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ORIGINAL PAPER

Existence of Solutions for a Functional Integro-Differential Equation with Infinite Point and Integral Conditions

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Abstract
In this article, we study the existence of solutions for two initial value problems of the functional integro-differential equation with nonlocal infinite-point and integral conditions. We study the continuous dependence of the solution. As some examples illustrate the importance of the results.

Keywords Existence of solutions · Continuous dependence · Nonlocal condition · Integral condition · Infinite point condition

Mathematics Subject Classification 34A12, 34k20, 34k25

Introduction

It is well-known that a lot of problems investigated in engineering, mechanics, mathematical physics, vehicular traffic theory [1,14], [2, pp. 157–167], queuing theory and also several real world problems can be described with help of various functional differential (integral) equations. The theory of functional differential (integral) equations is highly developed and constitutes a significant and important branch of nonlinear analysis. There have been published, up to now, numerous research papers; see [3–8,10,12,13,15–17].

In this paper, we are interested with the initial value problem (IVP) for the functional integro-differential equation

$$\frac{dx}{dt} = g(t, x(t), \quad \int_0^t f(s, x(s))ds), \quad a.e \quad t \in (0, T], \tag{1}$$

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with the nonlocal condition

$$x(0) + \sum_{j=1}^{m} p_j x(\tau_j) = x_0, \quad \sum_{j=1}^{m} p_j > 0, \quad \tau_j \in (0, T]. \tag{2}$$

The existence of at least and unique solution $x \in C[0, T]$, under certain conditions, will be proved. The continuous dependence of the solution on the nonlocal-data p_j, on x_0 and on the functional f, will be studied.

As applications, the IVP of Eq. (1) with integral condition

$$x(0) + \int_0^T x(s)dh(s) = x_0, \quad h : [0, T] \to \mathbb{R} \text{ increasing function} \tag{3}$$

will be studied. Also, if $\sum_{j=1}^{\infty} p_j$ is convergent, the IVP of Eq. (1) with infinite-point condition

$$x(0) + \sum_{j=1}^{\infty} p_j x(\tau_j) = x_0, \tag{4}$$

will be studied.

Integral Representation

Consider the IVP (1)–(2) with the assumptions:

1. $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory-condition. There exist a function $c_1 \in L^1[0, T]$ and a positive constant $b_1 > 0$, such that
$$|g(t, \alpha, \beta)| \leq c_1(t) + b_1|\alpha| + b_1|\beta|.$$
2. $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory-condition. There exist a function $c_2 \in L^1[0, T]$ and a positive constant $b_2 > 0$, such that
$$|f(t, \beta)| \leq c_2(t) + b_2|\beta|.$$
3. $$\sup_{t\in[0,1]} \int_0^t c_1(s)ds \leq M_1, \quad \sup_{t\in[0,1]} \int_0^t \int_0^s c_2(\theta)d\theta ds \leq M_2.$$
4. $\left(1 + E\sum_{j=1}^{m} p_j\right)\left(b_1 T + \frac{1}{2} b_1 b_2 T^2\right) < 1.$

Definition 2.1 By a solution of the IVP (1)–(2) we mean a function $x \in C[0, T]$ that satisfies (1)–(2).

Lemma 2.1 *The solution of IVP (1)–(2) if it exist, then it can be represented by the integral-equation*

$$x(t) = E\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right] + \int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)ds, \tag{5}$$

where $E = (1 + \sum_{j=1}^{m} p_j)^{-1}$.

Proof Let x be a solution of IVP (1)–(2). Integrating both sides of (1) we obtain

$$x(t) = x(0) + \int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds. \tag{6}$$

Using the nonlocal condition (2), we get

$$\sum_{j=1}^{m} p_j x(\tau_j) = x(0)\sum_{j=1}^{m} p_j + \sum_{j=1}^{m} p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds,$$

since, $\sum_{j=1}^{m} p_j x(\tau_j) = x_0 - x(0)$, we have

$$x_0 - x(0) = x(0)\sum_{j=1}^{m} p_j + \sum_{j=1}^{m} p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds,$$

then

$$x(0) = \frac{1}{1+\sum_{j=1}^{m} p_j}\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right]. \tag{7}$$

Using (6) and (7), we obtain

$$\begin{aligned} x(t) = &\frac{1}{1+\sum_{j=1}^{m} p_j}\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right] \\ &+ \int_0^t g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds. \end{aligned}$$

□

Existence of Solution

Theorem 3.1 *Let the assumptions 1–4 be satisfied. Then the IVP* (1)–(2) *has at least one solution* $x \in C[0, T]$.

