Int. J. Appl. Comput. Math (2017) 3 (Suppl 1):S1249–S1259
DOI 10.1007/s40819-017-0414-5

ORIGINAL PAPER

Periodic Solution for Strongly Nonlinear Oscillators by He's New Amplitude–Frequency Relationship

O. González-Gaxiola[1]

Published online: 4 September 2017

Abstract This paper applies He's new amplitude–frequency relationship recently established by He (Int J Appl Comput Math 3(2):1557–1560, 2017. doi:10.1007/s40819-016-0160-0) to study periodic solutions of strongly nonlinear systems with odd nonlinearities. Some examples are given to illustrate the effectiveness, ease and convenience of the method. In general, the results are valid for small as well as large oscillation amplitude. The method can be easily extended to other nonlinear systems with odd nonlinearities and can therefore be found widely applicable in engineering and other science. The method used in this paper can be applied directly to highly nonlinear problems without any discretization, linearization or additional requirements.

Keywords Nonlinear oscillators · Periodic solution · Approximate frequency · Conservative oscillator

Mathematics Subject Classification 34L30 · 34B15 · 34C15

Introduction

Nonlinear vibration arises everywhere in science, engineering and other disciplines, since most phenomena in our world today, are essentially nonlinear and are described by nonlinear equations. It is very important in applications to have a version of the frequency (or period) to have a better understanding of the phenomena modeled through differential equations that contain terms with high nonlinearities, and a simple mathematical method is very useful for practical applications.

Recently many analytical methods have appeared to obtain the approximate solutions of nonlinear systems, such as the parameter-expansion method [30], the harmonic balance

✉ O. González-Gaxiola
ogonzalez@correo.cua.uam.mx

1 Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa, Vasco de Quiroga 4871, Santa Fe, Cuajimalpa, 05300 Mexico City, Mexico

method [3,22,28,31], the energy balance method [23,36], the Hamiltonian approach [13,37], the use of special functions [6,7], the max–min approach [14,39], the Adomian decomposition technique [10], the variational iteration method [15,16,21,33,35] and homotopy perturbation [2,4,9,11,17–19] are some examples. An excellent study, in which many of these techniques can be found in detail to solve nonlinear problems of oscillatory type can be seen in [20].

Recently, In [12] an analytical approximate technique for large and small amplitudes oscillations of a class of conservative single degree-of-freedom systems with odd non-linearity is proposed. In this study, we have applied new method to find the approximate solutions of nonlinear differential equation governing strongly nonlinear oscillators and have made a comparison with the exact solution. The most interesting features of the used method are its simplicity and its excellent accuracy of both period and corresponding periodic solution for the entire range of oscillation amplitude. Finally, four examples are presented to describe the solution methodology and to illustrate the usefulness and effectiveness of the proposed technique.

He’s New Amplitude–Frequency Relationship

Consider a one-dimensional, free nonlinear oscillator (undamped and unforced) governed by

$$u'' + f(u) = 0, \tag{1}$$

with the initial conditions

$$u(0) = A, \quad u'(0) = 0 \tag{2}$$

where a prime denotes differentiation with respect to t, u is the displacement, and the nonlinear restoring force $f(u)$ is an odd function of u, i.e. $f(-u) = -f(u)$ and satisfies $f(u)/u > 0$ for $u \in [-A, A]$, $u \neq 0$. It is obvious that $u = 0$ is the equilibrium position. The system oscillates between the symmetric bounds $-A$ and A. If $f(u)$ is a nonlinear function, both period T and frequency $\omega = 2\pi/T$ of the corresponding oscillation are dependent upon the amplitude of oscillation A. The relationship between the frequency and amplitude is the main property of a nonlinear oscillator; see [5,8] and references therein.

A simple realization of the harmonic oscillator in classical mechanics is a particle which is acted upon by a restoring force proportional to its displacement from its equilibrium position. Considering motion in one dimension, this means

$$f = -ku \tag{3}$$

Such a force might originate from a spring which obeys Hooke’s law. The force constant k is a measure of the stiffness of the spring. Now applying Newton’s second law to the force from Eq. (3), we obtain

$$f = mu'' = -ku \tag{4}$$

where m is the mass of the body attached to the spring, which is itself assumed massless. This leads to a linear differential equation of familiar form

$$u'' + \omega^2 u = 0, \quad \omega^2 = k/m. \tag{5}$$

The square of its frequency can be easily obtained, which reads

$$\omega^2 = F'(u) \tag{6}$$

Table 1 Criterion for choosing a location point

Conditions	Location point for Eq. (10)
$uf''(u) < 0$	$0 < u_i < A/2$
$uf''(u) > 0$	$A/2 \le u_i < A$

where $F(u)$ is the restoring force, $F(u) = \omega^2 u$.

