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ORIGINAL PAPER

Solution for the Nonlinear Relativistic Harmonic Oscillator via Laplace-Adomian Decomposition Method

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Abstract In this paper, the Adomian decomposition method in combination with the Laplace transform is used to solve the nonlinear differential equation that governs the oscillation of the relativistic oscillator. The solution obtained is a power series of functions that have never been reported and which show a very good match when compared with other approximate solutions, obtained by different methods. The method proposed herein works with high degree of accuracy. Moreover, the proposed method requires less computational effort, and is therefore, very convenient for solving such types of nonlinear differential equations.

Keywords Relativistic harmonic oscillator · Nonlinear ordinary differential equations · Nonlinear oscillations · Adomian polynomials · Laplace transform

Mathematics Subject Classification 34L30 · 34C15 · 74G10

Introduction

Many of the phenomena that arise in the real world can be described by means of nonlinear partial and ordinary differential equations and, in some cases, by integro-differential equations. However, most of the mathematical methods developed thus far are only capable of solving linear differential equations. In the 1980's, George Adomian (1923–1996) introduced a powerful method to solve nonlinear differential equations, known as the Adomian decomposition method (ADM) [3,4]. The technique is based on the decomposition of a solution of a nonlinear differential equations into a series of functions. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytic function.

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The Adomian method is very simple in an abstract formulation; however, calculating the polynomials is difficult, which becomes a non-trivial task. This method has widely been used to solve equations that come from nonlinear models as well as to solve fractional differential equations [15,16,31]. The chaotic nature and nonlinearity of other systems, proposed in the past, have been studied through ADM in Ghosh et al. [21]. The advantage of this method is that it solves the problem directly without the need of linearization, perturbation, or any other transformation, and also, requires relatively lesser computational effort as compared to most other methods.

The relativistic nonlinear harmonic oscillator, was studied for the first time in the middle of the last century [9,10,22,29,30]. In spite of its importance in several models of physics, exact solutions of its equation of motion have not been obtained. In the present work, we will use the Adomian decomposition method in combination with the Laplace transform (LADM) [33] to determine the solution to the relativistic oscillator problem. This equation is a nonlinear ordinary differential equation that, in physics, is used to model a simple one dimensional harmonic oscillator with relativistic velocities. We decompose the nonlinear terms of this equation using the Adomian polynomials and then, in combination with the use of the Laplace transform, we obtain an algorithm to solve the problem subject to initial conditions. Finally, we illustrate our procedure and the quality of the algorithm obtained by solving two examples in which the nonlinear differential equation is solved for different initial conditions.

Our work is divided into several sections. In the "Adomian Decomposition Method Combined With Laplace Transform" section, we present, in a brief and self-contained manner, the LADM. Several references are given to delve deeper into the subject and to study its mathematical foundation, which is beyond the scope of the present work. In the "Relativistic Harmonic Oscillator" section, we present a brief introduction to the model described by the relativistic harmonic oscillator. In the "Main Result: Solution of the Relativistic Harmonic Oscillator Equation Through LADM" section, we establish that LADM can be used to solve this equation in a very simple way. In "Application to the Relativistic Harmonic Oscillator" section, we show by means of two examples, the quality and precision of our method, comparing the obtained results with existing approximate solutions available in the literature and obtained by other methods. Finally, in the "Conclusion and Summary" section, we present the conclusions and implications of this study.

The Adomian Decomposition Method Combined with Laplace Transform

The ADM is a method to solve ordinary and nonlinear differential equations. Using this method, it is possible to express analytic solutions in terms of a series [4]. In other words, the method identifies and separates the linear and nonlinear parts of a differential equation. By inverting and applying the highest order differential operator that is contained in the linear part of the equation, it is possible to express the solution in terms of the rest of the equation affected by the inverse operator. At this point, the solution is proposed by means of a series with terms that will be determined and that give rise to the Adomian Polynomials [32]. The nonlinear part can also be expressed in terms of these polynomials. The initial (or the border conditions) and the terms that contain the independent variables will be considered as the initial approximation. In this manner, and by means of recurrence relations, it is possible to find the terms of the series that give the approximate solution of the differential equation. In the next paragraph, we will see how to use the ADM in combination with the Laplace transform (LADM). Let us consider the following homogeneous differential equation of second order:

$$\frac{d^2x}{dt^2} + N(x) = 0 \tag{1}$$

with initial conditions

$$x(0) = \alpha, \quad x'(0) = \beta \tag{2}$$

where α, β are real constants and N is a nonlinear operator acting on the dependent variable x and some of its derivatives. In general, if we consider the second-order differential operator $L_{tt} = \frac{\partial^2}{\partial t^2}$, then the Eq. (1) can be written as

$$L_{tt}x(t) + N(x(t)) = 0. \tag{3}$$

Solving for $L_{tt}x(t)$, we have

$$L_{tt}x(t) = -N(x(t)). \tag{4}$$

The LADM consists of applying Laplace transform (denoted throughout this paper by $\mathscr{L}$) first on both sides of Eq. (4), thereby obtaining

$$\mathscr{L}\{L_{tt}x(t)\} = -\mathscr{L}\{N(x(t))\}. \tag{5}$$

An equivalent expression to (5) is

$$s^2x(s) - sx(0) - x'(0) = -\mathscr{L}\{Nx(t)\}, \tag{6}$$

using the initial conditions (2), we have

$$x(s) = \frac{\alpha}{s} + \frac{\beta}{s^2} - \frac{1}{s^2}\mathscr{L}\{N(x(t))\}. \tag{7}$$

Now, applying the inverse Laplace transform to Eq. (7)

$$x(t) = \alpha + \beta t - \mathscr{L}^{-1}\left[\frac{1}{s^2}\mathscr{L}\{N(x(t))\}\right]. \tag{8}$$

The ADM proposes a series of solutions $x(t)$, given by,

$$x(t) = \sum_{n=0}^{\infty} x_n(t). \tag{9}$$

The nonlinear term $N(x)$ is given by

$$N(x) = \sum_{n=0}^{\infty} A_n(x_0, x_1, \ldots, x_n) \tag{10}$$

where $\{A_n\}_{n=0}^{\infty}$ is the so-called Adomian polynomials sequence established in Wazwaz [5,32] and, in general, gives us term by term:

$$\begin{aligned}
&A_0 = N(x_0)\\
&A_1 = x_1N'(x_0)\\
&A_2 = x_2N'(x_0) + \tfrac{1}{2}x_1^2N''(x_0)\\
&A_3 = x_3N'(x_0) + x_1x_2N''(x_0) + \tfrac{1}{3!}x_1^3N^{(3)}(x_0)\\
&A_4 = x_4N'(x_0) + \left(\tfrac{1}{2}x_2^2 + x_1x_3\right)N''(x_0) + \tfrac{1}{2!}x_1^2x_2N^{(3)}(x_0) + \tfrac{1}{4!}x_1^4N^{(4)}(x_0)\\
&\vdots
\end{aligned}$$

Other polynomials can be generated in a similar manner. Some other approaches to obtaining Adomian's polynomials can be found in Duan [17,19].
Using (9) and (10) in Eq. (8), we obtain,

$$\sum_{n=0}^{\infty} x_n(t) = \alpha + \beta t - \mathscr{L}^{-1}\left[\frac{1}{s^2}\mathscr{L}\left\{\sum_{n=0}^{\infty} A_n(x_0, x_1, \ldots, x_n)\right\}\right]. \tag{11}$$

From equation (11), we deduce the recurrence formula, which is given as follows:

$$\begin{cases} x_0(t) = \alpha + \beta t, \\ x_{n+1}(t) = -\mathscr{L}^{-1}\left[\frac{1}{s^2}\mathscr{L}\{A_n(x_0, x_1, \ldots, x_n)\}\right], \quad n = 0, 1, 2, \ldots \end{cases} \tag{12}$$