Proof Let the operator F associated with the integral-equation (5) by

$$\begin{aligned} Fx(t) = &E\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right] \\ &+ \int_0^t g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds. \end{aligned}$$

Let $Q_r = \{x \in \mathbb{R} : ||x|| \le r\}$, where $r = \frac{E|x_0|+(1+E\sum_{j=1}^{m} p_j)(M_1+b_1M_2)}{1-((1+E\sum_{j=1}^{m} p_j)(b_1T+\frac{1}{2}b_1b_2T^2))}$, it clear that Q_r is nonempty, closed, bounded and convex subset of $C[0, T]$. Then we have, for $x \in Q_r$

$$\begin{aligned} |Fx(t)| \le &E\left[|x_0| + \sum_{j=1}^{m} p_j \int_0^{\tau_j} |g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)|ds\right] \\ &+ \int_0^t |g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)|ds \end{aligned}$$

$$\le E\left[|x_0| + \sum_{j=1}^{m} p_j \int_0^{\tau_j} (c_1(s) + b_1|x(s)| + b_1 \int_0^s |f(\theta, x(\theta))|d\theta)ds\right]$$
$$+ \int_0^t (c_1(s) + b_1|x(s)| + b_1 \int_0^s f(\theta, x(\theta))d\theta)ds$$
$$\le E\left[|x_0| + \sum_{j=1}^{m} p_j(M_1 + b_1Tr + b_1 \int_0^{\tau_j} \int_0^s |c_2(\theta) + b_2|x(\theta)|d\theta ds)\right]$$
$$+M_1 + b_1Tr + b_1 \int_0^t \int_0^s (c_2(\theta) + b_2|x(\theta)|)d\theta ds$$
$$\le E|x_0| + E\sum_{j=1}^{m} p_j(M_1 + b_1Tr + b_1M_2 + \frac{1}{2}b_1b_2T^2r)$$
$$+M_1 + b_1Tr + b_1M_2 + \frac{1}{2}b_1b_2T^2r$$
$$= E|x_0| + \left(1 + E\sum_{j=1}^{m} p_j\right)\left(M_1 + b_1Tr + b_1M_2 + \frac{1}{2}b_1b_2T^2r\right) = r.$$

Then $F : Q_r \to Q_r$ and the class of functions $\{Fx\}$ is uniformly bounded in Q_r. Now, let $t_1, t_2 \in (0, 1)$ s. t $|t_2 - t_1| < \delta$, then

$$|Fx(t_2) - Fx(t_1)| = \left|\int_0^{t_2} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right.$$
$$\left. - \int_0^{t_1} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right|$$
$$\le \int_{t_1}^{t_2} \left|g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)\right| ds$$
$$\le \int_{t_1}^{t_2} (c_1(s) + b_1|x(s)| + b_1 \int_0^s |f(\theta, x(\theta))d\theta|)ds$$
$$\le \int_{t_1}^{t_2} c_1(s)ds + (t_2 - t_1)b_1r + b_1 \int_{t_1}^{t_2} \int_0^s c_2(\theta)d\theta ds$$
$$+\frac{1}{2}b_1b_2r\left(t_2^2 - t_1^2\right).$$

Then the class of functions $\{Fx\}$ is equi-continuous in Q_r.

Let $x_n \in Q_r$, $x_n \to x(n \to \infty)$, then from Assumptions 1–2, we obtain $g(t, x_n(t), y_n(t)) \to g(t, x(t), y(t))$ and $f(t, x_n(t)) \to f(t, x(t))$ as $n \to \infty$. Also

$$\lim_{n\to\infty} Fx_n(t) = \lim_{n\to\infty}\Bigg[E\Bigg[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x_n(s), \int_0^s f(s, x_n(\theta))d\theta)ds\Bigg]$$
$$+ \int_0^t g\left(s, x_n(s), \int_0^s f(\theta, x_n(\theta))d\theta\right)ds\Bigg]. \tag{8}$$

Using assumptions 1–2 and Lebesgue Dominated convergence Theorem [11], from (8) we obtain

$$\lim_{n\to\infty} Fx_n(t) = \Bigg[E\Bigg[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} \lim_{n\to\infty} g(s, x_n(s), \int_0^s f(\theta, x_n(\theta))d\theta)ds\Bigg] + \int_0^t \lim_{n\to\infty} g\left(s, x_n(s), \int_0^s f(\theta, x_n(\theta))d\theta\right)ds\Bigg] = Fx(t).$$

