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ORIGINAL PAPER

Numerical Investigation of the Time Fractional Mobile-Immobile Advection-Dispersion Model Arising from Solute Transport in Porous Media

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Abstract

Evolution equations containing fractional derivatives can offer efficient mathematical models for determination of anomalous diffusion and transport dynamics in multi-faceted systems that cannot be precisely modeled by using normal integer order equations. In recent times, researches have found out that lots of physical processes illustrate fractional order characteristics that alters with time or space. The continuum of order in the fractional calculus permits the order of the fractional operator be accounted for as a variable. In the current research work, radial basis functions (RBFs) approximation is utilized for solving fractional mobile-immobile advection-dispersion (TF-MIM-AD) model in a bounded domain which is applied for explaining solute transport in both porous and fractured media. In this approach, firstly, the discretization process of the aforesaid equation with of convergence order $\mathcal{O}(\delta t)$ in the t-direction is described via the finite difference scheme for $0 < \alpha < 1$. Afterwards, by help of the meshless methods based on RBFs, we will illustrate how to obtain the approximated solution. The stability and convergence of time-discretized scheme are also theoretically discussed in detail throughout the paper. Finally, two numerical instances are included to clarify effectiveness and accuracy of our proposed concepts which is investigated in the current research work.

Keywords TF-MIM-AD model · Radial basis functions · RBF-PS · Collocation methods · Stability · Convergence

Mathematics Subject Classification 35R11 · 65M70 · 91G60

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Introduction

Solute transport in rivers, groundwater and streams is controlled by the physical features or heterogeneity in different reaches. While the advection-dispersion equation and its extensions (e.g., the mobile-immobile (MIM) or transient storage models based on a second order dispersion term) have been successfully used in the past, recent research highlights the need for transport models that can better describe the heterogeneity and connectivity of spatial features within a general network perspective of solute transport. The MIM approach is based on a simple hypothesis: not all the opening spaces in a geologic medium contribute to universal flow. Based on a literature survey done on recent works of [1], the idea of the MIM has become famous across hydrologists for studying transport in saturant and unsaturated zones, and in granular as well as fractured media. The transport flow in porous medium is mainly controlled by the processes of advection and dispersion (ADE) [2] that will predict a breakthrough curve (BTC). A Gaussian distribution function from an a straightly releasing solute source can be defined for predicting a breakthrough curve (BTC).

Investigation of numerical models of solute transport is a fundamental aspect in parameter reconnaissance at both the field and micro scales. The ADE is naturally applied to characterize the motion of solute transport in porous media. However, increasing evidence indicates that the ADE model is difficult to explain transport in heterogeneous, fractured or even homogeneous media. As mentioned in [3], the breakthrough curves (BTCs) have been fitted by both the ADE and MIM models and results are shown that the MIM model does premier than ADE in both porous and fractured media, especially to explain the peaks and long tails of the BTCs. Because of such mentioned reason, the MIM model can describe the BTCs better in a tough walled fracture than in a smooth-walled fracture. The single-rate MIM model appears ADE transport in the mobile region, and transport in the immobile region only by diffusion, making physical non-equilibrium. We refer the interested reader to previous studies [1,3–8] for finding more details.

Most phenomena in scientific world are described through nonlinear fractional partial differential equations. Mathematical models of fractional have become extremely useful and important to model complex phenomena in the fields of physics, chemistry,acoustics, viscoelasticity, electromagnetics, biology, and engineering [9–14]. It is worth to point out that the main characteristic of fractional derivatives, or more precisely derivatives of positive real order, is so called the memory effect, that is, future state of a physical system depends on present as well as past states. It is well known that the state of many systems at a given time depends on their configuration at previous times. The fractional derivative takes into account this history in its definition as a convolution with a function whose amplitude decays at earlier times as a power-law. That is because of the fact that, the fractional derivative is natural to use when modeling dynamical or physical systems in various bioengineering applications such as frequency dependent damping behavior of materials, motion of a large thin plate in a Newtonian fluid, creep and relaxation functions for viscoelastic materials in the past several decades.

The fractional mobile-immobile advection-dispersion equation is a generalization of the classical mobile-immobile advection-dispersion equation in which the first-order derivative is replaced with a fractional-order derivative [15]. The fractional mobile-immobile advection-dispersion model for solute transport presumes power law waiting times in the immobile zone, resulting to a fractional time derivative in the model equations. The equations are equipollent to previous models of MIM transport with power law memory functions and are the limiting equations that govern continuous time random walks with heavy tailed random waiting times.

In this paper, our aim is to investigate the mobile-immobile advection-dispersion (MIM-AD) model based on fractional order derivatives. After that, we are dealing the numerical approximation of the following time fractional mobile-immobile advection-dispersion (TF-MIM-AD) model order α ($0 < \alpha \leq 1$):

$$\beta_1 \frac{\partial u(x,t)}{\partial t} + \beta_2 \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mu_1 \frac{\partial^2 u(x,t)}{\partial x^2} - \mu_2 \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad a \leq x \leq b, \quad 0 \leq t \leq T, \quad (1)$$

with intial condition

$$u(x,0) = g(x), \qquad a \leq x \leq b, \quad (2)$$

and the boundary conditions

$$u(a,t) = g_1(t), \quad (b,t) = g_2(t), \qquad t > 0, \quad (3)$$

where $u(x,t)$ represents the solute concentration, $f(x,t)$ is the source term a, b, α, $g(x)$, $g_1(t)$ and $g_2(t)$ are given and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ represents the Caputo fractional derivative [9,10] which can be defined as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t-\xi)^\alpha} d\xi, & 0 < \alpha < 1, \\ \frac{\partial u(x,t)}{\partial t} & \alpha = 1. \end{cases}$$

Also β_1, β_2, μ_1 and μ_2 are constant coefficients. Here a source term $f(x,t)$ is selected for the purposes of confirmation in Section "Examples and Discussions". To determine an exact solution of this problems is extremely difficult thus many researchers are attempting procedures to approximate these problems [16–23]. There exist some numerical algorithms for solving fractional mobile-immobile advection-dispersion model such as implicit and explicit difference method [24–27], spectral method [28,29].

The Literature Review of Meshless Methods

A meshless (meshfree) method is a numerical method used to establish a system of algebraic equations for the whole domain of the problem without using a predefined mesh for the domain (or boundary) discretization. We will make use of meshfree methods for our scattered data approximation because mesh generation is one of the most time consuming part of any mesh-based numerical simulation. The meshfree method gives an economical alternative to methods such as those using wavelets, multivariate splines, finite elements, finite difference and finite volume, where all require the connectivity of nodes. In the last decade in order to omit mesh construction, scientists have employed meshless methods. In such approach an assortment of scattered data were used as instead of constructing a meshing paradigm. One of the most prominent meshless method is RBF method that seems to be a very well-organized system while facing interpolation of multidimensional scattered data.

