Int. J. Appl. Comput. Math (2018) 4:101
https://doi.org/10.1007/s40819-018-0532-8

ORIGINAL PAPER

Certain Sequences Involving Product of k-Bessel Function

M. Chand[1] · P. Agarwal[2] · Z. Hammouch[3]

Published online: 6 June 2018

Abstract Recently, operational techniques have drawn the attention of several researchers in the study of generating relations and summation formulae. In the present paper, here, we introduce a new sequence of functions involving the product of the generalized k-Bessel function. By using the operational techniques, some generating relations and finite summation formulae of the sequence presented here are also established.

Keywords Special function · Generating relations · Generalized k-Bessel function · Sequence of function · Finite summation formula

Introduction and Preliminaries

Bessel functions, first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel, are important special functions and these are widely used in physics and engineering such as Electromagnetic waves, Heat conduction, rotational flows, signal processing, Diffusion problems Dynamics of floating bodies, etc. Therefore, these are of interest to engineers and physicists as well as mathematicians. In this paper, we aim to introduce a new sequence of functions involving the product of the generalized k-Bessel function to establish the generating relations and summation formulae by using the operational techniques.

✉ Z. Hammouch
hammouch.zakia@gmail.com

M. Chand
mehar.jallandhra@gmail.com

P. Agarwal
goyal.praveen2011@gmail.com

1 Department of Mathematics, Fateh College for Women, Bathinda 151103, India

2 Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India

3 E3MI Department of Mathematics, FSTE Moulay Ismail University, BP 509, 52000 Errachidia, Morocco

Recently, Romero et al. [8] (see, also [1]) introduced the k-Bessel function of the first kind for $\lambda, \gamma, \nu \in \mathbb{C}, k \in \mathbb{R}$ and $\Re(\lambda) > 0, \Re(\nu) > 0$ as follows:

$$J_{k,\mu}^{(\gamma),(\lambda)}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \mu + 1)} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^n, \tag{1}$$

where $(\gamma)_{n,k}$ and $\Gamma_k(\gamma)$ are k-Pochhemmer symbol and k-gamma function. These are introduced by Diaz and Pariguan [3] and defined as:

$$(\gamma)_{n,k} := \begin{cases} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R}; \gamma \in \mathbb{C} \setminus \{0\}) \\ \gamma(\gamma+k)\ldots(\gamma+(n-1)k) & (n \in \mathbb{N}; \gamma \in \mathbb{C}), \end{cases} \tag{2}$$

They gave the relation with the classical Euler's gamma function(see [2,8]) as:

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1} \Gamma\left(\frac{\gamma}{k}\right), \tag{3}$$

when $k = 1$, (2) reduces to the classical Pochhammer symbol and Euler's gamma function, respectively (see [6]).

In terms of the k-Pochhamer symbol $(\gamma)_{n,k}$ defined by (2), we introduce more generalized form of k-Bessel function $\omega_{k,\nu,b,c}^{\gamma,\lambda}(z)$ as follows:

$$\omega_{k,\nu,b,c}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (\gamma)_{n,k}}{\Gamma_k(\nu + \lambda n + \frac{b+1}{2})} \frac{\left(\frac{z}{2}\right)^{\nu+2n}}{(n!)^2} \tag{4}$$

where $\lambda, \gamma, \nu, c, b \in \mathbb{C}$ and $\Re(\lambda) > 0, \Re(\nu) > 0$.

A new sequence of function $\left\{V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)\right\}_{n=0}^{\infty}$ is introduced in this paper as:

$$\begin{aligned} &V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s) \\ &= \frac{1}{n!}\xi^{-\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\left(T_\xi^{\sigma,s}\right)^n\left\{\xi^{\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\right\}, \end{aligned} \tag{5}$$

where $T_\xi^{\sigma,s} \equiv \xi^\sigma (s + \xi D)$, $D \equiv \dfrac{d}{dx}$, σ and s are constants, $k_1, \ldots, k_r$ are finite and non-negative integer, $p_{k_i}(\xi)$ is a polynomial in ξ of degree k_i (where $i = 1, \ldots, r$) and $\omega_{\mu,\nu,b,c}^{\gamma,\lambda}(\xi)$ is a generalized k-Bessel function, which is defined in (4). $T_\xi^{\sigma,s}$ is based on the work of Mittal [4], Patil and Thakare [5], Srivastava and Singh [9].