Then $Fx_n \to Fx$ as $n \to \infty$. Therefore F is continuous.

$$\lim_{t\to 0} x(t) = E\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)ds\right] \in C[0, T].$$

Then by Schauder fixed point Theorem [9] there exist at least one solution $x \in C[0, T]$ of the integral-equation (5).
To complete the proof, differentiation (5) we obtain

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\Bigg\{E\Bigg[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\Bigg] + \int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)ds\Bigg\}\\ &= \frac{d}{dt}\int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)ds\\ &= g\left(t, x(t), \int_0^t f(\theta, x(\theta))d\theta\right).\end{aligned}$$

Also, from the integral-equation (5), we get

$$\begin{aligned}x(\tau_j) &= E\Big[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g\Big(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\Big)ds\Big] + \int_0^{\tau_j} g\Big(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\Big)ds\\ x(0) &= E\Big[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\Big],\end{aligned} \tag{9}$$

and

$$\sum_{j=1}^{m} p_j x(\tau_j) = E\sum_{j=1}^{m} p_j\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right] + \sum_{j=1}^{m} p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)ds, \tag{10}$$

from (9) and (10) we have

$$x(0)+\sum_{j=1}^{m}p_jx(\tau_j)$$
$$=E(1+\sum_{j=1}^{m}p_j)\left[x_0-\sum_{j=1}^{m}p_j\int_0^{\tau_j}g\left(s,x(s),\int_0^s f(\theta,x(\theta))d\theta\right)ds\right]$$
$$+\sum_{j=1}^{m}p_j\int_0^{\tau_j}g\left(s,x(s),\int_0^s f(\theta,x(\theta))d\theta\right)ds.$$

Then

$$x(0)+\sum_{j=1}^{m}p_jx(\tau_j)=x_0.$$

Therefor there exist at least one solution $x\in C[0,T]$ of the IVP (1)–(2). □

Nonlocal Integral Condition

Let $x\in C[0,T]$ be the solution of the IVP (1)–(2). Let $p_j=h(t_j)-h(t_{j-1})$, h is increasing function, $\tau_j\in(t_{j-1},t_j)$, $0=t_0<t_1<t_2,\ldots<t_m=1$ then, as $m\to\infty$ the nonlocal-condition (2) will be

$$x(0)+\sum_{j=1}^{m}h(t_j)-h(t_{j-1})x(\tau_j)=x_0.$$

And

$$x(0)+\lim_{m\to\infty}\sum_{j=1}^{m}h(t_j)-h(t_{j-1})x(\tau_j)=x(0)+\int_0^T x(s)dh(s)=x_0.$$

Theorem 4.1 *Let the assumptions 1–4 be satisfied. Then the IVP of* (1)–(3) *has at least one solution* $x\in C[0,T]$.

Proof As $m\to\infty$, the solution of the IVP (1)–(2) will be

$$x(t)=\lim_{m\to\infty}\frac{1}{1+\sum_{j=1}^{m}p_j}\left[x_0-\sum_{j=1}^{m}p_j\int_0^{\tau_j}g\left(s,x(s),\int_0^s f(\theta,x(\theta))d\theta\right)ds\right]$$
$$+\int_0^t g(s,x(s),\int_0^s f(\theta,x(\theta))d\theta)ds$$
$$=\frac{1}{1+h(T)-h(0)}\left[x_0-\lim_{m\to\infty}\sum_{j=1}^{m}p_j\int_0^{\tau_j}g(s,x(s),\int_0^s f(\theta,x(\theta))d\theta)ds(h(t_j)\right.$$
$$\left.-h(t_{j-1}))\right]+\int_0^t g(s,x(s),\int_0^s f(\theta,x(\theta))d\theta)ds$$

$$
= \frac{1}{1+h(T)-h(0)}\left[x_0 - \int_0^T \int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds \cdot dh(t)\right]
+ \int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds.
$$

□

Infinite-Point Boundary Condition

Theorem 5.1 *Let the assumptions 1–4 be satisfied. Then the IVP of (1)–(4) has at least one solution $x \in C[0, T]$.*

Proof Let the assumptions of Theorem 3.1 be satisfied. Let $S_m,\ S_m = \sum_{j=1}^m p_j$ be convergent sequence, then

$$
x_m(t) = \frac{1}{1+\sum_{j=1}^m p_j}\left[x_0 - \sum_{j=1}^m p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds\right]
+ \int_0^t g\left(s, x_m(s), \int_0^s f(\theta, x_m(\theta))d\theta\right) ds. \tag{11}
$$

