac{D}{\alpha}\left[T+\dfrac{\theta t_1^2}{2}\right]$

$$+\frac{1}{2}C_b\frac{D}{\alpha}(T-t_1)^2 + \frac{C_h t_1^2}{2}\frac{D}{\alpha} + \frac{c_p I_p}{\theta^2}\frac{D}{\alpha}\left[\theta^2(t_1-M)^2\right]$$

$$+\frac{I_e}{2}D(1-\frac{p}{\alpha})[t_1^2-N^2]$$

$$\Rightarrow \frac{PT}{\alpha} - \frac{I_e}{2}(1-\frac{p}{\alpha})[t_1^2-N^2]$$

$$= c_p\frac{1}{\alpha}\left[T+\frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b\frac{1}{\alpha}(T-t_1)^2$$

$$+\frac{C_h t_1^2}{2}\frac{1}{\alpha} + \frac{c_p I_p}{\theta^2}\frac{1}{\alpha}\left[\theta^2(t_1-M)^2\right]$$

$$\Rightarrow P = \frac{c_p\frac{1}{\alpha}\left[T+\frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b\frac{1}{\alpha}(T-t_1)^2 + \frac{C_h t_1^2}{2}\frac{1}{\alpha} + \frac{c_p I_p}{\theta^2}\frac{1}{\alpha}\left[\theta^2(t_1-M)^2\right]}{\left[\frac{T}{\alpha}+\frac{I_e}{2\alpha}(1-\frac{p}{\alpha})[t_1^2-N^2]\right]} = P^*$$

$$\square$$

Proof of Theorem 2 By taking the second order derivative of $TP_{11}(p,t_1,T)$ with respect to t_1 which is strictly negative, concavity of total profit function will be proved. Considering $D = ae^{-\frac{p}{\alpha}}$.

$$\frac{\partial^2 TP_{11}}{\partial t_1^2} = -\frac{1}{T}\left[c_p D\theta + c_h D + c_b D + c_p I_p D\right] < 0$$

For *necessary condition*, $\dfrac{\partial TP_{11}}{\partial t_1} = 0$

$$\frac{\partial TP_{11}}{\partial t_1} = -\frac{1}{T}\left[c_p D\theta t_1 + c_h D t_1 - c_b D(T-t_1) + \frac{c_p I_p D}{\theta}\left[e^{\theta(t_1-M)} - \theta(t_1-M) - 1\right]\right] = 0$$

Using *the taylor expansion series and neglating the higher order term*

$$\Rightarrow t_1 \big[c_p D\theta + c_h D + c_b D + c_p I_p D \big] = c_b DT + c_p I_p DM$$

$$\Rightarrow t_1 = t_1^* = \frac{(c_b DT + c_p I_p DM)}{\big[c_p D\theta + c_h D + c_b D + c_p I_p D \big]}$$

$\square$

Proof of Theorem 3 By taking the second order derivative of $TP_{11}(p, t_1, T)$ with respect to T which is strictly negative, concavity of total profit function will be proved. Considering $D = ae^{-\frac{p}{\alpha}}$.

$$\frac{\partial^2 TP_{11}(p, t_1, T)}{\partial T^2} = -\frac{2}{T^3} \left[c_o + \frac{c_p D\theta t_1^2}{2} + \frac{C_h D t_1^2}{2} + \frac{c_p I_p D}{\theta^2} \left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1 \right] - \frac{p I_e D}{2} \left[t_1^2 - N^2 \right] \right]$$

$$= -\frac{2}{T^3} \frac{T^2}{2} C_b D = -\frac{1}{T} C_b D < 0$$

The root of the first derivative of total profit function $TP_{11}(p, t_1, T)$ with respect to T is optimal value of T which is T^*:

$$\frac{\partial TP_{11}(p, t_1, T)}{\partial T} = 0$$

i.e. $pD - \dfrac{1}{2} C_b D - \dfrac{1}{T^2} \left[c_o + \dfrac{c_p D\theta t_1^2}{2} + \dfrac{C_h D t_1^2}{2} + \dfrac{c_p I_p D}{\theta^2} \left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1 \right] - \dfrac{p I_e D}{2} \left[t_1^2 - N^2 \right] \right] = 0$

