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A Two-Warehouse Inventory Model for Deteriorating Items with Capacity Constraints and Back-Ordering Under Financial Considerations

R. Chakrabarty[1] · T. Roy[1] · K. S. Chaudhuri[2]

Abstract This paper investigated two warehouse inventory model for a single deteriorating items under the assumption of shortages and partial backlogging under payments delay and the effect of inflation and time value for money is also incorporated. Again, two warehouse capacity (owned and rented) have been considered over the finite horizon planning under which back-ordering is assumed. When the order quantity becomes higher than the storage capacity in own warehouse then the system involves two warehouse model. In this model, excess inventory are stored in rented warehouses which are assumed to charge higher unit holding costs than owned warehouses. Moreover, in rented warehouse item deteriorates with higher rates than in owned warehouse. At first own warehouse are filled out by its storage capacity and the rests are kept in rented warehouse. So, at first, the items in RW decreases due to demand and deterioration until it reaches to the level zero while on the other hand a portion on the inventory on OW is depleted due to deterioration only. Then, the inventory on OW decreases due to demand and deterioration until it reaches to the level zero. Finally, shortages start and backlogging rate is negative exponential of function waiting time. Here, the demand is used as exponential function of time and backlogging rate varies negative exponential function of waiting time. The total cost function is carried out under the effect of JIT setup cost. Two numerical examples are illustrated for the developed model and sensitivity analysis has been carried out to identify behaviour of model parameters.

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Introduction

For last twenty years, many researchers have been attracted by the effect of permissible
delay in payments on optimal inventory management. From an economic point of view, such
type of credit policies can be applied to get larger orders since this credit policies results
as an alternative solution of price discounts. In that case the main problem is to decide the
place where the items should get stock. Many researcher have worked considering the effect
of permissible delay in payments for a single warehouse with unlimited capacity. But in a
more practical sense, any warehouse has limited capacity. Hartely [1] developed an inventory
model involving two different warehouses, one of which is owned (OW) while the other is
rented (RW). The items are stored first in OW which has a fixed capacity W and the only
excess units are stored in RW which has higher holding cost than in OW. Goswami and
Chaudhuri [2] designed two-level storage model including both with and without shortages
in which demand was considered as linear increasing function of time. But in this model
deterioration was not taken into account. Sarma [3] first developed a two-warehouse model
for deteriorating items with an infinite replenishment rate and shortages. Other notable papers
in this direction are by Pakkala and Achary [4], Zhou and Yang [5], Lee [6], Yang [7]. Some
most recent research works in this direction are by Jaggi et al. [8], Tiwari et al. [9], Sett et
al. [10].

Again many researchers have been working on the effect of permissible delays in payments
on inventory management during last two decades. Shah [11] considered an inventory model
for exponentially decaying inventory under permissible delay in payment. Aggarwal and
Jaggi [12] developed economic order policies for deteriorating items under the effect of
permissible delay in payments. Then Jamal et al. [13] generalized the model of Aggarwal
and Jaggi considering shortages. Sarkar [14] developed an EOQ model with time varying
deterioration rate in which the effect of delay in payments were also taken into account.
Other numerous papers related to delay in payments are Chung and Liao [15,16], Shinn and
Hwang [17], Huang and Liao [18], Liao [19,20], Chen et al. [21].

Again, in real world problem, deterioration of many items which is defined as the decay,
damage of stored items can not be neglected and as a result of which it decreases usefulness
of the items. Ghare and Schrader [22] were the pioneers to establish an inventory model for
deteriorating items. Covert and Philip [23] extended Ghare and Schrader's constant deterio-
ration rate to a two parameter weibull distribution. Cheng et. al. [24] developed an inventory
control model for deteriorating items in which trapezoidal type demand and partial backlog-
ging were assumed. Chung and Lin [25] studied trade credit issue in a supply chain model
proposed to determine the order quantity of a deteriorating item. Some other related article
in this direction are Cardenas-Barron [26], Widyadana [27], Chung et. al. [28]. Recently
many researcher have worked on various types of deterioration like probabilistic deteriora-
tion, variable deterioration, deterioration on seasonal product and some most recent works
in this directions are by Sarkar and Sarkar [29], Sarkar and Sarkar [30], Sarkar et al. [31],
sarkar [32], Sarkar et al. [33], Sarkar et al. [34], Wu et al. [35], Sarkar et al. [36], Shah and
Cardenas-Barron [37].

Furthermore, both inflation and time value of money have main effect on financial markets.
Ray and Chaudhuri [38] developed a finite-horizon EOQ model with back ordering and

varying demand rate whereas the effects of inflation and time value of money were taken into consideration. Chen [39] considered the situation in which the demand rate is time-proportional and shortages are fully back ordered. Other articles in this direction come from Bose et.al. [40], Sarkar et al. [41], Roy and Chaudhuri [42], Sana [43], Roy and Chaudhuri [44]. Further more many researcher have considered credit policy in their article and some of which are by Ouyang et al. [45], Wu et al. [46], Chun et al. [47].

Yang [48] developed two-warehouse inventory management problem for perishable products with constant demand rate and back ordering shortages under inflation. Hou [49] developed an EOQ model over finite planning horizon for perishable products with stock-dependent demand rate and full back ordering under inflation and time value of money. Cardenas-Barron et. al. [50] also developed an EOQ model in honor of Ford Whitman Harris where as Liao and Huang [51] developed a two-warehouse deterministic inventory model for deteriorating items considering delay in payments and capacity constraints. But in this model partial backlogging and time value of money were not taken into consideration. Some recent works relating imperfect production, variable back order, inspection, trade credits and time dependent back-logging are by Sarkar and Saren [52], Sarkar [53], Sarkar et al. [54], Tripathi and Chaudhary [55], Das et al. [56], Sarkar et al. [57].