Take the limit to (11), as $m \to \infty$, we have

$$
\begin{aligned}
\lim_{m\to\infty} x_m(t) &= \lim_{m\to\infty}\left[\frac{1}{1+\sum_{j=1}^m p_j}\left[x_0 - \sum_{j=1}^m p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds\right]\right. \\
&\quad \left. + \int_0^t g\left(s, x_m(s), \int_0^s f(\theta, x_m(\theta))d\theta\right) ds\right] \\
&= \lim_{m\to\infty} \frac{1}{1+\sum_{j=1}^m p_j}\left[x_0 - \sum_{j=1}^m p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds\right] \\
&\quad + \lim_{m\to\infty} \int_0^t g\left(s, x_m(s), \int_0^s f(\theta, x_m(\theta))d\theta\right) ds.
\end{aligned} \tag{12}
$$

Now $|p_j x(\tau_j)| \leq |p_j| \|x\|$, therefore by comparison test $\sum_{j=1}^\infty p_j x(\tau_j)$ is convergent. Also

$$
\begin{aligned}
\left|\int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta) ds\right| &\leq \int_0^{\tau_j} (c_1(s) + b_1|x(s)| \\
&\quad + b_1 \int_0^s f(\theta, x(\theta))d\theta) ds \\
&\leq \int_0^{\tau_j} (c_1(s) + b_1|x(s)| + b_1 \int_0^s (c_2(s) + b_2|x(s)|) d\theta) ds \\
&\leq M_1 + b_1\|x\| + b_1 M_2 + \frac{1}{2} b_1 b_2 \|x\| \leq M,
\end{aligned}
$$

then $|p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta) ds| \leq |p_j| \cdot M$ and by the comparison test $\sum_{j=1}^\infty p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta) ds$ is convergent.

Now, $|g| \leq |c_1(s) + b_1\|x\| + b_1 M_2 + b_1 b_2 \|x\|$, using assumptions 1–2 and Lebesgue Dominated convergence Theorem [11], from (12) we obtain

$$x(t) = \frac{1}{1+\sum_{j=1}^{\infty} p_j}\left[x_0 - \sum_{j=1}^{\infty} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right] + \int_0^t g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds. \tag{13}$$

The Theorem proved. □

Uniqueness of the Solution

Let g and f satisfy the following assumptions

5. $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $\alpha, \beta \in \mathbb{R}$ and satisfies the lipschitz condition
$$|g(t, \alpha, \beta) - g(t, u, v)| \leq b_1|\alpha - u| + b_1|\beta - v|, \tag{14}$$
6. $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $\alpha \in \mathbb{R}$ and satisfies the lipschitz condition
$$|f(t, \alpha) - f(t, u)| \leq b_2|\alpha - u|, \tag{15}$$
7.
$$\sup_{t\in[0,T]} \int_0^t |f(s, 0, 0)|ds \leq L_1, \quad \sup_{t\in[0,T]} \int_0^t \int_0^s |g(\theta, 0)|d\theta ds \leq L_2.$$

Theorem 6.1 *Let the assumptions 5–7 be satisfied. Then the solution of the IVP (1)–(2) is unique.*

Proof From assumption 5 we have g is measurable in t for any $x, y \in \mathbb{R}$ and satisfies the lipschitz condition, then it is continuous in $\alpha, \beta \in \mathbb{R}\ \forall t \in [0, T]$, and

$$|g(t, \alpha, \beta)| \leq b_1|\alpha| + b_1|\beta| + |f(t, 0, 0)|.$$

Then condition 1 is satisfied. Also by the same way we can show that assumption 2 satisfied by Assumption 6. Now, from Theorem 3.1 the solution of the IVP (1)–(2) exists.
Let x, y be two the solution of (1)–(2), then

$$\begin{aligned}
|x(t) - y(t)| = |E &\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds\right] \\
&+ \int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds \\
&- E\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g\left(s, y(s), \int_0^s f(\theta, y(\theta))d\theta\right) ds\right] \\
&- \int_0^t g\left(s, y(s), \int_0^s f(\theta, y(\theta))d\theta\right) ds| \\
\leq E &\sum_{j=1}^{m} p_j \int_0^{\tau_j} |g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)
\end{aligned}$$

