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ORIGINAL PAPER

Wave Interactions in Non-ideal Isentropic Magnetogasdynamics

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Abstract In this paper, we consider the Riemann problem and wave interactions for a quasilinear hyperbolic system of partial differential equations governing the one dimensional unsteady simple wave flow of an isentropic, non-ideal, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field. This class of equations includes, as a special case of ideal isentropic magnetogasdynamics. We study the shock and rarefaction waves and their properties, and show the existence and uniqueness of the solution to the Riemann problem for arbitrary initial data under certain conditions and then we discuss the vacuum state in non-ideal isentropic magnetogasdynamics. We discuss numerical tests and study the solution influenced by the van der Waals excluded volume for different initial data along with all possible interactions of elementary waves.

Keywords Magnetogasdynamics · van der Waals gas · Riemann problem · Wave interactions

Introduction

In the recent past, analysis of magnetogasdynamics has been the subject of great interest both from mathematical and physical point of view due to its applications in the variety of fields such as astrophysics, nuclear science, engineering physics and plasma physics, etc (see, [1–6] and the references cited therein). Lax [7] solved the Riemann problem for the case when the initial data consisting of constant states U_l and U_r are such that U_l and U_r are sufficiently small; here U is the vector of conserved variable with U_l to the left of $x = 0$ and U_r to the right of $x = 0$ separated by a discontinuity at $x = 0$. Smoller [8] solved the

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Riemann problem by considering U_l and U_r to be arbitrary constant vectors; for details and methodologies, the reader is referred to the book by Smoller [9]. Smoller and Temple [10] demonstrated the existence of solutions with shocks for equations describing a perfect fluid in special relativity; this work was generalized by Chen [11] for the general isentropic relativistic gases. The striking features of wave interactions as well as the variety of engineering and physical situations where this phenomenon has to be faced gave rise over the years to a great and growing interest on this subject. For interaction of elementary waves in unsteady one-dimensional Euler equations, we refer to Smoller [9], and Chang and Hsiao [12]. Liu [13] is concerned with the interactions of the elementary waves for the nonlinear degenerate wave equations. Curro and Usco [14] investigated the nonlinear wave interactions for quasilinear hyperbolic 2×2 systems. Ji and Zheng [15] are concerned with classical solutions for the interaction of two arbitrary planar rarefaction waves for the self-similar Euler equations in two space dimensions. The interaction of steady rarefaction waves, and the interaction of a rarefaction wave in a supersonic jet stream out of an orifice into the atmosphere is presented by Chen and Qu [16]. Liu and Sun [17] discussed the existence and uniqueness of the solutions for the Riemann problem in magnetogasdynamics and investigated the interactions of the elementary waves. Kuila and Raja Sekhar [18] investigated the elementary waves of the Riemann problem in magnetogasdynamics and construct the exact solution for it in different approach and compare exact solutions with numerical solutions. Solution of the Riemann problem in magnetogasdynamics have been obtained by Singh and Singh [19]. Radha and Sharma [20] studied the interaction of a weak discontinuity wave with the elementary waves of the Riemann problem for the one-dimensional Euler equations. Interactions of forward and backward centered rarefaction waves for pressure-gradient equations are presented by Zhang et al. [21]. Using the theory of progressive waves and some related procedures, waves of finite and moderately small amplitudes, influenced by the effects of non-linear convection, attenuation and geometrical spreading are studied by Ambika et al. [22] in an imperfect gas modeled by the van der Waals equation of state. Chadha and Jena [23] used the Lie group of transformations and obtained the whole range of self-similar solutions to the problem of propagation of shock waves through a non-ideal dusty gas.

Recently, Raja Sekhar and Sharma [24] studied the Riemann problem and elementary wave interactions for the one-dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (p + \rho u^2 + B^2/2\mu)_x = 0, \end{aligned} \tag{1}$$

where ρ, u, p, B and μ denotes the density, velocity, pressure, transverse magnetic field and magnetic permeability respectively; $p = A\rho^\gamma$ for ideal polytropic gas, A is positive constant and γ is the adiabatic constant lies $1 < \gamma < 2$ for most gases and $B = k_2\rho$ where k_2 is a positive constant.

