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ORIGINAL PAPER

Generalization of Gegenbauer Wavelet Collocation Method to the Generalized Kuramoto–Sivashinsky Equation

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Abstract
Gegenbauer (Ultraspherical) wavelets operational matrices play an important role for numeric solution of differential equations. In this study, operational matrices of rth integration of Gegenbauer wavelets are presented and general procedures of these matrices are correspondingly given first time. The proposed method is based on the approximation by the truncated Gegenbauer wavelet series. Algebraic equation system has been obtained by using the Chebyshev collocation points and solved. Proposed method has been applied to the Generalized Kuramoto–Sivashinsky equation using quasilinearization technique. Numerical examples showed that the method proposed in this study demonstrates the applicability and the accuracy of the Gegenbauer wavelet collocation method.

Keywords Gegenbauer wavelets · Collocation method · Kuramoto–Sivashinsky equation · Quasilinearization technique

Introduction

Wavelets, known as very well-localized functions, a powerful and recognized tool used in image processing, quantum mechanics, signal processing, computer science and many more other areas. Wavelets are greatly useful for solving differential, fractional differential [1–3], integral, integro-differential and fractional Volterra integro-differential [4–6] equations and give accurate solutions. The wavelet technique allows the development of extremely fast algorithms when it is compared with the algorithms ordinarily used. Gu and Jiang [7] derived the Haar wavelets operational matrix of integration. Burgers and sine–Gordon equations in [8], nonlinear PDEs of fractional order in [9], Fisher's equation in [10], Fitz Hugh-Nagumo equation in [11], Convection–diffusion equations in [12], film-pore diffusion model in [13], nonlinear parabolic equations in [14] nonlinear boundary value problems in [15], generalized Burgers-Huxley equation in [16] and magnetohydrodynamic flow equations in [17] were solved by Haar wavelet method. In the literature, special attention has been given to the applications of Legendre wavelets [18, 19]. The Legendre and Chebyshev wavelets operational matrixes of integration and product operation matrix have been introduced in [20, 21] and

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in [22–24] respectively. These matrices can be used to solve problems such as identification, analysis and optimal control. Fredholm integral equations of the first kind in [25], fractional differential equations in [26], nonlinear fractional integrodifferential equations in [27], one dimensional heat equation in [28], Bratu's problem in [29] and water quality assessment model problem in [30] were solved by Chebyshev wavelet method. Çelik [31–33] solved differential equations, generalized Burgers-Huxley equation and Free vibration problems of non-uniform Euler–Bernoulli beam by Chebyshev wavelet collocation method. Gegenbauer wavelets have been introduced to solve numerically the Abel's integral equation in [34]. Fractional-order differential equations in [35–37], Lane–Emden type differential equations in [38, 39] and various 2nth-order initial and boundary value problems in [40, 41] were solved by Gegenbauer wavelets.

Operational matrices of rth integration of Gegenbauer wavelets have been presented first time in this study. Proposed method has been applied to the nonlinear partial differential equation called as generalized Kuramoto–Sivashinsky (GKS) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0 \tag{1}$$

where α, β and γ are nonzero real constants. It is noteworthy that the GKS equation retains the fundamental elements of any nonlinear process that involves wave evolution: the simplest possible nonlinearity uu_x, instability and energy production u_{xx}, stability and energy dissipation u_{xxxx} and dispersion u_{xxx}. In the context of thin-film flows, the terms u_{xx}, u_{xxx} and u_{xxxx} are due to the interfacial kinematics associated with inertia, viscosity and surface tension, respectively, with the corresponding parameters α, β and γ all positive and measuring the relative importance of these effects [42].

We consider nonlinear partial differential equations of the form

$$\dot{u}(x,t) = F(u, u', u'', \ldots, u^{(r)})$$

The quasilinearizations of these equations give a set of recurrence linear differential equations

$$\dot{u}_{s+1}(x,t) = F(u_s, u_s', u_s'', \ldots, u_s^{(r)}) + \sum_{i=0}^{r}\left(u_{s+1}^{(i)} - u_s^{(i)}\right) F_{u_s^{(i)}}(u_s, u_s', u_s'', \ldots, u_s^{(r)}) \tag{2}$$

where $F_{u_s^{(i)}}(u_s, u_s', u_s'', \ldots, u_s^{(r)}) = \frac{\partial}{\partial u_s^{(i)}}\left(F(u_s, u_s', u_s'', \ldots, u_s^{(r)})\right)$, $\dot{u}(x,t) = \frac{\partial u(x,t)}{\partial t}$, $u'(x,t) = \frac{\partial u(x,t)}{\partial x}$ and $u_0(x,t)$ is taken as a function satisfying initial/boundary conditions [43].

The method is based on the approximation by the truncated Gegenbauer wavelets series. By using the Chebyshev collocation points, algebraic equation system has been obtained. Solving this algebraic equation system, the coefficients of the Gegenbauer wavelet series can be found. Hence, we have the implicit form of the approximate solution of nonlinear partial differential equations. This method is applied to the three generalized Kuramoto–Sivashinsky (GKS) equations using quasilinearization technique. Calculations demonstrated that the accuracy of the Gegenbauer wavelet collocation method is quite high even in the case of a small number of grid points.

Gegenbauer (Ultraspherical) Polynomials

Gegenbauer polynomials [44], or ultra-spherical harmonics polynomials of order $m \in Z^+$ are defined as $C_m^\lambda(x)$ for $\lambda > -\frac{1}{2}$ on the interval $[-1,\ 1]$ and given by the following recurrence formulae,

$$C_0^\lambda(x) = 1, \quad C_1^\lambda(x) = 2\lambda x,$$

$$(m+1)C_{m+1}^\lambda(x) = 2(m+\lambda)xC_m^\lambda(x) - (m+2\lambda-1)C_{m-1}^\lambda(x), \quad m = 1, 2, 3, \ldots$$

These polynomials are also given by the generating function

$$\frac{1}{(1-2xt+t^2)^\lambda} = \sum_{m=0}^{\infty} C_m^\lambda(x)t^m. \tag{3}$$

The following relations of Gegenbauer polynomials can be derive by using generating function.

$$\frac{d}{dx}\left(C_m^\lambda(x)\right) = 2\lambda C_{m-1}^{\lambda+1}(x), \quad m \geq 1. \tag{4}$$

$$(m+\lambda)C_m^\lambda(x) = \lambda\left(C_m^{\lambda+1}(x) - C_{m-2}^{\lambda+1}(x)\right), \quad m \geq 2. \tag{5}$$

$$\frac{d}{dx}\left(C_{m+1}^\lambda(x) - C_{m-1}^\lambda(x)\right) = 2\lambda\left(C_m^{\lambda+1}(x) - C_{m-2}^{\lambda+1}(x)\right) = 2(m+\lambda)C_m^\lambda(x) \tag{6}$$

By integration of the Eq. (6) from -1 to x, the following relation can be obtained

$$\int_{-1}^{x} C_m^\lambda(t)dt = \frac{1}{2(m+\lambda)}\left(C_{m+1}^\lambda(x) - C_{m-1}^\lambda(x) - C_{m+1}^\lambda(-1) + C_{m-1}^\lambda(-1)\right), \quad m \geq 1.$$

The equation given as:

$$\int (1-x^2)^{\lambda-\frac{1}{2}} C_m^\lambda(x)dx = -\frac{2\lambda(1-x^2)^{\lambda+\frac{1}{2}}}{m(m+2\lambda)} C_{m-1}^{\lambda+1}(x), \quad m \geq 1 \tag{7}$$

can be obtained from the Rodrigues formula. Gegenbauer polynomials satisfy the following relations

$$C_m^\lambda(-1) = \frac{(-1)^m \Gamma(m+2\lambda)}{m!\,\Gamma(2\lambda)}, \quad C_m^\lambda(1) = \frac{\Gamma(m+2\lambda)}{m!\,\Gamma(2\lambda)}$$

Gegenbauer polynomials are orthogonal on $[-1,\ 1]$ with respect to the weight function $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$ as

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_m^\lambda(x)C_n^\lambda(x)dx = \begin{cases} L_m^\lambda, & m = n \\ 0, & m \neq n \end{cases}$$

where

$$L_m^\lambda = \begin{cases} \frac{\pi 2^{1-2\lambda}\Gamma(m+2\lambda)}{m!(m+\lambda)(\Gamma(\lambda))^2}, & \lambda \neq 0 \\ \frac{2\pi}{m^2}, & \lambda = 0 \\ \pi, & \lambda = 0, m = 0 \end{cases} \tag{8}$$

is the normalizing factor.