Liao and Huang [51] evaluated a two-warehouse deterministic inventory model for deteriorating items considering delay in payments and capacity constraints. But in this model shortages, partial backlogging and time value of money were not taken into account. In this article we extended Liao and Huang [51] model by developing a two-warehouse inventory model for deteriorating items with the consideration of shortages and partial backlogging under delay in payments and the effect of inflation and time value of money.

Notation and Assumptions

Based on the data of Liao and Huang [49] model, this paper is developed with the following *Notation* and *Assumptions*.

Notation

C_0: JIT setup cost per replenishment
D: demand rate per unit time
W: storage capacity of OW
T: length of each replenishment cycle
s: selling price per unit
c: purchasing cost per unit
$I_R(t)$: on hand inventory at time t in rented warehouse
$I_O(t)$: on hand inventory at time t in own warehouse
h_0: holding cost per unit time in OW
h_r: holding cost per unit time RW with $h_r \geq h_0$
α: deterioration rate in OW, where $0 < \alpha < 1$
β: deterioration rate in RW, where $0 < \beta < 1$ and $\beta > \alpha$
t_w: time at which the inventory level reaches zero in RW
M: credit period set by the supplier
I_0: capital opportunity cost (as a percentage)
I_e: earned interest rate (as a percentage)
I_c: payable interest rate (as a percentage)
H: length of planning horizon

i: constant inflation rate

r: discount rate, representing the time value of money

R: $r - i$, representing the net constant discount rate of inflation

t_j: time at which the inventory level in jth replenishment cycle drops to zero

T_j: total time that is elapsed up to and including the jth replenishment cycle where $T_0 = 0$, $T_1 = T$ and $T_N = H$

c_s: shortage cost per unit time

I_m: maximum inventory level at the beginning of each replenishment cycle

I_s: maximum shortage quantity at the end of each replenishment cycle

F: fraction of replenishment cycle where the net stock is positive (a decision variable)

N: number of replenishment during the planning horizon, $N = \frac{H}{T}$

Q: order quantity of the item.

Assumptions

(a) The planning horizon in this model is finite.

(b) The demand rate is a exponential increasing function of time $D(t) = ae^{bt}$, a,b >0. The deterioration rates are constant.

(c) The owned warehouse (OW) has a fixed capacity of W units.

(d) The rented warehouse (RW) has unlimited capacity.

(e) The items in OW are consumed only after consuming the items kept in RW.

(f) In shortage period $[t_1,T]$ the demand $De^{-\delta(T-t)}$ at any time t is satisfied whereas the remaining part of the demand $D(1 - e^{-\delta(T-t)})$ remains unsatisfied, where δ is a positive constant and (T − t) is the waiting time until the next replenishment begins at time T. So the backlogging rate is $De^{-\delta(T-t)}$ which varies as negative exponential function of waiting time.

(g) Single deteriorating type of product has been considered.

(h) The inventory level at the end of the planning horizon will be zero.

(i) The cost factors are deterministic, but will be inflated under constant inflation rate.

(j) The number of replenishment are restricted to integer one.

(k) The inventory costs (including holding cost and deterioration cost) in RW are higher than those in OW. That is $h_r + \beta c > h_0 + \alpha c$.

(l) The last order is only being placed to satisfy the shortage of last period.

Mathematical Formulation

The problem which we have discussed here is how retailers know whether or not to rent RW to hold the items. If the order quantity $Q \leq W$ then it is not necessary to rent the warehouse. But if $Q > W$ then W units are stored in OW and the remainder is dispatched in RW. Here we shall denote $t_a = \frac{1}{\alpha+b}log(1 + \frac{(\alpha+b)W}{a})$ (see "Appendix A"). The inequality $Q > W$ holds if and only if $t_a < t_1$ and then the situation of two warehouse inventory system will arise.

Development of Two-Warehouse Model ($t_a < t_1$)

If $Q > W$, the the system involves two warehouse model. At first W units are stored in OW and the rests are kept in RW. So, during the time interval $(0, t_w)$ the items in RW decreases due to demand and deterioration until it reaches to the level zero. Again on the other hand a portion on the inventory on OW is depleted due to deterioration only. Then, during (t_w, t_1) the inventory on OW decreases due to demand and deterioration. After $t = t_1$ shortages
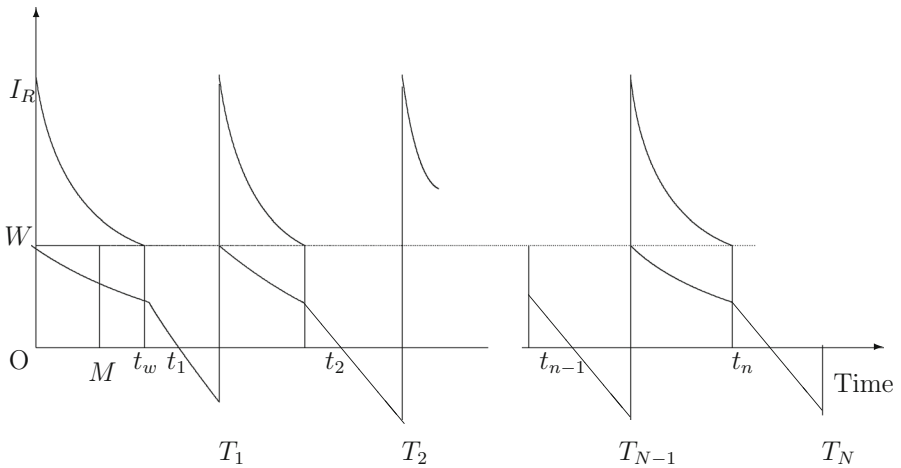
Inventory

Fig. 1 Pictorial representation of the inventory cycle

start and backlogging rate is negative exponential function of waiting time. The pictorial representation of the above inventory system can be represented by Fig. 1.

