Int. J. Appl. Comput. Math (2017) 3:2447–2466
DOI 10.1007/s40819-016-0247-7

ORIGINAL PAPER

Analysis of the Time Fractional 2-D Diffusion-Wave Equation via Moving Least Square (MLS) Approximation

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Published online: 15 September 2016

Abstract In this paper, the time fractional two-dimensional diffusion-wave equation defined by Caputo sense for $(1 < \alpha < 2)$ is analyzed by an efficient and accurate computational method namely meshless local Petrov–Galerkin (MLPG) method which is based on the Galerkin weak form and moving least squares (MLS) approximation. We consider a general domain with Dirichlet boundary conditions further to given initial values as continuous functions. Meshless Galerkin weak form is adopted to the interior nodes while the meshless collocation technique is applied for the nodes on the boundaries of the domain. Since Dirichlet boundary condition is imposed directly therefore the general domains are also applicable easily. In MLPG method, the MLS approximation is usually used to construct shape functions which plays important rule in the convergence and stability of the method. It is proved the method is unconditionally stable in some sense. Two numerical examples are presented, one of them with the regular domain and the other one with non-regular domain, and satisfactory agreements are achieved.

Keywords Local weak formulation · Meshless local Petrov–Galerkin (MLPG) method · Moving least squares (MLS) · Diffusion-wave equation · Time fractional derivative

Introduction

The fractional derivative and fractional differential equations have been used to describe many phenomena in physics and engineering, such as boundary layer effects in ducts, allometric scaling laws in biology and ecology, colored noise, dielectric polarization, electromagnetic waves, electrode–electrolyte polarization, fractional kinetics, quantitative finance, quantum evolution of complex systems, power-law phenomenon in fluid and complex network, viscoelastic mechanics, etc. [1,2]. On the other hand, an important class of fractional differential

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equations which has been studied widely in recent years is the time fractional diffusion-wave equation (FDWE). The time FDWE is obtained from the classical diffusion-wave equation by replacing the second-order time derivative term by a fractional derivative of order $(1 < \alpha < 2)$ [3]. Many of the universal electromagnetic, acoustic, mechanical responses can be described exactly by the FDWE [4]. It is also worth mentioning that fractional diffusion equation and diffusion wave equation have a lot in common, for example, they can behave like diffusion. To see some different kinds of fractional differential equations interest readers are referred to [5–7].

Strictly speaking, fractional diffusion-wave equation with damping is similar to the fractional Cattaneo equation where FDWE has more a term $f(x, t)$ rather than fractional Cattaneo equation. A considerable amount of papers have been appeared dealing with fractional diffusion-wave equation with damping (FDWE). It is well known that whereas diffusion equation describes a process, where a disturbance of the initial conditions spreads infinitely fast, the propagation velocity of the disturbance is constant for the wave equation. In a certain sense, the time-fractional diffusion-wave equation that is obtained from the diffusion equation by substituting the first derivative in time by the fractional derivative of order α, $1 < \alpha < 2$, interpolates between these two different behaviors.

The present paper considers the following time-fractional two-dimensional diffusion-wave equation of order $(1 < \alpha < 2)$:

$$\frac{\partial^\alpha u(\mathbf{x}, t)}{\partial t^\alpha} + \gamma_1 \frac{\partial u(\mathbf{x}, t)}{\partial t} = \gamma_2 \Delta u + f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega \subseteq \mathbb{R}^2, \ t \in [0, T], \tag{1}$$

subject to compatible initial conditions

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \ \frac{\partial u}{\partial t}(\mathbf{x}, 0) = \psi(\mathbf{x}), \ \mathbf{x} \in \Omega, \tag{2}$$

and the boundary condition

$$u(\mathbf{x}, t) = g(\mathbf{x}, t) \ \text{ for } \ \mathbf{x} \in \Gamma = \partial\Omega, \ t \in [0, T], \tag{3}$$

where $\mathbf{x} = (x, y)$ is spatial variable, Γ the boundary of domain and, γ_1 and γ_2 are constants. Also, $f(\mathbf{x}, t)$ is source function with sufficient smoothness and, $\varphi(\mathbf{x})$, $\psi(\mathbf{x})$ and $g(x, t)$ are given continuous functions. Furthermore, in Eq. (1), the time-fractional derivatives are in the sense of Caputo which is defined by

$$D_t^\alpha F(t) = \begin{cases} \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-\xi)^{k-\alpha-1} F^{(k)}(\xi) \mathrm{d}\xi, & k-1 < \alpha < k, \ t > 0, \\ F^{(k)}(t), & \alpha = k. \end{cases} \tag{4}$$

In the case $\alpha = 2$, this equation is the telegraph equation, which governs electrical transmission in a telegraph cable [4]. This equation could also be characterized as a wave equation, governing wave motion in a string, with a damping effect due to the term $\frac{\partial u(\mathbf{x},t)}{\partial t}$.

In the literature, several meshless weak form methods have been reported such as: Meshless methods based on weak forms such as the element free Galerkin (EFG) method [8,9], meshless methods based on collocation techniques (strong forms) such as the meshless collocation method based on radial basis functions(RBFs) [10–23] and meshless methods based on the combination of weak forms and collocation technique [24–35].

The weak forms are used to derive a set of algebraic equations through a numerical integration process using a set of quadrature domain that may be constructed globally or locally in the domain of the problem. In the global weak form methods, global background cells are needed for numerical integration in computing the algebraic equations. To avoid

the use of global background cells, a so-called local weak form is adopted to develop the meshless local Petrov–Galerkin (MLPG) method [36–46].

In the last few years, several numerical methods have been proposed for solving FDWE, see [4] and Refs. therein. In Ref. [4], the authors have applied a collocation method based on the Legendre wavelets (LWs) to solve the 1-D form of the problem (1)–(3) and obtained good results. In this paper, we focus on the numerical solution of the Eqs. (1)–(3) (which is two-dimensional) using a kind of MLPG method which is based on the Galerkin weak form and moving least squares (MLS) approximation and achieve still satisfactory results. Two illustrative examples are given so that the one of them possesses regular domain and the other one enjoys non-regular domain.

The MLS Approximation Procedure

A meshless method uses a local approximation to represent the trial function with the values of the unknown variable at some nodal points. In the current paper, the moving least squares (MLS) approximation is used. Consider a sub-domain Ω_s, the neighborhood of a point $\mathbf{x}$ and denoted as the support domain of the MLS approximation for the trial function at $\mathbf{x}$, which is located in the problem domain Ω (see Fig. 1). To approximate the function u in Ω_s, over a number of randomly located nodes $\mathbf{x}_i$, $i = 1, 2, \ldots, n$, the Moving Least Squares approximant $u^h(\mathbf{x})$ of u, $\forall \mathbf{x} \in \Omega_s$, could be defined by

$$u^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\, \mathbf{a}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_s, \tag{5}$$

where $\mathbf{p}^T(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_m(\mathbf{x})]$ is a complete monomial basis of order m, and $a(\mathbf{x})$ is a vector containing coefficients $a_j(\mathbf{x})$, $j = 1, 2, \ldots, m$ which are functions of the space coordinates $\mathbf{x}$. $p_j(\mathbf{x})$ is monomial in the space coordinate $x^T = [x, y]$, and m is the number of polynomial basis functions. The coefficient vector $\mathbf{a}(\mathbf{x})$ is discovered by minimizing a weighted discrete L_2 norm, defined as:

$$\begin{aligned} J(x) &= \sum_{i=1}^{n} w_i(\mathbf{x})[p^T(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - \hat{u}_i]^2 \\ &= [\mathbf{P}.\mathbf{a}(\mathbf{x}) - \hat{\mathbf{u}}]^T.W.[\mathbf{P}.\mathbf{a}(\mathbf{x}) - \hat{\mathbf{u}}], \end{aligned} \tag{6}$$

where $w_i(\mathbf{x})$ is the weight function associated with the node i, with $w_i(\mathbf{x}) > 0$ for all $\mathbf{x}$ in the support of $w_i(\mathbf{x})$, $\mathbf{x}_i$ denotes the value of $\mathbf{x}$ at node i, n is the number of nodes in Ω_s for which the weight functions $w_i(\mathbf{x}) > 0$, the matrices $\mathbf{P}$ and $\mathbf{W}$ are given as