Gegenbauer polynomials are generalized forms of the Legendre and Chebyshev polynomials. For $\lambda = 0$, $\lambda = 1$, and $\lambda = \frac{1}{2}$, we can get first kind Chebyshev polynomials as:

$$T_m(x) = \frac{m}{2} \lim_{\lambda \to 0} \frac{C_m^{\lambda}(x)}{\lambda} \quad (m \geq 1),$$

second kind Chebyshev polynomials as:

$$U_m(x) = C_m^1(x)$$

and Legendre polynomial as:

$$L_m(x) = C_m^{\frac{1}{2}}(x)$$

respectively.

Gegenbauer (Ultraspherical) Wavelet Method

Wavelets consist of a family of functions coming from dilation and translation of a single function named the mother wavelet. If a as a dilation parameter and b as translation parameter vary continuously, the following family of continuous wavelets may be obtained [45].

$$\psi_{a,b}(x) = |a|^{1/2} \psi\left(\frac{x-b}{a}\right), \quad a, b \in R, \quad a \neq 0. \tag{9}$$

Gegenbauer wavelets are written as

$$\psi_{nm}(x) = \psi\,(k, n, m, x).$$

where $k = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots, 2^k$, m is degree of Gegenbauer polynomials, λ is the known ultraspherical parameter and $x \in [0, 1)$. They are defined by:

$$\psi_{nm}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{L_m^{\lambda}}} C_m^{\lambda}(2^{k+1}x - 2n + 1), & \frac{n-1}{2^k} \leq x < \frac{n}{2^k}, \\ 0, & otherwise \end{cases} \tag{10}$$

where $C_m^{\lambda}(2^{k+1}x - 2n + 1)$ are Gegenbauer polynomials of degree m which are orthogonal with respect to the weight function $w_n(x) = w(2^{k+1}x - 2n + 1) = \left(1 - (2^{k+1}x - 2n + 1)^2\right)^{\lambda - \frac{1}{2}}$ on $[-1, \; 1]$.

A function $f(x) \in L_w^2[0, 1]$ may be expanded as:

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{nm} \psi_{nm}(x) \tag{11}$$

where

$$f_{nm} = \langle f(x), \psi_{nm}(x) \rangle \tag{12}$$

$\langle\, . \,,\, . \,\rangle$ denotes the inner product with weight function $w_n(x)$ in Eq. (12).

Truncated form of Eq. (11) can be written as:

$$f(x) \cong \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{nm} \psi_{nm}(x) = \mathbf{C}^{\mathbf{T}} \mathbf{\Psi}(x) \tag{13}$$

where $\mathbf{C}$ and $\boldsymbol{\Psi}(x)$ are $2^k M \times 1$ columns vectors given by:

$$\mathbf{C^T} = [f_{10}, f_{11}, \ldots, f_{1M-1}, f_{20}, \ldots, f_{2M-1}, \ldots, f_{2^k0}, \ldots, f_{2^kM-1}]$$

$$\boldsymbol{\Psi}(x) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^k0}, \ldots, \psi_{2^kM-1}]^T \tag{14}$$

The integration of the $\psi_{nm}(x)$ given in Eq. (10) can be shown as

$$p_{nm}(x) = \int_0^x \psi_{nm}(s)ds \tag{15}$$

For $m = 0$, $m = 1$ and $m > 1$, $p_{nm}(x)$ can be obtained as

$$p_{n0}(x) = \begin{cases} 0, & 0 \le x < \frac{n-1}{2^k} \\ \psi_{n0}(u) + \frac{2^{-1}}{\lambda}\sqrt{\frac{L_1^\lambda}{L_0^\lambda}}\psi_{n1}(u), & \frac{n-1}{2^k} \le x < \frac{n}{2^k} \\ 2, & \frac{n}{2^k} \le x < 1 \end{cases}$$

$$p_{n1}(x) = \begin{cases} 0, & 0 \le x < \frac{n-1}{2^k} \\ -\frac{\lambda(2\lambda+1)}{2(\lambda+1)}\sqrt{\frac{L_0^\lambda}{L_1^\lambda}}\psi_{n0}(u) + \frac{2^{-1}}{(\lambda+1)}\sqrt{\frac{L_2^\lambda}{L_1^\lambda}}\psi_{n2}(u), & \frac{n-1}{2^k} \le x < \frac{n}{2^k} \\ 0, & \frac{n}{2^k} \le x < 1 \end{cases}$$

$$p_{nm}(x) = \begin{cases} 0, & 0 \le x < \frac{n-1}{2^k} \\ \left(\frac{C_{m-1}^\lambda(-1) - C_{m+1}^\lambda(-1)}{2(\lambda+m)}\right)\sqrt{\frac{L_0^\lambda}{L_m^\lambda}}\psi_{n0}(u) - \frac{-2^{-1}}{(\lambda+m)}\sqrt{\frac{L_{m-1}^\lambda}{L_m^\lambda}}\psi_{nm-1}(u) & \\ \quad + \frac{2^{-1}}{(\lambda+m)}\sqrt{\frac{L_{m+1}^\lambda}{L_m^\lambda}}\psi_{nm+1}(u), & \frac{n-1}{2^k} \le x < \frac{n}{2^k} \\ \left(\frac{C_{m+1}^\lambda(1) - C_{m+1}^\lambda(-1) - C_{m-1}^\lambda(1) + C_{m-1}^\lambda(-1)}{2(\lambda+m)}\right)\sqrt{\frac{L_0^\lambda}{L_m^\lambda}}\psi_{n0}(u), & \frac{n}{2^k} \le x < 1 \end{cases}$$

where $u = 2^{k+1}x - 2n + 1$. The integration of the $\boldsymbol{\Psi}(x)$ can be represented as

$$\int_0^x \boldsymbol{\Psi}(s)ds = [p_{10}, p_{11}, \ldots, p_{1M-1}, p_{20}, \ldots, p_{2M-1}, \ldots, p_{2^k0}, \ldots, p_{2^kM-1}]^T = \mathbf{P_1\Psi_1}(x) \tag{16}$$

where

$$\boldsymbol{\Psi_1}(x) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M}, \psi_{20}, \ldots, \psi_{2M}, \ldots, \psi_{2^k0}, \ldots, \psi_{2^kM}]^T$$

$$\mathbf{L_1} = \begin{bmatrix} 1 & \frac{2^{-1}}{\lambda}\sqrt{\frac{L_1^\lambda}{L_0^\lambda}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\lambda(2\lambda+1)}{2(\lambda+1)}\sqrt{\frac{L_0^\lambda}{L_1^\lambda}} & 0 & \frac{2^{-1}}{(\lambda+1)}\sqrt{\frac{L_2^\lambda}{L_1^\lambda}} & 0 & \cdots & 0 & 0 & 0 \\ \left(\frac{C_1^\lambda(-1) - C_3^\lambda(-1)}{2(\lambda+2)}\right)\sqrt{\frac{L_0^\lambda}{L_2^\lambda}} & \frac{-2^{-1}}{(\lambda+2)}\sqrt{\frac{L_1^\lambda}{L_2^\lambda}} & 0 & \frac{2^{-1}}{(\lambda+2)}\sqrt{\frac{L_3^\lambda}{L_2^\lambda}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \left(\frac{C_{M-2}^\lambda(-1) - C_M^\lambda(-1)}{2(\lambda+M-1)}\right)\sqrt{\frac{L_0^\lambda}{L_{M-1}^\lambda}} & 0 & 0 & 0 & \cdots & \frac{-2^{-1}}{(\lambda+M-1)}\sqrt{\frac{L_{M-2}^\lambda}{L_{M-1}^\lambda}} & 0 & \frac{2^{-1}}{(\lambda+M-1)}\sqrt{\frac{L_M^\lambda}{L_{M-1}^\lambda}} \end{bmatrix}$$