Radial basis function (RBF) approaches were originally studied by Roland Hardy, an Iowa State geodesist, in 1968. This method has been considered as one of the first efficient techniques for the data interpolation in scattered space. The main disadvantage of previously used polynomial methods is their unisolvency characteristics for two-dimensional and developed dimensional scattered data. After a decade of research, Hardy established the method which was subsequently be recognized as the multiquadric (MQ) radial basis function which is just one of numerous existing RBFs [30]. At that time, in 1979, based on research work done by Richard Franke, MQ RBF method was introduced to be the best known methods for scattered data interpolation [31]. In honor of Frankes vast applied experiments with the MQ, he is frequently recognized for presenting the MQ into the field of mathematical science [32]. The subsequent important occurrence in RBF history was in 1986 when, an IBM mathematician named as Charles Micchelli, explained and established the concept behind the MQ technique. The invertible mode of system matrix for the MQ method was proven, which remarks the affectivity, precision and reliability in RBF scattered data interpolation problems [33]. Some years later, the MQ method was first utilized to explain partial differential equations by physicist Edward Kansa [34]. In 1992, results extracted by Wolodymyr Madych and Stuart Nelson [35] indicated the spectral convergence rate of MQ interpolation. All RBF methods using an infinitely differentiable RBF have been proven to be comprehensive layout of the polynomial based pseudo-spectral methods [36].

The use of such novel method for mathematical solution of partial differential equation is based on the collocation method. It is (conditionally) positive definite, translationally and rotationally invariant. The chief benefits of this method are straightforward programming process and probable spectral precision. On the other hand, ill-conditioning of the resulting linear system is considered to be the main difficulty. RBF approaches that use infinitely differentiable origin functions that include a free factor are theoretically spectrally accurate. The well utilization of such RBF methods comprises development of a linear arrangement that is extremely ill-conditioned when the parameters of the method are in situation that the best accurateness is ideally comprehended. Consequently, in several applications, RBF methods does not possess the potential to produce exact results as they are skilled theoretically. Just contrary to mesh based approaches such as finite element method, finite difference method finite volume method and meshless methods use a set of accidental or uniform points which are not interlinked in the arrangement named as mesh.

The RBF method can be taken into account as a category of compromise between the finite element (FE) and the Pseudo-spectral (PS) methods. On the one hand, the RBF method is based on an expansion into basis functions that have a spatial location just similar to FE method. In this point of view, these basis functions can be grouped in a definite section to locally increase the accuracy of the method. On the other hand, the basic functions exerted in the RBF expansion are high-order functions that conventionally cover the whole domain like with the PS technique. It was remarked that RBFs converge to PS methods in their at radial function boundary, making RBFs a generalized formulation and methodology to PS methods, for scattered nodes and non-flat radial functions [37]. RBFs have quite a lot of rewards over PS methods: in spite of subscription of pliability in terms of the domain shape , they permit a local node refinement, an easy singularity-free generalization to N dimensions and a shape parameter letting user extend the solution space to regions outside of the polynomial space, particularly susceptible to the Runge phenomenon [38]. We refer the interested reader to [32,33,35]. for discussing the existence, uniqueness, and convergence of the RBFs approximation. Many detailed discussions have been carried out regarding meshless methods and their related applications for solving complex PDEs [39–45], fractional equations [46–54] and integral equations [55–57].

The Organization of the Current Paper

In this regard, our aim is to construct the idea of the interpolation by RBFs to approximate numerically the TF-MIM-AD model. The layout of current research is arranged as follows. In Section "Construction of the Proposed Methods", the aforementioned equation is estimated in the temporal direction based on a scheme of $\mathcal{O}(\delta t^{2-\alpha})$. In this section, we also approximated by using the meshless methods based on RBFs in the spatial variable. Section "Theoretical Analysis of Time Discrete Scheme" is dedicated to stability and convergence the the time discrete scheme and the error estimates of this method. Also, in this section we will indicate that time order of convergence scheme is $\mathcal{O}(\delta t)$. Two numerical examples are provided in "Examples and Discussions" section to confirm the convergence behavior and to show accuracy of the proposed methods. Finally, we end with a concise conclusion in Section "Conclusion" and some references are brought together at the end.

Construction of the Proposed Methods

In this section, we discuss how the meshless methods based on RBFs can be exploited to solve the TF-MIM-AD equations. Firstly, let $\{x_j = jh | j = 1, 2, 3, \ldots, N\}$ in the bounded interval $[a, b]$, where x_1, x_N are the boundary points and $t_n = n\delta t, n = 0, 1, 2, 3, \ldots, M$ be an equidistant partition of $[0, T]$, where $h = (b-a)/N$, $\delta t = T/M$ and $u^n(x_i) = u(x_i, t_n)$.

Finite Difference Time Discretization

The time fractional derivative term in Eq. (1) is simulated by using the finite difference scheme:

$$\begin{aligned}\frac{\partial^\alpha u(x, t_{n+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+1}} \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t_{n+1}-\xi)^\alpha} d\xi \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{k\delta t}^{(k+1)\delta t} \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t_{n+1}-\xi)^\alpha} d\xi \\ &\approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{k\delta t}^{(k+1)\delta t} \frac{\partial u(x,\xi_k)}{\partial \xi} \frac{1}{(t_{n+1}-\xi)^\alpha} d\xi. \end{aligned} \tag{4}$$

Now, the first order temporal derivative can be approximated by the forward difference formula:

$$\frac{\partial u(x,\xi_k)}{\partial \xi} = \frac{u(x, t_{k+1}) - u(x, t_k)}{\delta t} + R_1^{k+1}(x),$$

where $\xi_k \in [t_k, t_{k+1}]$. By making use of Taylor's Theorem the truncation error can be calculated as:

$$|R_1^{k+1}(x)| \le C_1 \delta t, \quad \text{or} \quad R_1^{k+1} = \mathcal{O}(\delta t).$$

Therefore, the relation (4) can be discretized by the following statements:

$$\frac{\partial^\alpha u(x, t_{n+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \left(\frac{u(x, t_{k+1}) - u(x, t_k)}{\delta t} + \mathcal{O}(\delta t) \right) \int_{k\delta t}^{(k+1)\delta t} \frac{1}{(t_{n+1}-\xi)^\alpha} d\xi$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \left(\frac{u(x,t_{k+1}) - u(x,t_k)}{\delta t} + \mathcal{O}(\delta t) \right) \int_{k\delta t}^{(k+1)\delta t} \frac{dr}{r^{\alpha}} \\
&= \begin{cases} \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)} (u^{n+1} - u^n) + \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(k+1)^{1-\alpha} - k^{1-\alpha} \right] (u^{n+1-k} - u^{n-k}) & n \geq 1 \\ \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)} (u^1 - u^0) & n = 0 \end{cases} \\
&= \begin{cases} a_0 \left[(u^{n+1} - u^n) + \sum_{k=1}^{n} b_k (u^{n+1-k} - u^{n-k}) \right] & n \geq 1 \\ a_0 (u^1 - u^0) & n = 0 \end{cases} + R_2^{k+1},
\end{aligned} \tag{5}
$$

where $a_0 = \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)}$, $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$, $(k = 0, 1, \ldots, n)$, $u^0 = u(x, t = 0) = g(x)$. Also the truncation error R_2^{k+1} satisfy

$$
|R_2^{k+1}(x)| \leq C_2 \delta t^{2-\alpha}, \quad \text{or} \quad R_2^{k+1} = \mathcal{O}(\delta t^{2-\alpha}).
$$

