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ORIGINAL PAPER

Numerical Solution of Nonlinear Second Order Singular BVPs Based on Green's Functions and Fixed-Point Iterative Schemes

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Abstract
This article discusses a numerical iterative scheme for the solution of a class of nonlinear singular boundary value problems. It introduces a recent approach, based on Green's functions and Picard's and Mann's fixed-point iterations procedures, to tackle such problems. The convergence analysis of the proposed method is presented to verify its efficiency. A number of examples are given to demonstrate the applicability of the method. The numerical experiments show that this approach is better than many other existing techniques and that it is reliable, accurate and less time consuming.

Keywords Nonlinear singular boundary value problems · Green's function · Picard's and Mann's iterative scheme · Fixed point

Introduction

Nonlinear singular boundary value problems (SBVPs) have been studied by many mathematicians, physicists and engineers. They used different methods in order to achieve the most accurate numerical solutions and that require the least CPU time. In recent years, a wide spectrum of papers have been devoted to solve such problems. For instance, Motsa and Sibanda [25] presented a novel approach to solve nonlinear SBVP arising in physiology for the study of tumour growth. They used successive linearization method (SLM) and compared their numerical results to those obtained by other methods such as ordinary cubic spline method [16], finite differences (see Pandey and Singh [29] and the references therein), Adomian decomposition method (ADM) [33], third degree B-spline [3], non-polynomial cubic splines [20], and cubic B-spline collocation [16]. Moreover, other papers proposed alternate computational methods based on Bernstein polynomials, via the transformation of the original problem to an eigenvalue problem then applying an open domain MATLAB collocation code "bvpsuite" to solve the nonlinear SBVPs [30]. In [33], Singh and Kumar used a new technique based on Green's function and the Adomian decomposition method (ADM) for solving nonlinear

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singular boundary value problems (SBVPs). In [27] Niu et al. used a simplified reproducing kernel method and least squares approach for solving nonlinear singular boundary value problems. Other techniques include piecewise shooting reproducing kernel method [7,8], mixed decomposition-spline approach [22], variational iteration method [15,34], topological techniques [6], Padé approximation and collocation methods [1], a fourth order method [4], and other novel numerical methods such as those in [2,5,10,11,17–20,23,24,31,32].

Some applications of nonlinear singular boundary value problems (SBVPs) for ordinary differential equations arise in many branches of applied mathematics, engineering such as chemical reactions, sciences such as nuclear physics and many others. For instance, it arises in the theory of electro-hydrodynamics and in the radial stress on a rotationally symmetric shallow membrane cap. In addition, it describes the equilibrium of the isothermal gas sphere and finds the distribution of heat sources in the human head. Last but not least, it has application for finding the steady-state oxygen diffusion in a spherical cell (see [9] and [33] and the references therein).

In this paper, a recently introduced iterative method based on Green's functions and fixed-point iteration schemes, such as Picard's and Mann's procedures, is presented for the approximate solution of a generalized class of nonlinear SBVPs (see [13,14,18,21,22] and the references therein). Five examples are considered and the results are compared with other numerical methods. The objective is to show that the iterative procedure yields relatively highly accurate approximate solutions and converges rapidly. Proof of convergence as well as rate of convergence are also included in our study.

An outline of the paper is as follows. To begin with, we will present the definition and construction of the Green's function and then designate the fixed-point iteration method. A proof of convergence of the scheme as well as its rate of convergence will be included. Moving on, we will investigate five different nonlinear SVBPs to show the efficiency and high accuracy of the method. Finally, we will summarize our findings.