$$
\begin{aligned}
&-g\left(s, y(s), \int_0^s f(\theta, y(\theta))d\theta\right)\Big| ds\\
&+\int_0^t |g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)\\
&-g\left(s, y(s), \int_0^s f(\theta, y(\theta))d\theta\right)|ds\\
\leq\ & E\sum_{j=1}^m p_j \int_0^{\tau_j}\left(b_1\|x-y\| + b_1\int_0^s |f(\theta, x(\theta))\right.\\
&-f(\theta, y(\theta))|d\theta)\, ds + \int_0^t (b_1\|x-y\|\\
&\left.+b_1\int_0^s |f(\theta, x(\theta)) - f(\theta, y(\theta))|d\theta\right) ds\\
\leq\ & b_1 T\|x-y\| E\sum_{j=1}^m p_j + \frac{1}{2}b_1 b_2 T^2\|x-y\| E\sum_{j=1}^m p_j\\
&+b_1 T\|x-y\| + \frac{1}{2}b_1 b_2 T^2\|x-y\|\\
=\ & \left(1+E\sum_{j=1}^m p_j\right)(b_1 T + \frac{1}{2}b_1 b_2 T^2)\|x-y\|.
\end{aligned}
$$

Hence

$$\left(1-\left(1+E\sum_{j=1}^m p_j\right)\left(b_1 T+\frac{1}{2}b_1 b_2 T^2\right)\right)\|x-y\| \leq 0.$$

Since $\left(1+E\sum_{j=1}^m p_j\right)\left(b_1 T+\frac{1}{2}b_1 b_2 T^2\right) < 1$, then $x(t) = y(t)$ and the solution of the IVP (1)–(2) is unique. □

Continuous Dependence

Continuous Dependence on x_0

Definition 7.1 The solution $x \in C[0,1]$ of the IVP (1)–(2) continuously depends on x_0, if

$$\forall\epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad |x_0 - x_0^*| < \delta \Rightarrow ||x - x^*|| < \epsilon,$$

where x^* is the solution of the IVP

$$\frac{dx^*}{dt} = g(t, x^*(t), \int_0^t f(s, x^*(s))ds), \quad a.e \quad t \in (0, T], \tag{16}$$

with the nonlocal condition

$$x(0) + \sum_{j=1}^m p_j x^*(\tau_j) = x_0^*, \quad \sum_{j=1}^m p_j > 0, \ \tau_j \in (0, T]. \tag{17}$$

Theorem 7.1 *Let the assumptions of Theorem 6.1 be satisfied. Then the solution of the IVP (1)–(2) continuously depends on x_0.*

Proof Let x, x^* be two solutions of the IVP (1)–(2) and (16)–(17) respectively. Then

$$
\begin{aligned}
|x(t) - x^*(t)| &= |E\left[x_0 - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)ds\right] \\
&\quad + \int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds \\
&\quad - E\left[x_0^* - \sum_{j=1}^{m} p_j \int_0^{\tau_j} g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) ds\right] \\
&\quad - \int_0^t g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) ds| \\
&\le E|x_0 - x_0^*| + E\sum_{j=1}^{m} p_j \int_0^{\tau_j} |g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) \\
&\quad - g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)|ds + \int_0^t |g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) \\
&\quad - g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right)|ds, \\
&\le E|x_0 - x_0^*| + E\sum_{j=1}^{m} p_j \int_0^{\tau_j} (b_1\|x - x^*\| \\
&\quad + b_1 \int_0^s |f(\theta, x^*(\theta)) - f(\theta, x(\theta))|d\theta)ds + \int_0^t (b_1\|x - x^*\| \\
&\quad + b_1 \int_0^s |f(\theta, x(\theta)) - f(\theta, x^*(\theta))|d\theta)ds \\
&\le E|x_0 - x_0^*| + b_1 T\|x - x^*\| E\sum_{j=1}^{m} p_j + \frac{1}{2}b_1 b_2 T^2\|x - x^*\| E\sum_{j=1}^{m} p_j \\
&\quad + b_1 T\|x - x^*\| + \frac{1}{2}T^2 b_1 b_2\|x - x^*\| \\
&\le E\delta + \left(1 + E\sum_{j=1}^{m} p_j\right)(b_1 T + \frac{1}{2}b_1 b_2 T^2)\|x - x^*\|.
\end{aligned}
$$

Hence

$$
\|x - x^*\| \le \frac{E\delta}{\left[1 - \left(1 + E\sum_{j=1}^{m} p_j\right)\left(b_1 T + \frac{1}{2}b_1 b_2 T^2\right)\right]} = \epsilon.
$$