$$\mathbf{F_1} = \begin{bmatrix} 2 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \left(\frac{C_3^\lambda(1)-C_3^\lambda(-1)-C_1^\lambda(1)+C_1^\lambda(-1)}{2(\lambda+2)}\right)\sqrt{\frac{L_0^\lambda}{L_2^\lambda}} & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ \left(\frac{C_{M-1}^\lambda(1)-C_{M-1}^\lambda(-1)-C_{M-3}^\lambda(1)+C_{M-3}^\lambda(-1)}{2(\lambda+M-2)}\right)\sqrt{\frac{L_0^\lambda}{L_{M-2}^\lambda}} & 0 \cdots 0 \\ \left(\frac{C_M^\lambda(1)-C_M^\lambda(-1)-C_{M-2}^\lambda(1)+C_{M-2}^\lambda(-1)}{2(\lambda+M-1)}\right)\sqrt{\frac{L_0^\lambda}{L_{M-1}^\lambda}} & 0 \cdots 0 \end{bmatrix}$$

$$\mathbf{P_1} = \frac{1}{2^{k+1}} \begin{bmatrix} \mathbf{L_1} & \mathbf{F_1} & \mathbf{F_1} & \cdots & \mathbf{F_1} & \mathbf{F_1} \\ \mathbf{0} & \mathbf{L_1} & \mathbf{F_1} & \cdots & \mathbf{F_1} & \mathbf{F_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{L_1} & \mathbf{F_1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L_1} \end{bmatrix}$$

The second integrations of the $\boldsymbol{\Psi}(x)$ can be represented as

$$\int_0^x \int_0^{x_1} (s)\,ds\,dx_1 = \int_0^x \mathbf{P_1}\boldsymbol{\Psi_1}(x_1)dx_1 = \mathbf{P_1}\int_0^x \boldsymbol{\Psi_1}(x_1)dx_1 = \mathbf{P_1}\mathbf{P_2}\boldsymbol{\Psi_2}(x) \neq \mathbf{P_1^2}\boldsymbol{\Psi}(x)$$

The rth integrations of the $\boldsymbol{\Psi}(x)$ can be represented as

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_{r-1}} (s)\,ds\,dx_{r-1}dx_{r-2}\cdots dx_1 = \mathbf{P_1}\mathbf{P_2}\cdots\mathbf{P_r}\boldsymbol{\Psi_r}(x) \neq \mathbf{P_1^r}\boldsymbol{\Psi}(x)$$

where

$$\mathbf{L_r} = \begin{bmatrix} 1 & \frac{2^{-1}}{\lambda}\sqrt{\frac{L_1^\lambda}{L_0^\lambda}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\lambda(2\lambda+1)}{2(\lambda+1)}\sqrt{\frac{L_0^\lambda}{L_1^\lambda}} & 0 & \frac{2^{-1}}{(\lambda+1)}\sqrt{\frac{L_2^\lambda}{L_1^\lambda}} & 0 & \cdots & 0 & 0 & 0 \\ \left(\frac{C_1^\lambda(-1)-C_3^\lambda(-1)}{2(\lambda+2)}\right)\sqrt{\frac{L_0^\lambda}{L_2^\lambda}} & \frac{-2^{-1}}{(\lambda+2)}\sqrt{\frac{L_1^\lambda}{L_2^\lambda}} & 0 & \frac{2^{-1}}{(\lambda+2)}\sqrt{\frac{L_3^\lambda}{L_2^\lambda}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \left(\frac{C_{M+r-3}^\lambda(-1)-C_{M+r-1}^\lambda(-1)}{2(\lambda+M+r-2)}\right)\sqrt{\frac{L_0^\lambda}{L_{M+r-2}^\lambda}} & 0 & 0 & 0 & \cdots & \frac{-2^{-1}}{(\lambda+M+r-2}\sqrt{\frac{L_{M+r-3}^\lambda}{L_{M+r-2}^\lambda}} & 0 & \frac{2^{-1}}{(\lambda+M+r-2)}\sqrt{\frac{L_{M+r-1}^\lambda}{L_{M+r-2}^\lambda}} \end{bmatrix}$$

$$\mathbf{F_r} = \begin{bmatrix} 2 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \left(\frac{C_3^\lambda(1)-C_3^\lambda(-1)-C_1^\lambda(1)+C_1^\lambda(-1)}{2(\lambda+M-1)}\right)\sqrt{\frac{L_0^\lambda}{L_2^\lambda}} & 0 \cdots 0 \\ \vdots & \vdots \ddots 0 \\ \left(\frac{C_{M-1}^\lambda(1)-C_{M-1}^\lambda(-1)-C_{M-3}^\lambda(1)+C_{M-3}^\lambda(-1)}{2(\lambda+M-1)}\right)\sqrt{\frac{L_0^\lambda}{L_{M-2}^\lambda}} & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ \left(\frac{C_{M+r-1}^\lambda(1)-C_{M+r-1}^\lambda(-1)-C_{M+r-3}^\lambda(1)+C_{M+r-3}^\lambda(-1)}{2(\lambda+M+r-2)}\right)\sqrt{\frac{L_0^\lambda}{L_{M+r-2}^\lambda}} & 0 \cdots 0 \end{bmatrix}$$

$$\mathbf{P_r} = \frac{1}{2^{k+1}} \begin{bmatrix} \mathbf{L_r} & \mathbf{F_r} & \mathbf{F_r} & \cdots & \mathbf{F_r} & \mathbf{F_r} \\ \mathbf{0} & \mathbf{L_r} & \mathbf{F_r} & \cdots & \mathbf{F_r} & \mathbf{F_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{L_r} & \mathbf{F_r} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L_r} \end{bmatrix}$$

and

$$\mathbf{\Psi_r}(x) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M+r-1}, \psi_{20}, \ldots, \psi_{2M+r-1}, \ldots, \psi_{2^k 0}, \ldots, \psi_{2^k M+r-1}]^T \quad (17)$$

The matrices $\mathbf{L_r}$ and $\mathbf{F_r}$ have the dimension $(M + r - 1) \times (M + r)$. Hence $\mathbf{P_r}$ has the dimension $2^k(M + r - 1) \times 2^k(M + r)$.

Gegenbauer Wavelet Collocation Method for the (GKS) Equation

Consider Eq. (1) with initial and boundary conditions

$$u(x, 0) = f(x), \quad u(0, t) = g_1(t), \quad u(1, t) = g_2(t), \quad u_x(0, t) = g_3(t), \quad u_{xx}(0, t) = g_4(t)$$

It is assumed that $\dot{u}^{(4)}(x, t)$ can be expanded in terms of truncated Gegenbauer wavelet series as

$$\dot{u}^{(4)}(x, t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{nm} \psi_{nm}(x) = \mathbf{C^T} \mathbf{\Psi}(x). \quad (18)$$

where "$\cdot$" and "$^{(4)}$" means differentiation with respect to t and x.