Substituting Eq. (5) into Eq. (1), the time derivative of the TF-MIM-AD equation is descritized taking into account the classic finite difference formula and space derivatives between successive two time steps n and $n+1$ as :

$$
\begin{aligned}
&(\beta_1 + a_0 \delta t \beta_2) u^{n+1} - (\mu_1 \delta t) \nabla^2 u^{n+1} + (\mu_2 \delta t) \nabla u^{n+1} \\
&= \begin{cases} (\beta_1 + \beta_2 a_0 \delta t) u^n - \beta_2 a_0 \delta t \sum_{k=1}^{n} b_k (u^{n+1-k} - u^{n-k}) + \delta t f^{n+1}, & n \geq 1, \\ (\beta_1 + \beta_2 a_0 \delta t) u^0 + \delta t f^1, & n = 0, \end{cases} + R^{k+1},
\end{aligned} \tag{6}
$$

in which ∇ is the gradient differential operator and $f^{n+1} = f(x, t_{n+1})$; $n = 0, 1, \ldots, M$. The truncation error can be obtained as follows:

$$
|R^{k+1}(x)| \leq \hat{C} \delta t^{1+\alpha}, \quad \text{or} \quad R^{k+1} = \mathcal{O}(\delta t^{1+\alpha}),
$$

where $\hat{C}$ is a positive constant. We will prove that convergence order of semi-discrete scheme is $\mathcal{O}(\delta t)$. By omitting the small term R^{k+1} and denoting U^k as the approximation of u^k, then we gain the approximate implicit discrete scheme as follows :

$$
\begin{aligned}
&(\beta_1 + a_0 \delta t \beta_2) U^{n+1} - \mu_1 \delta t \nabla^2 U^{n+1} + \mu_2 \delta t \nabla U^{n+1} \\
&= \begin{cases} (\beta_1 + \beta_2 a_0 \delta t) U^n - \beta_2 a_0 \delta t \sum_{k=1}^{n} b_k (U^{n+1-k} - U^{n-k}) + \delta t f^{n+1}, & n \geq 1, \\ (\beta_1 + \beta_2 a_0 \delta t) U^0 + \delta t f^1, & n = 0. \end{cases}
\end{aligned} \tag{7}
$$

Now, we will describe the meshless methods based on RBFs to approximate the spatial derivatives in the next two subsection in details.

Discretization in Space: RBF-Collocation Method

In this section, the spatial discretization scheme is presented by the Kansa method, as the domain-type meshless method, that is obtained by direct collocating the RBFs. The approximate expansion of $u(x_i, t_n)$ at a point of interest x_i is as follows:

$$U_i^{n+1} = U(x_i, t_{n+1}) \simeq \sum_{j=1}^{N} \lambda_j^{n+1} \phi(r_{ij}) + \lambda_{N+1}^{n+1} x_j + \lambda_{N+2}^{n+1}, \tag{8}$$

where $\{\lambda_j^n\}$ are unknown vector of the nth time layer $\phi(r_{ij})$ radial basis function, $r_{ij} = |x_i - x_j|$. Besides N equations resulting from collocating Eq. (8) at N points, we need an additional condition (extra 2 conditions) for the polynomial part to guarantee a unique solution of the N linear equations:

$$\sum_{j=1}^{N} \lambda_j^{n+1} = \sum_{j=1}^{N} \lambda_j^{n+1} x_j = 0. \tag{9}$$

Substituting Eq. (8) into Eq. (9) in a matrix form, it is to represent that

$$\{U\}^{n+1} = A\{\lambda\}^{n+1}, \tag{10}$$

where $\{U\}^{n+1} = [U_1^{n+1}, \ldots, U_N^{n+1}, 0, 0]^T$ and $\{\lambda\}^{n+1} = [\lambda_1^{n+1}, \ldots, \lambda_N^{n+1}]^T$ and the matrix $A = (a_{ij})_{(N+2)\times(N+2)}$ has entries:

$$A = \begin{bmatrix} \Phi & P_{N\times 2} \\ P^T & \mathbf{0}_{2\times 2} \end{bmatrix}$$

where $\Phi = [\phi(r_{ij})]_{N\times N}$ and $P = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix}_{N\times 2}$.

Reconstruction of Eq. (6) in the matrix form can be exhibited as follows:

$$B\{\lambda\}^1 = \{b\}^1, \tag{11}$$

in which

$$B = \begin{bmatrix} L(\Phi) & L(P) \\ P^T & \mathbf{0} \end{bmatrix}_{(N+2)\times(N+2)}, \tag{12}$$

where operator L is defined by

$$L(*) = \begin{cases} \left[\beta_1 + \beta_2 a_0 \delta t - (\mu_1 \delta t)\nabla^2 + (\mu_2 \delta t)\nabla\right](*), & 1 < i < N, \\ \\ (*), & i = 1 \text{ or } N, \end{cases} \tag{13}$$

and $\{b\}^1 = [b_1^1, \ldots, b_N^1, 0, 0]^T$ where $b_1^1 = g_1^1$, $b_N^2 = g_2^1$ and $b_i^1 = (\beta_2 a_0 \delta t + \beta_1) U_i^0 + \delta t f_i^1, \quad i = 2, 3, \ldots, N-1$.

Also, for $n \geq 1$

$$B\{\lambda\}^{n+1} = \{b\}^{n+1} \tag{14}$$

$\{b\}^{n+1} = [b_1^{n+1}, \ldots, b_N^{n+1}, 0, 0]^T$ are calculated by Eq. (6) as follows:

$$b_i^{n+1} = \begin{cases} g_1^{n+1} & i = 1, \\ (\beta_1 + \beta_2 a_0 \delta t) U_i^n - \beta_2 a_0 \delta t \sum_{k=1}^{n} b_k (U_i^{n+1-k} - U_i^{n-k}) + \delta t f^{n+1}, & 1 < i < N, \\ g_2^{n+1}, & i = N. \end{cases} \tag{15}$$

Posterior to obtaining the accurate answers for the algebraic system of equations $B\{\lambda\}^{n+1} = \{b\}^{n+1}$ at each time level, the solution can be determined using Eq. (10).

Discretization in Space: RBF-PS Meshless Method

Fasshauer [58] linked the RBFs collocation method to the pseudo-spectral (PS) method, known as RBF-PS method. Fasshauer used the RBF-PS method to approximate the Allen-Cahn equation, 2D Helmholtz equation and 2D Laplace equation with piecewise boundary conditions [59]. The authors of [60,61] used the RBF-PS method for analyzing beams, plates and shells problems. Authors of [62,63] exploited the RBF-PS method for composite and sandwich plates problems, and Marjan Uddin and co-workers [64,65] utilized RBF-PS method to solve some wave-type PDEs. Now, we developed the approach of [43,58] and used a numerical scheme for the Eq. (1). First of all, we review the properties of differentiation matrices (DM). Assume $\phi_j,\ \ j=1,2,\ldots,N$ be an arbitrary linearly independent set of smooth functions that will apply as the basis for our investigation space and $x=\{x_1,x_2,\ldots,x_N\}$ be a set of distinct points in $\Omega\subseteq\mathbb{R}^d$. We suppose that the approximate expansion is as follows:

$$u^h(x)=\sum_{j=1}^{N}\lambda_j\phi_j(x),\qquad x\in\mathbb{R} \tag{16}$$

where $h=h_{x,\Omega}:=\sup_{x\in\Omega}\min_{1\le j\le N}\|x-x_j\|_2$. Collocating Eq. (16) at the grid points x_i, we get

$$u^h(x_i)=\sum_{j=1}^{N}\lambda_j\phi_j(x_i),\quad i=1,2,\ldots,N, \tag{17}$$

then we gain the following matrix-vector form:

$$\mathbf{u}=\mathbf{A}\lambda, \tag{18}$$

where

$$\lambda=[\lambda_1,\lambda_2,\ldots,\lambda_N]^T,$$

and $\mathbf{A}$ is the evaluation matrix with entries $A_{ij}=\phi_j(x_i)$

$$\mathbf{u}=[u^h(x_1),u^h(x_2),\ldots,u^h(x_N)]^T.$$

We can obtain the derivative of u^h by differentiating the basis function in (16)

$$\frac{\partial u^h(x)}{\partial x}=\sum_{j=1}^{N}\lambda_j\frac{\partial\phi_j(x)}{\partial x}, \tag{19}$$