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}^T(\mathbf{x}_1) \\ \mathbf{p}^T(\mathbf{x}_2) \\ \ldots \\ \mathbf{p}^T(\mathbf{x}_n) \end{pmatrix}_{n \times m}, \quad \mathbf{W} = \begin{pmatrix} w_1(\mathbf{x}) & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & w_n(\mathbf{x}), \end{pmatrix}$$

and $\hat{\mathbf{u}}^T = [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n]$. Here, it should be noted that $\hat{u}_i$, $i = 1, 2, \ldots, n$ in Eq. (6) are the fictitious nodal values, and not the nodal values of the unknown trial function $u^h(\mathbf{x})$ in general. The stationarity of J in Eq. (6) with respect to $\mathbf{a}(\mathbf{x})$ leads to the following linear system of equations between $\mathbf{a}(\mathbf{x})$ and $\hat{\mathbf{u}}$:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\hat{\mathbf{u}}, \tag{7}$$

where the matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are given by

$$\mathbf{A}(\mathbf{x}) = \mathbf{P}^T\mathbf{W}\mathbf{P} = \mathbf{B}(\mathbf{x})\mathbf{P} = \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i), \tag{8}$$

$$\mathbf{B}(\mathbf{x}) = \mathbf{P}^T\mathbf{W} = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \ldots, w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)]. \tag{9}$$

The MLS approximation is well defined only when the matrix $\mathbf{A}$ in Eq. (7) is non-singular. It can be seen that this is the case if and only if the rank of $\mathbf{P}$ equals m. A necessary condition for a well-defined MLS approximation is that at least m weight functions are non-zero (i.e. $n > m$) for each sample point $\mathbf{x} \in \Omega$ and that the nodes in Ω_s should not be arranged in a special pattern such as on a straight line. Here a sample point may be a nodal point under consideration or a quadrature point.

Solving for $\mathbf{a}(\mathbf{x})$ from Eq. (7) and substituting it into Eq. (5) yields a relation which may be written as the form of an interpolation function similar to that used in FEM, as

$$u^h(\mathbf{x}) = \Phi^T(\mathbf{x}).\hat{\mathbf{u}} = \sum_{i=1}^{n} \phi_i(\mathbf{x})\hat{u}_i, \quad \mathbf{x} \in \Omega_s, \tag{10}$$

where $u^h(\mathbf{x}_i) \equiv u_i$ is not essentially equal to $\hat{u}_i$ and,

$$\Phi^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \tag{11}$$

or

$$\phi_i(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})[\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})]_{ji}. \tag{12}$$

Here, $\phi_i(\mathbf{x})$ is usually called the shape function of the MLS approximation corresponding to nodal point $\mathbf{x}_i$. From Eqs. (9) and (11), it is easily seen that $\phi_i(\mathbf{x}) = 0$ when $w_i(\mathbf{x})$. In practical applications, $w_i(\mathbf{x})$ is generally chosen such that it is non-zero over the support of nodal points $\mathbf{x}_i$. The support of the nodal points $\mathbf{x}_i$ is usually taken to be a circle of radius r_s, centered at $\mathbf{x}_i$ (see Fig. 1). The fact that $\phi_i(\mathbf{x}) = 0$, for $\mathbf{x}$ not in the support of nodal point $\mathbf{x}_i$ preserves the local character of the Moving Least Squares approximation.

Let $C^q(\Omega)$ be the space of qth continuously differentiable functions on Ω. If $w_i(\mathbf{x}) \in C^q(\Omega)$ and $p_j(\mathbf{x}) \in C^s(\Omega)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, then $\phi_i(\mathbf{x}) \in C^r(\Omega)$ with $r = \min(q, s)$. The partial derivatives of $\phi_i(\mathbf{x})$ are obtained as

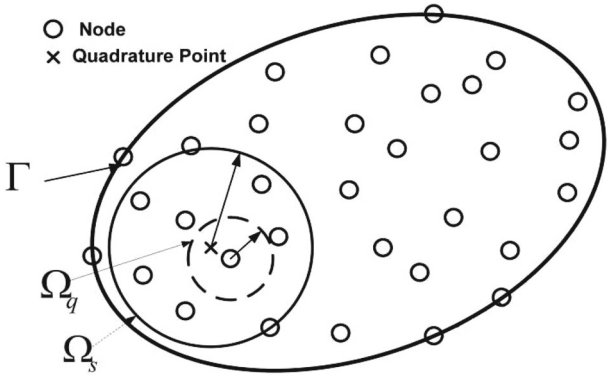

Fig. 1 Ω_s and Ω_q are local support and local quadrature domains, respectively

$$\phi_{i,k} = \sum_{j=1}^{m} [p_{j,k}(\mathbf{A}^{-1}\mathbf{B})_{ji} + p_j(\mathbf{A}^{-1}\mathbf{B}_{,k} + \mathbf{A}_{,k}^{-1}\mathbf{B})_{ji}] \tag{13}$$

in which $\mathbf{A}_{,k}^{-1} = (\mathbf{A}^{-1})_{,k}$ represents the derivative of the inverse of A with respect to x^k, which is given by $\mathbf{A}_{,k}^{-1} = -\mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1}$, where$(\ )_{,i}$ denotes $\partial(\ )/\partial x_i$.

In this paper, the Gaussian weight function is applied as

$$w_i(\mathbf{x}) = \begin{cases} \frac{exp\left[-\left(\frac{d_i}{c_i}\right)^2\right] - exp\left[-\left(\frac{r_s}{c_i}\right)^2\right]}{1 - exp\left[-\left(\frac{r_s}{c_i}^2\right)\right]}, & 0 \le d_i \le r_s, \\ 0, & d_i \ge r_s, \end{cases} \tag{14}$$

where $d_i = \| x - x_i \|$, c_i is a constant controlling the shape of the weight function w_i and r_s is the size of the support domain.

The size of support, r_s, of the weight function w_i associated with node i should be chosen such that r_s should be large enough to have sufficient number of nodes covered in the domain of definition of every sample point ($n \ge m$) to ensure the regularity of $\mathbf{A}$. A very small r_s may result in a relatively large numerical error in using Gauss numerical quadrature to calculate the entries in the system matrix. On the other hand, r_s should also be small enough to maintain the local character of the MLS approximation.

The Time Fractional Discretization of the Problem

According to Eq. (4), $\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}$ could be written as follows:

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(\mathbf{x},\xi)}{\partial \xi^2} (t-\xi)^{1-\alpha} \mathrm{d}\xi, & 1 < \alpha < 2, \\ \frac{\partial^2 u(\mathbf{x},t)}{\partial t^2}, & \alpha = 2. \end{cases} \tag{15}$$

In order to discretize the problem in the time direction for $1 < \alpha < 2$, we substitute $t^{(n+1)}$ into Eq. (15), then the integral can be partitioned as

$$\begin{aligned} \frac{\partial^\alpha u(\mathbf{x},t^{(n+1)})}{\partial t^\alpha} &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t^{(n+1)}} \frac{\partial^2 u(\mathbf{x},\xi)}{\partial \xi^2} (t^{(n+1)} - \xi)^{1-\alpha} \mathrm{d}\xi \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n} \int_{t^{(k)}}^{t^{(k+1)}} \frac{\partial^2 u(\mathbf{x},\xi)}{\partial \xi^2} (t^{(n+1)} - \xi)^{1-\alpha} \mathrm{d}\xi, \end{aligned} \tag{16}$$

where $t^{(0)} = 0$, $t^{(n+1)} = t^{(n)} + \triangle t$, $n = 0, 1, 2, \ldots, M$, and $M\triangle t = T$. Approximations of the first and second order derivatives due to the finite difference formulae are defined as

$$\begin{aligned} \frac{\partial^2 u(\mathbf{x},\sigma)}{\partial t^2} &= \frac{u(\mathbf{x},t^{(n+1)}) - 2u(\mathbf{x},t^{(n)}) + u(\mathbf{x},t^{(n-1)})}{\triangle t^2} \\ &\quad + o(\triangle_\sigma t + \triangle t^2), \end{aligned} \tag{17}$$