Let us now consider a general nonlinear oscillator of the form given by Eq. (1). In this case, the restoring force is given by $f(u)$. We extend Eq. (6) to nonlinear cases, that is

$$\omega^2 = f'(u). \tag{7}$$

According to He's new amplitude–frequency formulation, the approximate frequency as a function of A can be obtained as follows [12]:

$$\omega^2(A) = \frac{\sum_{i=1}^{N} \omega_i^2(A)}{N} \tag{8}$$

with each $\omega_i^2(A)$ defined by

$$\omega_i^2(A) = f'(u_i) \tag{9}$$

where u_i are location points, $0 < u_i < A$. Explicitly, $u_i = iA/N$ for every $i = 1, 2, \ldots, N-1$.

The simplest way to calculate the frequency is given by

$$\omega^2(A) = f'(u_i), \tag{10}$$

for some $0 < u_i < A$. The accuracy, however, depends greatly upon the location point.

In Table 1 we present the criteria suggested by He [12] for choosing a suitable location point u_i.

Therefore, according to Eq. (10) the analytical approximate frequency ω as a function of A is

$$\omega_{app}(A) = \sqrt{f'(u_i)}. \tag{11}$$

For conservative oscillations with an odd restoring force $f(u)$, there exists a periodic motion around the equilibrium point $u = 0$ with frequency ω and amplitude A. Therefore, a reasonable and the simplest initial approximation of $u(t)$ which satisfies the initial condition given by Eq. (2) is [25]:

$$u(t) = A\cos(\omega t). \tag{12}$$

From Eq. (11) we obtain the following approximate periodic solution to (1)

$$u_{app}(t) = A\cos\left(\sqrt{f'(u_i)} \cdot t\right). \tag{13}$$

Numerical Examples

In this section, we will give four examples to illustrate the use and the efectiveness of the present approach.

Nonlinear oscillators in physics, engineering, biology, mathematical and related fields have been the focus of attention for many years. The Duffing equation is a well-known nonlinear differential equation which is related to many practical engineering systems such as the classical nonlinear spring system with odd nonlinear restoring characteristics and more recently in different physical phenomena. There have been many variations of Duffing equation, one of them is the cubic–quintic Duffing equation. The systems modelled by the cubic–quintic Duffing equation include the nonlinear dynamics of a slender elastica, the compound Korteweg–de Vries (KdV) equation in nonlinear wave systems, or the propagation of a short electromagnetic pulse in a nonlinear medium, structural dynamics, among others [31,32]. In Examples 1 and 2, we will illustrate that the proposed method is remarkably effective and applicable in solving the generalized Duffing equation even with strong nonlinearities. Finally, in the Examples 3 and 4 He's new amplitude–frequency relationship will be used to solve the nonlinear differential equation related to plasma physics and oscillations of a mass attached to a stretched elastic wire, respectively.

Example 1 Consider the cubic–quintic Duffing nonlinear oscillator, which is modelled by the following second-order differential equation

$$u'' + u + u^3 + u^5 = 0, \tag{14}$$

with initial conditions

$$u(0) = A, \quad u'(0) = 0. \tag{15}$$

In the present example we have $f(u) = u + u^3 + u^5$, it is clear that f is an odd function and satisfies $f(u)/u > 0$.

Calculating we have $f'(u) = 1 + 3u^2 + 5u^4$ and $f''(u) = 6u + 20u^3$, hence $uf''(u) > 0$. Now, considering the criterion given in Table 1 we must take the location points $A/2 \le u_i < A$. If we take $u_i = 0.5772A$ and consider the proposed approach in Eq. (11), one can assume for the frequency–amplitude formulation

$$\omega_{app}(A) = \sqrt{1 + 3(0.5772)^2 A^2 + 5(0.5772)^4 A^4}. \tag{16}$$

We, therefore, obtain the following periodic solution:

$$u_{app}(t) = A\cos\left(\sqrt{1 + 3(0.5772)^2 A^2 + 5(0.5772)^4 A^4} \cdot t\right) \tag{17}$$

which has a high accuracy (see Figs. 1, 2).