Now, collocating Eq. (19) at the grid points x_i in the form matrix, yields

$$\mathbf{u}_x=\mathbf{A}_x\lambda, \tag{20}$$

in which matrix $\mathbf{A}_x$ has entries $\frac{\partial\phi_j(x)}{\partial x}$. Hence, we require to ensure invertibility of the evaluation of matrix $\mathbf{A}$ for determining the differentiation matrix $\mathbf{D}$. This relies on both the basis functions selected and the location of the grid points x_i. According to Bochner's theorem, the invertibility of the matrix $\mathbf{A}$ for any set of distinct grid points x_i is insured by using the positive definite radial basis functions. Now, by using Eq. (18) one gets

$$\lambda=\mathbf{A}^{-1}\mathbf{u}.$$

Considering Eq. (20) and the above result, we then obtain

$$\mathbf{u}_x=\mathbf{A}_x\mathbf{A}^{-1}\mathbf{u}. \tag{21}$$

Now, the approximate solution can be rewritten as follows:

$$u^{n+1}(x_i) = \sum_{j=1}^{N} \lambda_j \Phi(r_{ij}), \quad i = 1, 2, \ldots, N, \tag{22}$$

Then, the matrix-vector from Eq. (22) is below as:

$$\mathbf{u}^{n+1} = \mathbf{A}\Lambda, \tag{23}$$

where

$$\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)^T, \quad \mathbf{u}^{n+1} = (u_1^{n+1}, u_2^{n+1}, \ldots, u_N^{n+1})^T.$$

The following matrix-vector form is achieved by differentiating Eq. (22) with respect to x and evaluating it at the grid points(x_i):

$$\mathbf{u}_{xx}^{n+1} = \mathbf{A}_{xx}\Lambda, \tag{24}$$

where

$$\mathbf{u}_{xx}^{n+1} = \left(\frac{\partial^2 u_1^{n+1}}{\partial x^2}, \frac{\partial^2 u_2^{n+1}}{\partial x^2}, \ldots, \frac{\partial^2 u_N^{n+1}}{\partial x^2} \right)^T,$$

and elements of matrix $\mathbf{A}_{xx}$ are $A_{xx,ij} = \frac{\partial^2 \phi(\|x_i - x_j\|)}{\partial x^2}$. Now, from Eq. (23) one obtains:

$$\Lambda = \mathbf{A}^{-1}\mathbf{u}^{n+1},$$

and Eq. (24) yields

$$\mathbf{u}_{xx}^{n+1} = \mathbf{A}_{xx}\mathbf{A}^{-1}\mathbf{u}^{n+1}. \tag{25}$$

Now, by substituting Eqs. (21) and (25) in the Eq. (7), we can write

$$(\beta_1 + a_0\delta t\beta_2)\mathbf{u}^{n+1} - \mu_1\delta t\mathbf{A}_{xx}\mathbf{A}^{-1}\mathbf{u}^{n+1} + \mu_2\delta t\mathbf{A}_x\mathbf{A}^{-1}\mathbf{u}^{n+1}$$
$$= \begin{cases} (\beta_2 a_0\delta t + \beta_1)\mathbf{u}^n - \beta_2 a_0\delta t \sum\limits_{k=1}^{n} b_k(\mathbf{u}^{n+1-k} - \mathbf{u}^{n-k}) + \delta t f^{n+1}, & n \geq 1, \\ (\beta_2 a_0\delta t + \beta_1)\mathbf{u}^0 + \delta t f^1, & n = 0. \end{cases}$$

Consequently, the above relation can be rewritten as

$$\mathbf{D}\mathbf{u}^{n+1} = \begin{cases} (\beta_2 a_0\delta t + \beta_1)\mathbf{u}^n - \beta_2 a_0\delta t \sum\limits_{k=1}^{n} b_k(\mathbf{u}^{n+1-k} - \mathbf{u}^{n-k}) + \delta t f^{n+1}, & n \geq 1, \\ (\beta_2 a_0\delta t + \beta_1)\mathbf{u}^0 + \delta t f^1, & n = 0, \end{cases} \tag{26}$$

in which

$$\mathbf{D} = (\beta_1 + a_0\delta t\beta_2)\mathbf{u}^{n+1} - \mu_1\delta t\mathbf{A}_{xx}\mathbf{A}^{-1}\mathbf{u}^{n+1} + \mu_2\delta t\mathbf{A}_x\mathbf{A}^{-1}\mathbf{u}^{n+1},$$

where I is the identity matrix. By solving this linear system we can obtain the numerical solution at each time levels.

Theoretical Analysis of Time Discrete Scheme

First of all, we introduce the following preliminary of functional analysis that are used for discretization of time variable.

An Overview Preliminary of Applied Functional Analysis

Let Ω demonstrate a bounded and open domain in $\mathbb{R}^2$ let dx be the Lebesgue measure on $\mathbb{R}^2$. For $p < \infty$, we define by $L^p(\Omega)$ the space of the measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that $\int_\Omega |u(x)|^p dx \leq \infty$. More generally, we can denote Banach space by the norm

$$||u||_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

The space $L^p(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(x)v(x)dx,$$

with the endowed norm in L^2,

$$||u||_2 = [(u, u)]^{\frac{1}{2}} = \left[\int_\Omega u(x)u(x)dx \right]^{\frac{1}{2}}.$$

Also we suppose that Ω is an open domain in $\mathbb{R}^d$, $\gamma = (\gamma_1, \ldots, \gamma_d)$ is a d- tuple of non-negative integers and $|\gamma| = \sum_{i=1}^{p} \gamma_i$. Accordingly, we put

$$D^\gamma v = \frac{\partial^{|\gamma|} v}{\partial x_1^\gamma \partial x_2^\gamma \cdots \partial x_d^\gamma}.$$

In this regard, one can obtain:

$$\begin{aligned} H^1(\Omega) &= \{v \in L^2(\Omega),\ \frac{dv}{dx} \in L^2(\Omega)\}, \\ H_0^1(\Omega) &= \{v \in H^1(\Omega),\ v|_{\partial(\Omega)} = 0\ \}, \\ H^m(\Omega) &= \{v \in L^2(\Omega), \quad D^\gamma v \in L^2(\Omega) \quad \text{for all positive integer } |\gamma| \leq m\}. \end{aligned}$$

Now, we present the definition of inner product in Hilbert space:

$$(u, v)_m = \sum_{|\gamma| \leq m} \int_\Omega D^\gamma u(x) D^\gamma v(x) dx,$$

which induces the norm

$$||u||_{H^m(\Omega)} = \left(\sum_{|\gamma| \leq m} ||D^\gamma u||^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

The Sobolev space $W^{1,p}(I)$ is said to be

$$W^{1,p}(I) = \{u \in L^p(I); \quad \exists g \in L^P(I) : \int_I u\varphi^{'} = \int_I g\varphi^{'}, \quad \forall \varphi \in C^1(I)\}.$$

Also, in this paper, we define the following inner product and the associated energy norms in L^2 and H^1

$$||v|| = (v, v)^{1/2}, \quad ||v||_1 = (v, v)_1^{1/2}, \qquad |v|_1 = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial x}\right)^{1/2}.$$

by inner products of $L^2(\Omega)$ and $H^1(\Omega)$

$$(u, v) = \int u(x)v(x)dx, \qquad (u, v)_1 = (u, v) + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right),$$

respectively.