This bahaviour can be represented by following four differential equations:

$$\frac{dI_R(t)}{dt} + \beta I_R(t) = -ae^{bt}, \quad 0 \le t \le t_w \tag{1}$$

$$\frac{dI_0(t)}{dt} = -\alpha I_0(t), \quad 0 \le t \le t_w \tag{2}$$

$$\frac{dI(t)}{dt} + \alpha I(t) = -ae^{bt}, \quad t_w \le t \le t_1 \tag{3}$$

$$\frac{dI(t)}{dt} = -ae^{bt}e^{-\delta(T-t)}, \quad t_1 \le t \le T \tag{4}$$

with initial and boundary condition

$$I_R(t_w) = 0 = I(t_1), \quad I_0(0) = W, \tag{5}$$

The solution of the differential equations (1)–(4) subject to the condition (5) are given by

$$I_R(t) = \frac{a}{\beta + b}(e^{\beta(t_w-t)+bt_w} - e^{bt}), \quad 0 \le t \le t_w \tag{6}$$

$$I_0(t) = We^{-\alpha t}, \quad 0 \le t \le t_w \tag{7}$$

$$I(t) = \frac{a}{\alpha + b}(e^{\alpha(t_1-t)+bt_1} - e^{bt}), \quad t_w \le t \le t_1 \tag{8}$$

$$I_R(t) = \frac{ae^{-\delta T}}{b + \delta}(e^{(b+\delta)t_1} - e^{(b+\delta)t}), \quad t_1 \le t \le T \tag{9}$$

The maximum inventory at the beginning of each cycle

$$\begin{aligned} I_m &= I_R(0) + I_0(0) \\ &= \frac{a}{\beta + b}(e^{(\beta+b)t_w} - 1) + W \end{aligned} \tag{10}$$

The maximum shortage at the end of each cycle

$$
\begin{aligned}
I_s &= -I(T) \\
&= \frac{a}{\delta + b}(e^{bT} - e^{\delta(t_1 - T) + bt_1})
\end{aligned}
\tag{11}
$$

JIT Setup Cost

Since the number of replenishment or period is N, so the JIT setup cost over the planning horizon under consideration of inflation and time value of money is

$$
\begin{aligned}
C_{SET} &= \sum_{j=0}^{N} C_0 \frac{a(e^{bT} - 1)}{bI_m} e^{-JRT} \\
&= C_0 \frac{a(e^{bT} - 1)}{bI_m} \left(\frac{e^{-\frac{N+1}{N}RH} - 1}{e^{-\frac{RH}{N}} - 1} \right)
\end{aligned}
\tag{12}
$$

Holding Cost

Holding cost in RW

$$
\begin{aligned}
C_{RW} &= h_r \int_0^{t_w} I_R(t)dt \\
&= \frac{ah_r}{\beta + b} \left\{ \frac{1}{b}(1 - e^{bt_w}) + \frac{1}{\beta}(e^{(\beta + b)t_w} - e^{bt_w}) \right\}
\end{aligned}
\tag{13}
$$

Holding cost in OW

$$
\begin{aligned}
C_{OW} &= h_0 \left\{ \int_0^{t_w} I_0(t)dt + \int_{t_w}^{t_1} I_0(t)dt \right\} \\
&= \frac{h_0 W}{\alpha}(1 - e^{-\alpha t_w}) + \frac{h_0 a}{\alpha + b} \left\{ \frac{1}{\alpha}(e^{\alpha(t_1 - t_w) + bt_1} - e^{bt_1}) + \frac{1}{b}(e^{bt_w} - e^{bt_1}) \right\}
\end{aligned}
\tag{14}
$$

Total holding cost

$$
\begin{aligned}
C_{TOT} &= C_{RW} + C_{OW} \\
&= \frac{ah_r}{b\beta(\beta + b)} \left\{ \beta(1 - e^{bt_w}) + b(e^{(\beta + b)t_w} - e^{bt_w}) \right\} + \frac{h_0}{b\alpha(\alpha + b)} \{ b(\alpha + b)W \\
&\quad (1 - e^{-\alpha t_w}) + ab(e^{\alpha(t_1 - t_w) + bt_1} - e^{bt_1}) + a\alpha(e^{bt_w} - e^{bt_1}) \}
\end{aligned}
\tag{15}
$$