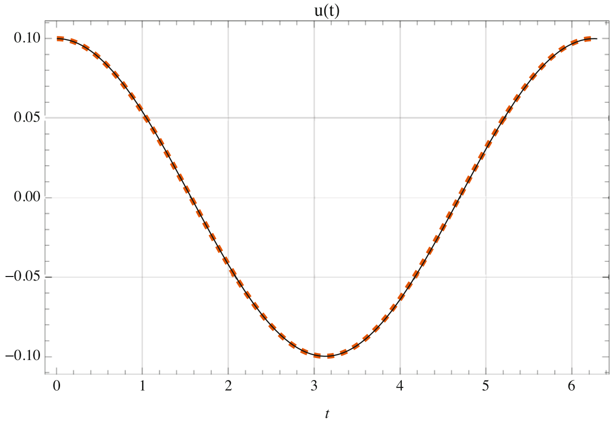

Fig. 1 Comparison of analytical approximation (*dashed*) and exact solution (*black*) for $A = 1/10$ in Example 1

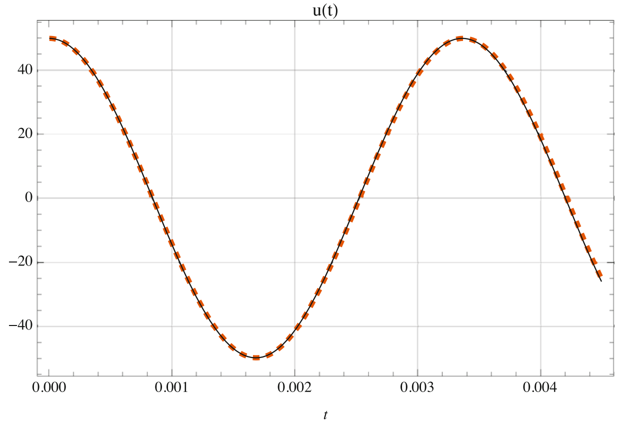

Fig. 2 Comparison of analytical approximation (*dashed*) and exact solution (*black*) for $A = 50$ in Example 1

Table 2 Comparison between frequencies $\omega_{app}(A)$ and $\omega_{ex}(A)$ for different values of A

A	$\omega_{app}(A)$ Eq. (16)	$\omega_{ex}(A)$ Eq. (18)	Relative error (%)
1/1000	1.0000004997	1.0000003750	0.0000124
1/100	1.0000499755	1.0000375023	0.0012401
1/10	1.0050125835	1.0037729382	0.1234941
10	75.171283755	75.177400632	0.0081365
50	1863.0910920	1867.5739782	0.2400379
100	7450.3513534	7468.8303066	0.2474142
1000	744,968.72043	746,834.68847	0.2498502

The exact frequency for the present example is given by [38]:

$$\omega_{ex}(A) = \frac{2\pi}{\displaystyle\int_0^{\pi/2} \frac{4d\theta}{\sqrt{1 + \frac{1}{2}\left(1 + \sin^2\theta\right)A^2 + \frac{1}{3}\left(1 + \sin^2\theta + \sin^4\theta\right)A^4}}}. \tag{18}$$

From Table 2, it can be observed that Eq. (16) yield excellent analytical approximate periods for both small and large values of oscillation amplitude A.

The obtained results in this example reveals that the presented method is very effective, simple and exact and is valid for small and large amplitudes.

Example 2 In this example, we consider the following nonlinear Duffing oscillator:

$$u'' + u + u^5 = 0, \tag{19}$$

subject to the initial conditions

$$u(0) = A, \quad u'(0) = 0. \tag{20}$$

For this problem,

$$f(u) = u + u^5,$$

it is clear that f is an odd function and satisfies $f(u)/u > 0$.