For our investigation the following operational techniques are required:

$$\exp\left(tT_\xi^{\sigma,s}\right)\left(\xi^\beta f(\xi)\right) = \xi^\beta\left(1-\sigma\xi^\sigma t\right)^{-\left(\frac{\beta+s}{\sigma}\right)} f\left(\xi\left(1-\sigma\xi^\sigma t\right)^{-1/\sigma}\right), \tag{6}$$

$$\exp\left(tT_\xi^{\sigma,s}\right)\left(\xi^{\alpha-\sigma n} f(\xi)\right) = \xi^\alpha (1+\sigma t)^{-1+\left(\frac{\alpha+s}{\sigma}\right)} f\left(\xi(1+\sigma t)^{1/\sigma}\right), \tag{7}$$

$$\left(T_\xi^{\sigma,s}\right)^n (\xi u v) = \xi \sum_{m=0}^{\infty} \binom{n}{m} \left(T_\xi^{\sigma,s}\right)^{n-m}(v)\left(T_\xi^{\sigma,1}\right)^m (u), \tag{8}$$

$$\begin{aligned} &(1+\xi D)(1+\sigma+\xi D)(1+2\sigma+\xi D)(1+3\sigma+\xi D)\ldots(1+(m-1)\sigma+\xi D)\xi^{\beta-1} \\ &= \sigma^m \left(\frac{\beta}{\sigma}\right)_m \xi^{\beta-1}, \end{aligned} \tag{9}$$

$$(1-\sigma t)^{\frac{-\alpha}{\sigma}} = (1-\sigma t)^{\frac{-\beta}{\sigma}} \sum_{m=0}^{\infty} \left(\frac{\alpha-\beta}{\sigma}\right)_m \frac{(\sigma t)^m}{m!}. \tag{10}$$

Generating Relations

In this section, we stablish here some generating relation involving the product of generalized k-Bessel function by employing the operational techniques.

Theorem 1 *Let* $\lambda_i, \gamma_i, \nu_i, b_i, c_i \in \mathbb{C}$; $\alpha, \mu_i \in \mathbb{R}$; σ *and* s *are constant; such that* $\Re(\lambda) > 0, \Re(\nu) > 0, \Re(\alpha) + s > 0, \sigma > 0$, *then we have the following formula:*

$$\begin{aligned}\sum_{n=0}^{\infty} & V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)\,\xi^{-\sigma n} t^n \\ &= (1-\sigma t)^{-\left(\frac{\alpha+s}{\sigma}\right)} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right] \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}\left(\xi\,(1-\sigma t)^{-1/\sigma}\right)\right].\end{aligned} \tag{11}$$

where $p_{k_i}(\xi)$ *is a polynomial in* ξ *of degree* k_i. $k_i (i = 1, \ldots, r)$ *are finite and non-negative integers.*

Proof To prove the result in Eq. (11), we start from new equation of function given in Eq. (5), from this equation we have:

$$\begin{aligned}\sum_{n=0}^{\infty} & V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)\, t^n \\ &= \xi^{-\alpha} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right] \exp(t T_{\xi}^{\sigma,s}) \left\{\xi^{\alpha} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_1}(\xi)\right]\right\},\end{aligned} \tag{12}$$

employing the operational technique given in Eq. (6), the above Eq. (12) reduces to:

$$\begin{aligned}\sum_{n=0}^{\infty} & V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)\, t^n \\ &= (1-\sigma\xi^{\sigma} t)^{-\left(\frac{\alpha+s}{\sigma}\right)} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right] \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}\left(\xi(1-\sigma\xi^{\sigma} t)^{-1/\sigma}\right)\right],\end{aligned} \tag{13}$$

after replacing t by $t\xi^{-\sigma}$ in Eq. (13), we have the desired result (11). □