Total holding cost over planning horizon under inflation and time value of money consideration is

$$
\begin{aligned}
C_{HOL} &= \sum_{J=0}^{N-1} \left[\frac{ah_r}{b\beta(\beta + b)} \{ \beta(1 - e^{bt_w}) + b(e^{(\beta + b)t_w} - e^{bt_w}) \} + \frac{h_0}{b\alpha(\alpha + b)} \{ b(\alpha + b) \ W \right. \\
&\quad \left. (1 - e^{-\alpha t_w}) + ab(e^{\alpha(t_1 - t_w) + bt_1} - e^{bt_1}) + a\alpha(e^{bt_w} - e^{bt_1}) \} \right] e^{-JRT} \\
&= \left[\frac{ah_r}{b\beta(\beta + b)} \{ \beta(1 - e^{bt_w}) + b(e^{(\beta + b)t_w} - e^{bt_w}) \} + \frac{h_0}{b\alpha(\alpha + b)} \{ b(\alpha + b)W \right. \\
&\quad \left. (1 - e^{-\alpha t_w}) + ab(e^{\alpha(t_1 - t_w) + bt_1} - e^{bt_1}) + a\alpha(e^{bt_w} - e^{bt_1}) \} \right] \left[\frac{e^{-RH} - 1}{e^{\frac{-RH}{N}} - 1} \right]
\end{aligned}
\tag{16}
$$

Now $T = \frac{H}{N}$ and $t_1 = \frac{FH}{N}$ yields to:

$$
\begin{aligned}
C_{HOL} = &\left[\frac{ah_r}{b\beta(\beta + b)} \left\{ \beta(1 - e^{bt_w}) + b(e^{(\beta+b)t_w} - e^{bt_w}) \right\} + \frac{h_0}{b\alpha(\alpha + b)} \right. \\
&\times \left\{ b(\alpha + b)W(1 - e^{-\alpha t_w}) + ab\left(e^{\alpha\left(\frac{FH}{N} - t_w\right) + b\frac{FH}{N}} - e^{b\frac{FH}{N}} \right) \right. \\
&\left. \left. + a\alpha \left(e^{bt_w} - e^{b\frac{FH}{N}} \right) \right\} \right] \left[\frac{e^{-RH} - 1}{e^{\frac{-RH}{N}} - 1} \right]
\end{aligned}
\tag{17}
$$

Shortage Cost

Average shortage S should be determined at first.

$$
S = \frac{1}{2} \int_{t_1}^{T} \frac{a}{\delta + b} \left(e^{bT} - e^{\delta(t_1 - T) + bt_1} \right) dt
\tag{18}
$$

$T = \frac{H}{N}$ and $t_1 = \frac{FH}{N}$ yields to:

$$
S = \frac{a}{2(\delta + b)} \left(e^{\frac{bH}{N}} - e^{\delta\left(\frac{FH}{N} - \frac{H}{N}\right) + b\frac{FH}{N}} \right) \left(\frac{H}{N} - \frac{FH}{N} \right)
\tag{19}
$$

Total shortage cost over planning horizon under time value of money and inflation is

$$
\begin{aligned}
C_{SHO} &= \sum_{J=0}^{N-1} c_s e^{-JRT} S \\
&= \frac{ac_s}{2(\delta + b)} \left(e^{\frac{bH}{N}} - e^{\delta\left(\frac{FH}{N} - \frac{H}{N}\right) + b\frac{FH}{N}} \right) \left(\frac{H}{N} - \frac{FH}{N} \right) \left[\frac{e^{-RH} - 1}{e^{\frac{-RH}{N}} - 1} \right]
\end{aligned}
\tag{20}
$$

Purchasing Cost

The purchasing cost of jth cycle C_{pj} is being calculated at first. $T = \frac{H}{N}$ and $t_1 = \frac{FH}{N}$ yields to:

$$
C_{pj} = \left\{ \frac{ca}{\beta + b}(e^{(\beta+b)t_w} - 1) + cW \right\} + \left\{ \frac{ca}{\delta + b} \left(e^{\frac{bH}{N}} - e^{\delta\left(\frac{FH}{N} - \frac{H}{N}\right) + b\frac{FH}{N}} \right) e^{\frac{-RH}{N}} \right\}
\tag{21}
$$

Total purchasing cost over the planning horizon under inflation and time value of money is

$$
\begin{aligned}
C_{PUR} &= \sum_{J=0}^{N-1} C_{pj} e^{-JRT} \\
&= \left\{ \frac{ca}{\beta + b}(e^{(\beta+b)t_w} - 1) + cW \right\} \left[\frac{e^{-RH} - 1}{e^{\frac{-RH}{N}} - 1} \right] \\
&\quad + \left\{ \frac{ca}{\delta + b} \left(e^{\frac{bH}{N}} - e^{\delta\left(\frac{FH}{N} - \frac{H}{N}\right) + b\frac{FH}{N}} \right) e^{\frac{-RH}{N}} \right\} \left[\frac{e^{-RH} - 1}{e^{\frac{-RH}{N}} - 1} \right]
\end{aligned}
\tag{22}
$$

Interest Earned and Charged $(M < t_w < t_1)$

Since $M < t_w < t_1$, the retailer sells the items and continues to accumulate sales revenue and earns interest with rate I_e during time 0 to M. Again, the retailer starts paying interest for the items in stock after time M with rate I_p.

Interest earned:

As the items are sold, so the sales revenue is used to earn interest. At the beginning of the time interval, the back ordered quantity which is I_s, should be replenished first and the maximum accumulated sold until M is equal to $\int_0^M Dt\,dt$. $T = \frac{H}{N}$ and $t_1 = \frac{FH}{N}$ yields to:

$$IE = I_e s \left[I_s M + \int_0^M Dt\,dt \right]$$

$$= I_e s \left[\frac{a}{\delta + b} \left(e^{\frac{bH}{N}} - e^{\delta \left(\frac{FH}{N} - \frac{H}{N} \right) + b \frac{FH}{N}} \right) M + \frac{a}{b^2} \left(M b e^{bM} - e^{bM} + 1 \right) \right] \quad (23)$$