Table 3 Comparison between frequencies $\omega_{app}(A)$ and $\omega_{ex}(A)$ for different values of A

A	$\omega_{app}(A)$ Eq. (21)	$\omega_{ex}(A)$ Eq. (22)	Relative error (%)
1/1000	1.0000000000	1.0000000000	0.0000000
1/100	1.0000000028	1.0000000031	0.0000000
1/10	1.0000278833	1.0000312493	0.0003365
1	1.2480683052	1.2647077571	1.3156756
10	74.684301857	74.690887847	0.0088176
100	7467.7607379	7468.3420769	0.0077840
500	186,694.01678	186,708.55006	0.0077839
1000	746,776.06710	746,834.20022	0.0077839
10,000	7.467760×10^7	7.468342×10^7	0.0077839

Derivating we have, $f'(u) = 1 + 5u^4$ and $f''(u) = 20u^3$, hence $uf''(u) = 20u^4 > 0$. Therefore, considering the criterion given in Table 1 we must take the location points $A/2 \leq u_i < A$. If we take $u_i = 0.5779A$ and consider the proposed approach in Eq. (11), one can assume for the frequency–amplitude formulation

$$\omega_{app}(A) = \sqrt{1 + 5(0.5779)^4 A^4}. \tag{21}$$

The exact frequency for the present problem was established in [16] and is given by

$$\omega_{ex}(A) = \frac{\pi\sqrt{A^4+3}}{2\sqrt{3}} \left(\int_0^{\pi/2} \frac{1}{\sqrt{1 + \left(\frac{A^4}{A^4+3}\right)(\sin^2\theta + \sin^4\theta)}} \, d\theta \right)^{-1}. \tag{22}$$

To illustrate and verify accuracy of these approximate analytical approach, a comparison of approximate frequencies $\omega_{app}(A)$ for different values of amplitude A and the exact frequencies $\omega_{ex}(A)$ is presented in Table 3. Note that the approximation is very accurate for small values and large values of A. From Table 3 we can see that

$$\lim_{A\to 0^+} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = 1 \quad \text{and} \quad \lim_{A\to\infty} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = 0.999922. \tag{23}$$

Considering the approximation for the frequency obtained in Eq. (21) the approximate solution of Eq. (19) becomes

$$u_{app}(t) = A\cos\left(\sqrt{1 + 5(0.5779)^4 A^4} \cdot t\right). \tag{24}$$

For this example we will not show graphs as we did in the previous example, because the high precision would not allow the distinction between them.

Example 3 A problem of some importance in plasma physics concerns an electron beam injected into a plasma tube where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius. The governing equation for the path u of the electrons is modelled by the following second-order differential equation [1,24]:

$$u'' + \frac{1}{u} = 0, \tag{25}$$

Table 4 Comparison between frequencies $\omega_{app}(A)$ and $\omega_{ex}(A)$ for different values of A

A	$\omega_{app}(A)$ Eq. (28)	$\omega_{ex}(A)$ Eq. (27)	Relative error (%)
1/1000	1251.5644556	1253.3141373	0.13960
1/100	125.15644556	125.33141373	0.13960
1/10	12.515644556	12.533141373	0.13960
1	1.2515644556	1.2533141373	0.13960
10	0.1251564456	0.1253314137	0.13960
100	0.0125156446	0.0125331414	0.13960
500	0.0025031289	0.0025066282	0.13960
1000	0.0012515644	0.0012533141	0.13960

with initial conditions

$$u(0) = A, \quad u'(0) = 0. \tag{26}$$

The exact solution for Eq. (25) as a function of A was obtained in [29] and this is

$$\omega_{ex}(A) = 2\pi\left[2\sqrt{2}A\int_0^1 \frac{ds}{\sqrt{\ln(1/s)}}\right]^{-1}. \tag{27}$$

To use the method presented in the "He's New Amplitude–Frequency Relationship" section, we will consider $f(u) = \frac{1}{u}$, it is clear that f is an odd function and satisfies $f(u)/u > 0$.