Theorem 2 *Let* $\lambda_i, \gamma_i, \nu_i, b_i, c_i \in \mathbb{C}$; $\alpha, \mu_i \in \mathbb{R}$; σ *and* s *are constant; such that* $\Re(\lambda) > 0, \Re(\nu) > 0, \Re(\alpha) + s > 0, \sigma > 0$, *then we have the following formula:*

$$\begin{aligned}\sum_{n=0}^{\infty} & V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha-\sigma n)}(\xi;\sigma,k_1,\ldots,k_r,s)\,\xi^{-\sigma n} t^n \\ &= (1+\sigma t)^{-1+\left(\frac{\alpha+s}{\sigma}\right)} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right] \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}\left(\xi\,(1+\sigma t)^{1/\sigma}\right)\right].\end{aligned} \tag{14}$$

where $p_{k_i}(\xi)$ *is a polynomial in* ξ *of degree* k_i. $k_i (i = 1, \ldots, r)$ *are finite and non-negative integers.*

Proof Again from Eq. (5), we have:

$$\begin{aligned} &\sum_{n=0}^{\infty} \xi^{-\sigma n} V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha-\sigma n)} (\xi;\sigma,k_1,\ldots,k_r,s)\, t^n \\ &= \xi^{-\alpha} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}(\xi)\right] \exp\left(t T_\xi^{\sigma,s}\right) \left\{\xi^{\alpha-\sigma n} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[-p_{k_i}(\xi)\right]\right\}, \end{aligned} \tag{15}$$

applying the operational technique given in Eq. (7), the above Eq. (15) reduces to

$$\begin{aligned} &\sum_{n=0}^{\infty} \xi^{-\sigma n} V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha-\sigma n)} (\xi;\sigma,k_1,\ldots,k_r,s)\, t^n \\ &= (1+\sigma t)^{\frac{\alpha+s}{\sigma}-1} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}(\xi)\right] \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[-p_{k_i}\left(\xi\,(1+\sigma t)^{1/\sigma}\right)\right], \end{aligned} \tag{16}$$

which is desired. □

Theorem 3 *Let $\lambda_i, \gamma_i, \nu_i, b_i, c_i \in \mathbb{C}$; $\alpha, \mu_i \in \mathbb{R}$; σ and s are constant; such that $\Re(\lambda) > 0, \Re(\nu) > 0, \Re(\alpha) + s > 0, \sigma > 0$, then we have the following formula:*

$$\begin{aligned} &\sum_{m=0}^{\infty} \binom{m+n}{n} V_{m+n}^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)} (\xi;\sigma,k_1,\ldots,k_r,s)\, \xi^{-\sigma m} t^m \\ &= (1-\sigma t)^{-\left(\frac{\alpha+s}{\sigma}\right)} \frac{\prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}(\xi)\right]}{\prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}\left(\xi\,(1-\sigma t)^{-1/\sigma}\right)\right]} \\ &\quad\times V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)} \left(\xi\,(1-\sigma t)^{-1/\sigma};\sigma,k_1,\ldots,k_r,s\right). \end{aligned} \tag{17}$$

where $p_{k_i}(\xi)$ is a polynomial in ξ of degree k_i. $k_i\,(i = 1,\ldots,r)$ are finite and non-negative integers.