Total interest earned over the planning horizon under inflation and time value of money is

$$TIE = \sum_{J=0}^{N-1} I E e^{-JRT}$$

$$= I_e s \left[\frac{a}{\delta + b} \left(e^{\frac{bH}{N}} - e^{\delta \left(\frac{FH}{N} - \frac{H}{N} \right) + b \frac{FH}{N}} \right) M + \frac{a}{b^2} \left(M b e^{bM} - e^{bM} + 1 \right) \right]$$

$$\times \left[\frac{e^{-RH} - 1}{e^{\frac{-RH}{N}} - 1} \right] \quad (24)$$

Interest charged

Interest payable is

$$IC = I_c c \int_M^{t_1} I(t)\,dt$$

$$= I_c c \left\{ \int_M^{t_w} I_R(t)\,dt + \int_M^{t_w} I_0(t)\,dt + \int_{t_w}^{t_1} I(t)\,dt \right\}$$

$$= I_c c \left[\frac{a}{(\beta + b)\beta b} \left\{ b \left(e^{\beta(t_w - M) + b t_w} - e^{b t_w} \right) + \beta \left(e^{bM} - e^{b t_w} \right) \right\} + \frac{W}{\alpha} \left(e^{-\alpha M} \right. \right.$$

$$\left. \left. - e^{-\alpha t_w} \right) + \frac{a}{(\alpha + b)\alpha b} \left\{ b \left(e^{\alpha(t_1 - t_w) + b t_1} - e^{b t_1} \right) + \alpha \left(e^{b t_w} - e^{b t_1} \right) \right\} \right] \quad (25)$$

Total interest payable over the planning horizon under inflation and time value money is (putting $T = \frac{H}{N}$ and $t_1 = \frac{FH}{N}$)

$$TIC = \sum_{J=0}^{N-1} IC e^{-JRT}$$

$$= I_c c \left[\frac{a}{(\beta + b)\beta b} \left\{ b \left(e^{\beta(t_w - M) + b t_w} - e^{b t_w} \right) + \beta \left(e^{bM} - e^{b t_w} \right) \right\} \right.$$

$$+ \frac{W}{\alpha} \left(e^{-\alpha M} - e^{-\alpha t_w} \right) + \frac{a}{(\alpha + b)\alpha b} \left\{ b \left(e^{\alpha \left(\frac{FH}{N} - t_w \right) + b \frac{FH}{N}} \right. \right.$$

$$\left. \left. \left. - e^{b \frac{FH}{N}} \right) + \alpha \left(e^{b t_w} - e^{b \frac{FH}{N}} \right) \right\} \right] \left[\frac{e^{-RH} - 1}{e^{\frac{-RH}{N}} - 1} \right] \quad (26)$$

So the total cost function is

$$TC(F) = C_{SET} + C_{HOL} + C_{SHO} + C_{PUR} + TIC - TIE$$

$$= C_0 \frac{a(e^{b\frac{H}{N}} - 1)}{b I_m} \left(\frac{e^{-\frac{N+1}{N} RH} - 1}{e^{-\frac{RH}{N}} - 1} \right) + \left[\frac{a h_r}{b\beta(\beta + b)} \left\{ \beta(1 - e^{b t_w}) + b \left(e^{(\beta + b) t_w} \right. \right. \right.$$

$$-e^{bt_w})\Big\} + \frac{h_0}{b\alpha(\alpha+b)}\Big\{b(\alpha+b)W(1-e^{-\alpha t_w}) + ab\Big(e^{\alpha\left(\frac{FH}{N}-t_w\right)+b\frac{FH}{N}}$$

$$-e^{b\frac{FH}{N}}\Big) + a\alpha\Big(e^{bt_w} - e^{b\frac{FH}{N}}\Big)\Big\} + \frac{ac_s}{2(\delta+b)}\Big(e^{\frac{bH}{N}} - e^{\delta\left(\frac{FH}{N}-\frac{H}{N}\right)+\frac{bFH}{N}}\Big)$$

$$\Big(\frac{H}{N} - \frac{FH}{N}\Big) + \Big\{\frac{ca}{\beta+b}\Big(e^{(\beta+b)t_w} - 1\Big) + cW\Big\} + \Big\{\frac{ca}{\delta+b}\Big(e^{\frac{bH}{N}}$$

$$- e^{\delta\left(\frac{FH}{N}-\frac{H}{N}\right)+b\frac{FH}{N}}\Big)e^{\frac{-RH}{N}}\Big\} + I_c c\Bigg[\frac{a}{(\beta+b)\beta b}\Big\{b\Big(e^{\beta(t_w-M)+bt_w} - e^{bt_w}\Big) + \beta$$

$$(e^{bM} - e^{bt_w})\Big\} + \frac{W}{\alpha}\Big(e^{-\alpha M} - e^{-\alpha t_w}\Big) + \frac{a}{(\alpha+b)\alpha b}\Big\{b\Big(e^{\alpha\left(\frac{FH}{N}-t_w\right)+b\frac{FH}{N}}$$

$$- e^{b\frac{FH}{N}}\Big) + \alpha\Big(e^{bt_w - e^{b\frac{FH}{N}}}\Big)\Big\}\Bigg] - I_e s\Bigg\{\frac{a}{\delta+b}\Big(e^{\frac{bH}{N}} - e^{\delta\left(\frac{FH}{N}-\frac{H}{N}\right)+b\frac{FH}{N}}\Big)M$$

$$+ \frac{a}{b^2}\Big(Mbe^{bM} - e^{bM} + 1\Big)\Big\}\Bigg]\Bigg[\frac{e^{-RH} - 1}{e^{\frac{-RH}{N}} - 1}\Bigg] \tag{27}$$

Notably, the values of $I_0(t_w)$ and $I(t_w)$ should coincide and thus

$I_0(t_w) = We^{-\alpha t_w} = \frac{a}{\alpha+b}(e^{\alpha(t_1-t_w)+bt_1} - e^{bt_w})$

Therefore we get

$t_w = \frac{1}{b}\log\frac{ae^{(\alpha+b)t_1} - (\alpha+b)W}{a}$

Since, the total cost function is very large in volume and too much complex to differentiate by hand, so we solved it numerically by Mathematica Software. So the sufficient condition for $\frac{\partial^2 TC(F)}{\partial F^2} > 0$ must be satisfied.