Calculating, we get $f'(u) = -\frac{1}{u^2}$ and $f''(u) = \frac{2}{u^3}$, hence $uf''(u) > 0$. Now, considering again the criterion given in Table 1 we must take the location points $A/2 \le u_i < A$. If we take $u_i = 0.799A$ and consider the proposed approach in Eq. (11), one can assume for the frequency–amplitude formulation

$$\omega_{app}(A) = \sqrt{\frac{1}{\left(\frac{799}{1000}\right)^2 A^2}} = \frac{1000}{799A}. \tag{28}$$

$$\lim_{A\to 0^+} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = \lim_{A\to\infty} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = 0.9986. \tag{29}$$

Finally, considering the approximation (28), we have obtain the following periodic solution of the Eq. (25)

$$u_{app}(t) = A\cos\left(\frac{1000}{799A}\cdot t\right). \tag{30}$$

The obtained solution is of remarkable accuracy, as shown in Table 4 and Fig. 3.

Example 4 The governing nonlinear differential equation of motion and the associated initial conditions for a mass attached to a stretched elastic wire are [28]:

$$u'' + u + \frac{u}{\sqrt{1+u^2}} = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{31}$$

Which, $f(u) = u + \frac{u}{\sqrt{1+u^2}}$. Its derivatives are:

$$f'(u) = 1 + \frac{1}{\sqrt{\left(1+u^2\right)^3}}, \quad f''(u) = -\frac{3u}{\sqrt{\left(1+u^2\right)^5}}. \tag{32}$$

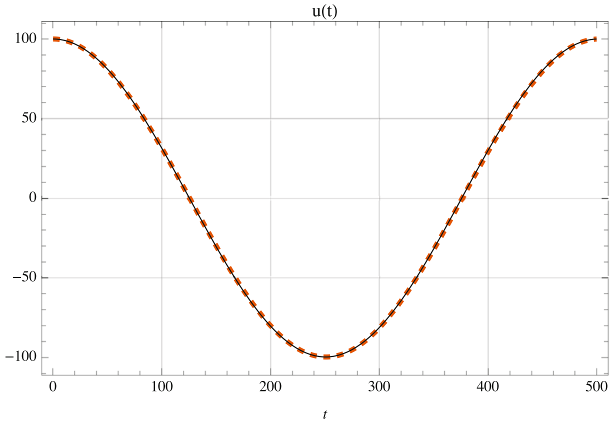

Fig. 3 Comparison of analytical approximation (*dashed*) and exact solution (*black*) for $A = 100$ in Example 3

Table 5 Comparison between frequencies $\omega_{app}(A)$ and $\omega_{ex}(A)$ for different values of A

A	$\omega_{app}(A)$ Eq. (33)	$\omega_{ex}(A)$ Eq. (34)	Relative error (%)
1/1000	1.4142134402	1.4142134298	0.0000007
1/100	1.4142013439	1.4142003049	0.0000734
1/10	1.4129946662	1.4128952474	0.0070365
1	1.3163234011	1.3273988465	0.8343720
10	1.0042330178	1.0606052889	5.3151037
100	1.0000045182	1.0063415277	0.6297076
1000	1.0000000045	1.0006363862	0.0635976
10,000	1.0000000000	1.0000636597	0.0063655

From Eq. (32) we have $uf''(u) < 0$. Considering the criterion given in Table 1 we must take the location points $A < u_i < A/2$. If we take $u_i = 0.48A$ and consider the proposed approach in Eq. (11), one can assume for the frequency–amplitude formulation

$$\omega_{app}(A) = \sqrt{1 + \frac{1}{\left(1 + \left(\frac{48}{100}\right)^2 A^2\right)^{3/2}}}. \tag{33}$$

The nonlinear oscillator described in Eq. (31) is a conservative system. By integrating Eq. (31) and using the initial conditions, we arrive at

$$\omega_{ex}(A) = \frac{1}{2}\pi \left(\int_0^{\frac{\pi}{2}} \frac{A\cos\theta}{\sqrt{A^2\cos^2\theta - 2\left(\sqrt{1 + A^2\sin^2\theta} - \sqrt{1 + A^2}\right)}} d\theta \right)^{-1} \tag{34}$$

By taking into account our approximation made through He's frequency–amplitude formulation Eq. (33) and $\omega_{ex}(A)$ from Eq. (34) we can calculate the Table 5 for small and large values of A.