Then the solution of the IVP (1)–(2) continuously depends on x_0. □

Continuous Dependence on the Nonlocal Data p_j

Definition 7.2 The solution $x \in C[0, 1]$ of the IVP (1)–(2) continuously depends on the nonlocal data p_j, if

$$
\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad |p_j - p_j^*| < \delta \Rightarrow ||x - x^*|| < \epsilon,
$$

where x^* is the solution of the IVP

$$\frac{dx^*}{dt} = g(t, x^*(t), \int_0^t f(s, x^*(s))ds), \quad a.e \quad t \in (0, 1), \tag{18}$$

with the nonlocal condition

$$x(0) + \sum_{j=1}^m p_j^* x^*(\tau_j) = x_0, \quad \sum_{j=1}^m p_j^* > 0, \quad \tau_j \in (0, 1). \tag{19}$$

Theorem 7.2 *Let the assumptions of Theorem 6.1 be satisfied. Then the solution of the IVP (1)–(2) continuously depends on the nonlocal data p_j.*

Proof Let $x,\ x^*$ be two the solutions of the IVP (1)–(2) and (18)–(19) respectively. Then

$$\begin{aligned}
|x(t) - x^*(t)| = {} & |E\left[x_0 - \sum_{j=1}^m p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds\right] \\
& + \int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds - E^*\left[x_0 - \sum_{j=1}^m p_j^* \int_0^{\tau_j} g(s, x^*(s),\right. \\
& \left.\int_0^s f(\theta, x^*(\theta))d\theta)ds\right] - \int_0^t g\left(s, x^*(s), \int_0^s f(s, x^*(\theta))d\theta\right) ds\Big| \\
\leq {} & EE^* m\delta|x_0| + |E^* \sum_{j=1}^m p_j^* \int_0^{\tau_j} g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) ds \\
& - E\sum_{j=1}^m p_j^* \int_0^{\tau_j} g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) ds \\
& + E\left[\sum_{j=1}^m p_j^* \int_0^{\tau_j} g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) ds\right. \\
& \left.- \sum_{j=1}^m p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds\right]\Big| \\
& + b_1 T\|x - x^*\| + \frac{1}{2}T^2 b_1 b_2 \|x - x^*\| \\
\leq {} & EE^* m\delta|x_0| + m\delta\left[\left(b_1 T + \frac{1}{2}b_1 b_2 T^2\right)\|x^*\| + TL_1 + \frac{1}{2}b_1 T^2 L_2\right]\sum_{j=1}^m p_j^* \\
& + E|\left[\sum_{j=1}^m p_j^* \int_0^{\tau_j} g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) ds\right. \\
& - \sum_{j=1}^m p_j \int_0^{\tau_j} g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) ds \\
& + \sum_{j=1}^m p_j \int_0^{\tau_j} g\left(s, x^*(s), \int_0^s f(\theta, x^*(\theta))d\theta\right) ds
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{m} p_j \int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right) ds \Bigg] |\\
&+ b_1 T\|x - x^*\| + \frac{1}{2}T^2 b_1 b_2 \|x - x^*\|\\
\le\ & EE^* m\delta|x_0| + m\delta\left[\left(b_1 T + \frac{1}{2}b_1 b_2 T^2\right)\|x^*\| + TL_1 + \frac{1}{2}b_1 T^2 L_2\right]\sum_{j=1}^{m} p_j^*\\
&+ E\Bigg[m\delta\left[\left(b_1 T + \frac{1}{2}b_1 b_2 T^2\right)\|x^*\| + TL_1 + \frac{1}{2}b_1 T^2 L_2\right]\\
&+\sum_{j=1}^{m} p_j \left(b_1 T\|x - x^*\| + \frac{1}{2}T^2 b_1 b_2 \|x - x^*\|\right)\Bigg]\\
&+ b_1 T\|x - x^*\| + \frac{1}{2}T^2 b_1 b_2 \|x - x^*\|\\
\le\ & EE^* m\delta|x_0| + m\delta\left[\left(b_1 T + \frac{1}{2}b_1 b_2 T^2\right)\|x^*\| + TL_1 + \frac{1}{2}b_1 T^2 L_2\right]\\
&\left(E + \sum_{j=1}^{m} p_j^*\right) + \left(1 + E\sum_{j=1}^{m} p_j\right)\left(b_1 T + \frac{1}{2}T^2 b_1 b_2\right)\|x - x^*\|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|x - x^*\|\\
&\le \frac{EE^* m|x_0| + m\left[\left(b_1 T + \frac{1}{2}b_1 b_2 T^2\right)\|x^*\| + TL_1 + \frac{1}{2}b_1 T^2 L_2\right]\left(E + \sum_{j=1}^{m} p_j^*\right)}{1 - \left(1 + E\sum_{j=1}^{m} p_j\right)\left(b_1 T + \frac{1}{2}T^2 b_1 b_2\right)}\delta = \epsilon,
\end{aligned}
$$

where $E^* = (1 + \sum_{j=1}^{m} p_j^*)^{-1}$. Then the solution of the IVP (1)–(2) continuously depends on the nonlocal data p_j. □