By integrating Eq. (18) with respect to t from t_s to t and four times with respect to x from 0 to x, following equations are obtained

$$u^{(4)}(x, t) = u^{(4)}(x, t_s) + (t - t_s)\mathbf{C^T}\mathbf{\Psi}(x) \quad (19)$$

$$u^{(3)}(x, t) = u^{(3)}(x, t_s) + u^{(3)}(0, t) - u^{(3)}(0, t_s) + (t - t_s)\mathbf{C^T}\mathbf{P_1}\mathbf{\Psi_1}(x) \quad (20)$$

$$u^{(2)}(x, t) = u^{(2)}(x, t_s) + u^{(2)}(0, t) - u^{(2)}(0, t_s) + x(u^{(3)}(0, t) - u^{(3)}(0, t_s)) + (t - t_s)\mathbf{C^T}\mathbf{P_1}\mathbf{P_2}\mathbf{\Psi_2}(x) \quad (21)$$

$$u_x(x, t) = u_x(x, t_s) + u_x(0, t) - u_x(0, t_s) + x(u^{(2)}(0, t) - u^{(2)}(0, t_s)) + \frac{x^2}{2}(u^{(3)}(0, t) - u^{(3)}(0, t_s)) + (t - t_s)\mathbf{C^T}\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(x) \quad (22)$$

$$u(x, t) = u(x, t_s) + u(0, t) - u(0, t_s) + x(u_x(0, t) - u_x(0, t_s)) + \frac{x^2}{2}(u^{(2)}(0, t) - u^{(2)}(0, t_s)) + \frac{x^3}{6}(u^{(3)}(0, t) - u^{(3)}(0, t_s)) + (t - t_s)\mathbf{C^T}\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{P_4}\mathbf{\Psi_4}(x) \quad (23)$$

From the initial and boundary conditions

$$u(x, 0) = f(x), \quad u(0, t) = g_1(t), \quad u(1, t) = g_2(t), \quad u_x(0, t) = g_3(t), \quad u_{xx}(0, t) = g_4(t) \quad (24)$$

we have the following equation as:

$$\frac{1}{6}(u^{(3)}(0,t)-u^{(3)}(0,t_s)) = -(g_1(t)-g_1(t_s))+(g_2(t)-g_2(t_s))-(g_3(t)-g_3(t_s)) - \frac{1}{2}(g_4(t)-g_4(t_s))-(t-t_s)\mathbf{C^T P_1 P_2 P_3 P_4 \Psi_4}(1) \tag{25}$$

If Eqs. (24, 25) are substituted into Eqs. (19–23), the following equations are obtained.

$$u^{(4)}(x,t) = (t-t_s)\mathbf{C^T \Psi}(x) + u^{(4)}(x,t_s) \tag{26}$$

$$u^{(3)}(x,t) = u^{(3)}(x,t_s)+(t-t_s)\mathbf{C^T}[\mathbf{P_1\Psi_1}(x)-6\mathbf{P_1P_2P_3P_4\Psi_4}(1)]-6(g_1(t)-g_1(t_s)) + 6(g_2(t)-g_2(t_s))-6(g_3(t)-g_3(t_s)) -3(g_4(t)-g_4(t_s)) \tag{27}$$

$$u_{xx}(x,t) = u_{xx}(x,t_s)+(t-t_s)\mathbf{C^T}[\mathbf{P_1P_2\Psi_2}(x)-6x\mathbf{P_1P_2P_3P_4\Psi_4}(1)]+g_4(t)-g_4(t_s) + 6x\left(-(g_1(t)-g_1(t_s))+(g_2(t)-g_2(t_s))-(g_3(t)-g_3(t_s))-\frac{1}{2}(g_4(t)-g_4(t_s))\right) \tag{28}$$

$$u_x(x,t) = u_x(x,t_s)+(t-t_s)\mathbf{C^T}\left[\mathbf{P_1P_2P_3\Psi_3}(x)-3x^2\mathbf{P_1P_2P_3P_4\Psi_4}(1)\right]-3x^2(g_1(t)-g_1(t_s)) + 3x^2(g_2(t)-g_2(t_s))+(1-3x^2)(g_3(t)-g_3(t_s))+\left(x-\frac{3x^2}{2}\right)(g_4(t)-g_4(t_s)) \tag{29}$$

$$u(x,t) = u(x,t_s)+(t-t_s)\mathbf{C^T}\left[\mathbf{P_1P_2P_3P_4\Psi_4}(x)-x^3\mathbf{P_1P_2P_3P_4\Psi_4}(1)\right] + (1-x^3)(g_1(t)-g_1(t_s))+x^3(g_2(t)-g_2(t_s)) + (x-x^3)(g_3(t)-g_3(t_s)) + (x^2-x^3)\frac{1}{2}(g_4(t)-g_4(t_s)) \tag{30}$$

$$u_t(x,t) = \mathbf{C^T}\left[\mathbf{P_1P_2P_3P_4\Psi_4}(x)-x^3\mathbf{P_1P_2P_3P_4\Psi_4}(1)\right] + (1-x^3)g_1'(t)+x^3g_2'(t)+(x-x^3)g_3'(t)+(x^2-x^3)\frac{1}{2}g_4'(t) \tag{31}$$

Nonlinear Eq. (1) is converted into a sequence of linear differential equations by quasilinearization technique. First approximate solution satisfying initial/boundary conditions is taken as

$$u^0(x,t) = f(x)+(1-x^3)(g_1(t)-g_1(t_s)) + x^3(g_2(t)-g_2(t_s)) + (x-x^3)(g_3(t)-g_3(t_s)) + (x^2-x^3)\frac{1}{2}(g_4(t)-g_4(t_s)) \tag{32}$$

and

$$(u(x,t)\,u_x(x,t))^{l+1} \cong u^{l+1}(x,t)\,u_x^l(x,t)+u^l(x,t)\,u_x^{l+1}(x,t)-u^l(x,t)\,u_x^l(x,t) \tag{33}$$

can be obtained by quasilinearization technique. Hence converted problem is obtained as

$$u_t^{l+1}(x,t)+u^{l+1}(x,t)\,u_x^l(x,t)+u^l(x,t)\,u_x^{l+1}(x,t)+\alpha u_{xx}^{l+1}(x,t) + \beta u_{xxx}^{l+1}(x,t)+\gamma u_{xxxx}^{l+1}(x,t)-u^l(x,t)\,u_x^l(x,t)=0 \tag{34}$$

where l is index of quasilinearization technique and $l = 0, 1, 2, \ldots$

Replacing Eqs. (26–31) into the Eq. (34), we have the following equation.

$$\mathbf{C^T}(t-t_s)\begin{bmatrix} \left(\frac{1}{t-t_s}+u_x^l(x,t)\right)\mathbf{P_1P_2P_3P_4\Psi_4}(x)+u^l(x,t)\mathbf{P_1P_2P_3\Psi_3}(x)+\alpha\mathbf{P_1P_2\Psi_2}(x) \\ +\beta\mathbf{P_1\Psi_1}(x)+\delta(x)-\left(\frac{x^3}{t-t_s}+x^3u_x^l(x,t)+3x^2u^l(x,t)+6\alpha x+6\beta\right)\mathbf{P_1P_2P_3P_4\Psi_4}(1)\end{bmatrix}$$
$$=u^l(x,t)u_x^l(x,t)-u_x^l(x,t)u(x,t_s)-u^l(x,t)u_x(x,t_s)-\alpha u^{(2)}(x,t_s)-\beta u^{(3)}(x,t_s)-\delta u^{(4)}(x,t_s)$$
$$-g_1'(t)-xg_3'(t)-\frac{x^2}{2}g_4'(t)-x^3\left(g_2'(t)-g_1'(t)-g_3'(t)-\frac{1}{2}g_4'(t)\right)$$
$$-\left(x^3u_x^l(x,t)+3x^2u^l(x,t)+6\alpha x+6\beta\right)(g_2(t)-g_2(t_s))$$
$$+\left((x^3-1)u_x^l(x,t)+3x^2u^l(x,t)+6\alpha x+6\beta\right)(g_1(t)-g_1(t_s))$$
$$+\left((x^3-x)u_x^l(x,t)+(3x^2-1)u^l(x,t)+6\alpha x+6\beta\right)(g_3(t)-g_3(t_s))$$
$$+\left((x^3-x^2)u_x^l(x,t)+(3x^2-2x)u^l(x,t)+6\alpha x-2\alpha+6\beta\right)\left(\frac{1}{2}g_4(t)-\frac{1}{2}g_4(t_s)\right) \quad (35)$$

The collocation points can be taken as $2^{k+1}x_{ni}-2n+1=\cos\frac{((M+1)-i)\pi}{(M+1)}$ or

$$x_{ni}=\frac{1}{2^{k+1}}\left(2n-1+\cos\frac{((M+1)-i)\pi}{(M+1)}\right),\quad i=1,2,\ldots,M,\quad n=1,2,\ldots,2^k. \quad (36)$$

Substituting the collocation points $x\rightarrow x_{ni}$ and time variable $t\rightarrow t_{s+1}$ into (35), a discretized form of the vectors $(x_{\mathbf{ni}})$, $\mathbf{\Psi_1}(x_{\mathbf{ni}})$ and $\mathbf{\Psi_r}(x_{\mathbf{ni}})$ can be obtained. Hence form Eq. (35), we obtain algebraic equation system whose matrix notation is

$$\mathbf{C^TU}=\mathbf{B} \quad (37)$$

where $\mathbf{U}$ is a $2^kM\times 2^kM$ matrix. $\mathbf{C}$ and $\mathbf{B}$ are $2^kM\times 1$ vectors. Hence, by solving algebraic equation system (37), we can find the coefficients of the Gegenbauer wavelet series that satisfied differential equation and given initial and boundary conditions.