Using (12), we can obtain an approximate solution of (1), (2) using

$$x(t) \approx \sum_{n=0}^{k} x_n(t), \quad \text{where} \quad \lim_{k\to\infty} \sum_{n=0}^{k} x_n(t) = x(t). \tag{13}$$

It is evident that, the Adomian decomposition method, combined with the Laplace transform requires less effort in comparison with the traditional Adomian decomposition method. This method considerably decreases the volume of calculations. The decomposition procedure of Adomian is easily set, without requiring the linearization of the problem. With this approach, the solution is found in the form of a convergent series with easily computed components; in many cases, the convergence of this series is very fast and only a few terms are needed to gain an understanding of how the solutions behave. Convergence conditions of this series are examined by several authors, mainly in Abbaoui and Cherrault and Cherruault [1,2,13,14]. Additional references related to the use of the ADM, combined with the Laplace transform, can be found in Wazwaz, Khuri and Yusufoğlu [25,33,34] and references therein.

The Relativistic Harmonic Oscillator

The equation of motion of the relativistic harmonic oscillator is given by the nonlinear differential Eq. [12,23]:

$$\frac{d^2x}{dt^2} + \left[1 - \left(\frac{dx}{dt}\right)^2\right]^{\frac{3}{2}} x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = \beta. \tag{14}$$

This normalized, dimensionless form of the equation is obtained by considering the rest mass m to be unity and the speed of light c to also be unity [26]. It is easy to verify that the dimensionless length x and the dimensionless time t are related to the dimensional variables $\bar{x}$ and $\bar{t}$ through $x = \omega_0\bar{x}/c$ and $t = \omega_0\bar{t}$, respectively, where $\omega_0 = \sqrt{k/m}$ is the angular frequency for the non-relativistic oscillator.
To the best of our knowledge, no exact solution of the nonlinear Eq. (14) has yet been published; therefore the research work about Eq. (14) has been intense. A fundamental result reported in Mickens [26] is that all the solutions of (14) are periodic functions with the period dependent on the initial velocity β. In the same work, an approximation solution of (14) was

determined using the harmonic balance method (HBM), which is given by

$$x_{\mathrm{HBM}}(t) = \frac{\beta}{\omega}\left(\frac{3\beta^4+8\beta^2+64}{64}\right)\sin(\omega t) - \frac{\beta^3}{24\omega}\left(\frac{3\beta^2+128}{128}\right)\sin(3\omega t) + \left(\frac{3\beta^5}{640\omega}\right)\sin(5\omega t), \quad \text{where } \omega = \sqrt[4]{\frac{2-2\beta^2}{2-\beta^2}} \text{ and } 0<\beta<1. \tag{15}$$

Some more detailed work in the same direction was reported 10 years later in Beléndez and Mickens [7,8,27,28]. Other mass-spring systems have been studied by the same method in Beléndez [11]. Thereafter, in Ebaid [20], using the differential transformation method (DTM), some periodic solutions were obtained. More recently, the relativistic harmonic oscillator was studied by using the homotopy perturbation method (HPM) [12], where a good approximation is obtained using the fact that the solutions are periodic functions. In the following section, we will develop an algorithm using the method described in the "Adomian Decomposition Method Combined with Laplace Transform" section in order to solve the nonlinear differential Eq. (14) without resorting to any truncation or linearization. Moreover the *a priori* assumption that the solutions are periodic functions is not required.