$$\frac{\partial u(\mathbf{x},t^{(n+1)})}{\partial t} = \frac{3u(\mathbf{x},t^{(n+1)}) - 4u(\mathbf{x},t^{(n)}) + u(\mathbf{x},t^{(n-1)})}{2\triangle t} + o(\triangle t^2), \tag{18}$$

where $\sigma \in [t^{(n)}, t^{(n+1)}]$ and $\triangle_\sigma t = \sigma - t^{(n)}$. Replacing Eq. (17) into Eq. (16), gives

$$\begin{aligned}\frac{\partial^\alpha u(\mathbf{x}, t^{(n+1)})}{\partial t^\alpha} &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t^{(n+1)}} \frac{\partial^2 u(\mathbf{x}, \xi)}{\partial \xi^2} (t^{(n+1)} - \xi)^{1-\alpha} \mathrm{d}\xi \\ &\cong \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n} \frac{u^{(k+1)} - 2u^{(k)} + u^{(k-1)}}{\triangle t^2} \int_{t^{(k)}}^{t^{(k+1)}} (t^{(n+1)} - \xi)^{1-\alpha} \mathrm{d}\xi,\end{aligned} \tag{19}$$

where $u^{(k)} = u(\mathbf{x}, t^{(k)})$, $k = 0, 1, 2, \ldots, M$. In the above equation, the integral is easily obtained as

$$\int_{t^{(k)}}^{t^{(k+1)}} (t^{(n+1)} - \xi)^{1-\alpha} \mathrm{d}\xi = \frac{1}{(2-\alpha)} \triangle t^{2-\alpha} [(n-k+1)^{2-\alpha} - (n-k)^{2-\alpha}]. \tag{20}$$

Rearrangement of Eqs. (19) and (18) by notation $b_k = (k+1)^{2-\alpha} - (k)^{2-\alpha}$ lead to

$$\begin{aligned}\frac{\partial^\alpha u(\mathbf{x}, t^{(n+1)})}{\partial t^\alpha} &= \frac{\triangle t^{-\alpha}}{\Gamma(3-\alpha)} \sum_{k=0}^{n} b_k [u^{(n-k+1)} - 2u^{(n-k)} + u^{(n-k-1)}] \\ &\cong a_0 \left\{ u^{(n+1)} - 2u^{(n)} + u^{(n-1)} + \sum_{k=1}^{n} b_k [u^{(n-k+1)} - 2u^{(n-k)} + u^{(n-k-1)}] \right\},\end{aligned} \tag{21}$$

and

$$\frac{\partial u(\mathbf{x}, t^{(n+1)})}{\partial t} \cong a_0' \left(3u^{(n+1)} - 4u^{(n)} + u^{(n-1)} \right), \tag{22}$$

where $a_0 = \frac{\triangle t^{-\alpha}}{\Gamma(3-\alpha)}$, $a_0' = \frac{1}{2\triangle t}$ and $n = 0, 1, 2, \ldots, M$. We note that Eq. (1) at $t = t^{(n+1)}$ due to θ-weighted finite difference formulation is as follows:

$$\frac{\partial^\alpha u(\mathbf{x}, t^{(n+1)})}{\partial t^\alpha} + \gamma_1 \frac{\partial u(\mathbf{x}, t^{(n+1)})}{\partial t} = \gamma_2 [\theta \Delta u^{(n+1)} + (1-\theta) \Delta u^{(n)}] + f^{(n+1)}, \tag{23}$$

where, $0 < \theta < 1$ is a constant, $\Delta u^{(n)} = \Delta u(\mathbf{x}, t^{(n)})$ and $f^{(n)} = f(\mathbf{x}, t^{(n)})$. We set $\theta = \frac{1}{2}$ for simplicity, and substitute Eqs. (21) and (22) into Eq. (23), then we obtain

$$\begin{aligned}&a_0 \left\{ u^{(n+1)} - 2u^{(n)} + u^{(n-1)} + \sum_{k=1}^{n} b_k [u^{(n-k+1)} - 2u^{(n-k)} + u^{(n-k-1)}] \right\} \\ &\quad + \gamma_1 a_0' \left(3u^{(n+1)} - 4u^{(n)} + u^{(n-1)} \right) = \frac{1}{2} \gamma_2 [\Delta u^{(n+1)} + \Delta u^{(n)}] + f^{(n+1)},\end{aligned}$$

or equivalently

$$\begin{aligned}\frac{1}{2}\gamma_2 \Delta u^{(n+1)} - (a_0 + 3\gamma_1 a_0') u^{(n+1)} &= -\frac{1}{2}\gamma_2 \Delta u^{(n)} - (2a_0 + 4\gamma_1 a_0') u^{(n)} \\ &\quad + \sum_{k=1}^{n} a_0 b_k [u^{(n-k+1)} - 2u^{(n-k)} + u^{(n-k-1)}] \\ &\quad + (a_0 + \gamma_1 a_0') u^{(n-1)} - f^{(n+1)}.\end{aligned} \tag{24}$$

The Local Weak Form of the MLPG

Instead of giving the global weak form, the meshless local Galerkin weak form method constructs the weak form over local quadrature cell such as Ω_q, which is a small region taken for each node in the global domain Ω (see Fig. 1). The local quadrature cells over lap each other and cover the whole global domain Ω. The local quadrature cells could be of any geometric shape and size. In this paper they are taken to be of circular shape. The local weak form of Eq. (24) for $\mathbf{x}_i = (x_i, y_i) \in \Omega_q^i$ can be written as

$$\begin{aligned}
&\int_{\Omega_q^i} \left[\frac{1}{2}\gamma_2 \Delta u^{(n+1)} - (a_0 + 3\gamma_1 a_0')u^{(n+1)}\right] v(\mathbf{x})\mathrm{d}\Omega \\
&= \int_{\Omega_q^i} \left[-\frac{1}{2}\gamma_2 \Delta u^{(n)} - (2a_0 + 4\gamma_1 a_0')u^{(n)} + (a_0 + \gamma_1 a_0')u^{(n-1)}\right] v(\mathbf{x})\mathrm{d}\Omega \\
&\quad + \int_{\Omega_q^i} \left(\sum_{k=1}^{n} a_0 b_k [u^{(n-k+1)} - 2u^{(n-k)} + u^{(n-k-1)}]\right) v(\mathbf{x})\mathrm{d}\Omega - \int_{\Omega_q^i} f^{(n+1)} v(\mathbf{x})\mathrm{d}\Omega,
\end{aligned} \tag{25}$$

where Ω_q^i is the local quadrature domain associated with the point i, i.e., it is a circle centered at $\mathbf{x}_i$ of radius r_q and, $v(\mathbf{x})$ is the Heaviside step function [47,48],

$$v(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega_q, \\ 0, & \mathbf{x} \notin \Omega_q, \end{cases} \tag{26}$$

as the test function in each local quadrature domain. Using the divergence theorem, Eq. (25) yields the following expression:

$$\begin{aligned}
&-(a_0 + 3\gamma_1 a_0') \int_{\Omega_q^i} u^{(n+1)} v(\mathbf{x})\mathrm{d}\Omega - \frac{1}{2}\gamma_2 \int_{\Omega_q^i} \nabla u^{(n+1)} \nabla v \mathrm{d}\Omega + \frac{1}{2}\gamma_2 \int_{\partial\Omega_q^i} v \frac{\partial u^{(n+1)}}{\partial n}\mathrm{d}\Gamma \\
&= \frac{1}{2}\gamma_2 \int_{\Omega_q^i} \nabla u^{(n)} \nabla v \mathrm{d}\Omega - \frac{1}{2}\gamma_2 \int_{\partial\Omega_q^i} v \frac{\partial u^{(n)}}{\partial n}\mathrm{d}\Gamma - (2a_0 + 4\gamma_1 a_0') \int_{\Omega_q^i} u^{(n)} v(\mathbf{x})\mathrm{d}\Omega \\
&\quad + \sum_{k=1}^{n} a_0 b_k \left[\int_{\Omega_q^i} u^{(n-k+1)} v(\mathbf{x})\mathrm{d}\Omega - 2\int_{\Omega_q^i} u^{(n-k)} v(\mathbf{x})\mathrm{d}\Omega + \int_{\Omega_q^i} u^{(n-k-1)} v(\mathbf{x})\mathrm{d}\Omega\right] \\
&\quad + (a_0 + \gamma_1 a_0') \int_{\Omega_q^i} u^{(n-1)} v(\mathbf{x})\mathrm{d}\Omega - \int_{\Omega_q^i} f^{(n+1)} v(\mathbf{x})\mathrm{d}\Omega,
\end{aligned} \tag{27}$$

where $\partial\Omega_q^i$ is the boundary of Ω_q^i, $n = (n_1, n_2)$ is the outward unit normal to the boundary $\partial\Omega_q^i$, and