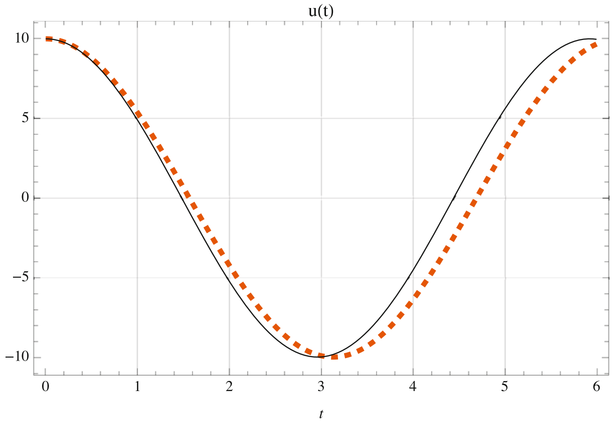

Fig. 4 Comparison of analytical approximation (*dashed*) and exact solution (*black*) for $A = 10$ in Example 4

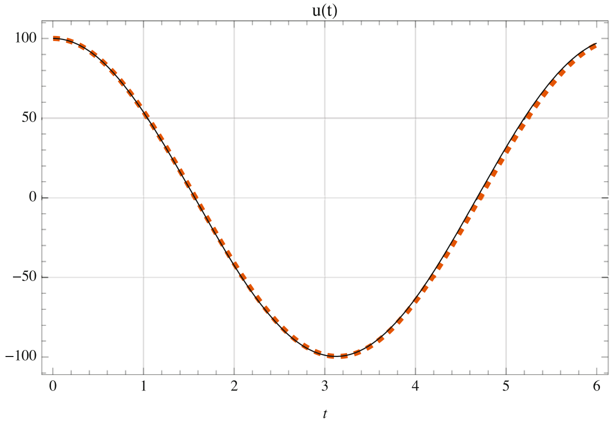

Fig. 5 Comparison of analytical approximation (*dashed*) and exact solution (*black*) for $A = 100$ in Example 4

Also, considering the approximation given by Eq. (33), we have obtain the following periodic solution of the Eq. (31)

$$u_{app}(t) = A\cos\left(\sqrt{1 + \frac{1}{\left(1 + \left(\frac{48}{100}\right)^2 A^2\right)^{3/2}}} \cdot t\right). \tag{35}$$

The obtained solution is very acceptable accuracy, as shown in Figs. 4 and 5.

We can conclude that formula (33) is valid for the whole range of values of amplitude of oscillation and its maximum relative error is 5.3% and this is obtained when $A = 10$. We can also see that, for very large or very small values of A, we have

$$\lim_{A \to 0^+} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = \lim_{A \to \infty} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = 1. \tag{36}$$

The study of nonlinear problems arisen in many areas of physics and also engineering is very complex issue for scientists. Since most phenomena in our world are essentially nonlinear and are described by nonlinear equations, it is very difficult to solve nonlinear problems and in general, it is often more difficult to get an analytic approximation than a numerical one for given nonlinear problems. One of the critical problems in materials science field is the behavior of elastic materials. The extensive literature on the topic is now available and we can only mention a few recent interesting investigations in [26,27,34].

Conclusions

He's new amplitude–frequency relationship recently established by He [12] is proved to be a powerful mathematical tool for use in the search for periodic solutions of nonlinear oscillators. It is simple, straightforward and effective. Moreover the approximate analytical solutions are valid for small as well as large amplitudes of oscillation.

The new method applied in this paper is of potential and can be applied to other strongly nonlinear oscillators with more general restoring forces provided that they meet the requirements established in "He's New Amplitude–Frequency Relationship" section. Finally, four examples have been presented to illustrate excellent accuracy of the analytical approximate periods and the corresponding periodic solutions; being our main contribution in the present study to find the location points involved in Eq. (10). All numerical work and graphics were performed with the Mathematica software package.

Acknowledgements The author would like to thank the anonymous referees for their constructive comments and suggestions that helped to improve the paper.

