ORIGINAL PAPER

Relative Perturbation Bounds for Matrix Eigenvalues and Singular Values

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Abstract

In this paper, some new relative perturbation bounds for the eigenvalues of diagonalizable matrices are derived. A relative perturbation bound for singular values is also obtained. The present results improve some existing results.

Keywords Perturbation bound · Eigenvalue · Diagonalizable matrix · Singular value

Mathematics Subject Classification 65F15 · 15A42 · 15A18

Introduction

Many problems in science and engineering deal with eigenvalues and singular values of matrices. Here our main concern is perturbation bounds for matrix eigenvalues and singular values. In general, eigenvalue perturbation problems comprise the method to estimate the error when the eigenvalues of a perturbed matrix are approximated by the eigenvalues of an unperturbed matrix. Earlier, the authors working on perturbation theory have concentrated mainly on the bounds on the absolute differences between the approximate eigenvalue and true eigenvalue. However, there are practical situations when small eigenvalues should be determined to high relative accuracy. In this case, bounds on relative error are required. The main theme of this work is to obtain some relative perturbation bounds for eigenvalues of diagonalizable matrices. A result on relative perturbation bounds for singular value is also derived. Some techniques used in this work can be found in [1–4].

The following notations are used in this paper. M_n denotes the set of all matrices of order $n \times n$ with complex entries and S_n is the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$. A^t indicates the transpose of the matrix A. For a matrix $A = (a_{ij}) \in M_n$ we use the following

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notations (see [5,6]),

$$\|A\|_p = \left(\sum_{i,j=1}^n |a_{ij}|^p \right)^{\frac{1}{p}} \quad \text{for } p \ge 1, \tag{1}$$

and

$$\|A\|_{q,p} = \left(\sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \quad \text{for } p \ge 1, \ \frac{1}{p} + \frac{1}{q} = 1. \tag{2}$$

For $p = 2$, Eq. (1) gives the norm $\|.\|_2$ which is known as Frobenius norm. The operator norm is denoted by $\|.\|$.

One of the well-known theorem in the direction of absolute perturbation bound for matrix eigenvalues is the Hoffman–Weildant Theorem [7], which states that, if A and B are normal $n \times n$ matrices and $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $\{\mu_1, \mu_2, \dots, \mu_n\}$ are their eigenvalues respectively then there exist a permutation $\pi \in S_n$ such that,

$$\left(\sum_{i=1}^n \left| \lambda_i - \mu_{\pi(i)} \right|^2 \right)^{\frac{1}{2}} \le \|A - B\|_2. \tag{3}$$

In 1998, Eisenstat and Ipsen [8] proved that a Hoffman–Weildant type absolute perturbation bound implies a relative bound for diagonalizable matrices. They proved that, if A and B both are diagonalizable matrices of order n and if A is non-singular then there exists a permutation $\pi \in S_n$ such that,

$$\left(\sum_{i=1}^n \left| \frac{\lambda_i - \mu_{\pi(i)}}{\lambda_i} \right|^2 \right)^{\frac{1}{2}} \le \kappa(X)\kappa(\tilde{X})\|A^{-1}(A - B)\|_2 \tag{4}$$

where X and $\tilde{X}$ are the invertible matrices which diagonalize the matrices A and B respectively and $\kappa(X) = \|X\|\|X^{-1}\|$ is the condition number of the matrix X. Furthermore, Li and Sun [9] obtained bounds by taking the matrix A as normal and non-singular and the matrix B arbitrary. Later Li and Chen [10] generalized the result obtained by Li and Sun [9] for diagonalizable matrices. Recently Chen [11] has proved a relative perturbation bound for singular values of $n \times n$ matrices. If $s_1 \ge s_2 \ge \dots \ge s_n$ and $\tilde{s}_1 \ge \tilde{s}_2 \ge \dots \ge \tilde{s}_n$ are singular values of $n \times n$ complex matrices A and B respectively, where A is non-singular, then he proved that, there exists a permutation $\tau \in S_n$ such that,

$$\sqrt{ \sum_{i=1}^n \left(\frac{s_i - \tilde{s}_{\tau(i)}}{s_i} \right)^2 } \le \sqrt{ \frac{\|A^{-1}E\|_2^2 + \|EA^{-1}\|_2^2}{2} }, \tag{5}$$

where $E = B - A$.