Continuous Dependence on the Functional *f*

Definition 7.3 The solution $x \in C[0, T]$ of the IVP (1)–(2) continuously depends on the functional f, if

$$
\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad |f - f^*| < \delta \Rightarrow ||x - x^*|| < \epsilon,
$$

where x^* is the solution of the IVP

$$
\frac{dx^*}{dt} = g\left(t, x^*(t), \int_0^t f^*(s, x^*(s))ds\right), \quad a.e \quad t \in (0, T], \tag{20}
$$

with the nonlocal condition

$$
x(0) + \sum_{j=1}^{m} p_j x^*(\tau_j) = x_0, \quad \sum_{j=1}^{m} p_j > 0, \quad \tau_j \in (0, T]. \tag{21}
$$

Theorem 7.3 *Let the assumptions of Theorem 6.1 be satisfied. Then the solution of the IVP (1)–(2) continuously depends on the functional f.*

Proof Let x, x^* be two solutions of the IVP (1)–(2) and (20)–(21) respectively. Then

$$
\begin{aligned}
|x(t)-x^*(t)| \le |, |E &\left[x_0-\sum_{j=1}^{m}p_j\int_0^{\tau_j} g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)ds\right]\\
&+\int_0^t g\left(s, x(s), \int_0^s f(\theta, x(\theta))d\theta\right)ds\\
&-E\left[x_0-\sum_{j=1}^{m}p_j\int_0^{\tau_j} g\left(s, x^*(s), \int_0^s f^*(\theta, x^*(\theta))d\theta\right)ds\right]\\
&-\int_0^t g\left(s, x^*(s), \int_0^s f^*(\theta, x^*(\theta))d\theta\right)ds\Big|\\
&\le E\sum_{j=1}^{m}p_j\int_0^{\tau_j}\Big|g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)\\
&-g\left(s, x^*(s), \int_0^s f^*(\theta, x^*(\theta))d\theta\right)\Big|ds\\
&+\int_0^t |g(s, x(s), \int_0^s f(\theta, x(\theta))d\theta)-g\left(s, x^*(s), \int_0^s f^*(\theta, x^*(\theta))d\theta\right)\Big|ds,\\
&\le E\sum_{j=1}^{m}p_j\int_0^{\tau_j}\left(b_1\|x-x^*\|+b_1\int_0^s|f^*(\theta, x^*(\theta))-f(\theta, x(\theta))|d\theta\right)ds\\
&+\int_0^t\left(b_1\|x-x^*\|+b_1\int_0^s|f(\theta, x(\theta))-f^*(\theta, x^*(\theta))|d\theta\right)ds\\
&\le b_1T\|x-x^*\Big\|E\sum_{j=1}^{m}p_j+\frac{1}{2}b_1T^2\delta E\sum_{j=1}^{m}p_j+\frac{1}{2}b_1b_2T^2\Big\|x-x^*\|E\sum_{j=1}^{m}p_j\\
&+\frac{1}{2}b_1T^2\delta+b_1T^2\|x-x^*\|+\frac{1}{2}b_1b_2T^2\|x-x^*\|\\
&\le\left(1+E\sum_{j=1}^{m}p_j\right)\frac{1}{2}b_1T^2\delta+\left(1+E\sum_{j=1}^{m}p_j\right)\left(b_1T+\frac{1}{2}b_1b_2T^2\right)\|x-x^*\|.
\end{aligned}
$$

Hence

$$
\|x-x^*\|\le\frac{\left(1+E\sum_{j=1}^{m}p_j\right)\frac{1}{2}b_1T^2\delta}{1-\left(1+E\sum_{j=1}^{m}p_j\right)\left(b_1T+\frac{1}{2}b_1b_2T^2\right)}=\epsilon.
$$

Then the solution of the IVP (1)–(2) continuously depends on the functional f. □

Examples

Example 8.1 Consider the nonlinear integro-differential equation

$$
\begin{aligned}
\frac{dx}{dt}&=t^4e^{-t}+\frac{\ln(1+x(t))}{4+t^3}\\
&+\int_0^t\frac{1}{9}\left(\sin(3s+3)+\frac{s^4\cos x(s)}{e^{|x(s)|}}\right)dt,\quad a.e\quad t\in(0,1],
\end{aligned}
\tag{22}
$$

with infinite point boundary condition

$$x(0)+\sum_{j=1}^{\infty}\frac{1}{j^2}x\left(\frac{j-1}{j}\right)=x_0. \tag{23}$$