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} n_1 + \frac{\partial u}{\partial y} n_2$$

is the normal derivative, i.e., the derivative in the outward normal direction to the boundary $\partial\Omega_q^i$. Because the derivative of the Heaviside step function $v(\mathbf{x})$ is equal to zero, then the local weak form Eq. (27) is changed into the following simple integral equation:

$$\begin{aligned}
&-(a_0 + 3\gamma_1 a_0') \int_{\Omega_q^i} u^{(n+1)}\mathrm{d}\Omega + \frac{1}{2}\gamma_2 \int_{\partial\Omega_q^i} \frac{\partial u^{(n+1)}}{\partial n}\mathrm{d}\Gamma = -\frac{1}{2}\gamma_2 \int_{\partial\Omega_q^i} \frac{\partial u^{(n)}}{\partial n}\mathrm{d}\Gamma \\
&\quad - (2a_0 + 4\gamma_1 a_0') \int_{\Omega_q^i} u^{(n)}\mathrm{d}\Omega + (a_0 + \gamma_1 a_0') \int_{\Omega_q^i} u^{(n-1)}\mathrm{d}\Omega
\end{aligned}$$

$$+\sum_{k=1}^{n} a_0 b_k \left[\int_{\Omega_q^i} u^{(n-k+1)}\mathrm{d}\Omega - 2\int_{\Omega_q^i} u^{(n-k)}\mathrm{d}\Omega + \int_{\Omega_q^i} u^{(n-k-1)}\mathrm{d}\Omega\right]$$
$$-\int_{\Omega_q^i} f^{(n+1)}\mathrm{d}\Omega. \tag{28}$$

Applying the moving least squares (MLS) approximation for the unknown functions, the local integral Eq. (28) is transformed into a system of algebraic equations with used unknown quantities, as described in the next section.

Numerical Implementation of MLPG: Reducing to Linear Algebraic System

In this section, we consider Eq. (28) to see how to obtain discrete equations. Consider N regularly located points on the boundary and the domain of the problem (i.e. $\Omega \subseteq \mathbb{R}^2$) so that the distance between to consecutive nodes in each direction is constant and equal to h. Assuming that $u(\mathbf{x}_i, k\triangle t)$ for all $k = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, N$ are known, our aim is to compute $u(\mathbf{x}_i, (n+1)\triangle t)$, $i = 1, 2, \ldots, N$. So, we have N unknowns and to compute these unknowns we need N equations. As it will be described, corresponding to each node we obtain one equation. For nodes which are located in the interior of the domain, i.e., for $\mathbf{x}_i \in$ interior Ω, to obtain the discrete equations from the locally weak forms (28), substituting approximation formula (10) into local integral equations (28) yields:

$$\begin{aligned}
&-(a_0+3\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega\right)u_j^{(n+1)} + \frac{1}{2}\gamma_2\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\mathrm{d}\Gamma\right)u_j^{(n+1)}\\
&= -\frac{1}{2}\gamma_2\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\mathrm{d}\Gamma\right)u_j^{(n)} - (2a_0+4\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega\right)u_j^{(n)}\\
&\quad+\sum_{k=1}^{n} a_0 b_k\left[\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega\right)u_j^{(n-k+1)}\right.\\
&\quad\left.-2\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega\right)u_j^{(n-k)} + \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega\right)u_j^{(n-k-1)}\right]\\
&\quad+(a_0+\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega\right)u_j^{(n-1)} - \int_{\Omega_q^i} f^{(n+1)}\mathrm{d}\Omega.
\end{aligned} \tag{29}$$

We had supposed $b_k = (k+1)^{2-\alpha} - (k)^{2-\alpha}$, $k = 1, 2, \ldots, n$ in the section 3, in addition assume $b_{-1} = 0$ and $b_0 = 1$. By these assumptions Eq. (29) is converted to the following equation

$$\begin{aligned}
&\left[-(a_0+3\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega\right) + \frac{1}{2}\gamma_2\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\mathrm{d}\Gamma\right)\right]u_j^{(n+1)}\\
&= \sum_{s=1}^{n}\left\{\left[a_0(b_{n-s-1} - 2b_{n-s} + b_{n-s+1})\right]\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega\right)u_j^{(s)}\right\}
\end{aligned}$$

$$
\begin{aligned}
&-4\gamma_1 a_0' \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(n)}\\
&-2a_0 b_n \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(0)} + a_0 b_n \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(-1)}\\
&-\frac{1}{2}\gamma_2 \sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\mathrm{d}\Gamma\right)u_j^{(n)} + \gamma_1 a_0' \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(n-1)} - \int_{\Omega_q^i} f^{(n+1)}\mathrm{d}\Omega.
\end{aligned}
\tag{30}
$$

According to the initial conditions that were introduced in Eq. (2), we apply the following assumptions:

$$
u_j^{(0)} = \varphi(\mathbf{x}_j), \quad u_j^{(-1)} = u_j^{(1)} - 2\triangle t\psi(\mathbf{x}_j), \tag{31}
$$

where, the second relation is the result of central finite difference formula, then we conclude the following

$$
\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(0)} = \int_{\Omega_q^i}\varphi(\mathbf{x})\mathrm{d}\Omega, \tag{32}
$$

$$
\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(-1)} = \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(1)} - 2\triangle t\int_{\Omega_q^i}\psi(\mathbf{x})\mathrm{d}\Omega. \tag{33}
$$

Therefore, applying Eqs. (32) and (33) into (30) yields

$$
\begin{aligned}
&\left[-(a_0+3\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right) + \frac{1}{2}\gamma_2 \sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\mathrm{d}\Gamma\right)\right]u_j^{(n+1)}\\
&= \sum_{s=1}^{n}\left\{\left[a_0(b_{n-s-1}-2b_{n-s}+b_{n-s+1})\right]\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(s)}\right\}\\
&\quad -4\gamma_1 a_0' \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(n)}\\
&\quad + a_0 b_n \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(1)} - \frac{1}{2}\gamma_2 \sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\mathrm{d}\Gamma\right)u_j^{(n)}\\
&\quad + \gamma_1 a_0' \sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j \mathrm{d}\Omega\right)u_j^{(n-1)}\\
&\quad - \int_{\Omega_q^i} f^{(n+1)}\mathrm{d}\Omega - 2a_0 b_n \int_{\Omega_q^i}\varphi(\mathbf{x})\mathrm{d}\Omega - 2a_0 b_n \triangle t\int_{\Omega_q^i}\psi(\mathbf{x})\mathrm{d}\Omega.
\end{aligned}
\tag{34}
$$

For nodes which are located on the boundary, we set

$$
u^{(n+1)}(\mathbf{x}_i) = g(\mathbf{x}_i, (n+1)\triangle t), \quad \mathbf{x}_i \in \Gamma. \tag{35}
$$

The matrix forms of Eqs. (34)–(35) for all N nodal points in the domain and the boundary of the problem are given below:

$$\left[-(a_0+3\gamma_1 a_0')\sum_{j=1}^{N}A_{ij}+\frac{1}{2}\gamma_2\sum_{j=1}^{N}B_{ij}\right]u_j^{(n+1)}=-4\gamma_1 a_0'\sum_{j=1}^{N}A_{ij}u_j^{(n)}$$
$$\sum_{s=1}^{n}\left\{\left[a_0(b_{n-s-1}-2b_{n-s}+b_{n-s+1})\right]\sum_{j=1}^{N}A_{ij}u_j^{(s)}\right\}+a_0b_n\sum_{j=1}^{N}A_{ij}u_j^{(1)}$$
$$+\gamma_1 a_0'\sum_{j=1}^{N}A_{ij}u_j^{(n-1)}-\frac{1}{2}\gamma_2\sum_{j=1}^{N}B_{ij}u_j^{(n)}-F_i^{(n+1)}-2a_0b_n\Phi_i-2a_0b_n\triangle t\Psi_i, \quad (36)$$

where

$$A_{ij}=\int_{\Omega_q^i}\phi_j\mathrm{d}\Omega,\quad B_{ij}=\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\mathrm{d}\Gamma,\quad F_i^{(n+1)}=\int_{\Omega_q^i}F(\mathbf{x},(n+1)\triangle t)\mathrm{d}\Omega, \quad (37)$$

$$\Phi_i=\int_{\Omega_q^i}\varphi(\mathbf{x})\mathrm{d}\Omega,\ \Psi_i=\int_{\Omega_q^i}\psi(\mathbf{x})\mathrm{d}\Omega. \quad (38)$$