Proof To obtained the result (17), we can write Eq. (5) as:

$$\left(T_\xi^{\sigma,s}\right)^n \left[\xi^{\alpha} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[-p_{k_i}(\xi)\right]\right] = n!\,\xi^{\alpha} \frac{V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)} (\xi;\sigma,k_1,\ldots,k_r,s)}{\prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}(\xi)\right]}, \tag{18}$$

multiplying both sides of the above Eq. (18) by $\exp\left(t\left(T_\xi^{\sigma,s}\right)\right)$, we have

$$\begin{aligned} &\exp\left(t\left(T_\xi^{\sigma,s}\right)\right) \left\{\left(T_\xi^{\sigma,s}\right)^n \left[\xi^{\alpha} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[-p_{k_i}(\xi)\right]\right]\right\} \\ &= n! \exp\left(t T_\xi^{\sigma,s}\right) \left[\xi^{\alpha} \frac{V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)} (\xi;\sigma,k_1,\ldots,k_r,s)}{\prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}(\xi)\right]}\right] \end{aligned} \tag{19}$$

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}\left(T_{\xi}^{\sigma,s}\right)^{m+n}\left\{\xi^{\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}\left(\xi\right)\right]\right\}$$
$$= n!\exp\left(tT_{\xi}^{\sigma,s}\right)\left\{\xi^{\alpha}\frac{V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}\left(\xi;\sigma,k_1,\ldots,k_r,s\right)}{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}\left(\xi\right)\right]}\right\}, \tag{20}$$

employing the operational technique (6), the above Eq. (20) can be written as:

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}\left(T_{\xi}^{\sigma,s}\right)^{m+n}\left[\xi^{\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}\left(\xi\right)\right]\right]$$
$$= n!\xi^{\alpha}\left(1-\sigma\xi^{\sigma}t\right)^{-\left(\frac{\alpha+s}{\sigma}\right)}\frac{V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}\left(\xi\left(1-\sigma\xi^{\sigma}t\right)^{-1/\sigma};\sigma,k_1,\ldots,k_r,s\right)}{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}\left(\xi\left(1-\sigma\xi^{\sigma}t\right)^{-1/\sigma}\right)\right]}, \tag{21}$$

now using Eq. (19) in the above Eq. (21), we have:

$$\sum_{m=0}^{\infty}\frac{t^m\left(m+n\right)!}{m!n!}\xi^{\alpha}\frac{V_{m+n}^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}\left(\xi;\sigma,k_1,\ldots,k_r,s\right)}{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}\left(\xi\right)\right]}$$
$$= \xi^{\alpha}\left(1-\sigma\xi^{\sigma}t\right)^{-\left(\alpha+\frac{s}{\sigma}\right)}\frac{V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}\left(\xi\left(1-\sigma\xi^{\sigma}t\right)^{-1/\sigma};\sigma,k_1,\ldots,k_r,s\right)}{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}\left(\xi\left(1-\sigma\xi^{\sigma}t\right)^{-1/\sigma}\right)\right]}, \tag{22}$$

therefore, we can write the above Eq. (22) as:

$$\sum_{m=0}^{\infty}\binom{m+n}{n}V_{m+n}^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}\left(\xi;\sigma,k_1,\ldots,k_r,s\right)t^m = \left(1-\sigma\xi^{\sigma}t\right)^{-\left(\alpha+\frac{s}{\sigma}\right)}$$
$$\times\frac{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}\left(\xi\right)\right]V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}\left(\xi\left(1-\sigma\xi^{\sigma}t\right)^{-1/\sigma};\sigma,k_1,\ldots,k_r,s\right)}{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}\left(\xi\left(1-\sigma\xi^{\sigma}t\right)^{-1/\sigma}\right)\right]}, \tag{23}$$

replacing t by $t\xi^{-\sigma}$ in above Eq. (23), this gives the required result (17). □