Error Analysis

Theorem 1 *Let $f(x)\in L_w^2[0,1]$ with bounded second order derivative $\left|f''(x)\right|\leq N$, can be expanded as an infinite sum of Gegenbauer wavelets, and the series converges uniformly to $f(x)$* [34]. *That is*

$$f(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}f_{nm}\psi_{nm}(x),\quad \forall\lambda>-\frac{1}{2}.$$

Theorem 2 *Let $f(x)\in L_w^2[0,1]$ with bounded second order derivative $\left|f''(x)\right|\leq N$, then we have the following accuracy estimation* [34]:

$$\int_0^1\left[f(x)-\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}f_{nm}\psi_{nm}(x)\right]^2\omega_n(x)dx<N^2\sum_{n=2^k+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5m(m-1)(m+2\lambda)(m+2\lambda+1)}.$$

Table 1 Absolute error of Example 1 with Gegenbauer wavelets collocation method for various collocation points

Chebyshev collocation points				Equal collocation points			
x	$M=4, k=1$	x	$M=8, k=1$	x	$M=4, k=1$	x	$M=8, k=1$
0.04774575	2.61586997e−9	0.01507684	2.26598740e−11	0.0625	1.75964698e−8	0.03125	1.95919947e−10
0.17274575	3.32663401e−8	0.05848889	1.25805988e−9	0.1875	2.41808250e−7	0.09375	5.00250996e−9
0.32725425	5.59853190e−7	0.12500000	1.15190100e−8	0.3125	1.06522922e−6	0.15625	2.18296501e−8
0.45225425	1.36318556e−6	0.20658796	4.81409800e−8	0.4375	2.77156372e−6	0.21875	5.64671301e−8
0.54774575	2.23206888e−6	0.29341204	1.26832590e−7	0.5625	5.68628948e−6	0.28125	1.12998580e−7
0.67274575	3.20302778e−6	0.37500000	2.43217430e−7	0.6875	8.20304648e−6	0.34375	1.93642540e−7
0.82725425	3.03586403e−6	0.44151111	3.67198115e−7	0.8125	7.95086998e−6	0.40625	2.98326525e−7
0.95225425	1.17876960e−6	0.48492316	4.59499284e−7	0.9375	3.73244576e−6	0.46875	4.24031297e−7
		0.51507684	5.27235111e−7			0.53125	5.63969164e−7
		0.55848889	6.27074241e−7			0.59375	7.06768080e−7
		0.62500000	7.74684882e−7			0.65625	8.35644521e−7
		0.70658796	9.14607526e−7			0.71875	9.27868457e−7
		0.79341204	9.49834378e−7			0.78125	9.54575951e−7
		0.87500000	7.95381554e−7			0.84375	8.81225633e−7
		0.94151111	4.72415206e−7			0.90625	6.68849318e−7
		0.98492316	1.40561170e−7			0.96875	2.76358340e−7

Table 2 Absolute error of Example 1 with Gegenbauer wavelets collocation method for various values of λ

x	$\lambda = -0.49$	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 10$
0.01507684	2.23632224e−11	2.26598740e−11	2.26598740e−11	2.22200036e−11	2.03799200e−11	2.02398098e−11
0.05848889	1.25783450e−9	1.25805988e−9	1.25906996e−9	1.25869004e−9	1.25500987e−9	1.25745991e−9
0.12500000	1.15185088e−8	1.15190100e−8	1.15191698e−8	1.15185899e−8	1.15175098e−8	1.15162999e−8
0.20658796	4.81393385e−8	4.81409800e−8	4.81402800e−8	4.81393401e−8	4.81400699e−8	4.81388800e−8
0.29341204	1.26833607e−7	1.26832590e−7	1.26833810e−7	1.26833120e−7	1.26833950e−7	1.26833330e−7
0.37500000	2.43218429e−7	2.43217430e−7	2.43219410e−7	2.43218140e−7	2.43218830e−7	2.43218340e−7
0.44151111	3.67197280e−7	3.67198115e−7	3.67197116e−7	3.67198186e−7	3.67199425e−7	3.67196257e−7
0.48492316	4.59501065e−7	4.59499284e−7	4.59501474e−7	4.59501545e−7	4.59500644e−7	4.59502484e−7
0.51507684	5.27235048e−7	5.27235111e−7	5.27235181e−7	5.27235342e−7	5.27236371e−7	5.27236237e−7
0.55848889	6.27073456e−7	6.27074241e−7	6.27074622e−7	6.27073572e−7	6.27073053 e-7	6.27074604 e-7
0.62500000	7.74684942e−7	7.74684882e−7	7.74683294e−7	7.74684452e−7	7.74684726e−7	7.74683753e−7
0.70658796	9.14607159e−7	9.14607526e−7	9.14608106e−7	9.14607386e−7	9.14605466e−7	9.14606537e−7
0.79341204	9.49833959e−7	9.49834378e−7	9.49834616e−7	9.49833727e−7	9.49833326e−7	9.49833098e−7
0.87500000	7.95381846e−7	7.95381554e−7	7.95382092e−7	7.95381534e−7	7.95381557e−7	7.95383352e−7
0.94151111	4.72416114e−7	4.72415206e−7	4.72416711e−7	4.72416034e−7	4.72419674e−7	4.72416787e−7
0.98492316	1.40563018e−7	1.40561170e−7	1.40564510e−7	1.40563150e−7	1.40563220e−7	1.40561050e−7

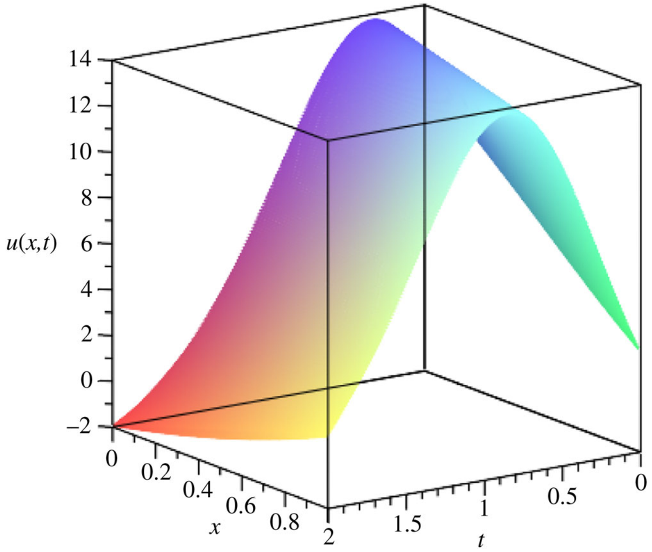

Fig. 1 Approximate solution of Example 1 for $\delta t = 0.01$

Table 3 Absolute error of Example 1 with Gegenbauer wavelets collocation method for various values of c