The results mentioned earlier involve only Frobenius norm. Ikramov [12] partially generalized the the Hoffman–Weildant theorem with the norm $\|.\|_p$ and obtained the following result. If A and B are hermitian matrices, then for $1 < p \le 2$

$$d_p(\lambda(A), \lambda(B)) \le \|A - B\|_p \tag{6}$$

where $d_p(\lambda(A), \lambda(B))$ is the "Holder distance" between the eigenvalues of A and B and it is given by

$$d_p(\lambda(A), \lambda(B)) = \min_{\pi \in S_n} \left(\sum_{i=1}^{n} \left| \lambda_i - \mu_{\pi(i)} \right|^p \right)^{\frac{1}{p}}. \tag{7}$$

Recently Zhou et al. [4] generalized the above result for diagonalizable matrices.

Similar as Holder distance $d_p(\lambda(A), \lambda(B))$ we define $\tilde{d}_p(\lambda(A), \lambda(B))$ for relative perturbation as,

$$\tilde{d}_p(\lambda(A), \lambda(B)) = \min_{\pi \in S_n} \left(\sum_{i=1}^{n} \left| \frac{\lambda_i - \mu_{\pi(i)}}{\lambda_i} \right|^p \right)^{\frac{1}{p}}, \quad p \geq 1, \tag{8}$$

where the matrix A is non-singular. A similar quantity $\tilde{d}_p(s(A), s(B))$ can be defined for the singular values $s(A) = \{s_1, \ldots, s_n\}$ of a non-singular matrix A and $s(B) = \{\tilde{s}_1, \ldots, \tilde{s}_n\}$ of a matrix B. In this paper we investigated whether a bound similar as (6) holds for $\tilde{d}_p(\lambda(A), \lambda(B))$ and $\tilde{d}_p(s(A), s(B))$. In "Relative Perturbation Bounds" section we will prove some upper and lower bounds for $\tilde{d}_p(\lambda(A), \lambda(B))$ of diagonalizable matrices A and B. In "Relative Perturbation Bound for Singular Values" section we will derive an upper bound for $\tilde{d}_p(s(A), s(B))$.

Relative Perturbation Bounds

Before going to the main results we mention some basic relations which are useful. Let $A = (a_{ij})$ and $B = (b_{ij})$ are in M_n and let $p \geq 1$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$, then we have the following relations (see [6])

$$\|AB\|_p \leq \|A\|_p \|B\|_{q,p}, \tag{9}$$

$$\|AB\|_p \leq \|A^t\|_{q,p} \|B\|_p. \tag{10}$$

If B is non-singular, then

$$\frac{\|A\|_p}{\|B^{-1}\|_{q,p}} \leq \|AB\|_p. \tag{11}$$

If A is non-singular, then

$$\frac{\|B\|_p}{\|(A^{-1})^t\|_{q,p}} \leq \|AB\|_p. \tag{12}$$

For $0 < p \leq 2$

$$\|AB\|_p \leq \|A\|_p \|B\|_p. \tag{13}$$

A square matrix of non-negative real numbers is called a doubly stochastic matrix if each row sum and each column sum is 1. A permutation matrix is a square matrix that has exactly one entry is 1 in each row and each column and 0 elsewhere. The Hadamard Product of two matrices with same dimension is defined by entrywise multiplication. If $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices with same dimension then the Hadamard Product of A and B denoted by $A \circ B$ and defined by

$$A \circ B = (a_{ij} b_{ij}).$$

For any two $n \times n$ real matrices A and B, $A \leq_e B$ means $(B - A)$ is entrywise non-negative. Also if $A = (a_{ij})$ be a $n \times n$ matrix then we denote $A^{|\circ|p} = (|a_{ij}|^p)$ for a real $p > 0$. We denote the singular values of A as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. The next lemma involves entry wise inequality between two matrices.

Lemma 1 [13, Theorem 3.36] *Let $A \in M_n$ and let p, q be real numbers with $0 < p \leq 2$ and $q \geq 2$. Then there exist two doubly stochastic matrices B, $C \in M_n$ such that,*

$$s_n^p(A)B \leq_e A^{|\circ|p}$$

and

$$A^{|\circ|q} \leq_e s_1^q(A)C.$$

We prove the following lemma which is useful in this section.