Numerical Example

To illustrate the proposed model, we consider the two examples given below.

Example 1 Let us take the parameter values in the inventory system as $h_0 = \$0.2$, $h_r = \$0.5$, $c_s = \$0.3$, $c = \$5$, $C_0 = \$50$, $W = 30$, $\alpha = 0.02$, $\beta = 0.8$, $R = 0.5$, $H = 3$, $N = 7$, $\delta = 0.9$, $I_c = \$0.1$, $I_e = \$0.06$, $M = \frac{1}{12}$, $a = 300$, $b = 2$, $s = \$5$ in appropriate units. The optimal solution is $TC(F^*) = 4045.14$, $F^* = 0.438$ (Fig. 2).

$T = \frac{H}{N} = \frac{3}{7}$, $T_w = 0.115$, $t_a = 0.091$, $I_m = 70.783$, $I_b = 122.533$, $t_1 = F^*T = 0.188$.
In this case, some other numerical examples have been prepared based on changing α, β, R, M with different values of N and taking one parameter at a time and keeping the remaining parameters at their original levels. The result of this numerical analysis are shown in Table 1.

Example 2 Let us take the parameter values in the inventory system as $h_0 = \$0.2$, $h_r = \$0.5$, $c_s = \$0.3$, $c = \$5$, $C_0 = \$150$, $W = 30$, $\alpha = 0.02$, $\beta = 0.8$, $R = 0.3$, $H = 5$, $N = 9$, $\delta = 0.9$, $I_c = \$0.1$, $I_e = \$0.08$, $M = \frac{1}{15}$, $a = 200$, $b = 2$, $s = \$7$ in appropriate units. The optimal solution is $TC(F^*) = 6357.7$, $F^* = 0.638$ (Figs. 3, 4, 5).

$T = \frac{H}{N} = \frac{5}{9}$, $T_w = 0.278$, $t_a = 0.131$, $I_m = 114.08$, $I_b = 92.57$, $t_1 = F^*T = 0.354$.
In this case, some other numerical examples have been prepared based on changing α, β, R, M with different values of N and taking one parameter at a time and keeping the remaining parameters at their original levels. The result of this numerical analysis are shown in Table 2.
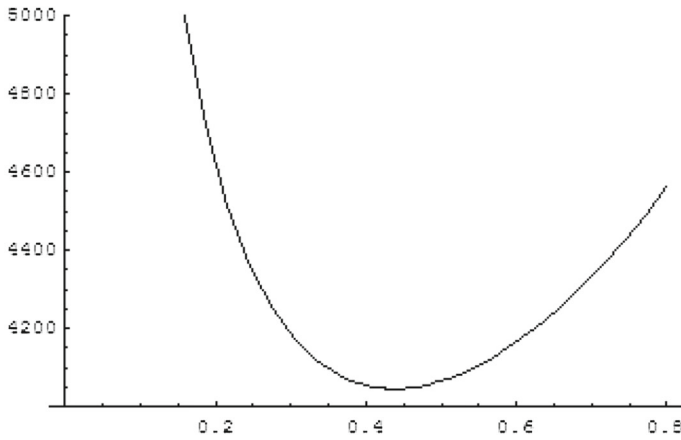
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Fig. 2 Total cost function TC versus F of Example 1

Table 1 Some numerical examples on the data set of Example 1

Parameter	Values	Different values of N	F^*	t_1^*	t_w^*	t_a	$TC(F^*)$
α	0.05	7	0.429	0.184	0.113	0.091	4054.95
	0.05	6	0.387	0.194	0.124	0.091	4243.13
β	0.7	7	0.441	0.189	0.117	0.091	4042.23
	0.7	6	0.397	0.199	0.128	0.091	4231.59
R	0.4	7	0.450	0.193	0.121	0.091	4558.19
	0.4	6	0.404	0.202	0.132	0.091	4783.3
M	$\frac{1}{14}$	7	0.437	0.187	0.115	0.091	4048.34
	$\frac{1}{14}$	6	0.394	0.197	0.126	0.091	4237.81
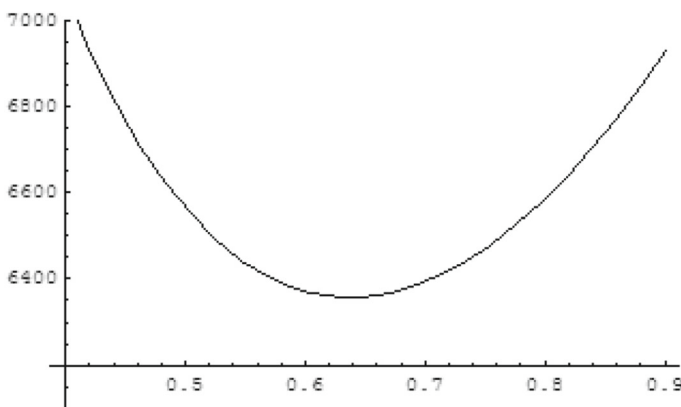
Fig. 3 Total cost function TC versus F of Example 2