Description of the Iteration Method

Green's Function

To construct the Green's function for certain SBVPs, consider first the following linear second order equation:

$$L[u] = u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \tag{1}$$

for $a < x < b$ with boundary conditions

$$\begin{aligned} B_a[u] &= a_0 u(a) + a_1 u'(a) = \alpha, \\ B_b[u] &= b_0 u(b) + b_1 u'(b) = \beta. \end{aligned} \tag{2}$$

The general solution is given by $u = u_h + u_p$ where u_h is the solution to $L[u] = 0$ subject to the boundary conditions (2), and u_p is the solution to $L[u] = f(x)$ satisfying the corresponding homogeneous boundary conditions

$$B_a[u] = B_b[u] = 0. \tag{3}$$

To find u_p, we first seek a solution for

$$L[u] = \delta(x - s), \tag{4}$$

subject to the conditions (3); this solution is referred to as the Green's function $G(x|s)$. Then

$$u_p = \int_a^b G(x|s)\ f(s)\ ds. \tag{5}$$

Let u_1, u_2 be two linearly independent solutions of $L[u] = 0$. The Green's function satisfies the homogeneous equation for $x \neq s$ and hence will be a linear combination of the solutions u_1, u_2 :

$$G(x|s) = \begin{cases} c_1u_1(x) + c_2u_2(x), & a < x < s \\ d_1u_1(x) + d_2u_2(x), & s < x < b \end{cases}.$$

The constants c_1, c_2, d_1, d_2 are determined using the following conditions:

(i) Homogeneous BCs:

$$B_a[G(x|s)] = B_b[G(x|s)] = 0.$$

(ii) Continuity of G at $x = s$:

$$c_1u_1(s) + c_2u_2(s) = d_1u_1(s) + d_2u_2(s).$$

(iii) Jump discontinuity of G' at $x = s$:

$$d_1u_1'(s) + d_2u_2'(s) - c_1u_1'(s) - c_2u_2'(s) = 1.$$

For nonlinear SBVPs

$$u''(x) + p(x)u'(x) + q(x)u(x) = f(x, u(x), u'(x)), \tag{6}$$

the particular solution satisfies

$$u_p = \int_a^b G(x|s)\ f(s, u_p(s), u_p'(s))\ ds, \tag{7}$$

where G is the Green's function corresponding to (6).

Picard's Green's Scheme (PGS)

In this section, we will describe and detail our proposed method. Let's consider a class of SVBPs of the form:

$$L[u] = u'' + \frac{p}{x}u' = f(x, u, u'), \tag{8}$$

with the boundary conditions (2). Let G be the Green's function for the linear term and define the integral operator

$$K[u_p] = \int_a^b G(x|s)\ L[u_p]\ ds. \tag{9}$$

Using (7), we can rewrite the latter equation as:

$$K[u_p] = \int_a^b G(x|s)\left[L[u_p] - f(s, u(s), u'(s))\right] ds + u_p. \tag{10}$$

For convenience, let's drop u_p and denote it by u. It follows that

$$K[u] = \int_a^b G(x|s)\left[L[u] - f(s, u(s), u'(s))\right] ds + u. \tag{11}$$

Applying Picard's iteration on $K[u]$, namely

$$u_{n+1} = K[u_n], \qquad n \geq 0,$$

yields the following iterative procedure:

$$u_{n+1} = u_n + \int_a^b G(x|s)\left[L[u_n] - f(s, u_n(s), u_n'(s))\right] ds, \tag{12}$$

where $L[u_n]$ is the linear term for the second order differential equation. The initial iterate u_0 is chosen to satisfy the corresponding homogenous equation in (8), $L[u] = 0$, and the specified boundary conditions.

Mann's Green's Embedded Method (MGS)

Next, we apply the following Mann's iterative algorithm for the approximation of fixed points, using the operator defined in (9):

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n K[u_n], \qquad n \geq 0.$$

Following the very similar steps as in the previous subsection, this results in the iterative scheme (MGS):

$$u_{n+1} = u_n + \alpha_n \int_a^b G(x|s)\left[L[u_n] - f(s, u_n(s), u_n'(s))\right] ds, \tag{13}$$

where $\{\alpha_n\}$ is a sequence of numbers that control the stability and speed up the convergence of the scheme. The starting function u_0 is chosen to be the solution for the corresponding homogenous equation $L[u] = 0$ subject to the specified boundary conditions (2).