In this paper, we consider a van der Waals gas obeying the equation of state [25,26]:

$$p = k_1 \left(\frac{\rho}{1 - a\rho} \right)^\gamma, \tag{2}$$

where k_1 is the constant and a the van der Waals excluded volume, which lies in the range $0 \le a \le 0.05$ [25] and satisfying $1 \gg a\rho \ge 0$ [26]. It may be noticed that the case $a = 0$ corresponds to the ideal gas. Our main purpose is to discuss all possible interactions of the elementary waves obtained in solving the Riemann problem for (3). By analyzing the explicit expressions of the shock waves and the rarefaction waves of the left state U_l and the right state U_r in the (ρ, u) plane, we discuss the interactions of shock waves and rarefaction waves from the different family and the same family.

This paper is organized as follows. In Sect. 2, we study the elementary wave solutions and their properties of the Riemann problem, i.e., shock waves and rarefaction waves. In Sect. 3, we consider the Riemann problem for arbitrary initial data and show the existence and uniqueness of the solution under certain conditions and then we discuss the vacuum state in isentropic non ideal magnetogasdynamics. In Sect. 4, we present numerical examples for different initial data and discuss the solutions under the effect of van der Waals excluded volume a. We study all possible interaction of elementary waves in Sect. 5. Section 6 is devoted to some concluding remarks and ideas for further future work.

Elementary Waves and Their Properties

We consider the Riemann problem for the system (1), with piecewise constant initial data, separated by a discontinuity at $x = x_0$, in conservation form

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \tag{3}$$

where $U = (\rho, \rho u)^{tr}$, $F(U) = (\rho u, p + \rho u^2 + B^2/2\mu)^{tr}$, tr denote the transformation and

$$U(x, t_0) = \begin{cases} U_l, & x < x_0, \\ U_r, & x > x_0. \end{cases} \tag{4}$$

To carry out the characteristic analysis of (3), it is convenient to use the primitive variables $V = (\rho, u)^{tr}$. Then for smooth solution, system (3) is equivalent to

$$\frac{\partial V}{\partial t} + M(V)\frac{\partial V}{\partial x} = 0, \tag{5}$$

where $M(V)$ is a 2×2 matrix having components M_{ij} with nonzero entries $M_{11} = M_{22} = u$, $M_{12} = \rho$, $M_{21} = \frac{w^2}{\rho}$, where $w = \sqrt{b^2 + c^2}$; $b^2 = k_2^2\rho/\mu$ and $c^2 = k_1\gamma\rho^{\gamma-1}/(1 - a\rho)^{\gamma+1}$. The eigenvalues of the system (5) are $\lambda_1 = u - w$ and $\lambda_2 = u + w$. As the eigenvalues of M are real and distinct when $w > 0$; so it is strictly hyperbolic. Let $\mathrm{r}^{(1)} = (-\rho, w)^{tr}$, $\mathrm{r}^{(2)} = (\rho, w)^{tr}$, be the right eigenvectors corresponding to eigenvalues λ_1 and λ_2, respectively. Since $\nabla\lambda_i.\mathrm{r}^{(i)} \neq 0$ for $i = 1, 2$, so both the characteristic fields are genuinely nonlinear. Thus the elementary wave solutions of the system (3) consists of shocks or centred rarefaction waves.

Shock Waves

Suppose U is a weak solution of (3) such that U_l and U are C^1 and extend continuously to the shock $x = x(t)$. Let $[U] = U_l - U$ be the jump discontinuity across the shock and $s = dx/dt$ the shock speed. Then, the following Rankine–Hugoniot (RH) jump conditions hold across the shock

$$s[\rho] = [\rho u], \tag{6}$$
$$s[\rho u] = [p + \rho u^2 + B^2/2\mu]. \tag{7}$$

We have the following lemma.

Lemma 1 *Let the states U_l and U satisfy the Rankine–Hugoniot jump conditions (6) and (7). Let $S_1 = S_1(U_l)$ and $S_2 = S_2(U_l)$ respectively denote 1-shock and 2-shock curves associated with λ_1 and λ_2 characteristic fields. Then the shock curves satisfy*

$$u = u_l - h(\rho_l, \rho), \tag{8}$$

where $h(\rho_l, \rho) = \sqrt{\left(p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu}\right)\left(\frac{\rho-\rho_l}{\rho\rho_l}\right)}$ such that for $1 < \gamma < 2$, we have for $\rho > \rho_l$, $u' < 0$ and $u'' > 0$ on S_1, whilst for $\rho < \rho_l$ we have $u' > 0$ and $u'' < 0$ on S_2.