The Main Result: Solution of the Relativistic Harmonic Oscillator Equation Through LADM

Comparing (14) with Eq. (4) we have that L_{tt} and N becomes:

$$L_{tt}x = \frac{d^2}{dt^2}x, \quad Nx = \left[1-\left(\frac{dx}{dt}\right)^2\right]^{\frac{3}{2}} x. \tag{16}$$

Now, by using Eq. (12) through the LADM method, we recursively obtain

$$\begin{cases} x_0(t) = \beta t, \\ x_{n+1}(t) = -\mathscr{L}^{-1}\left[\frac{1}{s^2}\mathscr{L}\{A_n(x_0, x_1, \ldots, x_n)\}\right], \quad n = 0, 1, 2, \ldots \end{cases} \tag{17}$$

In addition, the nonlinear term is decomposed as

$$Nx = \left[1-\left(\frac{dx}{dt}\right)^2\right]^{\frac{3}{2}} x = \sum_{n=0}^{\infty} A_n(x_0, x_1, \ldots, x_n) \tag{18}$$

where $\{A_n\}_{n=0}^{\infty}$ is the so-called Adomian polynomials sequence, the terms are calculated according to Duan [17–19]. The first few polynomials are given by

$$\begin{aligned} &A_0(x_0) = x_0\left(1-x_0'^2\right)^{\frac{3}{2}}, \\ &A_1(x_0, x_1) = x_1\left(1-x_0'^2\right)^{\frac{3}{2}}, \\ &A_2(x_0, x_1, x_2) = x_2\left(1-x_0'^2\right)^{\frac{3}{2}}, \\ &A_3(x_0, x_1, x_2, x_3) = x_3\left(1-x_0'^2\right)^{\frac{3}{2}}, \\ &A_4(x_0, x_1, x_2, x_3, x_4) = x_4\left(1-x_0'^2\right)^{\frac{3}{2}}, \\ &\vdots \\ &A_m(x_0, x_1, \ldots, x_m) = x_m\left(1-x_0'^2\right)^{\frac{3}{2}} \quad \text{for every } m \geq 0. \end{aligned}$$

Now, recursively using (17) with the Adomian polynomials given by the later sequence $\{A_n\}_{n=0}^{\infty}$, we obtain, for a given initial velocity β:

$$x_0(t) = \beta t, \tag{19}$$

$$\begin{aligned} x_1(t) &= -\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\{\beta(1-\beta^2)^{\frac{3}{2}}t\}\Big] = -\mathscr{L}^{-1}\Big[\frac{1}{s^4}\beta(1-\beta^2)^{\frac{3}{2}}\Big] \\ &= -\beta(1-\beta^2)^{\frac{3}{2}}\frac{t^3}{3!}, \end{aligned} \tag{20}$$

$$\begin{aligned} x_2(t) &= -\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{-\beta(1-\beta^2)^3\frac{t^3}{3!}\Big\}\Big] = \mathscr{L}^{-1}\Big[\frac{1}{s^6}\beta(1-\beta^2)^3\Big] \\ &= \beta(1-\beta^2)^3\frac{t^5}{5!}, \end{aligned} \tag{21}$$

$$\begin{aligned} x_3(t) &= -\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{\beta(1-\beta^2)^{\frac{9}{2}}\frac{t^5}{5!}\Big\}\Big] = -\mathscr{L}^{-1}\Big[\frac{1}{s^8}\beta(1-\beta^2)^{\frac{9}{2}}\Big] \\ &= -\beta(1-\beta^2)^{\frac{9}{2}}\frac{t^7}{7!}, \end{aligned} \tag{22}$$

$$\begin{aligned} x_4(t) &= -\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{-\beta(1-\beta^2)^6\frac{t^7}{7!}\Big\}\Big] = \mathscr{L}^{-1}\Big[\frac{1}{s^{10}}\beta(1-\beta^2)^6\Big] \\ &= \beta(1-\beta^2)^6\frac{t^9}{9!}, \end{aligned} \tag{23}$$

$$\begin{aligned} x_5(t) &= -\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{\beta(1-\beta^2)^{\frac{15}{2}}\frac{t^9}{9!}\Big\}\Big] = -\mathscr{L}^{-1}\Big[\frac{1}{s^{12}}\beta(1-\beta^2)^{\frac{15}{2}}\Big] \\ &= -\beta(1-\beta^2)^{\frac{15}{2}}\frac{t^{11}}{11!}, \end{aligned} \tag{24}$$