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N=6, F=0.387, TC= 4243.13

N=7, F=0.441, TC= 4042.23

N=6, F=0.404, TC= 4783.3

N=7, F= 0.437, TC=4048.34

Fig. 4 Showing some best results of Example 1

N=8, F=0.580, TC=6700.62

N=9, F=0.659, TC=6320.44

N=8, F= 0.618, TC= 10079.3
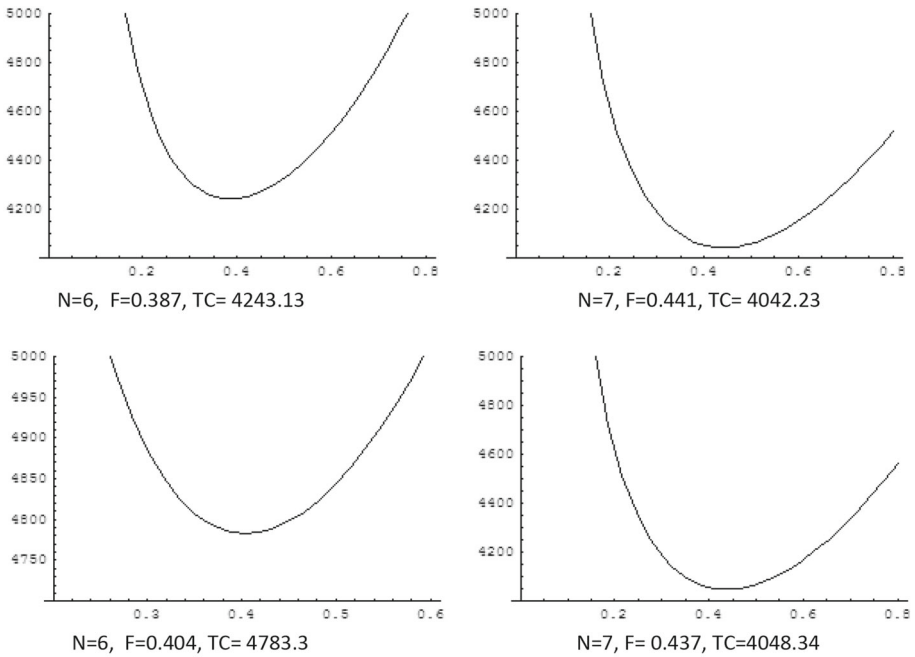
N=8, F= 0.588, TC= 6689.41

Fig. 5 Showing some best results of Example 2

Table 2 Some numerical examples on the data set of Example 2

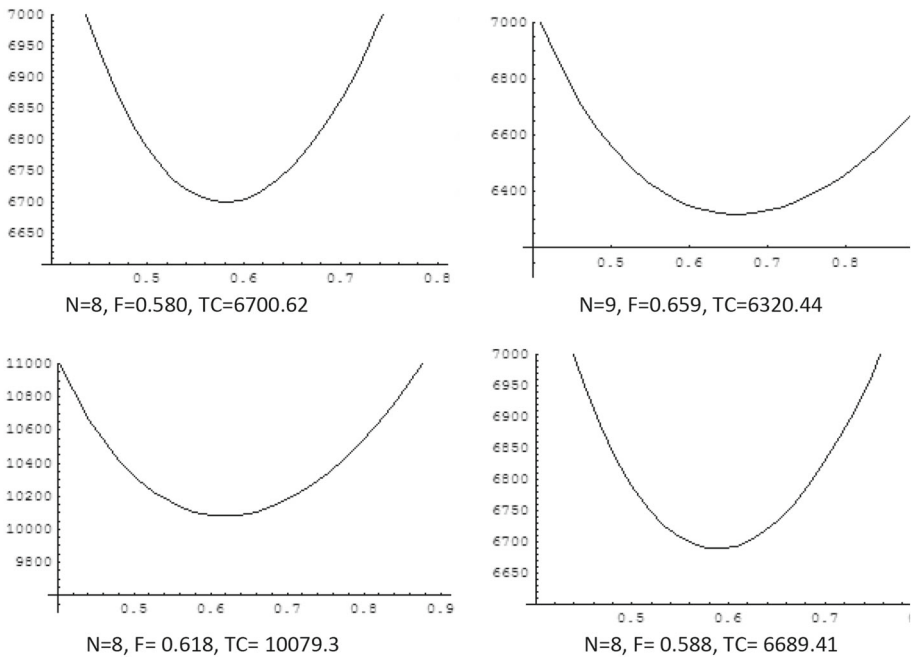| Parameter	Values	Different values of N	F^*	t_1^*	t_w^*	t_a	$TC(F^*)$
α	0.0 4	9	0.629	0.349	0.275	0.131	6377.24
	0.04	8	0.580	0.363	0.291	0.131	6700.62
β	0.6	9	0.659	0.366	0.292	0.131	6320.44
	0.6	8	0.607	0.412	0.345	0.131	6647.58
R	0.1	9	0.671	0.373	0.300	0.131	9545.05
	0.1	8	0.618	0.386	0.315	0.131	10079.3
M	$\frac{1}{18}$	9	0.637	0.354	0.277	0.131	6363.79
	$\frac{1}{18}$	8	0.588	0.368	0.293	0.131	6689.41