The optimal values of the sequence $\{\alpha_n\}$ is found by minimizing the L^2-norm of the residual error, $R_n(x; \alpha_n)$, of the n^{th} iteration u_n, namely

$$\|R_n(x; \alpha_n)\|_{L^2}^2 = \frac{1}{b-a} \int_a^b R_n^2(x; \alpha_n)\, dx, \tag{14}$$

where for each n, $R_n(x; \alpha_n)$ is given by

$$R_n(x; \alpha_n) = L[u_n] - f(x, u_n(x), u_n'(x)). \tag{15}$$

It is worth mentioning that with the proper choice of the parameters α_n's, the stability of the scheme can be controlled. For more details on the stability see [11].

Convergence Analysis of the PGS

This section includes the convergence analysis of the Picard's scheme. The analysis is based on the contraction principle [28]. Without loss of generality, we prove convergence of the PGS that applies to the following boundary value problem:

$$u''(x) + \frac{p}{x} u'(x) = f(x, u(x), u'(x), \tag{16}$$

where $p \geq 2$, and complimented with the boundary conditions:

$$u'(0) = \alpha, \quad u(1) = \beta. \tag{17}$$

First, we construct the Green's function for (16) using the properties detailed in Sect. 2.1. Solving the corresponding homogeneous equation of (16), which is a Cauchy–Euler equation, we have

$$G(x|s) = \begin{cases} A + B\,x^{1-p}, & 0 < x < s \\ C + D\,x^{1-p}, & s < x < 1 \end{cases}. \tag{18}$$

Applying the corresponding homogenous BCs of (17), that is $u'(0) = u(1) = 0$, we get the two equations

$$B = 0, \qquad C + D = 0. \tag{19}$$

The continuity of the Green's function gives the equation

$$A + B\,s^{1-p} = C + D\,s^{1-p}. \tag{20}$$

The unit jump discontinuity of the first derivative of the Green's function results in the equation

$$D(1-p)s^{-p} - B(1-p)s^{-p} = 1. \tag{21}$$

Solving the system of equations in (19)–(21), we get the Green's function

$$G(x|s) = \begin{cases} \frac{1}{p-1}\left(s^p - s\right), & 0 < x < s \\ \frac{s^p}{p-1}\left(1 - x^{1-p}\right), & s < x < 1 \end{cases}. \tag{22}$$

Substituting this latter Green's function in the PGS iterative procedure given in (12), we get the following PGS procedure that corresponds to the BVP in (16), (17):

$$\begin{aligned} u_{n+1} = u_n &+ \int_0^x \frac{s^p}{p-1}\left(x^{1-p} - 1\right)\left[u_n''(s) + \frac{p}{s}\,u_n'(s) - f\left(s, u_n(s), u_n'(s)\right)\right] ds \\ &+ \int_x^1 \frac{1}{p-1}\left(s - s^p\right)\left[u_n''(s) + \frac{p}{s}\,u_n'(s) - f\left(s, u_n(s), u_n'(s)\right)\right] ds. \end{aligned} \tag{23}$$

The next theorem gives convergence of the scheme.

Theorem 1 *Assume that $f\left(x, u, u'\right)$ is a continuous function whose derivative is bounded with respect to u. Assume that*

$$K := \frac{1}{2(p-1)} L_c < 1,$$

where

$$L_c = \max_{[0,1]\times R^2} \left|\frac{\partial f}{\partial u}\right|.$$

Then, the iterative sequence $\{u_n(x)\}_{n=1}^{\infty}$, given by (23), where $x \in [0, 1]$ and using any bounded starting function on $[0, 1]$, converges uniformly to the exact solution, $u(x)$, of problem (16)–(17).

Proof In order to prove the convergence, we will use the function space $C[0, 1]$ equipped with the maximum norm defined by $\|u\| = \max_{0\le x\le 1} |u(x)|$.