$$\vdots$$

In view of Eqs. (19)–(24), the series solution is

$$\begin{aligned} x(t) &= \beta t - \beta(1-\beta^2)^{\frac{3}{2}}\frac{t^3}{3!} + \beta(1-\beta^2)^3\frac{t^5}{5!} - \beta(1-\beta^2)^{\frac{9}{2}}\frac{t^7}{7!} \\ &\quad + \beta(1-\beta^2)^6\frac{t^9}{9!} - \beta(1-\beta^2)^{\frac{15}{2}}\frac{t^{11}}{11!} + \beta(1-\beta^2)^9\frac{t^{13}}{13!}\cdots \end{aligned} \tag{25}$$

$$\begin{aligned} &= \beta\Big(t - (1-\beta^2)^{\frac{3}{2}}\frac{t^3}{3!} + (1-\beta^2)^3\frac{t^5}{5!} - (1-\beta^2)^{\frac{9}{2}}\frac{t^7}{7!} + (1-\beta^2)^6\frac{t^9}{9!} - + \cdots\Big) \\ &= \beta\sum_{n=0}^{\infty}\Big((1-\beta^2)^{\frac{3}{2}}\Big)^n(-1)^n\frac{t^{2n+1}}{(2n+1)!}. \end{aligned} \tag{26}$$

From (26), we conclude that the solution of the Eq. (14), that is, the position of the relativistic harmonic oscillator is given by the series of power of functions with $0 < \beta < 1$

$$x(t) = \beta\sum_{n=0}^{\infty}\Big((1-\beta^2)^{\frac{3}{2}}\Big)^n(-1)^n\frac{t^{2n+1}}{(2n+1)!}. \tag{27}$$

According to Bartle [6], it is evident that the power series (27) converges in all $\mathbb{R}$. Moreover, it converges uniformly in any compact subinterval of $\mathbb{R}$. Using the expressions obtained above for the solution of Eq. (14), we illustrate, with two examples, the effectiveness of LADM to solve the nonlinear relativistic harmonic oscillator.

Application to the Relativistic Harmonic Oscillator

Example 1 In this first example, we consider the particular case of (14) such that $\beta = 0.1$; this case was studied in Ebaid [20] via differential transformation method (DTM) and also in Biazar and Hosami and Mickens [12] through the homotopy perturbation method (HPM). Good approximations were obtained in both works in comparison with the first known approximation solution of (14) obtained in Mickens [26] using the harmonic balance method (HBM). We will use formula (27) considering only the first fourteen terms (since the subsequent terms will be negligible)

$$x(t) = 0.1\sum_{n=0}^{13}(0.9850375)^n(-1)^n\frac{t^{2n+1}}{(2n+1)!} = 0.1t - 0.0985037\frac{t^3}{3!} + 0.0970299\frac{t^5}{5!} - 0.095578\frac{t^7}{7!} + 0.094148\frac{t^9}{9!} - 0.0927393\frac{t^{11}}{11!} + \cdots - 0.0822027\frac{t^{27}}{27!} \quad (28)$$

The approximations obtained for $\beta = 0.1$ through DTM in Ebaid [20] by using HPM in Biazar and Hosami [12] are as follows:

$$x_{\text{DTM}}(t) = 0.10033\sin(0.998t) - 0.000047097\sin(2.997t) + 0.00000008254\sin(4.841t) \quad (29)$$

$$x_{\text{HPM}}(t) = 0.10010\sin(0.999t) - 0.00004689\sin(2.997t) + 0.00000005062\sin(4.995t) \quad (30)$$

Moreover, using $\beta = 0.1$ in (15) we find

$$x_{\text{HBM}}(t) = 0.10025\sin(0.998t) - 0.00004173\sin(2.996t) + 0.00000004369\sin(4.944t) \quad (31)$$

The results obtained are shown in Table 1, in which results obtained in Ebaid, Biazar and Hosami and Mickens [20], [12], and [26] using DTM, HPM, and HBM, respectively, are compared. We also display this comparison in Figs. 1, 2 and 3. All numerical work was performed using the Mathematica software package.