Finite Summation Formulas

Theorem 4 *Let $\lambda_i, \gamma_i, \nu_i, b_i, c_i \in \mathbb{C}$; $\alpha, \mu_i \in \mathbb{R}$; σ and s are constant; such that $\Re(\lambda) > 0, \Re(\nu) > 0, \Re(\alpha) + s > 0, \sigma > 0$, then we have the following formula:*

$$V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}\left(\xi;\sigma,k_1,\ldots,k_r,s\right)$$
$$= \sum_{m=0}^{n}\frac{1}{m!}\left(\sigma\xi^{\sigma}\right)^m\left(\frac{\alpha}{\sigma}\right)_m V_{n-m}^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;0)}\left(\xi;\sigma,k_1,\ldots,k_r,s\right). \tag{24}$$

where $p_{k_i}(\xi)$ is a polynomial in ξ of degree k_i. $k_i(i = 1,\ldots,r)$ are finite and non-negative integers.

Proof The Eq. (5) can be written as:

$$V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s) = \frac{1}{n!}\xi^{-\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\left(T_\xi^{\sigma,s}\right)^n\left\{\xi\xi^{\alpha-1}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\right\}, \tag{25}$$

now applying the operational technique (8), we have:

$$V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s) = \frac{1}{n!}\xi^{-\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\xi\sum_{m=0}^{\infty}\binom{n}{m}$$
$$\times\left(T_\xi^{\sigma,s}\right)^{n-m}\left\{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\right\}\left(T_\xi^{\sigma,1}\right)^m\left(\xi^{\alpha-1}\right)$$

$$\begin{aligned}&= \frac{1}{n!}\xi^{-\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\xi\sum_{m=0}^{\infty}\frac{n!}{m!\,(n-m)!}\xi^{\sigma(n-m)}\\&\quad\times\left[(s+\xi D)(s+\sigma+\xi D)(s+2\sigma+\xi D)\ldots(s+(n-m-1)\sigma+\xi D)\right]\\&\quad\times\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\xi^{\sigma m}\\&\quad\times\left[(1+\xi D)(1+\sigma+\xi D)(1+2\sigma+\xi D)\ldots(1+(m-1)\sigma+\xi D)\right]\left(\xi^{\alpha-1}\right),\end{aligned} \tag{26}$$

using the result given in Eq. (9), the above Eq. (26) reduces to the following form:

$$\begin{aligned}V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s) &= \prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\sum_{m=0}^{n}\frac{1}{m!\,(n-m)!}\xi^{\sigma n}\\&\times\prod_{i=0}^{n-m-1}(s+i\sigma+\xi D)\left\{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\right\}\sigma^m\left(\frac{\alpha}{\sigma}\right)_m.\end{aligned} \tag{27}$$

Putting $\alpha = 0$ and replacing n by $n-m$ in (26), we get:

$$\begin{aligned}&V_{n-m}^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;0)}(\xi;\sigma,k_1,\ldots,k_r,s)\\&\quad= \frac{1}{(n-m)!}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\\&\quad\left(T_\xi^{\sigma,s}\right)^{n-m}\left\{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\right\}\end{aligned} \tag{28}$$

$$\begin{aligned}&\Rightarrow \frac{1}{(n-m)!}\left(T_\xi^{\sigma,s}\right)^{n-m}\left\{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\right\}\\&\quad= \frac{V_{n-m}^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;0)}(\xi;\sigma,k_1,\ldots,k_r,s)}{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]},\end{aligned} \tag{29}$$

the above Eq. (29) gives:

$$\frac{1}{(n-m)!}\prod_{i=0}^{n-m-1}(s+i\sigma+\xi D)\left\{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\right\}$$
$$=\xi^{\sigma(m-n)}\frac{V_{n-m}^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;0)}(\xi;\sigma,k_1,\ldots,k_r,s)}{\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]}, \tag{30}$$

from the Eqs. (27) and (30), we have the desired result. □

Theorem 5 *Let $\lambda_i,\gamma_i,\nu_i,b_i,c_i\in\mathbb{C}$; $\alpha,\mu_i\in\mathbb{R}$; σ and s are constant; such that $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\alpha)+s>0$, $\sigma>0$, then we have the following formula:*

$$V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)$$
$$=\sum_{m=0}^{n}\frac{1}{m!}\left(\sigma\xi^{\sigma}\right)^m\left(\alpha-\frac{\beta}{\sigma}\right)_m V_{n-m}^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\beta)}(\xi;\sigma,k_1,\ldots,k_r,s). \tag{31}$$

where $p_{k_i}(\xi)$ is a polynomial in ξ of degree k_i. $k_i\,(i=1,\ldots,r)$ are finite and non-negative integers.