By considering the following notations

$$\begin{aligned}
\mathbf{A}_{ij}&=-(a_0+3\gamma_1 a_0')\sum_{j=1}^{N}A_{ij}+\frac{1}{2}\gamma_2\sum_{j=1}^{N}B_{ij},\\
\alpha_{n,s}&=a_0(b_{n-s-1}-2b_{n-s}+b_{n-s+1}),\\
\beta_n&=a_0b_n,\\
\lambda_1&=-\frac{1}{2}\gamma_2,\\
\lambda_2&=-\gamma_1 a_0',\\
\delta_n&=-2a_0b_n,\\
\mu_n&=-2a_0b_n\triangle t,\\
\mathbf{F}^{n+1}&=\left[F_1^{(n+1)},F_2^{(n+1)},\ldots,F_N^{(n+1)}\right]^T,\\
\Phi&=[\Phi_1,\Phi_2,\ldots,\Phi_N]^T,\\
\Psi&=[\Psi_1,\Psi_2,\ldots,\Psi_N]^T,\\
U^{n+1}&=\left[u_1^{(n+1)},u_2^{(n+1)},\ldots,u_N^{(n+1)}\right]^T,
\end{aligned}$$

Equation (36) changes to the following matrix form

$$\mathbf{A}U^{(n+1)}=[\lambda_1 B+4\lambda_2 A]U^{(n)}-\lambda_2 AU^{(n-1)}+\sum_{s=1}^{n}\left\{\alpha_{n,s}AU^{(s)}\right\}$$
$$+\beta_n AU^{(1)}+\delta_n\Phi+\mu_n\Psi-\mathbf{F}^{n+1}. \quad (39)$$

Furthermore, to satisfy Eq. (35), for all nodes belong to the boundary, i.e. $\mathbf{x}_i\in\Gamma$, we set

$$\Phi_i=\Psi_i=0,\ \ \forall j: A_{ij}=B_{ij}=0,\quad \mathbf{F}_i^{(n+1)}=-g(\mathbf{x}_i,(n+1)\triangle t),\ \ \mathbf{A}_{ij}=\begin{cases}1, & j=i\\ 0, & j\neq i\end{cases} \quad (40)$$

for each step. We notice here that, when $n = 0$, we use directly (29) and then for $n > 0$ it is straightforward to use Eq. (39).

Error Analysis

Given positive N, let $\tau = T/N$, $t_n = n\tau (0 \le n \le N)$. The time domain $[0, T]$ is covered by $\{t_n | 0 \le n \le N\}$. Given grid function $\mathbb{v} = \{\mathbb{v}^n | 0 \le n \le N\}$, we denote:

$$\mathbb{v}^{n-1/2} = \frac{1}{2}(\mathbb{v}^n + \mathbb{v}^{n-1}), \quad \partial_t \mathbb{v}^{n-1/2} = \frac{\mathbb{v}^n - \mathbb{v}^{n-1}}{\tau} \tag{41}$$

Lemma 1 *Suppose* $1 < \alpha < 2, \quad \in C^2[0, T]$. *It holds*

$$\begin{aligned} &\left| \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\mathscr{G}'(\xi)}{(t_n - \xi)^{\alpha-1}} d\xi \right. \\ &\left. - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[a_0 \mathscr{G}(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \mathscr{G}(t_k) - a_{n-1} \mathscr{G}(0) \right] \right| \\ &\le \frac{1}{\Gamma(3-\alpha)} \left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1 + 2^{1-\alpha}) \right] \max_{0 \le t \le t_n} |\mathscr{G}''(t)| \tau^{3-\alpha} \end{aligned} \tag{42}$$

where

$$a_k = (k+1)^{2-\alpha} - k^{2-\alpha}. \tag{43}$$

Proof See [35] for more details.

Theorem 1 *The scheme* (24) *is unconditionally stable in the sense that for all* $\tau > 0$, *it holds*

$$\|\mathcal{U}^n\|^2 \le \mathbb{C} \left(\|\nabla \mathcal{U}^0\|^2 + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|U_t^0\|^2 + 2 \frac{t_n}{\gamma_1} \max_{1 \le k \le n} \|f^{k-1/2}\|^2 \right). \tag{44}$$

Proof Choosing $\mathbb{v} = \partial_t \mathcal{U}^{n-1/2}$ in (42) and noticing $a_0 = 1$ and $\mathbb{p}_1 = \Gamma(3-\alpha)\tau^{\alpha-1}$ then we have the following equation for $1 \le n \le N$:

$$\begin{aligned} &\frac{1}{\mathbb{p}_1} \|\partial_t \mathcal{U}^{n-1/2}\|^2 + (\nabla \mathcal{U}^{n-1/2}, \nabla \partial_t \mathcal{U}^{n-1/2}) + \gamma_1 (\partial_t \mathcal{U}^{n-1/2}, \partial_t \mathcal{U}^{n-1/2}) \\ &= \frac{1}{\mathbb{p}_1} \left[\sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) (\partial \mathcal{U}_t^{k-1/2}, \partial_t \mathcal{U}^{n-1/2}) - a_{n-1} (U_t^0, \partial_t \mathcal{U}^{n-1/2}) \right] \\ &\quad + (f^{n-1/2}, \partial_t \mathcal{U}^{n-1/2}), \quad 1 \le n \le N, \end{aligned} \tag{45}$$

Since

$$\begin{aligned} (\nabla \mathcal{U}^{n-1/2}, \nabla \partial_t \mathcal{U}^{n-1/2}) &= \left(\frac{\nabla \mathcal{U}^n - \nabla \mathcal{U}^{n-1}}{2}, \frac{\nabla \mathcal{U}^n + \nabla \mathcal{U}^{n-1}}{\tau} \right) \\ &= \frac{1}{2\tau} (\|\nabla \mathcal{U}^n\|^2 - \|\nabla \mathcal{U}^{n-1}\|^2). \end{aligned} \tag{46}$$

and noticing a_{k-1} and $(a_{n-k-1} - a_{n-k})$ are positive, hence

$$\begin{aligned}
&\frac{1}{\mathbb{p}_1}\|\partial_t \mathcal{U}^{n-1/2}\|^2 + \frac{1}{2\tau}(\|\nabla \mathcal{U}^n\|^2 - \|\nabla \mathcal{U}^{n-1}\|^2) + \gamma_1(\partial_t \mathcal{U}^{n-1/2}, \partial_t \mathcal{U}^{n-1/2}) \\
&\quad = \frac{1}{\mathbb{p}_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})|(\partial_t \mathcal{U}^{k-1/2}, \partial_t \mathcal{U}^{n-1/2})| - a_{n-1}|(U_t^0, \partial_t \mathcal{U}^{n-1/2})|\right] \\
&\qquad + |(f^{n-1/2}, \partial_t \mathcal{U}^{n-1/2})|, \quad 1 \le n \le N, \qquad (47)
\end{aligned}$$