Example 2 In the second example, we consider the particular case of (14) such that $\beta = 0.2$; this case was studied in [20] via DTM and also in [12] using HPM. Once again, in both works, good approximations were found in comparison with the first approximations obtained in (14) and the one obtained in [26] by HBM. As before, using formula (27), and taking the first fourteen terms, we obtain

$$x(t) = 0.2\sum_{n=0}^{13}(0.940604)^n(-1)^n\frac{t^{2n+1}}{(2n+1)!} = 0.2t - 0.1881208\frac{t^3}{3!} + 0.1769472\frac{t^5}{5!} - 0.1664372\frac{t^7}{7!} + 0.1565515\frac{t^9}{9!} - 0.1472253\frac{t^{11}}{11!} + \cdots - 0.0902233\frac{t^{27}}{27!} \quad (32)$$

Table 1 Table of x_{our} and $x_{\text{HBM}}, x_{\text{HPM}}, x_{\text{DTM}}$ for $\beta = 0.1$ and $t \in [0, 5]$

$\beta = 0.1$					
t	x_{our}	x_{HBM} [26]	x_{HPM} [12]	x_{DTM} [20]	Maximum error
0.5	0.0479729590	0.0479328162	0.0478998303	0.0479657729	7.3129×10^{-5}
1.0	0.0843725830	0.0842428703	0.0841703143	0.0843093300	2.0227×10^{-4}
1.5	0.1004175702	0.1000179266	0.0998843534	0.1001029926	5.3322×10^{-4}
2.0	0.0922371146	0.0913351729	0.0911171475	0.0914094983	1.1200×10^{-3}
2.5	0.0618047271	0.0603586490	0.0600634967	0.0604017614	1.7412×10^{-3}
3.0	0.0164621313	0.0147248839	0.0144036691	0.0147345492	2.0585×10^{-3}
3.5	−0.0328519050	−0.0344716193	−0.0347441361	−0.0344944258	1.8922×10^{-3}
4.0	−0.0742405162	−0.0753198191	−0.0754679268	−0.0753771430	1.2274×10^{-3}
4.5	−0.0977188217	−0.0978363382	−0.0977921545	−0.0979187422	1.9992×10^{-4}
5.0	−0.0976227495	−0.0964395012	−0.0961601232	−0.0965198814	1.4626×10^{-3}