Stability and Convergence

The purpose of the current section is to evaluate the stability and convergence behaviour of the proposed numerical solution. The Eq. (7) can be restated according to below expression:

$$(\beta_1\mu + \beta_2)U^{k+1} - \nu_1\nabla^2 U^{k+1} - \nu_2\nabla U^{k+1} = (\beta_1\mu + \beta_2(1 - b_1))U^k + \beta_2 \sum_{j=1}^{k}(b_j - b_{j+1})U^{k-j} + \beta_2 b_k U^0 + F^{k+1}, \tag{27}$$

where $\mu = \frac{\Gamma(2-\alpha)}{\delta t^{1-\alpha}}$, $\nu_1 = (\delta t\mu)\mu_1$, $\nu_2 = -(\delta t\mu)\mu_2$, $F = (\delta t\mu) f$. For the reader's convenience, we firstly mention the three following Lemmas for discretization of the time fractional derivative.

Lemma 1 (See [66,67].) *Let* $g(t) \in C^2[0, t_k]$ *and* $0 < \alpha < 1$ *then*

$$\left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{g(t)}{(x-t)^\alpha} dt - \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)} \left[(1 - b_0)g(t_k) + \sum_{j=1}^{k-1}(b_{k-j-1} - b_{k-j})g(t_j) + b_{k-1}g(t_0) \right] \right| \leq \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha}) \right] \max_{0\leq t\leq t_k} |g^{''}(t)|\delta t^{2-\alpha},$$

where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$.

Proof For the proof, see the references [66,67]. □

Lemma 2 *The sequence* b_j $(j = 0, 1, 2, \ldots)$ *of real numbers in the difference scheme defined by* (6) *satisfies the properties::*

- $b_0 = 1, b_j > 0,\ j = 0, 1, 2, \ldots, b_n \to 0\ \ as\ \ n \to \infty$;

- *we have*

$$b_j > b_{j+1}, \quad j = 0, 1, 2, \ldots;$$
$$\sum_{j=0}^{k-1}(b_{j+1} - b_j) + b_k = (1 - b_1) + \sum_{j=1}^{k-1}(b_{j+1} - b_j) + b_k = 1;$$

- *there exists a positive constant* $C > 0$ *such that:*

$$\delta t < C b_j \delta t^{\alpha}, \quad j = 0, 1, 2, \ldots,$$
$$\sum_{j=0}^{k} b_j \delta t^{\alpha} = (k+1)^{\alpha} \delta t^{\alpha} \le T^{\alpha}.$$

Proof It can be verified as obvious from the definition $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, where $0 < \alpha < 1$. Here we will not cover the detail. □

Lemma 3 *If* $U^k(x) \in H^1(\Omega)$ $k = 0, 1, \ldots, M$ *is the solution of time-discrete scheme* (27), *then*

$$\|U^k\| \le \|U^0\| + \beta_1^{-1} b_{k-1}^{-1} \max_{0 \le l \le M} ||F^l||,$$

Proof To prove the inequality, we use principle of mathematical induction on k as the counter of induction. When $k = 0$, we get

$$(\beta_1 + \mu\beta_2)U^1 = \nu_1 \nabla^2 U^1 + \nu_2 \nabla U^1 + (\beta_1 + \mu\beta_2)U^0 + F^0. \tag{28}$$

Multiplying above equation by U^1 and integrating on Ω, one can obtain:

$$(\beta_1 + \mu\beta_2)||U^1||^2 - \nu_1(\nabla^2 U^1, U^1) - \nu_2(\nabla U^1, U^1) = (\beta_1 + \mu\beta_2)(U^0, U^1) + (F^1, U^0)\ .$$

Employing the Cauchy–Schwarz inequality and $U^k(x) \in H^1(\Omega)$, it can be deduced that:

$$(\beta_1 + \mu\beta_2)||U^1|| \le (\beta_1 + \mu\beta_2)|U^0|| + ||F^1|| \le (\beta_1 + \mu\beta_2)||U^0|| + \max_{0 \le l \le M} ||F^l||,$$

notice that $\frac{1}{\beta_1\mu + \beta_2} \le \frac{1}{\beta_2}$ $(\beta_1 \ge 0,\ \beta_2 > 0)$ yields

$$||U^1|| \le ||U^0|| + \beta_1^{-1} b_0^{-1} \max_{0 \le l \le M} ||F^l||, \tag{29}$$

which is obviously true. Suppose that the theorem is true for all j

$$||U^j|| \le ||U^0|| + \beta_1^{-1} b_{j-1}^{-1} \max_{0 \le l \le M} ||F^l||, \quad j = 1, \ldots, k. \tag{30}$$

Multiplying Eq. (27) by U^{k+1} and integrating on Ω, we get the following equation

$$\begin{aligned}
&(\beta_1 + \mu\beta_2)||U^{k+1}||^2 - \nu_1(\nabla^2 U^{k+1}, U^{k+1}) - \nu_2(\nabla U^{k+1}, U^{k+1}) \\
&\quad = (\beta_1\mu + \beta_2(1 - b_1))(U^k, U^{k+1}) + \\
&\beta_2 \sum_{j=1}^{k}(b_j - b_{j+1})U^{k-j} + \beta_2 b_k(U^0, U^{k+1}) + (F^{k+1}, U^{k+1}).
\end{aligned}$$

From $U^k(x) \in H^1(\Omega)$, $b_{j+1} < b_j < 1$ and taking into account the Cauchy–Schwarz inequality, it follows that

$$(\beta_1 + \mu\beta_2)||U^{k+1}|| \leq (\beta_1\mu + \beta_2(1-b_1))||U^k|| + \beta_2 \sum_{j=1}^{k} (b_j - b_{j+1})||U^{k-j}|| + \beta_2 b_k ||U^0|| + ||F^{k+1}|| \,. \tag{31}$$

It should be noticed that in view of Eq. (30)

$$||U^j|| \leq ||U^j|| + \beta_1^{-1} b_{j-1}^{-1} \max_{0\leq l\leq M} ||F^l|| \leq ||U^j|| + \beta_1^{-1} b_j^{-1} \max_{0\leq l\leq M} ||F^l||. \tag{32}$$

Noting Lemma 2, we get $b_j < b_i < 1;\ \ 1 \leq i \leq j$ and easily it results

$$\begin{aligned}(1-b_1)||U^k|| + \sum_{j=1}^{k}(b_j - b_{j+1})||U^{k-j}|| &= \sum_{j=0}^{k-1}(b_j - b_{j+1})||U^{k-j}|| \\ &\leq \sum_{j=0}^{k-1}(b_j - b_{j+1})\left[||U^0|| + \beta_1^{-1} b_{k-j-1}^{-1} \max_{0\leq l\leq M} ||F^l||\right] \\ &\leq (1-b_k)||U^0|| + \beta_1^{-1}(1-b_k) b_k^{-1} \max_{0\leq l\leq M} ||F^l|| \\ &= (1-b_k)||U^0|| + \beta_1^{-1}(b_k^{-1} - 1) \max_{0\leq l\leq M} ||F^l||. \end{aligned} \tag{33}$$

In view of the expressions (31)–(33), one obtains :

$$||U^{k+1}|| \leq ||U^0|| + \beta_1^{-1} b_k^{-1} \max_{0\leq l\leq M} ||F^l||,$$

which concludes the proof of Lemma 3. □

Theorem 1 *The time -discrete numerical approach defined by Eq. (27) is un-conditionally stable.*