Then we can rewrite:

$$\begin{aligned}
&\frac{2\tau}{\mathbb{p}_1}\|\partial_t \mathcal{U}^{n-1/2}\|^2 + (\|\nabla \mathcal{U}^n\|^2 - \|\nabla \mathcal{U}^{n-1}\|^2) + 2\gamma_1\tau\|\partial_t \mathcal{U}^{n-1/2}\|^2 \\
&\quad \le \frac{\tau}{\mathbb{p}_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(\|\partial_t \mathcal{U}^{k-1/2}\|^2 + \|\partial_t \mathcal{U}^{n-1/2}\|^2) - a_{n-1}(\|U_t^0\|^2 + \|\partial_t \mathcal{U}^{n-1/2}\|^2)\right] \\
&\qquad + 2\tau|(f^{n-1/2}, \partial_t \mathcal{U}^{n-1/2})|, \quad 1 \le n \le N. \qquad (48)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\frac{\tau}{\mathbb{p}_1}\|\partial_t \mathcal{U}^{n-1/2}\|^2 + \|\nabla \mathcal{U}^n\|^2 - \|\nabla \mathcal{U}^{n-1}\|^2 + 2\gamma_1\tau\|\mathcal{U}^{n-1/2}\|^2 \\
&\quad = \frac{\tau}{\mathbb{p}_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\|\partial_t \mathcal{U}^{k-1/2}\|^2 - a_{n-1}\|U_t^0\|^2\right] \\
&\qquad + 2\tau|(f^{n-1/2}, \partial_t \mathcal{U}^{n-1/2})|, \quad 1 \le n \le N. \qquad (49)
\end{aligned}$$

We may consider without loss of generality $0 < 2\gamma_1\tau < 1$, then we obtain

$$\begin{aligned}
&\frac{\tau}{\mathbb{p}_1}\sum_{k=1}^{n} a_{n-k}\|\partial_t \mathcal{U}^{k-1/2}\|^2 + \|\nabla \mathcal{U}^n\|^2 \\
&\quad = \frac{\tau}{\mathbb{p}_1}\sum_{k=1}^{n-1} a_{n-k-1}\|\partial_t \mathcal{U}^{k-1/2}\|^2 + \|\nabla \mathcal{U}^{n-1}\|^2 - a_{n-1}\|U_t^0\|^2 + \tau\frac{\|f^{n-1/2}\|}{2\gamma_1}, \quad 1 \le n \le N,
\end{aligned}$$

where

$$2\tau|\left(f^{n-1/2}, \partial_t \mathcal{U}^{n-1/2}\right)| \le 2\tau\frac{\|f^{n-1/2}\|^2}{\gamma_1} + 2\gamma_1\tau\|\partial_t \mathcal{U}^{n-1/2}\|^2. \qquad (50)$$

Denoting

$$\mathbb{E}^n = \|\nabla \mathcal{U}^n\|^2 + \frac{\tau}{\mathbb{p}_1}\sum_{k=1}^{n} a_{n-k}\|\partial_t \mathcal{U}^{k-1/2}\|^2, \quad n \ge 1, \qquad (51)$$

consequently, we reach to the following inequality

$$\begin{aligned}
\mathbb{E}^n &\le \mathbb{E}^{n-1} + \frac{\tau a_{n-1}}{\mathbb{p}_1}\|U_t^0\|^2 + \frac{2\tau}{\gamma_1}\|f^{n-1/2}\|^2 \\
&\le \mathbb{E}^0 + \frac{\tau}{\mathbb{p}_1}\sum_{k=1}^{n} a_{k-1}\|U_t^0\|^2 + \frac{2\tau}{\gamma_1}\sum_{k=1}^{n}\|f^{k-1/2}\|^2 \\
&\le \mathbb{E}^0 + \frac{\tau}{\mathbb{p}_1}\sum_{k=1}^{n} a_{k-1}\|U_t^0\|^2 + \frac{2\tau n}{\gamma_1}\max_{1\le k\le n}\|f^{k-1/2}\|^2
\end{aligned}$$

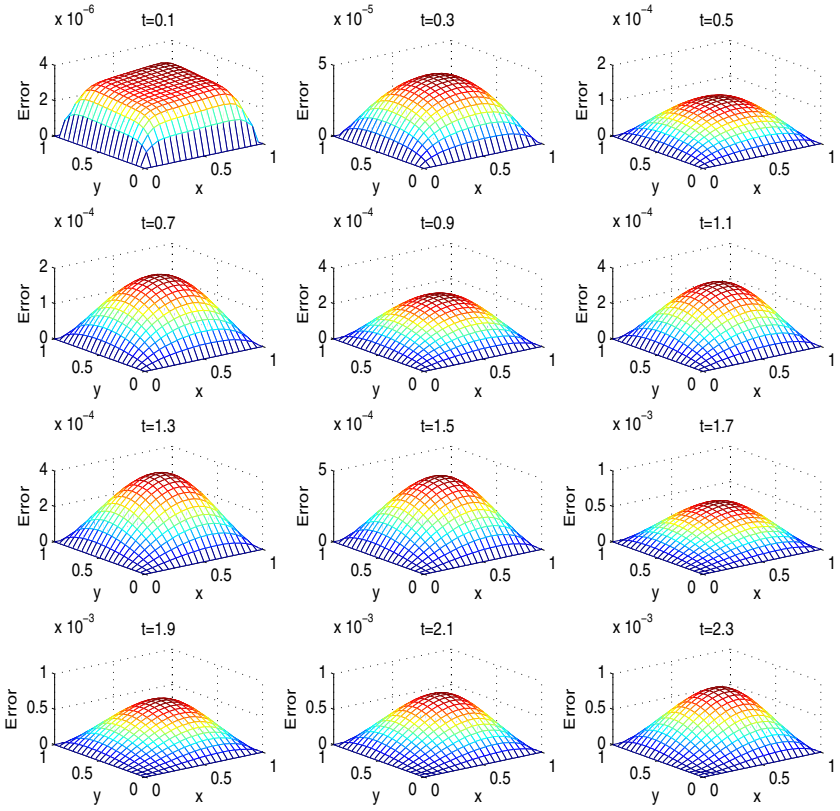

Fig. 2 Absolute error of MLPG solutions with $\triangle t = 0.001$, $N = 441$ and $\alpha = \frac{3}{2}$ for Example 1

It is easy to verify that $\sum_{k=1}^{n} a_{k-1} = n^{2-\alpha}$, therefore, we have

$$\frac{\tau}{\mathbb{J}_1} \sum_{k=1}^{n} a_{k-1} \|U_t^0\|^2 = \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|U_t^0\|^2. \tag{52}$$

It could be rewritten

$$\|\mathcal{U}^n\|^2 \leq \mathbb{C}\left(\|\nabla \mathcal{U}^0\|^2 + \frac{\tau}{\mathbb{J}_1} \sum_{k=1}^{n} a_{k-1} \|U_t^0\|^2 + 2\frac{n\tau}{\gamma_1} \max_{1\leq k\leq n} \|f^{k-1/2}\|^2. \right) \tag{53}$$

Hence the proof is complete. □

Two Numerical Experiments

In this section, we show the results obtained for two examples using the meshless method described above. In both examples, the domain integrals are evaluated with 16 points Gaussian quadrature rule while the boundary integrals are evaluated with 7 points Gaussian quadrature rule. To show the behavior of the solution and the efficiency of the proposed method, the following absolute error is applied to make comparison

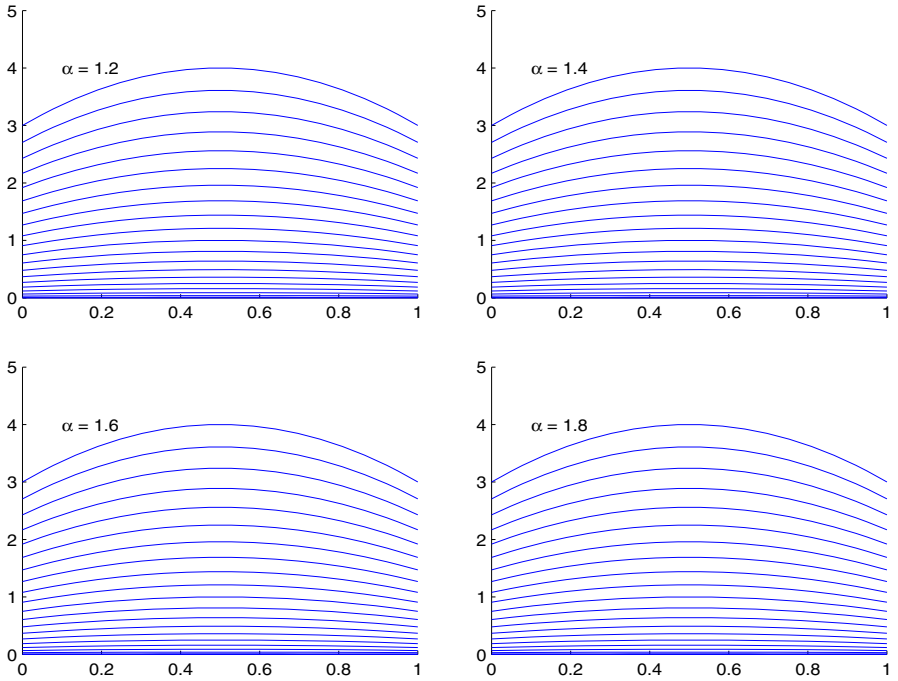

Fig. 3 Diagram of MLPG solutions $u(x, 0.5, t)$ at $t = 0, 0.1, 0.2, \ldots, 2$ with $\triangle t = 0.001$ and $N = 441$ for Example 1