Direct integration leads to

$$\begin{aligned}\int_0^x \frac{s^p}{1-p}\left(x^{1-p}-1\right)\left[u_n''(s)+\frac{p}{s}\,u_n'(s)\right]ds &= \frac{x^{1-p}-1}{1-p}\int_0^x \left[s^p u_n''(s)+ps^{p-1}\,u_n'(s)\right]ds\\ &= \frac{x^{1-p}-1}{1-p}\int_0^x \left(s^p\,u_n'(x)\right)'\,ds\\ &= \frac{x-x^p}{1-p}\,u_n'(x).\end{aligned} \tag{24}$$

Integrating twice by parts we get

$$\begin{aligned}&\int_x^1 \frac{1}{p-1}\left(s^p-s\right)u_n''(s)\,ds\\ &= \frac{1}{p-1}\Big[(1-p)u_n(1)+\left(px^{p-1}-1\right)u_n(x)+\left(x-x^p\right)u_n'(x)\\ &\quad + p(p-1)\int_x^1 s^{p-2}\,u_n(s)\,ds\Big].\end{aligned} \tag{25}$$

Integrating once by parts we get

$$\begin{aligned}&\int_x^1 \frac{1}{p-1}\left(s^p-s\right)\frac{p}{s}\,u_n'(s)\,ds\\ &= \frac{p}{p-1}\left[\left(1-x^{p-1}\right)u_n(x)-(p-1)\int_x^1 s^{p-2}\,u_n(s)\,ds\right].\end{aligned} \tag{26}$$

Substituting the results of (24)–(26) into the iterative scheme (PGS) given in (23), we have

$$\begin{aligned}u_{n+1} = u_n(1) &+ \int_0^x \frac{s^p}{1-p}\left(x^{1-p}-1\right)\,f\left(s,u_n(s),u_n'(s)\right)ds\\ &+\int_x^1 \frac{1}{p-1}\left(s^p-s\right)\,f\left(s,u_n(s),u_n'(s)\right)ds.\end{aligned} \tag{27}$$

Equivalently, we have

$$u_{n+1} = \beta + \int_0^1 G(x|s)\,f\left(s,u_n(s),u_n'(s)\right)ds, \tag{28}$$

where $\beta = u_n(1)$, from (17), and

$$G(x|s) = \begin{cases} \dfrac{s^p}{1-p}\left(x^{1-p}-1\right), & 0<s<x\\ \dfrac{1}{p-1}\left(s^p-s\right), & x<s<1 \end{cases}. \tag{29}$$

Define $T_G : C[0,1] \to C[0,1]$ to be the right side of Eq. (28):

$$T_G(u) \equiv \beta + \int_0^1 G(x|s)\,f\left(s,u(s),u'(s)\right)ds. \tag{30}$$

According to Banach-Picard fixed point theorem, to prove convergence it suffices to show that T_G is a contractive mapping. Therefore, we have

$$|T_G(u)-T_G(v)| = \left|\int_0^1 G(x|s)\,\left[f(s,u,u')-f(s,v,v')\right]\,ds\right|. \tag{31}$$

Simple integration gives

$$\int_0^1 G(x|s)\, ds = \frac{1}{2(p+1)}(x^2-1) \equiv g(x). \tag{32}$$

The maximum value of the absolute value of the function $g(x)$ on the interval [0, 1] occurs either at the critical points or endpoints.

$$|g(x)| \leq \frac{1}{2(p+1)}. \tag{33}$$

Using (32) and (33), we have from (31)

$$|T_G(u) - T_G(v)| \leq \frac{1}{2(p+1)} \int_0^1 \left| f(s,u,u') - f(s,v,v') \right| ds. \tag{34}$$

Applying the Mean Value Theorem for f, we obtain

$$\begin{aligned} |T_G(u) - T_G(v)| &\leq \frac{1}{2(p+1)} \max_{0\leq x\leq 1} \left| f(x,u(x),u'(x)) - f(x,v(x),v'(x)) \right| \\ &\leq \frac{1}{2(p+1)} L_c \, \|u-v\| \end{aligned} \tag{35}$$

where $\|u-v\| = \max_{0\leq x\leq 1} |u(x)-v(x)|$ and $L_c = \max_{[0,1]\times R^3} \left| \frac{\partial}{\partial u} f(x,u,u') \right|$. From the hypothesis of the theorem, namely that $K := \frac{1}{2(p+1)} L_c < 1$, it follows that

$$\|T_G(u) - T_G(v)\| \leq K \, \|u-v\|, \tag{36}$$

with $0 < K < 1$. This proves that T_G is a contraction mapping.
In regard to the rate of convergence, we have