x	$c = 0.1$	$c = 0.01$	$c = 0.001$
0.01507684	2.4993341e−12	9.0594199e−14	4.9737992e−14
0.05848889	1.3639934e−10	3.2187586e−12	7.1054274e−14
0.12500000	1.2436310e−9	3.0460967e−11	4.4941828e−13
0.20658796	5.1759201e−9	1.2735946e−10	1.3598012e−12
0.29341204	1.3475519e−8	3.3730974e−10	3.8999914e−12
0.37500000	2.5280230e−8	6.4892980e−10	7.0308204e−12
0.44151111	3.7133320e−8	9.8397024e−10	1.0990320e−11
0.48492316	4.5406010e−8	1.2361596e−9	1.3800516e−11
0.51507684	5.1139519e−8	1.4226904e−9	1.5740298e−11
0.55848889	5.8993770e−8	1.7003199e−9	1.8970603e−11
0.62500000	6.8938700e−8	2.1177997e−9	2.3390179e−11
0.70658796	7.4917600e−8	2.5288700e−9	2.8109959e−11
0.79341204	7.0018120e−8	2.6597502e−9	2.9360070e−11
0.87500000	5.2327940e−8	2.2532900e−9	2.4680036e−11
0.94151111	2.8093070e−8	1.3508560e−9	1.4901969e−11
0.98492316	7.8064701e−9	4.0413994e−10	4.4140247e−12

Numerical Results

Example 1 Consider generalized Kuramoto–Sivashinsky Eq. (1) with $\alpha = \gamma = 1$ and $\beta = 4$. Analytic solution is given in [46, 47] as:

$$u(x,t) = 2c + 9 + 15\Big(\tanh(ct - \frac{x}{2}) - \tanh^2(ct - \frac{x}{2}) - \tanh^3(ct - \frac{x}{2})\Big)$$

Table 4 Comparisons of the maximal absolute error of Example 1 for various values of c

c	Present Method	Method in [48]	Method in [47]	Method in [46]
0.1	7.4917600e−08	6.8e−05	7.7e−07	2.6e−04
0.01	2.6597502e−09	6.2e−07	1.8e−06	3.2e−05
0.001	2.9360070e−11	9.2e−09	1.6e−06	3.2e−05

Table 5 Absolute error of Example 2 with Gegenbauer wavelets collocation method for various collocation points

x	$M=4, k=0$	x	$M=4, k=1$	x	$M=4, k=2$
0.09549150	2.9354396e−7	0.04774575	2.0281300e−9	0.02387288	4.1450000e−11
0.34549150	3.7211226e−5	0.17274575	1.0908598e−7	0.08637288	2.641830e−9
0.65450850	3.7502062e−5	0.32725425	8.4619394e−7	0.16362712	1.5980410e−8
0.90450850	2.4764269e−6	0.45225425	2.1113651e−6	0.22612712	4.0483570e−8
		0.54774575	3.5793555e−6	0.27387288	7.0787060e−8
		0.67274575	5.4805196e−6	0.33637288	1.2715851e−7
		0.82725425	5.8101975e−6	0.41362712	2.2564562e−7
		0.95225425	2.5247867e−6	0.47612712	3.2826200e−7
				0.52387288	4.1854614e−7
				0.58637288	5.4450088e−7
				0.66362712	6.8993780e−7
				0.72612712	7.7449934e−7
				0.77387288	8.0216595e−7
				0.83637288	7.6478157e−7
				0.91362712	5.5035677e−7
				0.97612712	1.9107950e−7

The required initial and boundary conditions can be obtained from the exact solution. This nonlinear differential equation is converted into a sequence of linear differential equation generated by quasilinearization technique in (34). Replacing initial boundary conditions into the Eq. (35) and solving algebraic equation system in Eq. (37), we have coefficients $\mathbf{C}^{\mathbf{T}}$ of the Chebyshev wavelet series. By substituting the Gegenbauer wavelet coefficients into Eq. (30), we have the implicit form of the approximate solution satisfied differential equation and whose boundary conditions. Table 1 shows the absolute errors in Chebyshev and equal collocation points for $c=1$ $\lambda=0$, $t=2$, $M=4$, $k=1$, $M=8$, $k=1$. We can see that if M or k increase; approximate results are converged to the exact solution and Chebyshev collocation points give better results from equal collocation points. Table 2 shows the absolute errors in collocation points for $c=1$, $M=8$, $k=1$, $t=2$, $\delta t=(t_{s+1}-t_s)=0.02$ and various values of λ. Graphical presentation of the approximate solution is given in Fig. 1 for $\lambda=0$, $c=1$, $M=16$, $k=1$, $t=2$ and $\delta t=0.01$. Table 3 shows the absolute errors in collocation points for $M=8$, $k=1$, $t=1$, $\delta t=0.05$ and various values of c. Comparisons of the maximal absolute error of present method [46–48] are given in the Table 4. As can be seen in Table 4, it is clear that the results obtained by the presented method are superior respect to [46–48].

Table 6 Absolute error of Example 2 with Gegenbauer wavelets collocation method for various values of λ

x	$a = -0.49$	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 10$
0.01507684	6.2474470e−12	6.2600000e−12	6.1900000e−12	6.1800000e−12	6.3600000e−12	6.0700000e−12
0.05848889	3.6389203e−10	3.6396000e−10	3.6377000e−10	3.6395000e−10	3.6385000e−10	3.6408000e−10
0.12500000	3.5479326e−9	3.5477300e−9	3.5479300e−9	3.5479300e−9	3.5478600e−9	3.5480600e−9
0.20658796	1.5928221e−8	1.5928120e−8	1.5928280e−8	1.5928200e−8	1.5928100e−8	1.5928120e−8
0.29341204	4.5004827e−8	4.5004490e−8	4.5004810e−8	4.5004720e−8	4.5004650e−8	4.5004750e−8
0.37500000	9.1577141e−8	9.1577110e−8	9.1577250e−8	9.1577070e−8	9.1577220e−8	9.1577590e−8
0.44151111	1.4444164e−7	1.4444186e−7	1.4444186e−7	1.4444181e−7	1.4444150e−7	1.4444142e−7
0.48492316	1.8558520e−7	1.8558524e−7	1.8558543e−7	1.8558526e−7	1.8558525e−7	1.8558508e−7
0.51507684	2.1667357e−7	2.1667353e−7	2.1667359e−7	2.1667350e−7	2.1667371e−7	2.1667380e−7
0.55848889	2.6388339e−7	2.6388330e−7	2.6388328e−7	2.6388331e−7	2.6388343e−7	2.6388307e−7
0.62500000	3.37119470e−7	3.3711949e−7	3.3711926e−7	3.3711939e−7	3.3711959e−7	3.3711919e−7
0.70658796	4.13096678e−7	4.1309673e−7	4.1309674e−7	4.1309679e−7	4.1309718e−7	4.1309651e−7
0.79341204	4.44710428e−7	4.4471045e−7	4.4471068e−7	4.4471051e−7	4.4471037e−7	4.4471018e−7
0.87500000	3.84363197e−7	3.8436325e−7	3.8436337e−7	3.8436313e−7	3.8436344e−7	3.8436297e−7
0.94151111	2.34122231e−7	2.3412232e−7	2.3412256e−7	2.3412220e−7	2.3412196e−7	2.3412201e−7
0.98492316	7.08286318e−8	7.0828640e−8	7.0828790e−8	7.0828690e−8	7.0828980e−8	7.0828130e−8

Table 7 Comparisons of the maximal absolute error of Example 2 for various values of t

t	Present Method	Method in [48]
2	2.1419919e−7	2.44e−6
3	7.9603120e−8	2.69e−6
5	5.6990440e−8	1.55e−6
7	9.8184860e−8	1.68e−7
9	5.4814700e−8	4.49e−7
10	3.0481850e−8	3.35e−7