Proof Begins from Eq. (5), which can be written as:

$$\sum_{n=0}^{\infty}V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)\,t^n$$
$$=\xi^{-\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\exp\left(tT_{\xi}^{(\sigma,s)}\right)\left\{\xi^{\alpha}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[-p_{k_i}(\xi)\right]\right\}, \tag{32}$$

applying the operational technique given in Eq. (6), the Eq. (32) reduced to:

$$\sum_{n=0}^{\infty}V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)\,t^n$$
$$=\left(1-\sigma\xi^{\sigma}t\right)^{-(\alpha+\frac{s}{\sigma})}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}$$
$$\left[-p_{k_i}\left(\xi\left(1-\sigma\xi^{\sigma}t\right)^{-1/\sigma}\right)\right], \tag{33}$$

applying the result (10); the Eq. (33) gives:

$$\sum_{n=0}^{\infty}V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)\,t^n$$
$$=\left(1-\sigma\xi^{\sigma}t\right)^{-(\beta+\frac{s}{\sigma})}\sum_{m=0}^{\infty}\left(\alpha-\frac{\beta}{\sigma}\right)_m\frac{(\sigma\xi^{\sigma}t)^m}{m!}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]$$
$$\times\,\omega_{\mu,\nu,b,c}^{\gamma,\lambda}\left[-p_k\left(\xi\left(1-\sigma\xi^{\sigma}t\right)^{-1/\sigma}\right)\right]$$
$$=\sum_{m=0}^{\infty}\left(\alpha-\frac{\beta}{\sigma}\right)_m\frac{(\sigma\xi^{\sigma}t)^m}{m!}\xi^{-\beta}\prod_{i=1}^{r}\omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i}\left[p_{k_i}(\xi)\right]\exp\left(tT_{\xi}^{\sigma,s}\right)$$

$$\times \left\{ \xi^{\beta} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[-p_{k_i}(\xi) \right] \right\}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\alpha - \frac{\beta}{\sigma} \right)_m \frac{(\sigma \xi^{\sigma})^m \, t^{n+m}}{m! n!} \xi^{-\beta} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}(\xi) \right] \left(T_{\xi}^{\sigma,s} \right)^n$$

$$\times \left\{ \xi^{\beta} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[-p_{k_i}(\xi) \right] \right\}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\alpha - \frac{\beta}{\sigma} \right)_m \frac{(\sigma \xi^{\sigma})^m \, t^{n}}{m! \, (n-m)!} \xi^{-\beta} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}(\xi) \right] \left(T_{\xi}^{\sigma,s} \right)^{n-m}$$

$$\times \left\{ \xi^{\beta} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[-p_{k_i}(\xi) \right] \right\}. \tag{34}$$

Now equating the coefficient of t^n , we get:

$$V_n^{(\gamma_i,\lambda_i;\mu_i,\nu_i,b_i,c_i;\alpha)}(\xi;\sigma,k_1,\ldots,k_r,s)$$

$$= \sum_{m=0}^{n} \left(\alpha - \frac{\beta}{\sigma} \right)_m \frac{(\sigma \xi^{\sigma})^m}{m! \, (n-m)!} \xi^{-\beta} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[p_{k_i}(\xi) \right] \left(T_{\xi}^{\sigma,s} \right)^{n-m} \tag{35}$$