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|T_G(u_n) - T_G(u_{n-1})\| \\ &\leq K \, \|u_n - u_{n-1}\| \\ &\leq K^n \, \|u_1 - u_0\|. \end{aligned} \tag{37}$$

If $m > n > 0$, then

$$\begin{aligned} \|u_m - u_n\| &= \|u_m - u_{m-1}\| + \ldots + \|u_{n+1} - u_n\| \\ &\leq \left(K^{m-1} + \ldots + K^n\right) \|u_1 - u_0\| \\ &\leq K^n \left(1 + K + K^2 + \ldots\right) \|u_1 - u_0\| \\ &= \frac{K^n}{1-K} \|u_1 - u_0\|. \end{aligned} \tag{38}$$

If we let $m \to \infty$, we get the error estimate:

$$\left\| u^{\star} - u_n \right\| \leq \frac{K^n}{1-K} \|u_1 - u_0\|. \tag{39}$$

□

Numerical Examples

In this section, we will implement the Picard's Green's scheme for the solution of a nonlinear second order SBVPs. We will compare our numerical results with existing numerical solutions to confirm the validity and high accuracy of the strategy.

Table 1 Maximum absolute errors for Example 1 using PGS, compared with the methods in [8,27,33]

Method	E_3	V_3 [33]	E_5	V_5 [33]	E_{10}	V_{10} [33]	E_{15}	W_{16} [8]	Z_{16} [27]	E_{20}
Max. Err.	9.1(−3)	1.4(−2)	5.03(−3)	1.7(−3)	2.3(−5)	4.9(−5)	3.5(−7)	3.6(−4)	1.5(−5)	5.0(−9)

Table 2 Maximum absolute errors for Example 1 using MGS with $\alpha = 1.43$, compared with the methods in [8,27,33]

Method	E_3	V_3 [33]	E_5	V_5 [33]	E_{10}	V_{10} [33]	W_{16} [8]	Z_{16} [27]	E_{20}
Max. Err.	6.8(−4)	1.4(−2)	2.8(−5)	1.7(−3)	2.3(−7)	4.9(−5)	3.6(−4)	1.5(−5)	2.4(−11)

Example 1 Consider the following nonlinear SBVP describing the equilibrium of isothermal gas sphere [4], which is taken from Singh and Kumar [33]:

$$u''(x) = -\frac{2}{x}u'(x) + u^5(x), \tag{40}$$

where $0 < x < 1$ and subject to

$$u'(0) = 0, \quad u(1) = \sqrt{\frac{3}{4}}. \tag{41}$$

The exact solution is given by $u(x) = \sqrt{\frac{3}{3+x^2}}$.

Constructing the Green's function for the linear equation $L[u] = u'' = 0$ and complimented with the homogeneous BCs $u'(0) = 0$ and $u(1) = 0$, results in the subsequent form of the PGS (12).

$$\begin{aligned} u_{n+1} = u_n &- \int_0^x s^2\left(1 - \frac{1}{x}\right)\left[u_n''(s) + \frac{2}{s}u_n'(s) - u_n^5(s)\right]ds \\ &- \int_x^1 s(s-1)\left[u_n''(s) + \frac{2}{s}u_n'(s) - u_n^5(s)\right]ds. \end{aligned} \tag{42}$$

The initial iterate is the solution of $L[u] = 0$ subject to the BCs (41), which is found to be $u_0 = \sqrt{\frac{3}{4}}$.