References

1. Acton, J.R., Squire, P.T.: Solving Equations with Physical Understanding. Adam Hilger Ltd, Bristol (1985)
2. Beléndez, A., Hernández, A., Beléndez, T., Fernández, E., Álvarez, M.L., Neipp, C.: Application of He's homotopy perturbation method to the Duffing harmonic oscillator. Int. J. Nonlinear Sci. Numer. Simul. **8**(1), 79–88 (2007). doi:10.1515/IJNSNS.2007.8.1.79
3. Beléndez, A., Pascual, C.: Harmonic balance approach to the periodic solutions of the (an)harmonic relativistic oscillator. Phys. Lett. A **371**(4), 291–299 (2007). doi:10.1016/j.physleta.2007.09.010
4. Beléndez, A., Pascual, C., Gallego, S., Ortuño, M., Neipp, C.: Application of a modified He's homotopy perturbation method to obtain higher-order approximations of a $x^{1/3}$ force nonlinear oscillator. Phys. Lett. A **371**(5–6), 421–426 (2007). doi:10.1016/j.physleta.2007.06.042
5. Cai, X.C., Liu, J.F.: Application of the modified frequency formulation to a nonlinear oscillator. Comput. Math. Appl. **61**(8), 2237–2240 (2011). doi:10.1016/j.camwa.2010.09.025
6. Elías-Zúñiga, A.: Exact solution of the cubic–quintic Duffing oscillator. Appl. Math. Model. **37**(4), 2574–2579 (2013). doi:10.1016/j.apm.2012.04.005
7. Elías-Zúñiga, A.: Solution of the damped cubic–quintic Duffing oscillator by using Jacobi elliptic functions. Appl. Math. Comput. **246**, 474–481 (2014). doi:10.1016/j.amc.2014.07.110
8. Ganji, S.S., Ganji, D.D., Ganji, Z.Z., Karimpour, S.: Periodic solution for strongly nonlinear vibration systems by He's energy balance method. Acta Appl. Math. **106**(1), 79–92 (2009). doi:10.1007/s10440-008-9283-6
9. Ganji, D.D., Sadighi, A.: Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction–diffusion equations. Int. J. Nonlinear Sci. Numer. Simul. **7**(4), 411–418 (2006). doi:10.1515/IJNSNS.2006.7.4.411
10. González-Gaxiola, O., Santiago, J.A., de Chávez, J.Ruiz: Solution for the nonlinear relativistic harmonic oscillator via Laplace–Adomian decomposition method. Int. J. Appl. Comput. Math. **3**(3), 2627–2638 (2017). doi:10.1007/s40819-016-0267-3
11. Gorji, M., Ganji, D.D., Soleimani, S.: New application of He's homotopy perturbation method. Int. J. Nonlinear Sci. Numer. Simul. **8**(3), 319–328 (2007). doi:10.1515/IJNSNS.2007.8.3.319