Lemma 2 *Let $Y = (y_{ij})$ be $n \times n$ doubly stochastic matrix and $M = (m_{ij}) \in M_n$. Then there exist permutations τ and $\bar{\tau}$ in S_n such that for $p > 0$,*

$$\sum_{i=1}^{n} |m_{i,\tau(i)}|^p \leq \sum_{i,j=1}^{n} |m_{ij}|^p y_{ij} \leq \sum_{i=1}^{n} |m_{i,\bar{\tau}(i)}|^p.$$

Proof Since Y is doubly stochastic, so by well-known Birkhoff's Theorem it can be expressed as

$$Y = \sum_{k=1}^{n!} \alpha_k P_k,$$

where P_k are the permutation matrices and $\sum_{k=1}^{n!} \alpha_k = 1$, $\alpha_k \geq 0$. Let τ_k are the corresponding permutations of the permutation matrices P_k. For $M = (m_{ij}) \in M_n$

$$M^{|\circ|p} \circ Y = \left(|m_{ij}|^p y_{ij} \right).$$

Let e denotes the $n \times 1$ column vector whose components are all 1. Then

$$\sum_{i,j=1}^{n} |m_{ij}|^p y_{ij} = e^t (M^{|\circ|p} \circ Y)e.$$

Let

$$e^t (M^{|\circ|p} \circ P)e = \min\{e^t (M^{|\circ|p} \circ P_k)e : 1 \leq k \leq n!\}$$
$$\text{and } e^t (M^{|\circ|p} \circ \bar{P})e = \max\{e^t (M^{|\circ|p} \circ P_k)e : 1 \leq k \leq n!\}.$$

Also let τ and $\bar{\tau}$ are the permutations corresponding P and $\bar{P}$ respectively. Therefore we have,

$$\sum_{i,j=1}^{n} |m_{ij}|^p y_{ij} = e^t (M^{|\circ|p} \circ Y)e = e^t \left(M^{|\circ|p} \circ \left(\sum_{k=1}^{n!} \alpha_k P_k \right) \right) e$$

$$= e^t \left(\sum_{k=1}^{n!} \alpha_k \left(M^{|\circ|p} \circ P_k \right) \right) e$$

$$= \sum_{k=1}^{n!} \alpha_k \left(e^t (M^{|\circ|p} \circ P_k) e \right)$$

$$\geq \left(\min_{1 \leq k \leq n!} e^t (M^{|\circ|p} \circ P_k) e \right) \sum_{k=1}^{n!} \alpha_k$$

$$= e^t (M^{|\circ|p} \circ P) e.$$

Since P is a permutation matrix and τ be the corresponding permutation, so from above relation

$$\sum_{i,j=1}^{n} |m_{ij}|^p y_{ij} \geq e^t (M^{|\circ|p} \circ P) e = \sum_{i=1}^{n} |m_{i\tau(i)}|^p.$$

Similarly for other part we have

$$\sum_{i,j=1}^{n} |m_{ij}|^p y_{ij} = e^t (M^{|\circ|p} \circ Y) e = e^t \left(M^{|\circ|p} \circ \left(\sum_{k=1}^{n!} \alpha_k P_k \right) \right) e$$

$$= \sum_{k=1}^{n!} \alpha_k \left(e^t (M^{|\circ|p} \circ P_k) e \right)$$

$$\leq \left(\max_{1 \leq k \leq n!} e^t (M^{|\circ|p} \circ P_k) e \right) \sum_{k=1}^{n!} \alpha_k$$

$$= e^t (M^{|\circ|p} \circ \bar{P}) e = \sum_{i=1}^{n} |m_{i\bar{\tau}(i)}|^p.$$

$\square$

Let us consider the matrices A and B in M_n with eigenvalues $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ respectively and $E = A - B$. Then there are invertible matrices P, Q and diagonal matrices D_1, D_2 such that,

$$A = P D_1 P^{-1} \tag{14}$$

$$B = Q D_2 Q^{-1} \tag{15}$$

where D_1, D_2 are diagonal matrices containing the eigenvalues of A, B respectively.