For quantitative comparison, we now define E_n as the results obtained via the Picard Green's function approach (PGS), while V_n[19], W_n[6], and Z_n[23] are those obtained by the techniques proposed by Singh and Kumar [33], Geng [8], and Niu et al. [27] respectively. Numerical results of this SBVP, as reported in Table 1 below, confirm that our strategy is more accurate than the latter three methods combined.

It can be shown that the contraction constant for the corresponding PGS is $K = \frac{L_c}{2(p+1)}$ which is equal to 5/6. This yield slow convergence since K is close to 1. Thus, the results by the PGS may be improved if we use the MGS (23). The best choice for the value of α to minimize the absolute error in E_3 is found to be $\alpha^* = 1.43$; for simplicity this value is kept constant for the other iterations. The results are displayed in Table 2.

Example 2 Consider the following nonlinear SBVP, which is taken from Singh and Kumar [33]:

$$u''(x) = -\frac{2}{x}u'(x) - e^{-u(x)}, \tag{43}$$

Table 3 Residual errors for Example 2 using our scheme and methods in [27,33]

t	E_5	V_6	V_8	E_{10}	V_{10}	E_{15}	Z_{10}
0.1	4.3973(−4)	7.4909(−3)	2.3987(−3)	1.3290(−7)	8.2261(−4)	4.0165(−11)	1.1755(−5)
0.2	4.3290(−4)	7.1398(−3)	2.2757(−3)	1.3083(−7)	7.7789(−4)	3.9541(−11)	5.5947(−6)
0.3	4.2159(−4)	6.7221(−3)	2.1306(−3)	1.2741(−7)	7.2546(−4)	3.8507(−11)	1.7789(−6)
0.4	4.0586(−4)	6.1745(−3)	1.9426(−3)	1.2266(−7)	6.5808(−4)	3.7070(−11)	7.1507(−7)
0.5	3.8584(−4)	5.5303(−3)	1.7246(−3)	1.1661(−7)	5.8074(−4)	3.5242(−11)	4.0110(−7)
0.6	3.6169(−4)	4.8262(−3)	1.4905(−3)	1.0931(−7)	4.9869(−4)	3.3036(−11)	1.3760(−6)
0.7	3.3362(−4)	4.0999(−3)	1.2536(−3)	1.0082(−7)	4.1676(−4)	3.0472(−11)	2.7037(−6)
0.8	3.0187(−4)	3.3860(−3)	1.0257(−3)	9.1231(−8)	3.3896(−4)	2.7572(−11)	6.8831(−6)
0.9	2.6677(−4)	2.7139(−3)	8.1549(−4)	8.0622(−8)	2.6816(−4)	2.4366(−11)	1.8340(−5)
1.0	2.2868(−4)	2.1054(−3)	6.2888(−4)	6.9111(−8)	2.0607(−4)	2.0887(−11)	2.5312(−5)

where $0 < x \leq 1$ and subject to

$$u'(0) = 0, \quad 2u(1) + u'(1) = 0. \tag{44}$$

This problem is known as the Emden-Fowler equation of the second kind and arises in the study of distribution of heat sources in the human head [9]. The exact solution is not known explicitly.

The Green's function for the linear equation $L[u] = u'' = 0$ subject to homogeneous BCs, results in the subsequent form of the PGS (12).

$$\begin{aligned} u_{n+1} = u_n &- \int_0^x s^2\left(\frac{1}{2} - \frac{1}{x}\right)\left[u''(s) + \frac{2}{s}u'(s) + e^{-u(s)}\right]ds \\ &- \int_x^1 s\left(\frac{s}{2} - 1\right)\left[u''(s) + \frac{2}{s}u'(s) + e^{-u(s)}\right]ds, \end{aligned} \tag{45}$$

where the initial iterate is found to be $u_0 = 0$. Table 3 confirms that the PGS strategy is more accurate, when comparing the numerical results E_n of this SBVP using our introduced procedure and the numerical results V_n obtained by Singh and Kumar [33] method and Z_{10} obtained by Niu et al. approach [27].