$$= \left[c_o + \frac{c_p D\theta t_1^2}{2} + \frac{C_h D t_1^2}{2} + \frac{c_p I_p D}{\theta^2} \left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1 \right] - \frac{p I_e D}{2} \left[t_1^2 - N^2 \right] \right] = pD - \frac{T^2}{2} C_b D$$

i.e. $T = T^* = -\sqrt{\dfrac{2 \left[c_o - pD + \frac{c_p D\theta t_1^2}{2} + \frac{C_h D t_1^2}{2} + \frac{c_p I_p D}{\theta^2} \left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1 \right] - \frac{p I_e D}{2} \left[t_1^2 - N^2 \right] \right]}{C_b D}}$

$\square$

Appendix C

Proof of Theorem 4 By taking the second order derivative of $TP_{21}(p, t_1, T)$ with respect to p which is strictly negative, concavity of total profit function will be proved. Considering $D = ap^{-\alpha}$.

$$\frac{\partial^2 TP_{11}(p, t_1, T)}{\partial p^2} = \frac{1}{T} \left[\alpha DT(p^2 - p - 1) - \left[\begin{array}{l} c_p \alpha D(1 - \alpha p) \left[T + \frac{\theta t_1^2}{2} \right] + \frac{1}{2} C_b \alpha D(1 - \alpha p)(T - t_1)^2 \\ + \frac{C_h t_1^2}{2} \alpha D(1 - \alpha p) + \frac{c_p I_p}{\theta^2} \alpha D(1 - \alpha p) \left[\theta^2 (t_1 - M)^2 \right] \\ + \frac{\alpha D I_e}{2} \left[t_1^2 - N^2 \right] (1 + p - \alpha p^2) \end{array} \right] \right] < 0$$

For *necessary condition*, $\dfrac{\partial TP_{11}}{\partial p} = 0$

$$\frac{\partial TP_{11}(p, t_1, T)}{\partial p} = \frac{1}{T} \left[DT - \alpha p DT + \left[\begin{array}{l} c_p \alpha p D \left[T + \frac{\theta t_1^2}{2} \right] + \frac{1}{2} C_b \alpha p D(T - t_1)^2 + \frac{C_h t_1^2}{2} \alpha p D \\ + \frac{c_p I_p}{\theta^2} \alpha p D \left[\theta^2 (t_1 - M)^2 \right] + \frac{I_e}{2} D(1 - \alpha p) \left[t_1^2 - N^2 \right] \end{array} \right] \right] = 0$$

i.e. $D(T - \alpha p T) = -D \left[\begin{array}{l} c_p \alpha p \left[T + \frac{\theta t_1^2}{2} \right] + \frac{1}{2} C_b \alpha p(T - t_1)^2 + \frac{C_h t_1^2}{2} \alpha p + \frac{c_p I_p}{\theta^2} \alpha p \left[\theta^2 (t_1 - M)^2 \right] \\ \qquad\qquad\qquad\qquad\qquad\qquad + \frac{I_e}{2} (1 - \alpha p) \left[t_1^2 - N^2 \right] \end{array} \right]$

$$\Rightarrow T - \frac{I_e}{2}(1-\alpha p)[t_1^2 - N^2] = -p\left[c_p\alpha\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b\alpha(T - t_1)^2 + \frac{C_h t_1^2}{2}\alpha + \frac{c_p I_p}{\theta^2}\alpha[\theta^2(t_1 - M)^2] + \alpha T\right]$$

$$\Rightarrow \frac{I_e}{2}\alpha p[t_1^2 - N^2] + p\left[c_p\alpha\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b\alpha(T - t_1)^2 + \frac{C_h t_1^2}{2}\alpha + \frac{c_p I_p}{\theta^2}\alpha[\theta^2(t_1 - M)^2] + \alpha T\right]$$

$$= T - \frac{I_e}{2}[t_1^2 - N^2]$$

$$\Rightarrow p\left[\left[\frac{I_e}{2}\alpha[t_1^2 - N^2]\right] + \left[c_p\alpha\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b\alpha(T - t_1)^2 + \frac{C_h t_1^2}{2}\alpha + \frac{c_p I_p}{\theta^2}\alpha[\theta^2(t_1 - M)^2] + \alpha T\right]\right]$$