Table 8 Absolute error of Example 3 with Gegenbauer wavelets collocation method for various collocation points

x	$M=4, k=0$	x	$M=4, k=1$	x	$M=4, k=2$
0.09549150	1.70e−13	0.04774575	8.10e−15	0.02387288	4.00e−14
0.34549150	2.50e−13	0.17274575	3.00e−14	0.08637288	0
0.65450850	2.90e−13	0.32725425	1.20e−13	0.16362712	1.20e−13
0.90450850	6.00e−14	0.45225425	1.50e−13	0.22612712	8.00e−14
		0.54774575	1.00e−14	0.27387288	1.00e−14
		0.67274575	1.30e−13	0.33637288	1.30e−13
		0.82725425	0	0.41362712	1.10e−13
		0.95225425	7.00e−14	0.47612712	0
				0.52387288	0
				0.58637288	7.00e−14
				0.66362712	0
				0.72612712	8.00e−14
				0.77387288	1.40e−13
				0.83637288	1.90e−13
				0.91362712	7.00e−14
				0.97612712	1.20e−13

Example 2 Consider generalized Kuramoto–Sivashinsky Eq. (1) with $\alpha=1$ $\beta=0$ and $\gamma=0.5$. Analytic solution is given in [47] as:

$$u(x,t)=-\frac{0.1}{K}+\frac{60}{19}K(-38K^2\gamma+\alpha)\tanh(0.1t+Kx)+120\gamma K^3\tanh^3(0.1t+Kx)$$

where $K=\frac{1}{2}\left(\frac{11\alpha}{19\gamma}\right)^{\frac{1}{2}}$. The required initial and boundary conditions can be obtained from the exact solution. Table 5 shows the absolute errors in collocation points for $\lambda=0$, $t=2$ $M=4$, $k=0$, $M=4$, $k=1$ and $M=4$, $k=2$. Table 6 shows the absolute errors in collocation points for $M=8$, $k=1$, $t=2$, $\delta t=0.02$ and various values of λ. Comparisons of the maximal absolute error of present method and [48] are given in the Table 7 for $M=8$, $k=0$ and $\delta t=0.01$. As can be seen in Table 7, it is clear that the results obtained by the presented method are superior respect to [48].

Table 9 Absolute error of Example 3 with Gegenbauer wavelets collocation method for various values of λ

x	$\lambda = -0.49$	$\lambda = 0$	$a = 0.5$	$a = 1$	$\lambda = 2$	$\lambda = 10$
0.01507684	3.96557787e−15	1.500e−13	5.000e−14	7.000e−14	4.600e−13	1.690e−14
0.05848889	2.21211938e−14	4.000e−14	1.100e−13	2.320e−14	5.000e−13	4.100e−13
0.12500000	3.26405569e−14	6.000e−14	3.200e−13	9.000e−14	2.900e−13	4.700e−13
0.20658796	3.05623582e−14	2.000e−14	2.290e−14	6.000e−14	6.000e−14	1.800e−13
0.29341204	1.33226763e−14	0	9.000e−14	0	5.000e−14	9.000e−14
0.37500000	3.32026073e−14	6.000e−14	2.300e−14	1.600e−13	2.000e−13	2.300e−14
0.44151111	1.09981468e−14	6.000e−14	1.000e−14	1.380e−14	4.100e−13	1.200e−13
0.48492316	2.62741218e−14	5.000e−14	1.400e−13	5.000e−14	2.100e−13	1.700e−13
0.51507684	1.69586567e−14	6.000e−14	1.200e−13	0	1.400e−13	4.300e−13
0.55848889	5.06261699e−14	1.500e−13	9.000e−14	2.170e−14	1.500e−13	2.170e−14
0.62500000	6.32827124e−14	1.280e−14	7.000e−14	1.100e−13	3.100e−13	1.700e−13
0.70658796	8.03801470e−14	2.300e−13	2.800e−13	1.500e−13	7.000e−14	3.500e−13
0.79341204	5.81756865e−14	1.800e−13	1.200e−13	3.200e−14	2.000e−13	4.400e−13
0.87500000	8.48973670e−15	1.200e−13	7.000e−14	1.940e−14	5.000e−14	3.500e−13
0.94151111	8.88178420e−16	7.000e−14	4.000e−14	9.100e−15	2.200e−13	9.000e−14
0.98492316	3.81916720e−14	3.000e−14	1.700e−13	2.000e−14	2.100e−13	9.000e−14

Table 10 Comparisons of the maximal absolute error of Example 3 for various values of t

t	Present Method	Method in [48]	Method in [46]
0.2	1.06e−12	4.41e−10	2.38e−07
0.4	9.93e−13	2.38e−10	2.38e−07
0.6	8.10e−13	6.36e−10	2.38e−07
0.8	7.20e−13	1.05e−09	1.19e−07
1	5.70e−13	5.95e−10	2.38e−07
3	5.80e−13	3.01e−09	–
5	7.30e−13	3.05e−09	–
10	3.10e−13	6.88e−10	–

Example 3 Consider generalized Kuramoto–Sivashinsky Eq. (1) with $\alpha = 1$ $\beta = \frac{12}{\sqrt{47}}$ and $\gamma = 1$. Analytic solution is given in [46] as:

$$u(x,t) = -\frac{396.8}{47\sqrt{47}} + \frac{15}{47\sqrt{47}}\left(3\tanh(\theta) - 3\tanh^2(\theta) + \tanh^3(\theta)\right)$$

where $\theta = 0.1\,t + \frac{1}{2\sqrt{47}}x$. The required initial and boundary conditions can be obtained from the exact solution. Table 8 shows the absolute errors in collocation points for $\lambda = 0$, $t = 2$, $M = 4$, $k = 0$, $M = 4$, $k = 1$ and $M = 4$, $k = 2$. Table 9 shows the absolute errors in collocation points for $M = 8$, $k = 1$, $t = 2$, $\delta t = 0.02$ and various values of λ. Comparisons of the maximal absolute error of present method and [46, 48] are given in the Table 10 for $M = 8$, $k = 0$ and $\delta t = 0.01$. As can be seen in Table 10, it is clear that the results obtained by the presented method are superior respect to [46, 48].

Conclusion

Gegenbauer wavelet collocation method is proposed to obtain approximate solution of generalized Kuramoto–Sivashinsky equation. The method has been applied to the three nonlinear differential equations by using quasilinearization technique. Approximate and exact solutions of examples are correspondingly compared. For all examples, comparisons of the maximal absolute errors given in Tables 4,7 and 10 show that the results obtained by the proposed method are better than the represented in [46–48]. As can be seen from all tables, the present method is highly efficient and accurate. All of the calculations have been made by Maple program with 15 digits. These calculations demonstrated that the accuracy of the Gegenbauer wavelet collocation method is quite high even in the case of a small number of grid points. Application of proposed method is very simple because there are no complex integrals or methodology. Moreover, the this method is reliable, simple, fast, minimal computation costs, flexible, and convenient alternative method.