Example 3 Consider the following nonlinear SBVP, which is taken from Khuri and Sayfy [18]:

$$u''(x) = -\frac{1}{x}u'(x) - e^{u(x)}, \tag{46}$$

where $0 < x < 1$ and subject to

$$u'(0) = 0, \quad u(1) = 0. \tag{47}$$

The exact solution is given by $u(x) = 2\ln\left(\frac{A+1}{Ax^2+1}\right)$, where $A = 3 - 2\sqrt{2}$.

Similar to the previous example, the problem is also known as the Emden-Fowler equation of the second kind. The Green's function for the linear equation $L[u] = u'' = 0$, results in the subsequent form of the PGS (12).

Table 4 Maximum errors for Example 3 using our scheme and methods in [3,8,27,33]

t	E_5	T_5	E_{10}	T_{10}	V_{60} [3]	W_{64} [8]	Z_{64} [27]
Max Err	1.9707(−5)	2.5395(−5)	1.0445(−8)	2.1007(−6)	3.5011(−6)	1.2800(−5)	5.0(−6)

$$u_{n+1} = u_n - \int_0^x s \ln x \left[u''(s) + \frac{1}{s} u'(s) + e^{u(s)} \right] ds \\ - \int_x^1 s \ln s \left[u''(s) + \frac{1}{s} u'(s) + e^{u(s)} \right] ds, \tag{48}$$

where the initial iterate is found to be $u_0 = 0$.

Again E_n is defined as the results of our PGS approach, while T_n, V_n[3], W_n[6] and Z_{64}[23] are the results obtained by the techniques proposed by Singh and Kumar [33], Caglar and Caglar and Ozer [3], Geng [8], and Niu et al. [27] respectively. A comparison is summarized in Table 4.

Example 4 Consider the following nonlinear SBVP arising in the study of steady-state oxygen diffusion in spherical cell [34], which is taken from Singh and Kumar [33]:

$$u''(x) = -\frac{\alpha}{x} u'(x) + \frac{nu(x)}{u(x) + k}, \tag{49}$$

subject to

$$u'(0) = 0, \quad 5u(1) + u'(1) = 5, \tag{50}$$

where $n = 0.76129$ is the reaction rate and $k = 0.03119$ is the Michaelis constant (see [16,18,34]).

This above nonlinear SBVP arises in the study of steady-state oxygen diffusion in a spherical cell. The exact solution is not known explicitly. The Green's function for the linear equation $L[u] = u'' = 0$ subject to homogeneous BCs, results in the subsequent form of the PGS (12):

$$u_{n+1} = u_n - \int_0^x s^2 \left(\frac{4}{5} - \frac{1}{x} \right) \left[u''(s) + \frac{\alpha}{s} u'(s) - \frac{nu(s)}{u(s) + k} \right] ds \\ - \int_x^1 s \left(\frac{4}{5} s - 1 \right) \left[u''(s) + \frac{\alpha}{s} u'(s) - \frac{nu(s)}{u(s) + k} \right] ds, \tag{51}$$

where initial iterate $u_0 = 1$, and $\alpha = 1$. E_n is defined as the maximum absolute error of our PGS approach while V_n is the maximum absolute error obtained by the technique proposed by Singh and Kumar [33]. The results in Table 5 below confirm that the Green's function approach is more accurate than the other existing method.

Example 5 Finally we consider the following nonlinear SBVP, which is also taken from Singh and Kumar [33]:

$$u''(x) = -\frac{3}{x} u'(x) + \frac{1}{2} - \frac{1}{8u^2(x)}, \tag{52}$$

where $0 < x \leq 1$ and subject to

$$u'(0) = 0, \quad u(1) = 1. \tag{53}$$

Table 5 Residual errors for Example 4 using our scheme and method in [33]

t	E_2	V_2	V_4	V_6	E_4	E_{10}
0.2	2.2818(−5)	7.0398(−4)	1.6620(−5)	3.6849(−7)	4.5829(−10)	3.2336(−24)
0.4	1.9366(−5)	5.7469(−4)	1.1146(−5)	1.9897(−7)	3.8118(−10)	2.6859(−24)
0.6	1.4433(−5)	3.9211(−4)	5.1923(−6)	5.9014(−8)	2.7586(−10)	1.9404(−24)
0.8	8.9991(−6)	2.0025(−4)	1.2866(−6)	4.1663(−9)	1.6676(−10)	1.1731(−24)
1.0	4.0391(−6)	5.1477(−5)	2.7901(−8)	1.1166(−9)	7.3579(−11)	9.3710(−23)