12. He, J.H.: Amplitude–frequency relationship for conservative nonlinear oscillators with odd nonlinearities. Int. J. Appl. Comput. Math. **3**(2), 1557–1560 (2017). doi:10.1007/s40819-016-0160-0
13. He, J.H.: Hamiltonian approach to nonlinear oscillators. Phys. Lett. A **374**(23), 2312–2314 (2010). doi:10.1016/j.physleta.2010.03.064
14. He, J.H.: Max–min approach to nonlinear oscillators. Int. J. Nonlinear Sci. Numer. Simul. **9**(2), 207–210 (2008). doi:10.1515/IJNSNS.2008.9.2.207
15. He, J.H.: Variational approach for nonlinear oscillators. Chaos Solitons Fractals **34**(5), 1430–1439 (2007). doi:10.1016/j.chaos.2006.10.026
16. He, J.H.: Variational iteration method a kind of non-linear analytical technique: some examples. Int. J. Nonlinear Mech. **34**(4), 699–708 (1999). doi:10.1016/S0020-7462(98)00048-1
17. He, J.H.: Homotopy perturbation method for solving boundary value problems. Phys. Lett. A **350**(1–2), 87–88 (2006). doi:10.1016/j.physleta.2005.10.005
18. He, J.H.: Homotopy perturbation method for bifurcation on nonlinear problems. Int. J. Nonlinear Sci. Numer. Simul. **6**(2), 207–208 (2005). doi:10.1515/IJNSNS.2005.6.2.207
19. He, J.H.: The homotopy perturbation method for nonlinear oscillators with discontinuities. Appl. Math. Comput. **151**(1), 287–292 (2004). doi:10.1016/S0096-3003(03)00341-2
20. He, J.H.: Some asymptotic methods for strongly nonlinear equations. Int. J. Mod. Phys. B **20**(10), 1141–1199 (2006). doi:10.1142/S0217979206033796
21. He, J.H., Wu, X.H.: Construction of solitary solution and compact on-like solution by variational iteration method. Chaos Solitons Fractals **29**(1), 108–113 (2006). doi:10.1016/j.chaos.2005.10.100
22. Hu, H., Tang, J.H.: Solution of a Duffing-harmonic oscillator by the method of harmonic balance. J. Sound Vib. **294**(3), 637–639 (2006). doi:10.1016/j.jsv.2005.12.025
23. Khan, Y., Mirzabeigy, A.: Improved accuracy of He's energy balance method for analysis of conservative nonlinear oscillator. Neural Comput. Appl. **25**(3), 889–895 (2014). doi:10.1007/s00521-014-1576-2
24. Lan, X.: A Hamiltonian approach for a plasma physics problem. Comput. Math. Appl. **61**(8), 1909–1911 (2011). doi:10.1016/j.camwa.2010.06.028
25. Liao, S.J., Chwang, A.T.: Application of homotopy analysis method in nonlinear oscillations. J. Appl. Mech. **65**(4), 914–922 (1998). doi:10.1115/1.2791935
26. Marin, M.: On weak solutions in elasticity of dipolar bodies with voids. J. Comput. Appl. Math. **82**(1–2), 291–297 (1997). doi:10.1016/S0377-0427(97)00047-2
27. Marin, M.: Harmonic vibrations in thermoelasticity of microstretch materials. J. Vib. Acoust. ASME **132**(4), 044501-1 (2010). doi:10.1115/1.4000971
28. Mickens, R.E.: Oscillations in Planar Dynamics Systems. World Scientific, Singapore (1996)
29. Mickens, R.E.: Harmonic balance and iteration calculations of periodic solutions to $y'' + y^{-1} = 0$. J. Sound Vib. **306**, 968–972 (2007). doi:10.1016/j.jsv.2007.06.010
30. Mohyud-Din, S.T., Noor, M.A., Noor, K.I.: Parameter-expansion techniques for strongly nonlinear oscillators. Int. J. Nonlinear Sci. Numer. **10**(5), 581–583 (2009). doi:10.1515/IJNSNS.2009.10.5.581
31. Nayfeh, A.H.: Problems in Perturbation. Wiley, New York (1985)
32. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillations. Wiley, New York (1979)
33. Rafei, M., Ganji, D.D., Daniali, H., Pashaei, H.: The variational iteration method for nonlinear oscillators with discontinuities. J. Sound Vib. **305**(4–5), 614–620 (2007). doi:10.1016/j.jsv.2007.04.020
34. Sharma, K., Marin, M.: Effect of distinct conductive and thermodynamic temperatures on the reflection of plane waves in micropolar elastic half-space. U. P. B. Sci. Bull. Ser. A **75**(2), 121–132 (2013)
35. Wazwaz, A.M.: The variational iteration method: a powerful scheme for handling linear and nonlinear diffusion equations. Comput. Math. Appl. **54**(7–8), 933–939 (2007). doi:10.1016/j.camwa.2006.12.039
36. Yildirim, A., Askari, H., Saadatnia, Z., Kalami-Yazdi, M., Khand, Y.: Analysis of nonlinear oscillations of a punctual charge in the electric field of a charged ring via a Hamiltonian approach and the energy balance method. Comput. Math. Appl. **62**(1), 486–490 (2011). doi:10.1016/j.camwa.2011.05.029
37. Yildirim, A., Saadatnia, Z., Askari, H.: Application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms. Math. Comput. Model. **54**(1–2), 697–703 (2011). doi:10.1016/j.mcm.2011.03.012
38. Younesian, D., Askari, H., Saadatnia, Z., KalamiYazdi, M.: Frequency analysis of strongly nonlinear generalized Duffing oscillators using He's frequency–amplitude formulation and He's energy balance method. Comput. Math. Appl. **59**(9), 3222–3228 (2010). doi:10.1016/j.camwa.2010.03.013
39. Zeng, D.Q.: Nonlinear oscillator with discontinuity by the max–min approach. Chaos Solitons Fractals **42**(15), 2885–2889 (2009). doi:10.1016/j.chaos.2009.04.029