Theorem 1 *If A, B are diagonalizable matrices and follow the above mentioned decompositions then there exists permutation π in S_n such that*

$$\left[\sum_{i=1}^{n} \left| \frac{\alpha_i - \beta_{\pi(i)}}{\alpha_i} \right|^p \right]^{\frac{1}{p}} \leq \|Q^{-1}\| \|P\| \|(P^{-1})^t\|_{q,p} \|Q\|_{q,p} \|A^{-1} E\|_p$$

and

$$\left[\sum_{i=1}^{n} \left| \frac{\alpha_i - \beta_{\pi(i)}}{\alpha_i} \right|^p \right]^{\frac{1}{p}} \leq \|Q^{-1}\| \|P\| \|P^{-1}\|_p \|Q\|_p \|A^{-1} E\|_p,$$

where $1 < p \leq 2$.

Proof From Eqs. (14) and (15) we have

$$-A^{-1}E = A^{-1}B - I = (PD_1^{-1}P^{-1})(QD_2Q^{-1}) - I.$$

Multiplying P^{-1} from left and Q from right in the above relation we get,

$$-P^{-1}A^{-1}EQ = D_1^{-1}P^{-1}QD_2 - P^{-1}Q.$$

Let $P^{-1}Q = Z = (z_{ij})$. Then

$$-P^{-1}A^{-1}EQ = D_1^{-1}ZD_2 - Z. \tag{16}$$

Taking the norm $\|.\|_p$ on both sides of the above equation where $1 \le p < 2$, we get

$$\|P^{-1}A^{-1}EQ\|_p^p = \sum_{i,j=1}^{n} |z_{ij}|^p |\alpha_i^{-1}\beta_j - 1|^p.$$

From Lemma 1 there exists a doubly stochastic matrix $B = (b_{ij})$ such that for $1 < p \le 2$

$$s_n^p(Z)b_{ij} \le |z_{ij}|^p \;\; \forall i,j.$$

Therefore,

$$\sum_{i,j=1}^{n} b_{ij} |\alpha_i^{-1}\beta_j - 1|^p \le \|Z^{-1}\|^p \|P^{-1}A^{-1}EQ\|_p^p.$$

Again by Lemma 2 there exists a permutation $\pi \in S_n$ such that,

$$\sum_{i=1}^{n} |\alpha_i^{-1}\beta_{\pi(i)} - 1|^p \le \|Z^{-1}\|^p \|P^{-1}A^{-1}EQ\|_p^p.$$

i.e., $$\left[\sum_{i=1}^{n} |\alpha_i^{-1}\beta_{\pi(i)} - 1|^p\right]^{\frac{1}{p}} \le \|Q^{-1}P\| \|P^{-1}A^{-1}EQ\|_p. \tag{17}$$

Using (9) and (10) we have

$$\left[\sum_{i=1}^{n} \left|\frac{\alpha_i - \beta_{\pi(i)}}{\alpha_i}\right|^p\right]^{\frac{1}{p}} \le \|Q^{-1}\| \|P\| \|(P^{-1})^t\|_{q,p} \|Q\|_{q,p} \|A^{-1}E\|_p.$$

Also from (17) using (13) we have

$$\left[\sum_{i=1}^{n} \left|\frac{\alpha_i - \beta_{\pi(i)}}{\alpha_i}\right|^p\right]^{\frac{1}{p}} \le \|Q^{-1}\| \|P\| \|P^{-1}\|_p \|Q\|_p \|A^{-1}E\|_p.$$

$\square$

Theorem 2 *Under the same assumption of Theorem 1 there is permutation σ in S_n such that*

$$\left[\sum_{i=1}^{n} \left|\frac{\alpha_i - \beta_{\sigma(i)}}{\alpha_i}\right|^q\right]^{\frac{1}{q}} \ge \frac{\|A^{-1}E\|_q}{\|P^{-1}\| \|Q\| \|P^t\|_{p,q} \|Q^{-1}\|_{p,q}}.$$

where $q \ge 2$.