Table 6 Residual errors for Example 5 using our scheme and the method in [33]

t	E_2	V_2	V_4	V_6	E_6	E_9	E_{12}
0.1	2.9218(−4)	5.8369(−4)	1.1407(−6)	1.5209(−9)	3.9886(−11)	2.6148(−16)	1.7125(−21)
0.2	2.7762(−4)	5.4431(−4)	1.0645(−6)	1.0287(−9)	3.7464(−11)	2.4557(−16)	1.6084(−21)
0.3	2.5426(−4)	4.8182(−4)	9.4177(−7)	3.9594(−10)	3.3672(−11)	2.2067(−16)	1.4452(−21)
0.4	2.2341(−4)	4.0091(−4)	7.8006(−7)	1.7401(−10)	2.8843(−11)	1.8898(−16)	1.2377(−21)
0.5	1.8685(−4)	3.0800(−4)	5.9227(−7)	5.2101(−10)	2.3383(−11)	1.5316(−16)	1.0031(−21)
0.6	1.4670(−4)	2.1125(−4)	3.9735(−7)	5.8980(−10)	1.7720(−11)	1.1604(−16)	7.5996(−22)
0.7	1.0536(−4)	1.2033(−4)	2.1960(−7)	4.4440(−10)	1.2259(−11)	8.0257(−17)	5.2563(−22)
0.8	6.5391(−5)	4.6368(−5)	8.4986(−8)	2.2468(−10)	7.3396(−12)	4.8040(−17)	3.1463(−22)
0.9	2.9413(−5)	1.7565(−6)	1.2708(−8)	6.1702(−11)	3.2067(−12)	2.0986(−17)	1.3744(−22)
1.0	6.7102(−33)	0.0000(00)	0.0000(00)	0.0000(00)	1.3699(−31)	1.3703(−31)	1.3703(−31)

This nonlinear SBVP arises in the radial stress on a rotationally symmetric shallow membrane cap [15]. The exact solution is not known explicitly. The Green's function for the linear equation $L[u] = u'' = 0$ subject to homogeneous BCs, results in the subsequent form of the PGS (12).

$$\begin{aligned} u_{n+1} = u_n &- \int_0^x s^3 \left(\frac{1}{2} - \frac{1}{2x^2}\right) \left[u''(s) + \frac{3}{s}\, u'(s) + \frac{1}{8u^2} - \frac{1}{2}\right] ds \\ &- \int_x^1 s \left(\frac{s^2}{2} - \frac{1}{2}\right) \left[u''(s) + \frac{3}{s}\, u'(s) + \frac{1}{8u^2} - \frac{1}{2}\right] ds, \end{aligned} \tag{54}$$

where the initial iterate is $u_0 = 1$. E_n is defined as the maximum absolute error of Green's function approach while R_n is the maximum absolute error obtained by the proposed technique in Singh and Kumar [33]. After comparing the results in the Table 6, we assure that the PGS strategy is more accurate.

Conclusion

In this paper, a recent approach based on embedding Green's function into fixed-point iteration, is used to solve an extended class of second order nonlinear singular boundary value problems. Five test problems have been considered that demonstrate the efficiency of the scheme. The results confirmed the convergence of the scheme numerically. This claim has been justified by proving convergence of the proposed scheme as well as its rate of convergence. Moreover, the scheme seems to be computationally highly accurate for solving the given class of nonlinear SBVPs, when it compared with other existing methods. In future work, we plan to apply the proposed approach to optimal control problems (see [12,26]).

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