Proof We assume that $\widehat{U}^k(x),\ k = 1, \ldots, M$ is the solution of the approach (27) with the initial condition $\widehat{U}^0 = u(x, 0)$, then the error $\varepsilon^k = U^k(x) - \widehat{U}^k(x)$ satisfies

$$(\beta_1 + \mu\beta_2)\varepsilon^{k+1} - \nu_1 \nabla^2 \varepsilon^{k+1} + \nu_2 \nabla \varepsilon^{k+1} = (\beta_1\mu + \beta_2(1-b_1)\varepsilon^k + \beta_1 \sum_{j=1}^{k}(b_j - b_{j+1})\varepsilon^{k-j} + b_k\varepsilon^0 + F^{k+1},$$

and $\varepsilon^{k+1}|_{\partial\Omega} = 0$. In view of Lemma 3 and above-mentioned equation, the following inequality is obtained:

$$\|\varepsilon^k\| \leq \|\varepsilon^0\|, \qquad k = 1, \ldots, M.$$

This completes the proof of Theorem 1. □

Theorem 2 *Assume that $\{u(x, t_k)\}_{k=1}^{M}$ is the exact solution of Eqs. (1)–(3) and $\{U^k(x)\}_{k=1}^{M}$ be the time-discrete solution of Eq. (27) with initial condition $U^0(x) = u(x, 0)$. Then we get the following error estimation*

$$||u(x, t_k) - U^k(x)|| \leq C\delta t,$$

where C is a positive constant.

Proof Suppose the error term is defined as $\zeta^k = u(x, t_k) - U^k(x)$ at $t = t_k$; $k = 1, 2, \ldots, M$. From Eqs. (7) and (6), we simply write the following round-off error equation

$$(\beta_1 + \mu\beta_2)\zeta^{k+1} - \nu_1\nabla^2\zeta^{k+1} + \nu_2\nabla\zeta^{k+1} = (\beta_1 + \mu\beta_2)\zeta^k - \beta_2\sum_{j=1}^{k} b_j(\zeta^{k+1-j} - \zeta^{k-j}) + R^{k+1},$$

and $\zeta^0(x) = 0$, $\zeta^0(x)|_{\partial\Omega} = 0$. Returning again to Lemma 3 leads to

$$||\zeta^k||_2 \le b_{k-1}^{-1} \max_{0\le l\le M} ||R^l|| \le b_{k-1}^{-1}\beta_1^{-1}\delta t^{\alpha+1}.$$

Since $b_{k-1}^{-1}\beta_1^{-1}\delta t^{\alpha}$ is bounded [68], it holds that

$$||\zeta^k|| = ||u(x, t_k) - U^k(x)|| \le C\delta t,$$

and the proof of Theorem 2 is finished. In the next section, we will evaluate the convergence order in the time approximation by some numerical experiments. □

Examples and Discussions

To demonstrate the effectiveness of our approach, we current the numerical results of the suggested approach on two test problems in this section. In our computation, we applied the RBFs based on multiquadric (MQ) $\sqrt{c^2 + r^2}$ where c is MQ shape parameter. It should be noted that generally the choice of the optimal shape parameter in RBFs is still an open problem. Determination of suitable shape parameter is extracted experimentally for the each types of RBFs. In our experiments the optimal value of c is to be found numerically for the each time step separately. We obtain the accurateness and stability of the methods defined in this paper for different values of h, δt and c. To show the accuracy of method, we calculate the following error norm:

$$L_\infty = \max_{1\le j\le N-1} |U(x_j, T) - u(x_j, T)|.$$

The computational orders (denoted by C_1-order and C_2-order) in time variable and in space variable respectively can be computed as below

$$C_1 - \text{order} = \log_2\left(\frac{||L_\infty(2\delta t, h)||}{||L_\infty(\delta t, h)||}\right),$$
$$C_2 - \text{order} = \log_2\left(\frac{||L_\infty(16\delta t, 2h)||}{||L_\infty(\delta t, h)||}\right).$$

Matlab programming has been used for calculation in this paper.

Example 1 Let us consider the time fractional mobile-immobile advection-dispersion equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mu_1\frac{\partial^2 u(x,t)}{\partial x^2} - \mu_2\frac{\partial u(x,t)}{\partial x} + f(x,t), \qquad x \in [0, 1], \quad 0 \le t \le T,$$

Table 1 Time order of convergence for Eample 1 using MQ-RBF with $\gamma = 0.1$ and $h = 0.1$

δt	c	$\alpha = 0.4$		$\alpha = 0.9$	
		L_∞	C_1-order	L_∞	C_1-order
1/10	0.50	1.452×10^{-2}	–	1.596×10^{-2}	–
1/20	0.50	7.426×10^{-3}	0.9673	8.126×10^{-3}	0.9736
1/40	0.65	3.753×10^{-3}	0.9884	4.106×10^{-3}	0.9852
1/80	0.90	1.872×10^{-3}	1.0035	2.098×10^{-3}	0.9895
1/160	0.90	8.759×10^{-4}	1.0951	9.726×10^{-4}	1.0833
1/320	1.0	4.073×10^{-4}	1.1050	4.567×10^{-4}	1.0906
TCO		1			1

Table 2 Space order of convergence for Example 1 using MQ-RBF with $\gamma = 1/30$ and $c = 0.5$

h	δt	$\alpha = 0.2$		$\alpha = 0.6$	
		L_∞	C_2-order	L_∞	C_2-order
1/4	1/4	9.816×10^{-2}	–	2.381×10^{-2}	–
1/8	1/64	6.034×10^{-3}	4.0239	1.052×10^{-3}	4.5000
1/16	1/1024	2.823×10^{-4}	4.4178	4.826×10^{-5}	4.4462
1/8	1/8	1.028×10^{-2}	–	1.854×10^{-2}	–
1/16	1/128	4.852×10^{-3}	4.4051	9.073×10^{-4}	4.3529

where $f(x,t) = \exp(\frac{-(x-0.5)^2}{\gamma})\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 1 - \frac{2(x-0.5)}{\gamma} + \frac{2}{\gamma} - \frac{4(x-0.5)}{\gamma^2}\right]$, and analytic solution is $u(x,t) = t\exp\left(\frac{-(x-0.5)^2}{\gamma}\right)$ which is a Gaussian distribution solution with t height centered at $x = 0.5$ [26].

The initial and boundary conditions can be achieved from the exact solution. This example is solved by using the method introduced in this paper with various values of h, δt, for $a = 0$, $b = 1$, c at final time $T = 1$. The L_∞-error, C_1-order and C_2-order of applied method are shown in Tables 1 and 2, respectively. Computational orders in Table 1 verify the first order of accuracy in the temporal variable. Table 1 shows that the proposed numerical approach gives results with good accuracy. Also, in view of Table 2, we conclude that the convergence order of our proposed numerical approach in space is good agreement with [27]. In view of Table 3, the error achieved by RBF collocation technique is relatively close to the error achieved by RBF-PS collocation method but the coefficient matrix RBF-PS collocation approach is more well-posed than the coefficient matrix of the RBF collocation method. It is worth to mention that "Cond(M)" denotes the coefficient matrix of the presented techniques. Figure 1 shows plots of approximate solution and error for different values of γ using MQ-RBF scheme with parameters $h = 0.01$, $\delta t = 0.01$, $\alpha = 0.65$ and $c = 0.5$. Figure 2 displays the approximation solution of Example 1 using MQ with $c = 0.5$.