Proof For $2 \leq q$, taking norm $\|.\|_q$ on both sides of (16) we have,

$$\| P^{-1}A^{-1}EQ \|_q^q = \sum_{i,j=1}^{n} |z_{ij}|^q |\alpha_i^{-1}\beta_j - 1|^q.$$

From Lemma 1 there exists a doubly stochastic matrix $C = (c_{ij})$ such that

$$|z_{ij}|^q \leq s_1^q(Z) c_{ij} \ \forall i, j,$$

where $Z = P^{-1}Q$. Therefore,

$$\sum_{i,j=1}^{n} c_{ij} |\alpha_i^{-1}\beta_j - 1|^q \geq \left(\|Z\|^q \right)^{-1} \| P^{-1}A^{-1}EQ \|_q^q.$$

By Lemma 2 there exists a permutation $\sigma \in S_n$ such that,

$$\sum_{i,j=1}^{n} |\alpha_i^{-1}\beta_{\sigma(i)} - 1|^q \geq \frac{\| P^{-1}A^{-1}EQ \|_q^q}{\|Z\|^q}.$$

Using (11) and (12) we get

$$\left[\sum_{i=1}^{n} \left| \frac{\alpha_i - \beta_{\sigma(i)}}{\alpha_i} \right|^q \right]^{\frac{1}{q}} \geq \frac{\| P^{-1}A^{-1}EQ \|_q}{\| P^{-1}Q \|}$$

$$\geq \frac{\| A^{-1}E \|_q}{\| P^{-1} \| \| Q \| \| P^t \|_{p,q} \| Q^{-1} \|_{p,q}}.$$

$\square$

Relative Perturbation Bound for Singular Values

Let $A, B \in M_n$ where A is non-singular, the singular value decompositions (SVD)

$$A = U \Sigma V^* \quad \text{and} \quad B = U_1 \tilde{\Sigma} V_1^*$$

respectively, where U, V^*, U_1, V_1^* are $n \times n$ unitary matrices, and

$$\Sigma = diag(s_1, \ldots, s_n) \quad \text{and} \quad \tilde{\Sigma} = diag(\tilde{s}_1, \ldots, \tilde{s}_n),$$

and $s_1 \geq s_2 \geq \cdots \geq s_n$ and $\tilde{s}_1 \geq \tilde{s}_2 \geq \cdots \geq \tilde{s}_n$ are singular values of A and B respectively. In the next Theorem we use the following relation,

$$(a^p + b^p) \geq 2^{1-p}(a+b)^p, \qquad a, b \geq 0, \ p \geq 1.$$

Theorem 3 *Let $A, B \in M_n$ with A is non-singular, follow the above mentioned singular value decomposition. Then there exists permutation π of S_n such that for $p > 1$,*

$$\left[\sum_{i=1}^{n} \left| \frac{s_i - \tilde{s}_{\pi(i)}}{s_i} \right|^p \right]^{\frac{1}{p}} \leq \left[2^{p-1} (\| V^*A^{-1}EV_1 \|_p^p + \| U^*(EA^{-1})^*U_1 \|_p^p) \right]^{\frac{1}{p}},$$

where $E = B - A$.

Proof Given $A = U \Sigma V^*$ and $B = U_1 \tilde{\Sigma} V_1^*$, where U, V, U_1, V_1 are $n \times n$ unitary matrices. Since A is non-singular so,

$$
\begin{aligned}
A^{-1}E &= A^{-1}(A+E) - I \\
&= (V\Sigma^{-1}U^*)(U_1\tilde{\Sigma}V_1^*) - I, \\
\Rightarrow V^*A^{-1}EV_1 &= \Sigma^{-1}U^*U_1\tilde{\Sigma} - V^*V_1 \\
&= \Sigma^{-1}P\tilde{\Sigma} - Q,
\end{aligned}
$$

where $U^*U_1 = P = (p_{ij})$ and $V^*V_1 = Q = (q_{ij})$. Therefore,

$$
\|V^*A^{-1}EV_1\|_p^p = \sum_{i,j=1}^{n} \left| \frac{\tilde{s}_j}{s_i} p_{ij} - q_{ij} \right|^p. \tag{18}
$$

Also

$$
\begin{aligned}
EA^{-1} &= (A+E)A^{-1} - I \\
&= (U_1\tilde{\Sigma}V_1^*)(V\Sigma^{-1}U^*) - I.
\end{aligned}
$$