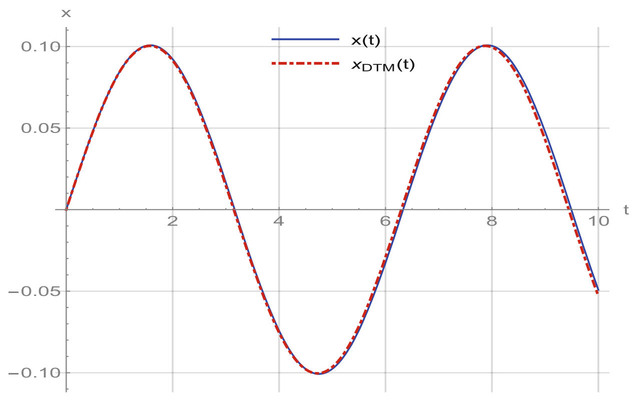

Fig. 1 Graph of the values of x_{our} and x_{DTM} for $\beta = 0.1$

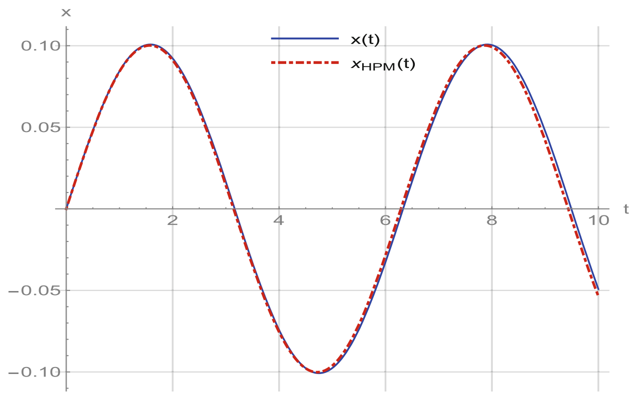

Fig. 2 Graph of the values of x_{our} and x_{HPM} for $\beta = 0.1$

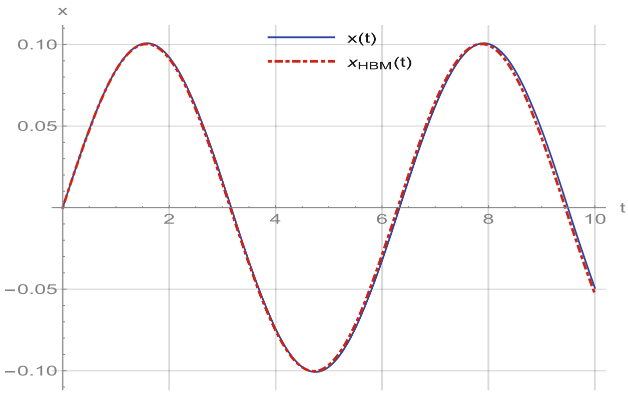

Fig. 3 Graph of the values of x_{our} and x_{HBM} for $\beta = 0.1$

Table 2 Table of x_{our} and $x_{\text{HBM}}, x_{\text{HPM}}, x_{\text{DTM}}$ for $\beta = 0.2$ and $t \in [0, 5]$

$\beta = 0.2$					
t	x_{our}	x_{HBM} [26]	x_{HPM} [12]	x_{DTM} [20]	Maximum error
0.5	0.0961266393	0.0960670231	0.0955486075	0.0962486620	5.7803×10^{-4}
1.0	0.1700884958	0.1693755477	0.1685301931	0.1698938305	1.5583×10^{-3}
1.5	0.2048315199	0.2017087375	0.2007521973	0.2026720850	4.0793×10^{-3}
2.0	0.1923447004	0.1846120791	0.1837113487	0.1859888431	8.6334×10^{-3}
2.5	0.1355072355	0.1225945408	0.1219478146	0.1243489902	1.3559×10^{-2}
3.0	0.0474246500	0.0313518348	0.0311772419	0.0333654695	1.6247×10^{-2}
3.5	−0.0515930631	−0.0672523781	−0.0668828424	−0.0655019642	1.5659×10^{-2}
4.0	−0.1387145001	−0.1500050664	−0.1492370936	−0.1491851170	1.1291×10^{-2}
4.5	−0.1938512888	−0.1967092103	−0.1957682647	−0.1971050236	3.2537×10^{-3}
5.0	−0.2042900444	−0.1953114965	−0.1943750213	−0.1969655785	9.9150×10^{-3}

The approximations obtained in the case of $\beta = 0.2$ vía DTM in [20] through HPM in [12] are, respectively:

$$x_{\text{DTM}}(t) = 0.203\sin(0.992t) - 0.0003695\sin(3.051t) + 0.000009257\sin(4.29t) \quad (33)$$

$$x_{\text{HPM}}(t) = 0.201\sin(0.995t) - 0.0003768\sin(2.985t) + 0.000001652\sin(4.974t). \quad (34)$$

And also using $\beta = 0.2$ in (15) we obtain

$$x_{\text{HBM}}(t) = 0.202\sin(0.995t) - 0.0003354\sin(2.985t) + 0.000001508\sin(4.974t) \quad (35)$$

Comparison of our results with the ones obtained in [20], [12], and [26] using DTM, HPM, and HBM, respectively, are shown in Table 2 and displayed in Figs. 4, 5, and 6, respectively. In this example, we can also see that the approximation accuracy depends on the initial velocity of the oscillator. All numerical work was performed using the Mathematica software package.