The initial and boundary conditions can be achieved from the analytic solution. This example is solved by help of the method exhibited in this paper with various values of h, δt, for $a = 0$ $b = 1$, c at final time $T = 1$. The L_∞-error, C_1-order and C_2-order of applied method are shown in Tables 4 and 5, respectively. Computational orders in Table 4 verify the first order of accuracy in the time variable. Table 4 indicates that the proposed numerical

Table 3 Errors and condition number evaluated for present methods with $\gamma = 0.1$, $\delta t = 1/100$ and $\alpha = 0.9$

h	MQ-RBF($c = 1$)		MQ-RBF-PS ($c = 1$)	
	L_∞	Cond (M)	L_∞	Cond (M)
1/5	3.174×10^{-3}	7.564×10^{2}	3.174×10^{-3}	1.00532
1/10	6.180×10^{-4}	4.330×10^{7}	6.182×10^{-4}	1.03468
1/15	4.361×10^{-4}	6.090×10^{9}	4.362×10^{-4}	1.09272
1/20	3.597×10^{-4}	9.146×10^{12}	3.595×10^{-4}	1.18542
1/25	2.401×10^{-4}	1.272×10^{16}	2.401×10^{-4}	1.32217

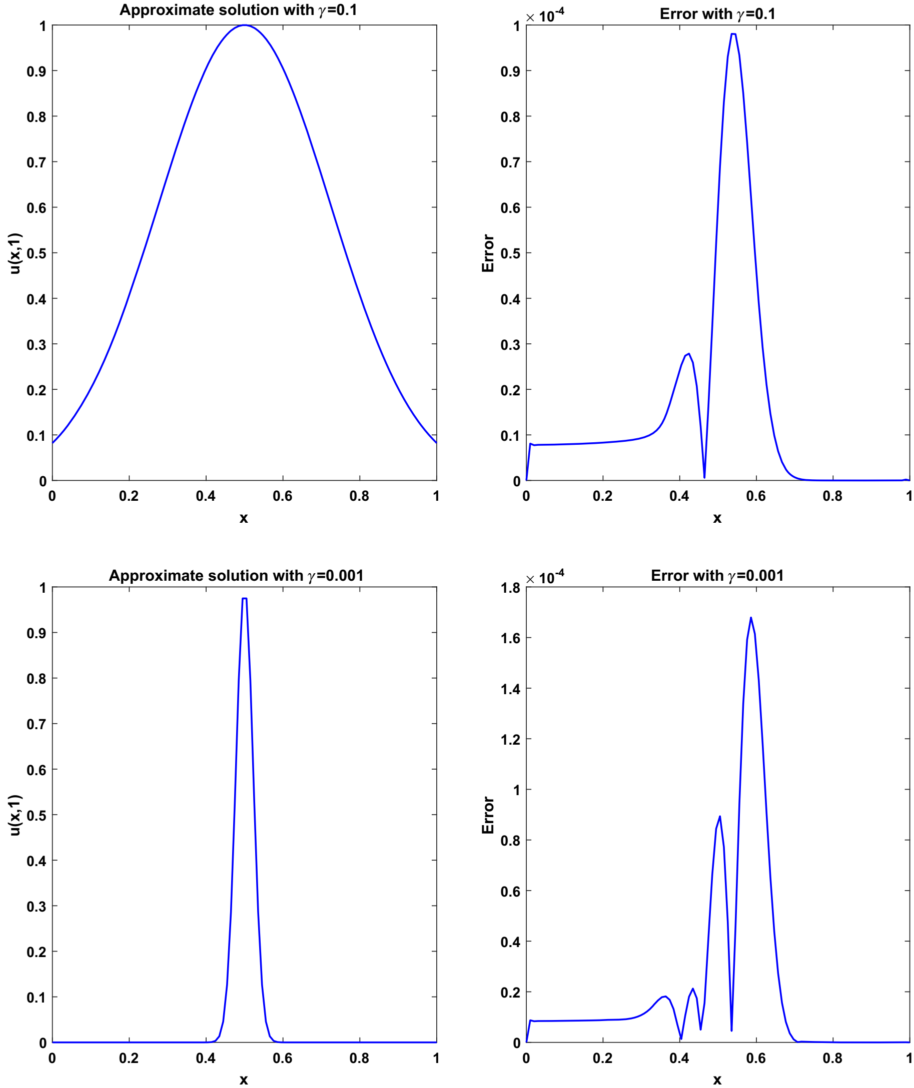

Fig. 1 Graphs of approximation solution and errors obtained for different values of γ using MQ-RBF scheme

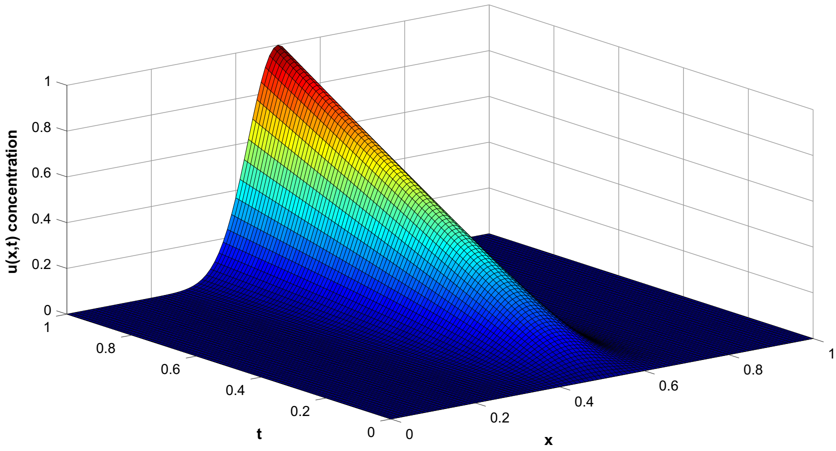

Fig. 2 Surface plot of numerical solution of numerical solution of Example 1 with $\gamma = 0.01$, $\alpha = 0.7$, $\delta t = 0.001$ and $h = 0.01$

Table 4 Time order of convergence for Example 2 using MQ-RBF and $h = 1/10$

δt	c	$\alpha = 0.4$		$\alpha = 0.9$	
		L_∞	C_1-order	L_∞	C_1-order
1/10	0.50	1.253×10^{-2}	–	1.053×10^{-2}	–
1/20	0.65	6.318×10^{-3}	0.9878	5.242×10^{-3}	1.0063
1/40	0.55	3.016×10^{-3}	1.0668	2.549×10^{-3}	1.0402
1/80	0.55	1.423×10^{-3}	1.0837	1.237×10^{-3}	1.0431
1/160	0.95	7.019×10^{-4}	1.0196	6.025×10^{-4}	1.0378
1/320	0.90	3.364×10^{-4}	1.0611	2.963×10^{-4}	1.0239
TCO		1			1

scheme produces results with good accuracy. Also, in view of Table 5, we conclude that the convergence order of our proposed numerical approach in space is good agreement with [27]. Table 6 shows that the results of this paper are better in comparison with the results of [26]. In view of Table 7, we can observe clearly that the coefficient matrix of RBF-PS collocation technique is more well conditioned than the coefficient matrices of the RBF collocation scheme. Figure 3 shows graphs of approximate solution and error using MQ-RBF scheme with parameters $h = 0.01$, $\delta t = 0.01$, $\alpha = 0.5$ and $c = 0.5$, which demonstrates that the numerical results agree well with the exact solution. Figure 4 displays the approximation solution and error of Example 2 using MQ with $c = 0.5$.