In addition

$$
\begin{aligned}
(EA^{-1})^* &= U\Sigma^{-1}V^*V_1\tilde{\Sigma}U_1^* - I \\
\Rightarrow U^*(EA^{-1})^*U_1 &= \Sigma^{-1}Q\tilde{\Sigma} - P.
\end{aligned}
$$

Therefore

$$
\|U^*(EA^{-1})^*U_1\|_p^p = \sum_{i,j=1}^{n} \left| \frac{\tilde{s}_j}{s_i} q_{ij} - p_{ij} \right|^p. \tag{19}
$$

Adding the equations (18) and (19) we get,

$$
\begin{aligned}
\|V^*A^{-1}EV_1\|_p^p + \|U^*(EA^{-1})^*U_1\|_p^p &= \sum_{i,j=1}^{n} \left[\left| \frac{\tilde{s}_j}{s_i} p_{ij} - q_{ij} \right|^p + \left| \frac{\tilde{s}_j}{s_i} q_{ij} - p_{ij} \right|^p \right] \\
&\geq 2^{1-p} \sum_{i,j=1}^{n} \left[\left| \frac{\tilde{s}_j}{s_i} p_{ij} - q_{ij} \right| + \left| \frac{\tilde{s}_j}{s_i} q_{ij} - p_{ij} \right| \right]^p \\
&\geq 2^{1-p} \sum_{i,j=1}^{n} \left[\left| \frac{\tilde{s}_j}{s_i} |p_{ij}| - |q_{ij}| \right| + \left| \frac{\tilde{s}_j}{s_i} |q_{ij}| - |p_{ij}| \right| \right]^p \\
&\geq 2^{1-p} \sum_{i,j=1}^{n} \left[\left| \frac{\tilde{s}_j}{s_i} |p_{ij}| - |q_{ij}| + \frac{\tilde{s}_j}{s_i} |q_{ij}| - |p_{ij}| \right| \right]^p \\
&= 2^{1-p} \sum_{i,j=1}^{n} \left[\left| \frac{\tilde{s}_j}{s_i} (|p_{ij}| + |q_{ij}|) - (|p_{ij}| + |q_{ij}|) \right| \right]^p \\
&= 2^{1-p} \sum_{i,j=1}^{n} \left[\left| \frac{\tilde{s}_j}{s_i} - 1 \right|^p (|p_{ij}| + |q_{ij}|)^p \right] \\
&\geq 2^{1-p} \sum_{i,j=1}^{n} \left[\left| \frac{\tilde{s}_j}{s_i} - 1 \right|^p |p_{ij}|^p \right]
\end{aligned}
$$

From Lemma 1 there exists double stochastic matrix $B = (b_{ij})$ such that,

$$
\|V^*A^{-1}EV_1\|_p^p + \|U^*(EA^{-1})^*U_1\|_p^p \geq 2^{1-p} s_n^p(P) \sum_{i,j=1}^{n} \left[\left| \frac{\tilde{s}_j}{s_i} - 1 \right|^p b_{ij} \right].
$$

Again from Lemma 2 there exists a permutation π of S_n such that,

$$\|V^*A^{-1}EV_1\|_p^p + \|U^*(EA^{-1})^*U_1\|_p^p \geq 2^{1-p} \sum_{i=1}^{n} \left[\left| \frac{\tilde{s}_{\pi(i)}}{s_i} - 1 \right|^p \right],$$

which yields the desired result.　　　　　　　　　　　　　　　　　　　　　　□

Remark 1 The results on relative perturbation bounds obtained in "Relative Perturbation Bounds" section generalize Corollary 5.2 of [8] and the result for singular values, proved in "Relative Perturbation Bound for Singular Values" section generalize Theorem 2.2 of [11].

As we have mentioned previously, relative error bounds are required when there are eigenvalues of small magnitude. There are practical situations where small eigenvalues have physical meaning and such situations include computing modes of vibration in a finite element context, and computing energy levels in quantum mechanical systems [14]. Also, relative perturbation bounds for eigenvalues and singular values have an applications in developing high accuracy numerical methods for eigenvalues and singular value problems [15,16].

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