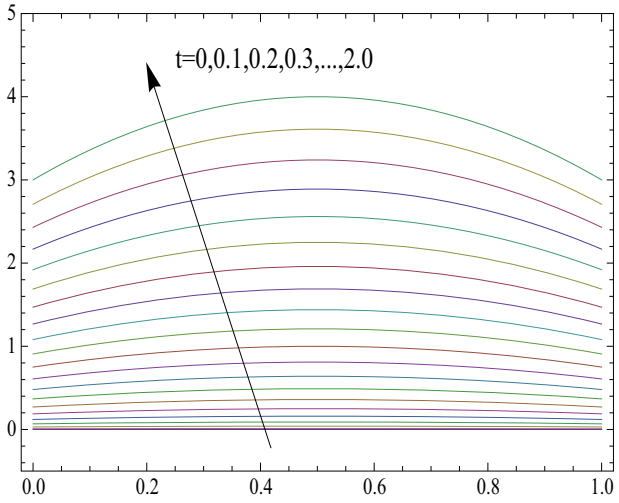

Fig. 4 Diagram of exact solutions $u(x, 0.5, t)$ at $t = 0, 0.1, 0.2, \ldots, 2$ for Example 1

$$Error = \max_{1 \le i \le N} \left| U_{exact}(\mathbf{x}_i) - U_{approx}(\mathbf{x}_i) \right|$$

where $U_{exact}(\mathbf{x}_i)$ and $U_{approx}(\mathbf{x}_i)$ are achieved by exact and approximate solution and N is number of nodal points. In both problems the regular node distribution is used. Also in

Table 1 The L_∞ error with fixed $N = 441$, $\alpha = 1.5$ and different time-steps for Example 1

t	$\triangle t = 0.1$	$\triangle t = 0.01$	$\triangle t = 0.001$
$t = 0.000000$	0.00000000	0.00000000	0.00000000
$t = 0.100000$	0.00021788	0.00003173	0.00000349
$t = 0.200000$	0.00139839	0.00016792	0.00001750
$t = 0.300000$	0.00364989	0.00041713	0.00004267
$t = 0.400000$	0.00673507	0.00073569	0.00007486
$t = 0.500000$	0.01024647	0.00108049	0.00010982
$t = 0.600000$	0.01384541	0.00142623	0.00014512
$t = 0.700000$	0.01735184	0.00176128	0.00017966
$t = 0.800000$	0.02070118	0.00208286	0.00021318
$t = 0.900000$	0.02389343	0.00239296	0.00024589
$t = 1.000000$	0.02696163	0.00269532	0.00027814

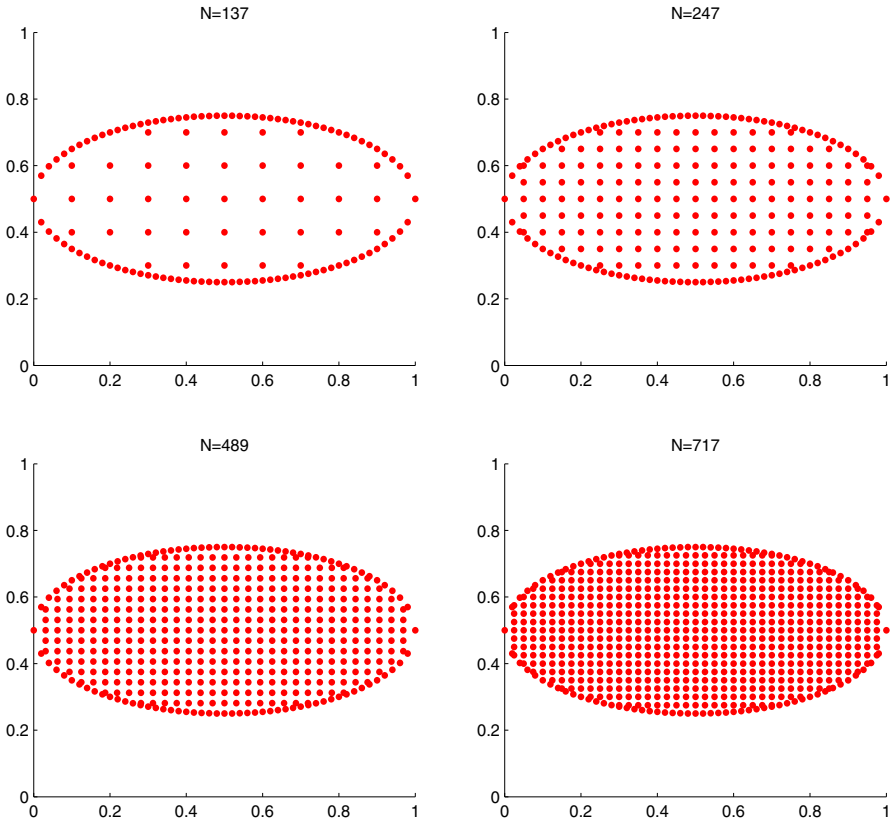

Fig. 5 The approximation of the domain of Example 2 by different number of nodal points

order to implement the meshless local weak form, the radius of the local quadrature domain $r_q = 0.7h$ is selected, where h is the distance between the nodes in x or y direction. The size of r_q is such that the union of these sub-domains must cover the whole global domain.

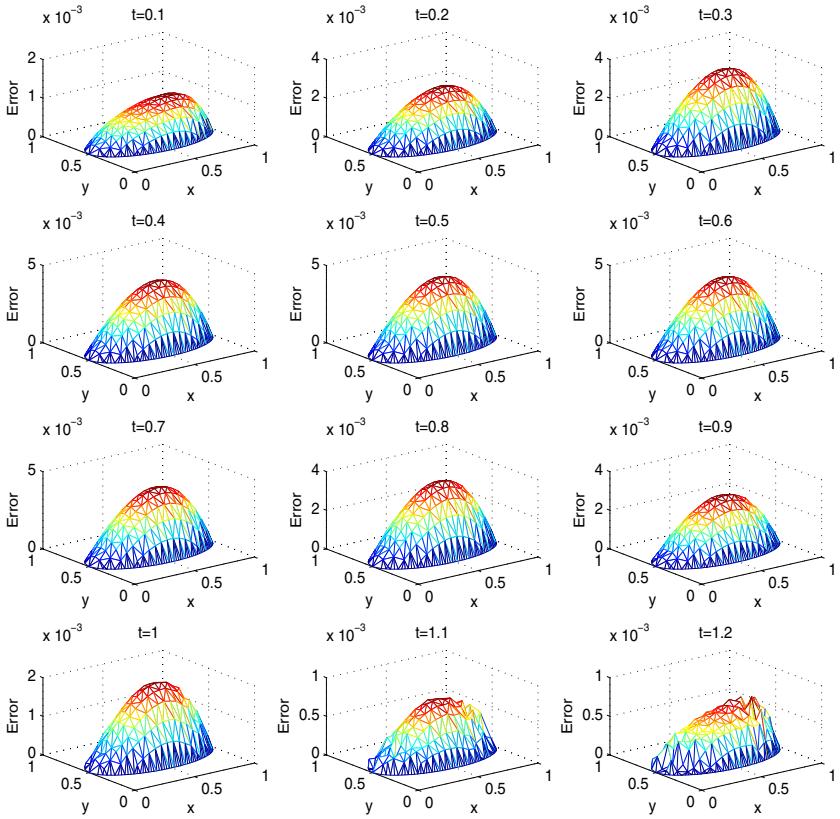

Fig. 6 Absolute error of MLPG solutions with $\triangle t = 0.05$, $N = 247$ and $\alpha = 1.2$ for Example 2

The radius of support domain to moving least squares approximation is $r_s = 4r_q$ and all shape parameters in (14) are chosen $c_i = c = 1.2h$. This size is significant enough to have sufficient number of nodes (n) and gives an appropriate shape functions. Also, the quadratic basis functions is used i.e. $m = 6$ is taken.