References

1. Cattani, C.: Fractional calculus and shannon wavelets. Math. Probl. Eng. **2012**, 25, Article ID 502812 (2012)
2. Heydari, M.H., Hooshmandasl, M.R., Ghaini, F.M.M., Mohammadi, F.: Wavelet collocation method for solving multiorder fractional differential equations. J. Appl. Math. **2012**, 19, Article ID 542401 (2012)
3. Heydari, M.H., Hooshmandasl, M.R., Ghaini, F.M.M., Fereidouni, F.: Two-dimensional Legendre wavelets forsolving fractional poisson equation with dirichlet boundary conditions. Eng. Anal. Bound. Elem. **37**, 1331–1338 (2013)
4. Cattani, C.: Shannon wavelets for the solution of integro-differential equations. Math. Probl. Eng. **2010**, 22, Article ID 408418 (2010)
5. Cattani, C., Kudreyko, A.: Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind. Appl. Math. Comput. **215**(12), 4164–4171 (2010)
6. Heydari, M.H., Hooshmandasl, M.R., Mohammadi, F., Cattani, C.: Wavelets method for solving systems of nonlinear singular fractional volterra integro-differential equations. Commun. Nonlinear Sci. Numer. Simul. **19**(1), 37–48 (2014)
7. Gu, J.S., Jiang, W.S.: The Haar wavelets operational matrix of integration. Int. J. Syst. Sci. **27**(7), 623–628 (1996)
8. Lepik, U.: Numerical solution of evolution equations by the Haar wavelet method. Appl. Math. Comput. **185**, 695–704 (2007)
9. Geng, W., Chen, Y., Li, Y., Wang, D.: Wavelet method for nonlinear partial differential equations of fractional order. Comput. Inf. Sci. **4**(5), 28–35 (2011)
10. Hariharan, G., Kannan, K., Sharma, K.R.: Haar wavelet method for solving Fisher's equation. Appl. Math. Comput. **211**, 284–292 (2009)
11. Hariharan, G., Kannan, K.: Haar wavelet method for solving FitzHugh-Nagumo equation. Int. J. Math. Stat. Sci. **2**(2), 59–63 (2010)
12. Hariharan, G., Kannan, K.: A comparative study of a Haar wavelet method and a restrictive Taylor's series method for solving convection-diffusion equations. Int. J. Comput. Methods Eng. Sci. Mech. **11**(4), 173–184 (2010)
13. Hariharan, G.: An efficient wavelet analysis method to film-pore diffusion model arising in mathematical chemistry. J. Membr Biol. **247**(4), 339–343 (2014)
14. Hariharan, G., Kannan, K.: Haar wavelet method for solving nonlinear parabolic equations. J. Math. Chem. **48**(4), 1044–1061 (2010)
15. Kaur, H., Mittal, R.C., Mishra, V.: Haar wavelet quasilinearization approach for solving nonlinear boundary value problems. Am. J. Comput. Math. **1**, 176–182 (2011)
16. Çelik, İ.: Haar wavelet method for solving generalized Burgers-Huxley equation. Arab. J. Math. Sci. **18**, 25–37 (2012)
17. Çelik, İ.: Haar wavelet approximation for magnetohydrodynamic flow equations. Appl. Math. Model. **37**, 3894–3902 (2013)

18. Maleknejad, K., Kajani, M.T., Mahmoudi, Y.: Numerical solution of linear Fredholm and Volterra integral equation of the second kind by using Legendre wavelets. Kybernetes **32**(9/10), 1530–1539 (2003)
19. Kajani, M.T., Vencheh, A.H.: Solving linear integro-differential equation with Legendre wavelet. Int. J. Comput. Math. **81**(6), 719–726 (2004)
20. Razzaghi, M., Yousefi, S.: Legendre wavelets direct method for variational problems. Math. Comput. Simul. **53**, 185–192 (2000)
21. Razzaghi, M., Yousefi, S.: Legendre wavelets operational matrix of integration. Int. J. Syst. Sci. **32**(4), 495–502 (2001)
22. Babolian, E., Fattahzadeh, F.: Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration. Appl. Math. Comput. **188**, 417–426 (2007)
23. Babolian, E., Fattahzadeh, F.: Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration. Appl. Math. Comput. **188**(1), 1016–1022 (2007)
24. Kajania, M.T., Vencheha, A.H., Ghasemib, M.: The Chebyshev wavelets operational matrix of integration and product operation matrix. Int. J. Comput. Math. **86**(7), 1118–1125 (2009)
25. Adibi, H., Assari, P.: Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind. Math. Probl. Eng. **2010**, 17, Article ID 138408 (2010)
26. Wang, Y.X., Fan, Q.B.: The second kind Chebyshev wavelet method for solving fractional differential equations. Appl. Math. Comput. **218**, 8592–8601 (2012)
27. Heydari, M.H., Hooshmandasl, M.R., Ghaini, F.M.M., Li, M.: Chebyshev wavelets method for solution of nonlinear fractional integrodifferential equations in a large interval. Adv. Math. Phys. **2013**, 12, Article ID 482083 (2013)
28. Hooshmandasl, M.R., Heydari, M.H., Ghaini, F.M.M.: Numerical solution of the one dimensional heat equation by using chebyshev wavelets method. Appl. Comput. Math. **1**(6), 1–7 (2012)
29. Yang, C., Hou, J.: Chebyshev wavelets method for solving Bratu's problem. Bound. Value Probl. **2013**, 142 (2013)
30. Hariharan, G.: An efficient wavelet based approximation method to water quality assessment model in a uniform channel. Ain Shams Eng. J. **5**(2), 525–532 (2014)
31. Çelik, İ.: Numerical solution of differential equations by using Chebyshev wavelet collocation method. Cankaya Univ. J. Sci. Eng. **10**(2), 169–184 (2013)
32. Çelik, İ.: Chebyshev Wavelet collocation method for solving generalized Burgers-Huxley equation. Math. Methods Appl. Sci. **39**, 366–377 (2016)
33. Çelik, İ.: Free vibration of non-uniform Euler-Bernoulli beam under various supporting conditions using Chebyshev wavelet collocation method. Appl. Math. Model. **54**, 268–280 (2018)
34. Pathak, A., Singh, R.K., Mandal, B.N.: Solution of Abel's integral equation by using Gegenbauer wavelets. Investig. Math. Sci. **4**(1), 43–52 (2014)
35. Abd-Elhameed, W.M., Youssri, Y.H.: New spectral solutions of multi-term fractional order initial value problems with error analysis. Comput. Model. Eng. Sci. **105**(5), 375–398 (2015)
36. Abd-Elhameed, W.M., Youssri, Y.H.: New ultraspherical wavelets spectral solutions for fractional Riccati differential equations. In: Abstract and Applied Analysis, vol. 2014 Hindawi (2014)
37. Rehman, M., Saeed, U.: Gegenbauer wavelets operational matrix method for fractional differential equations. J. Korean Math. Soc. **52**(5), 1069–1096 (2015)
38. Abd-Elhameed, W.M., Youssri, Y.H., Doha, E.H.: New solutions for singular Lane–Emden equations arising in astrophysics based on shifted ultraspherical operational matrices of derivatives. Comput. Methods Differ. Equ. **2**(3), 171–185 (2014)
39. Youssri, Y.H., Abd-Elhameed, W.M., Doha, E.H.: Ultraspherical wavelets method for solving Lane–Emden type equations. Rom. J. Phys. **60**(9), 1298–1314 (2015)
40. Youssri, Y.H., Abd-Elhameed, W.M., Doha, E.H.: Accurate spectral solutions of first-and second-order initial value problems by the ultraspherical wavelets-Gauss collocation method. Appl. Appl. Math. Int. J. **10**(2), 835–851 (2015)
41. Doha, E.H., Abd-Elhameed, W.M., Youssri, Y.H.: New ultraspherical wavelets collocation method for solving 2nth-order initial and boundary value problems. J. Egypt. Math. Soc. **24**(2), 319–327 (2016)
42. Schmuck, M., Pradas, M., Pavliotis, G.A., Kalliadasis, S.: A new mode reduction strategy for the generalized Kuramoto–Sivashinsky equation. IMA J. Appl. Math. **80**(2), 273–301 (2013)
43. Mandelzweig, V.B., Tabakin, F.: Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs. Comput. Phys. Commun. **141**(2), 268–281 (2001)
44. Szegö, G.: Orthogonal Polynomials, 4th edn. American Mathematical Society, Providence (1975)
45. Daubechies, I.: Ten Lectures on Wavelets. SIAM, Philadelphia (1992)
46. Khater, A.H., Temsah, R.S.: Numerical solutions of the generalized Kuramoto–Sivashinsky equation by Chebyshev spectral collocation methods. Comput. Math Appl. **56**(6), 1465–1472 (2008)

47. Lakestani, M., Dehghan, M.: Numerical solutions of the generalized Kuramoto–Sivashinsky equation using B-spline functions. Appl. Math. Model. **36**, 605–617 (2012)
48. Rashidinia, J., Jokar, M.: Polynomial scaling functions for numerical solution of generalized Kuramoto–Sivashinsky equation. Appl. Anal. **96**(2), 293–306 (2017)