Table 3 Sensitivity analysis of Examples 1 and 2

Parameter	% change in the parameter	F^*	% change in $TC(F^*)$ Example 1	F^*	%change in $TC(F^*)$ Example 2
h_o	$+50$	0.437	0.045	0.637	0.075
	$+20$	0.437	0.018	0.637	0.038
	-20	0.438	-0.018	0.638	-0.038
	-50	0.438	-0.045	0.638	-0.075
h_r	$+50$	0.436	0.059	0.633	0.239
	$+20$	0.437	0.024	0.636	0.096
	-20	0.438	-0.024	0.639	-0.097
	-50	0.440	-0.060	0.642	-0.246
α	$+50$	0.435	0.082	0.633	0.155
	$+20$	0.436	0.033	0.636	0.062
	-20	0.439	-0.033	0.639	-0.063
	-50	0.441	-0.084	0.642	-0.159
β	$+50$	0.425	0.263	0.604	0.967
	$+20$	0.432	0.110	0.623	0.415
	-20	0.444	-0.116	0.655	-0.462
	-50	0.454	-0.307	0.687	-1.27
C_0	$+50$	0.507	6.83	0.716	10.08
	$+20$	0.468	2.89	0.672	4.27
	-20	0.403	-3.18	0.596	-4.69
	-50	0.335	-8.79	0.515	-12.94
R	$+50$	0.413	-23.58	0.618	-22.43
	$+20$	0.427	-10.64	0.629	-10.10
	-20	0.450	12.68	0.646	12.01
	-50	0.471	36.60	0.662	34.60
c_s	$+50$	0.440	0.220	0.639	0.110
	$+20$	0.439	0.088	0.638	0.045
	-20	0.437	-0.089	0.637	-0.045
	-50	0.435	-0.222	0.636	-0.112

Table 3 continued

Parameter	% change in the parameter	F^*	% change in $TC(F^*)$ Example 1	F^*	%change in $TC(F^*)$ Example 2
δ	$+50$	0.416	-2.36	0.619	-1.29
	$+20$	0.428	-0.947	0.630	-0.513
	-20	0.448	0.947	0.645	0.507
	-50	0.464	2.36	0.658	1.25
I_c	$+50$	0.436	0.055	0.633	0.288
	$+20$	0.437	0.022	0.636	0.116
	-20	0.438	-0.022	0.639	-0.117
	-50	0.439	-0.056	0.642	-0.294
I_e	$+50$	0.437	-0.170	0.637	-0.149
	$+20$	0.437	-0.068	0.637	-0.059
	-20	0.438	0.068	0.638	0.059
	-50	0.438	0.170	0.638	0.148
a	$+50$	0.366	42.15	0.550	38.59
	$+20$	0.404	17.04	0.597	15.70
	-20	0.483	-17.36	0.691	-16.18
	-50	0.595	-44.35	0.817	-41.85
b	$+50$	0.443	25.27	0.618	32.88
	$+20$	0.440	9.22	0.629	11.69
	-20	0.435	-8.20	0.647	-10.09
	-50	0.430	-18.83	0.663	-22.69

Sensitivity Analysis

Using the numerical Examples 1 and 2 the sensitivity of the decision variable F^* and the total cost function $TC(F^*)$ to changes in each of the 12 parameters h_o, h_r, α, β, C_0, R, c_s, δ, I_c, I_e, a, b is examined in Table 3. The sensitivity analysis is performed by changing each of the parameters by -50, -20, $+20$ and $+50$ %, taking one parameter at a time and keeping the remaining parameters unchanged. It is observed that parameters C_0, R, a and b are highly sensitive and the parameters h_o, h_r, α, β, c_s, δ, I_c, I_e are slightly sensitive on total cost $TC(F^*)$. Again, the total cost function $TC(F^*)$ increases or decreases as the parameters h_o, h_r, α, β, C_0, c_s, I_c, a, b increases or decreases whereas the other parameters have reverse effect on total cost function.

Conclusions

Practically, we see that credit policies is an attractive feature in inventory management in order to purchase more items than in a rented warehouse capacity. Thus our research work deals with the optimal replenishment cycle for a two warehouse inventory model where delay in payment is permissible. Some realistic features like effect of inflation and time value of money, different deterioration rate in OW and RW, shortages and partial backlogging are investigated. Also, holding cost in RW has been assumed higher than in OW. The present value

of total cost during planning horizon under exponential increasing demand in this inventory system has been developed. Finally, the proposed model has been illustrated through several numerical examples and sensitivity analysis.

1. We could extend this model considering time-price or quadratic price dependent demand or stochastically fluctuating demand pattern.
2. We could extend this model again by considering fuzzy, fuzzy-stochastic environment and other asMonday, February 5, 2018 at 11:22 amsumption.

Appendix A

From Eq. (3) we obtain order quantity $Q = I(0) = \frac{a}{\alpha+b}(e^{(\alpha+b)t_1} - 1)$ and if the order quantity $Q \leq W$ then it is unnecessary to rent any warehouse. Thus $\frac{a}{\alpha+b}(e^{(\alpha+b)t_1} - 1) \leq W$ implies $t_1 \leq \frac{1}{\alpha+b}log(1 + \frac{(\alpha+b)W}{a})$. For simplicity, let $t_a = \frac{1}{\alpha+b}log(1 + \frac{(\alpha+b)W}{a})$. The inequality $Q \leq W$ holds if and only if $t_a \geq t_1$. Therefore, the inequality $Q > W$ holds if and only if $t_a < t_1$ which implies that there are W units of items stored in OW and the remainder are despatched in RW.

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