As can be seen from the last examples, the solutions we have obtained are periodic functions and the amplitude depends of the initial velocity, as found by the author in [26]. The main

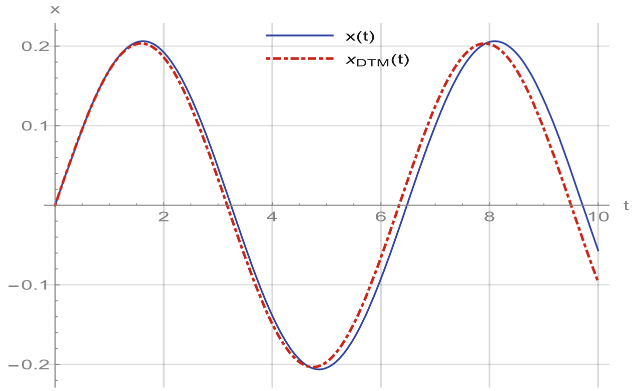

Fig. 4 Graph of the values of x_{our} and x_{DTM} for $\beta = 0.2$

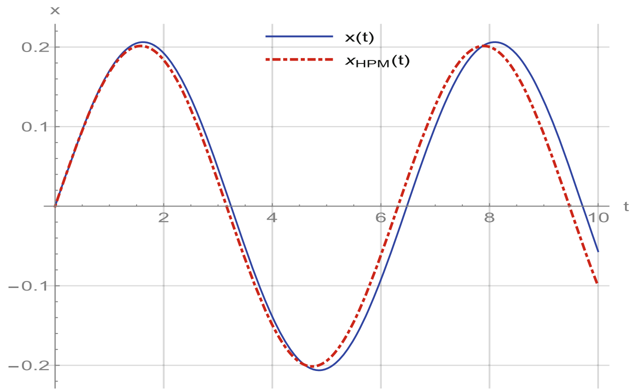

Fig. 5 Graph of the values of x_{our} and x_{HPM} for $\beta = 0.2$

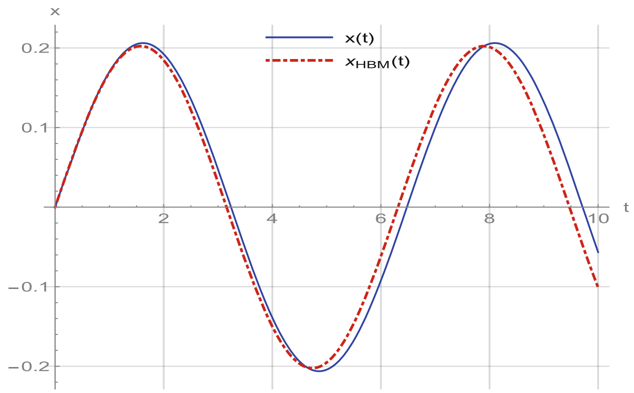

Fig. 6 Graph of the values of x_{our} and x_{HBM} for $\beta = 0.2$

distinction between our results and the ones reported previously is that the final series is uniformly convergent in any compact subset of the real line; therefore, we can obtain the results with the required accuracy.

Summary and Conclusions

To the best of our knowledge, there is no exact solution to the equation of motion for a relativistic harmonic oscillator. In this work, we have obtained a solution of the problem without the *a priori* assumption that the solutions are periodic functions; the solution that we have obtained is a series of powers of functions that uniformly converge on compact subsets of $\mathbb{R}$, never before reported. The problem of the limit function of the series solution is an open question that we are currently working on.
In order to show the accuracy and efficiency of our method, we have solved two examples and compared our results with the ones obtained with three different methods [12,20,26]. Our results show that LADM produces highly accurate solutions in complicated nonlinear problems. We, therefore, conclude that the Laplace-Adomian decomposition method is a notable non-sophisticated powerful tool that produces high quality approximate solutions for nonlinear ordinary differential equations using simple calculations and that reaches convergence with only a few terms. Finally, the Laplace-Adomian decomposition method would be a powerful mathematical tool for solving other nonlinear differential equations related with mathematical physics models. All numerical work and graphics were performed with the Mathematica software package.

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