Example 2 Now, we consider the time fractional mobile-immobile advection-dispersion equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mu_1 \frac{\partial^2 u(x,t)}{\partial x^2} - \mu_2 \frac{\partial u(x,t)}{\partial x} + f(x,t), \qquad x \in [0,1], \quad 0 \le t \le T,$$

Table 5 Space order of convergence for Example 1 using MQ-RBF and $c = 0.65$

h	δt	$\alpha = 0.1$		$\alpha = 0.8$	
		L_∞	C_2-order	L_∞	C_2-order
1/4	1/4	8.716×10^{-2}	–	6.318×10^{-2}	–
1/8	1/64	5.194×10^{-3}	4.0687	2.826×10^{-3}	4.4826
1/16	1/1024	2.065×10^{-4}	4.6526	1.143×10^{-4}	4.6279
1/8	1/8	2.318×10^{-2}	–	3.465×10^{-2}	–
1/16	1/128	1.302×10^{-3}	4.1541	1.852×10^{-3}	4.2257

Table 6 Comparison of numerical solutions and obtained errors with $h = \delta t = 0.01$ for Example 2

x	Exact solution	Method of [26]		Our method	
		Numerical solution	L_∞	Numerical solution	L_∞
0.1	0.1620000	0.16184371	1.56290×10^{-4}	0.161999	2.4791×10^{-7}
0.2	0.5120000	0.51059931	1.40069×10^{-3}	0.511999	6.2301×10^{-6}
0.3	0.8820000	0.87902481	2.97519×10^{-3}	0.881999	1.5352×10^{-6}
0.4	1.1520000	1.14770234	4.29766×10^{-3}	1.151999	8.8700×10^{-6}
0.5	1.2500000	1.24502781	4.97219×10^{-3}	1.249999	6.2301×10^{-6}
0.6	1.1520000	1.14719659	4.80341×10^{-3}	1.151999	2.4792×10^{-6}
0.7	0.8820000	0.87818473	3.81527×10^{-3}	0.881999	9.6736×10^{-6}
0.8	0.5120000	0.50972531	2.27469×10^{-3}	0.511999	3.7014×10^{-6}
0.9	0.1620000	0.16127920	7.20750×10^{-4}	0.161999	1.3887×10^{-7}

Table 7 Errors and condition number evaluated for present methods with $\delta t = 0.01$

h	MQ-RBF($c = 0.5$)		MQ-RBF-PS ($c = 0.5$)	
	L_∞	Cond (M)	L_∞	Cond (M)
1/5	3.254×10^{-3}	8.195×10^{2}	3.253×10^{-3}	1.22629
1/10	8.371×10^{-4}	3.396×10^{6}	8.371×10^{-4}	1.03451
1/15	6.041×10^{-4}	6.137×10^{9}	6.042×10^{-4}	1.09225
1/20	4.519×10^{-4}	9.204×10^{12}	4.518×10^{-4}	1.18444
1/25	3.427×10^{-4}	1.284×10^{16}	3.427×10^{-4}	1.43034

where

$$f(x,t) = 10x^2(1-x)^2\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) + 10(t+1)(-2 + 14x - 18x^2 + 4x^3).$$

In this case the analytic solution is as follows $u(x,t) = 10(1+t)x^2(1-x)^2$ [26].

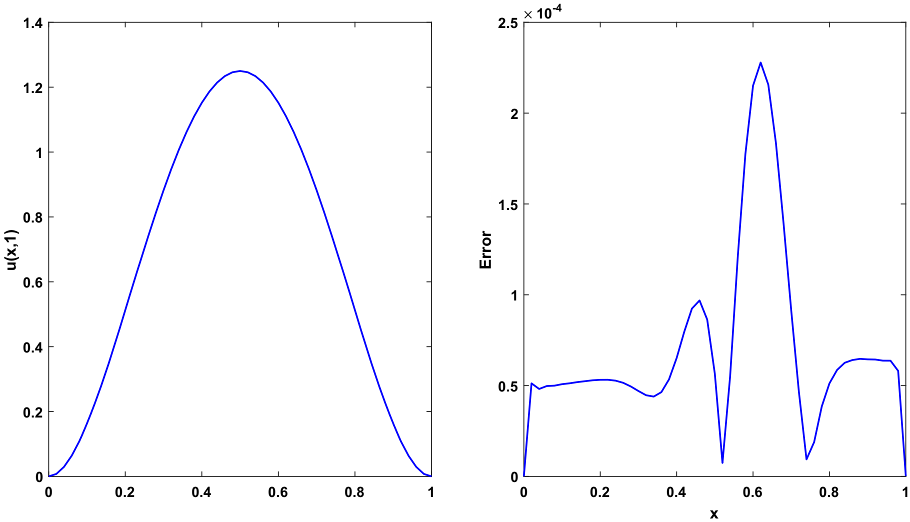

Fig. 3 Graphs of approximate solution and error obtained of Example 2 with $\alpha = 0.5$, $\delta t = 0.01$ and $h = 0.01$

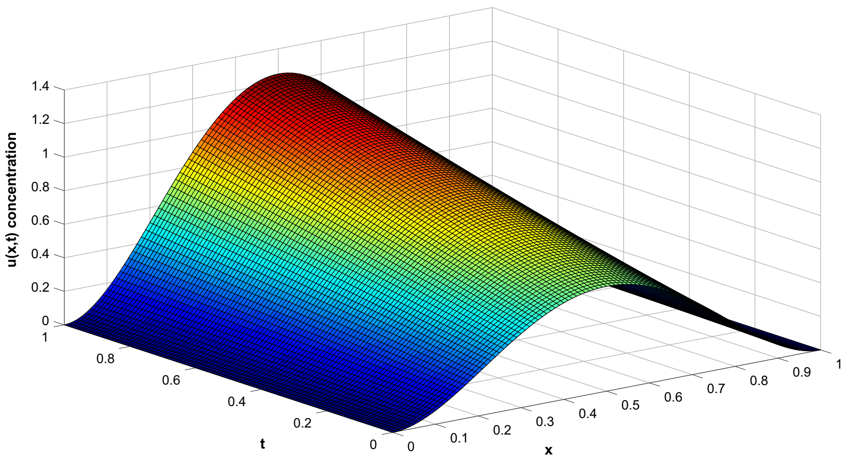

Fig. 4 Surface plot of numerical solution of Example 2 with $\alpha = 0.5$, $\delta t = 0.001$ and $h = 0.01$

Conclusion

In the research, an attempt was made to extend a meshless approach based on RBF collocation method for numerical solution of time fractional mobile-immobile advection-dispersion models, which is a category of fractional partial differential equation (FPDE). Also, we assessed the stability and also convergence of the proposed meshless approach hypothetically and mathematically. Two numerical instances with different problem domains are utilized to investigate the developed meshless model effectiveness and accuracy. As can be inferred from mentioned investigations, the convergence order of this current approach concerning to time is $\mathcal{O}(\delta t)$. We would like to mention that well-known RBF-PS technique is nothing else that a generalized finite difference scheme and also the obtained numerical results using RBFs collocation approach and RBF-PS method are equal but the condition number of the coefficient matrix of RBF-PS technique is less than the condition number of the coefficient matrix of RBF collocation method. All in all, the current meshless formulation is very operative to model and simulate of fractional differential equations, and it has well prospective to advance a robust simulation tool for models in science and engineering which are appeared by the numerous kinds of fractional differential equations. In the future studies, we will focus on problems with much more complexity.

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