$$\times \left\{ \xi^{\beta} \prod_{i=1}^{r} \omega_{\mu_i,\nu_i,b_i,c_i}^{\gamma_i,\lambda_i} \left[-p_{k_i}(\xi) \right] \right\},$$

employing the result (5) in Eq. (35), we have the desired formula (31). □

Concluding Remarks

1. If we choose $b = c = 1$ then generalized k-Bessel function reduced to the following form:

$$\omega_{k,\mu,1,1}^{\gamma,\lambda}(z) = \left(\frac{z}{2}\right)^{\mu} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{n,k}}{\Gamma_k(\lambda n + \mu + 1)} \frac{\left(\frac{z^2}{4}\right)^n}{(n!)^2} = \left(\frac{z}{2}\right)^{\mu} J_{k,\mu}^{(\gamma),(\lambda)} \left(\frac{z^2}{2}\right), \tag{36}$$

where $\lambda, \gamma, \mu, \in \mathbb{C}$ and $\Re(\lambda) > 0, \Re(\mu) > 0$. All the results in Section 2 reduced to involving the product of $J_{k,\mu}^{(\gamma),(\lambda)}(.)$.

2. If we choose $b = -1, c = 1$ then generalized k-Bessel function reduced to the k-Wright function [7] associated with the following relation:

$$\omega_{k,\mu,-1,1}^{\gamma,\lambda}(z) = \left(\frac{z}{2}\right)^{\mu} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{n,k}}{\Gamma_k(\lambda n + \mu)} \frac{\left(\frac{z^2}{4}\right)^n}{(n!)^2} = \left(\frac{z}{2}\right)^{\mu} W_{k,\lambda,\mu}^{\gamma} \left(\frac{-z^2}{2}\right) \tag{37}$$

where $\lambda, \gamma, \mu, \in \mathbb{C}$ and $\Re(\lambda) > 0, \Re(\mu) > 0$. All the results in Section 2 reduced to the involving the product of $W_{k,\lambda,\mu}^{\gamma}\left(\frac{-z^2}{2}\right)$.

In this way, with the help of our main sequence formula, some generating relations and finite summation formula of the sequence are also established in the present paper.

A new sequence of functions is important due to presence of generalized k-Bessel function $\omega^{\gamma,\lambda}_{k,\nu,b,c}(z)$. On account of the most general nature of the generalized k-Bessel function $\omega^{\gamma,\lambda}_{k,\nu,b,c}(z)$ a large number of sequences, generating relations and summation formulae involving simpler functions can be easily obtained as their special cases by assigning the values to the parameters.

References

1. Cerutti, R.A.: On the *k*-Bessel functions. Int. Math. Forum **7**(38), 1851–1857 (2012)
2. Choi, J., Kumar, D.: Solutions of generalized fractional kinetic equations involving Aleph functions. Math. Commun. **20**, 113–123 (2015)
3. Diaz, R., Pariguan, E.: On hypergeometric functions and *k*-Pochhammer symbol. Divulg. Math. **15**(2), 179–192 (2007)
4. Mittal, H.B.: Bilinear and Bilateral generating relations. Am. J. Math. **99**, 23–45 (1977)
5. Patil, K.R., Thakare, N.K.: Operational formulas for a function defined by a generalized Rodrigues formula-II. Sci. J. Shivaji Univ. **15**, 1–10 (1975)
6. Rainville, E.D.: Special Functions. Macmillan, New York (1960)
7. Romero, L., Cerutti, R.: Fractional calculus of a *k*-Wright type function. Int. J. Contemp. Math. Sci. **7**(31), 1547–1557 (2012)
8. Romero, L.G., Dorrego, G.A., Cerutti, R.A.: The *k*-Bessel function of first kind. Int. Math. Forum **38**(7), 1854–1859 (2014)
9. Srivastava, A.N., Singh, S.N.: Some generating relations connected with a function defined by a Generalized Rodrigues formula. Indian J. Pure Appl. Math. **10**(10), 1312–1317 (1979)