$$= T - \frac{I_e}{2}[t_1^2 - N^2]$$

$$\Rightarrow p = \frac{\left[T - \frac{I_e}{2}[t_1^2 - N^2]\right]}{\left[\left[\frac{I_e}{2}\alpha[t_1^2 - N^2]\right] + \left[c_p\alpha\left[T + \frac{\theta t_1^2}{2}\right] + \frac{1}{2}C_b\alpha(T - t_1)^2 + \frac{C_h t_1^2}{2}\alpha + \frac{c_p I_p}{\theta^2}\alpha[\theta^2(t_1 - M)^2] + \alpha T\right]\right]} = p^*$$

$\square$

Proof of Theorem 5 By taking the second order derivative of $TP_{21}(p, t_1, T)$ with respect to t_1 which is strictly negative, concavity of total profit function will be proved. Considering $D = ap^{-\alpha}$.

$$\frac{\partial^2 TP_{21}}{\partial t_1^2} = -\frac{1}{T}\left[c_p D\theta + c_h D + c_b D + c_p I_p D\right] < 0$$

For *necessary condition*, $\dfrac{\partial TP_{11}}{\partial t_1} = 0$

$$\frac{\partial TP_{21}}{\partial t_1} = -\frac{1}{T}\left[c_p D\theta t_1 + c_h Dt_1 - c_b D(T - t_1) + \frac{c_p I_p D}{\theta}\left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1\right]\right] = 0$$

Using *the taylor expansion series and neglating the higher order term*

$$\Rightarrow t_1\left[c_p D\theta + c_h D + c_b D + c_p I_p D\right] = c_b DT + c_p I_p DM$$

$$\Rightarrow t_1 = t_1^* = \frac{(c_b DT + c_p I_p DM)}{\left[c_p D\theta + c_h D + c_b D + c_p I_p D\right]}$$

$\square$

Proof of Theorem 6 By taking the second order derivative of $TP_{21}(p, t_1, T)$ with respect to T which is strictly negative, concavity of total profit function will be proved. Considering $D = ap^{-\alpha}$.

$$\frac{\partial^2 TP_{21}(p, t_1, T)}{\partial T^2} = -\frac{2}{T^3}\left[c_o + \frac{c_p D\theta t_1^2}{2} + \frac{C_h Dt_1^2}{2} + \frac{c_p I_p D}{\theta^2}\left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1\right] - \frac{pI_e D}{2}[t_1^2 - N^2]\right]$$

$$= -\frac{2}{T^3}\frac{T^2}{2}C_b D = -\frac{1}{T}C_b D < 0$$

The root of the first derivative of total profit function $TP_{21}(p, t_1, T)$ with respect to T is optimal value of T which is T^*:

$$\frac{\partial TP_{21}(p, t_1, T)}{\partial T} = 0$$

i.e. $pD - \dfrac{1}{2}C_b D - \dfrac{1}{T^2}\left[c_o + \dfrac{c_p D\theta t_1^2}{2} + \dfrac{C_h Dt_1^2}{2} + \dfrac{c_p I_p D}{\theta^2}\left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1\right] - \dfrac{pI_e D}{2}[t_1^2 - N^2]\right] = 0$

$$= \left[c_o + \frac{c_p D \theta t_1^2}{2} + \frac{C_h D t_1^2}{2} + \frac{c_p I_p D}{\theta^2} \left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1 \right] - \frac{p I_e D}{2} \left[t_1^2 - N^2 \right] \right] = pD - \frac{T^2}{2} C_b D$$

$$i.e.\ T = T^* = -\sqrt{ \frac{ 2 \left[c_o - pD + \frac{c_p D \theta t_1^2}{2} + \frac{C_h D t_1^2}{2} + \frac{c_p I_p D}{\theta^2} \left[e^{\theta(t_1 - M)} - \theta(t_1 - M) - 1 \right] - \frac{p I_e D}{2} \left[t_1^2 - N^2 \right] \right] }{ C_b D } }$$

$\square$

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