Set

$$g\left(t,x(t),\int_0^t f(s,x(s))ds\right)=t^4e^{-t}+\frac{\ln(1+x(t))}{4+t^3}+\int_0^t\frac{1}{9}\left(\sin(3s+3)+\frac{s^4\cos x(s)}{e^{|x(s)|}}\right)dt.$$

Then

$$\left|g\left(t,x(t),\int_0^t f(s,x(s))ds\right)\right|\le t^4e^{-t}+\frac{1}{4}\left(|x|+\frac{1}{4}\int_0^t\frac{4}{9}\left|\left(\cos(3s+3)+\frac{s^4\cos x(s)}{e^{|x(s)|}}\right)dt\right|\right),$$

and also

$$|f(s,x(s))|\le\frac{4}{9}|\cos(3s+3)|+\frac{4}{9}|x(s)|.$$

The assumptions 1–4 of Theorem 3.1 are satisfied with $c_1(t)=t^3e^{-t}\in L^1[0,1]$, $c_2(t)=\frac{1}{2}|\cos(3t+3)|\in L^1[0,1]$, $b_1=\frac{1}{3}$, $b_2=\frac{4}{9}$, $\left(1+\frac{\sum_{j=1}^{\infty}\frac{1}{j^2}}{1+\sum_{j=1}^{\infty}\frac{1}{j^2}}\right)\left(b_1+\frac{1}{2}b_1b_2\right)=\left(1+\frac{\frac{\pi^2}{6}}{1+\frac{\pi^2}{6}}\right)\left(\frac{1}{3}+\frac{2}{27}\right)<1$, and the series: $\sum_{j=1}^{\infty}\frac{1}{j^2}$ is convergent. Therefore, by applying to Theorem 3.1, the given IVP (22)–(23) has a solution $x\in[0,1]$.

Example 8.2 Consider the nonlinear integro-differential equation

$$\frac{dx}{dt}=t^5+t^2+1+\frac{x(t)}{\sqrt{t+9}}+\int_0^t\frac{1}{4}\left(\sin^2(3s+3)+\frac{sx(s)}{2(1+x(s))}\right)dt,\quad a.e\quad t\in(0,1], \tag{24}$$

with infinite point boundary condition

$$x(0)+\sum_{j=1}^{\infty}\frac{1}{j^4}x\left(\frac{j^2+j-1}{j^2+j}\right)=x_0. \tag{25}$$

Set

$$g\left(t,x(t),\int_0^t f(s,x(s))ds\right)=t^5+t^2+1+\frac{x(t)}{\sqrt{t+9}}+\frac{1}{4}\int_0^t\left(\sin^2(3s+3)+\frac{sx(s)}{2(1+x(s))}\right)dt.$$

Then

$$\left|g(t, x(t), \int_0^t f(s, x(s))ds)\right| \leq t^5 + t^2 + 1 + \frac{1}{3}|x| + \frac{1}{3}\int_0^t \frac{3}{4}\left|\left(\sin^2(3s+3) + \frac{sx(s)}{2(1+x(s))}\right)\right| dt,$$

and also

$$|f(s, x(s))| \leq \frac{3}{4}\left|\left(\sin^2(3s+3)\right)\right| + \frac{3}{8}|x(s)|.$$

The assumptions 1–4 of Theorem 3.1 are satisfied with $c_1(t) = t^5 + t^2 + 1 \in L^1[0, 1]$, $c_2(t) = \frac{3}{4}|(\sin^2(3s+3)| \in L^1[0, 1]$, $b_1 = \frac{1}{3}$, $b_2 = \frac{3}{8}$, $\left(1 + \frac{\sum_{j=1}^{\infty} \frac{1}{j^4}}{1+\sum_{j=1}^{\infty} \frac{1}{j^4}}\right)\left(b_1 + \frac{1}{2}b_1 b_2\right) = \left(1 + \frac{\frac{\pi^4}{90}}{1+\frac{\pi^4}{90}}\right)\left(\frac{1}{3} + \frac{1}{16}\right) < 1$, and the series: $\sum_{j=1}^{\infty} \frac{1}{j^4}$ is convergent Therefore, by applying to Theorem 3.1, the given IVP (24)–(25) has a solution $x \in [0, 1]$.

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