Example 1 (Regular domain) We set $\gamma_1 = \gamma_2 = 1$, the exact solution of problem (1)–(3) is taken as

$$u(x, y, t) = t^2(2 - x - y)(x + y),$$

and the domain of the problem is $\Omega = [0, 1] \times [0, 1]$. The functions $\varphi(x, y)$, $\psi(x, y)$ and $g(x, y, t)$ are defined accordingly, and also $f(x, y, t)$ is given by

$$f(x, y, t) = \frac{2t^{2-\alpha}(x + y - 2)(x + y)}{(\alpha - 2)\Gamma(2 - \alpha)} + 4t^2 + 2t(-x - y + 2)(x + y),$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\mathrm{d}t. \tag{54}$$

Figure 2 presents the absolute error of approximate MLPG solutions at different time levels with $\triangle t = 0.001$ and $N = 441(h = 0.05)$ for $\alpha = 1.5$. Also, the approximate MLPG

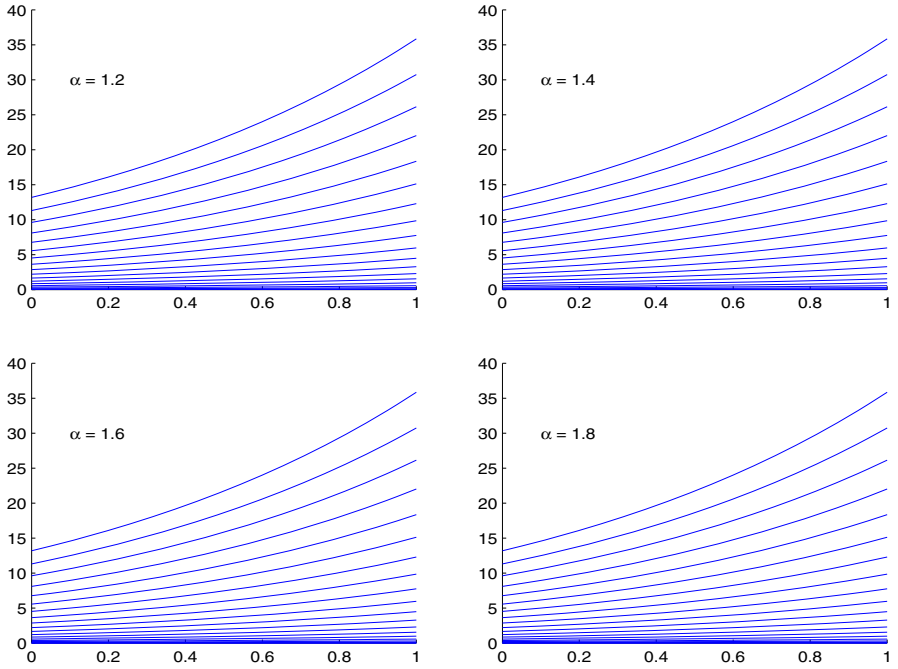

Fig. 7 Diagram of MLPG solutions $u(x, 0.5, t)$ at $t = 0, 0.1, 0.2, \ldots, 2$ with $\triangle t = 0.05$ and $N = 247$ for Example 2

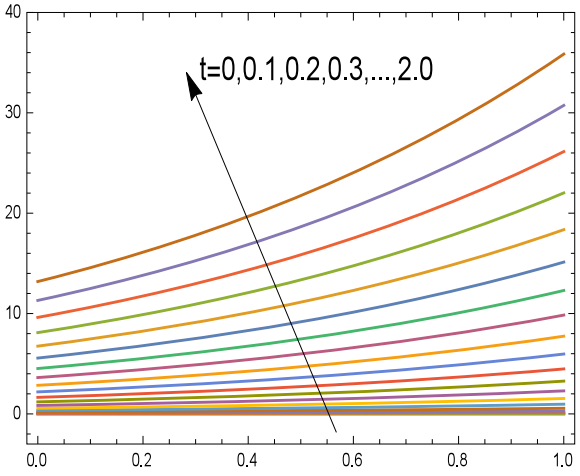

Fig. 8 Diagram of exact solutions $u(x, 0.5, t)$ at $t = 0, 0.1, 0.2, \ldots, 2$ for Example 2

solutions $u(x, 0.5, t)$ at many different time levels and different values of time fractional order i.e. α have been plotted in Fig. 3, while the corresponding exact solutions have been shown in Fig. 4. Furthermore, Table 1 shows the convergence with respect to time discretization.

Example 2 (Non-regular domain) In this example, we take again $\gamma_1 = \gamma_2 = 1$ and assume that

$$u(x, y, t) = t^3 \exp(x + y),$$

Table 2 The L_∞ error with fixed $N = 441$, $\alpha = 1.2$ and different time-steps for Example 2

t	$\triangle t = 0.1$	$\triangle t = 0.05$	$\triangle t = 0.01$
$t = 0.000000$	0.00000000	0.00000000	0.00000000
$t = 0.100000$	0.00148778	0.00104865	0.00022082
$t = 0.200000$	0.00547895	0.00264969	0.00048595
$t = 0.300000$	0.00800747	0.00358613	0.00066444
$t = 0.400000$	0.00899216	0.00411414	0.00077587
$t = 0.500000$	0.00941117	0.00435965	0.00083423
$t = 0.600000$	0.00935067	0.00434740	0.00084615
$t = 0.700000$	0.00882321	0.00408782	0.00081763
$t = 0.800000$	0.00780302	0.00358998	0.00075120
$t = 0.900000$	0.00630339	0.00286859	0.00064914
$t = 1.000000$	0.00435741	0.00192515	0.00051515

is the exact solution of the problem (1)–(3). Moreover, the domain of the problem is considered as

$$\Omega = \left\{ (x, y) : (x - \frac{1}{2})^2 + 4(y - \frac{1}{2})^2 \leq \frac{1}{4} \right\},$$

the approximation of this domain by different number of nodal points, while the regular distributed nodes are used for interior of the domain, are shown in Fig. 5.

The functions $\varphi(x, y)$, $\psi(x, y)$, and $g(x, y, t)$ are defined accordingly and also $f(x, y, t)$ is given by

$$f(x, y, t) = \frac{6t^{3-\alpha} e^{x+y}}{\left(\alpha^2 - 5\alpha + 6\right) \Gamma(2 - \alpha)} - 2t^3 e^{x+y} + 3t^2 e^{x+y}. \tag{55}$$

As previous example, Fig. 6 presents the absolute error of approximate MLPG solutions at different time levels with $\triangle t = 0.05$ and $N = 247$ for $\alpha = 1.2$. Also, the approximate MLPG solutions $u(x, 0.5, t)$ at many different time levels and different values of time fractional order i.e. α have been plotted in Fig. 7, while the corresponding exact solutions have been shown in Fig. 8. Moreover, Table 2 shows the convergence with respect to $\triangle t$.

Conclusions

In this paper, an efficient and accurate computational method namely meshless local Petrov–Galerkin (MLPG) method, which is based on the Galerkin weak form and moving least squares (MLS) approximation, has been applied to the time fractional two-dimensional diffusion-wave equation. The time fractional derivative has been defined by Caputo sense for $(1 < \alpha < 2)$. We have considered a arbitrary 2-D domain, which can be non-regular in general, while Dirichlet boundary conditions are prescribed to the boundaries of the domain. We have used meshless Galerkin weak form for the interior nodes whereas the meshless collocation technique has been applied to the nodes on the boundaries of the domain. As a consequence of imposing Dirichlet boundary conditions directly, the general domains are also applicable easily. In the proposed MLPG method, the moving least square (MLS) approximation has been used to construct shape functions which plays important rule in the convergence

and stability of the method. Two numerical experiments have been presented and satisfactory agreements are obtained.

Acknowledgements The author is very grateful to two anonymous reviewers for carefully reading the paper and for their comments and suggestions which have improved the paper very much.

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