Proof Using (6) and (7), we get (8). Let $\phi(\rho) = h^2(\rho_l, \rho)$; then differentiating (8) with respect to ρ we obtain $u' = -\frac{\phi'(\rho)}{2\sqrt{\phi(\rho)}}$. Since ϕ and ϕ' are positive for $\rho > \rho_l$ and therefore, $u' < 0$. Let $\psi(\rho) = (\phi'(\rho))^2 - 2\phi(\rho)\phi''(\rho)$ and so $\psi'(\rho) = -2\phi(\rho)\phi'''(\rho)$. Further, for $1 < \gamma < 2$, $\phi''(\rho) > 0$, whilst $\phi'''(\rho) < 0$ and hence $\psi'(\rho) > 0$. Thus $\psi(\rho) > \psi(\rho_l)$ for $\rho > \rho_l$ and $\psi(\rho_l) = 0$. For $1 < \gamma < 2$, we have $u'' = \frac{(\phi'(\rho))^2 - 2\phi(\rho)\phi''(\rho)}{4\phi(\rho)^{3/2}} > 0$ on S_1. In a similar manner, it follows that for $\rho < \rho_l$ and $1 < \gamma < 2$, we have $u' > 0$ and $u'' < 0$ on S_2.

Here we are going to prove that the shock curves satisfy the Lax entropy conditions.

Theorem 2 *Across 1-shock (respectively, 2-shock), $\rho_l < \rho$ and $u_l > u$ (respectively, $\rho_l > \rho$ and $u_l > u$) if, and only if, the Lax conditions hold, i.e.,*

$$s_1 < \lambda_1(U_l), \quad \lambda_1(U) < s_1 < \lambda_2(U), \tag{9}$$

and

$$\lambda_1(U_l) < s_2 < \lambda_2(U_l), \quad \lambda_2(U) < s_2. \tag{10}$$

Proof First, let us consider 1-shocks and prove $\lambda_1(U_l) > s_1$. Since $p' > 0$ and $p'' > 0$, by Lagrange's mean value theorem, there exists a $\xi \in (\rho_l, \rho)$ such that $p'(\xi) = (p - p_l)/(\rho - \rho_l)$. Furthermore, since $p'' > 0$, we have $p'(\xi) > p_l' = c_l^2$ and thus $c_l^2 < p'(\xi)\rho/\rho_l$, which implies that

$$c_l^2 < \frac{(p - p_l)\rho}{\rho_l(\rho - \rho_l)}. \tag{11}$$

Since $(\rho + \rho_l)/2 > \rho_l$, we have $k_2^2(\rho + \rho_l)/2\mu > k_2^2\rho_l/\mu$ thereby implies that $(B^2 - B_l^2)\rho/2\mu\rho_l(\rho - \rho_l) > (B^2 - B_l^2)/2\mu(\rho - \rho_l) > B^2/\mu\rho_l$, and therefore

$$b_l^2 < \frac{(B^2 - B_l^2)\rho}{2\mu\rho_l(\rho - \rho_l)}. \tag{12}$$

From (11) and (12), we obtain

$$w_l^2 < \frac{\rho}{\rho_l(\rho - \rho_l)}\left(p - p_l + \frac{B^2 - B_l^2}{2\mu}\right), \tag{13}$$

this implies

$$\frac{-\rho\sqrt{(p - p_l + (B^2 - B_l^2)/2\mu)((1/\rho_l) - (1/\rho))}}{\rho - \rho_l} < -w_l. \tag{14}$$

In view of (9), the above inequality yields $\rho(u - u_l)/(\rho - \rho_l) < -w_l$, and hence $s_1 < \lambda_1(U_l) = u_l - w_l$.

Next, since $p'' > 0$ and $\rho_l < \rho$ for 1-shock wave, we have $p'(\eta) = (p-p_l)/(\rho-\rho_l) < p'$ for some $\eta \in (\rho_l, \rho)$, and hence

$$c_l^2 > \frac{(p-p_l)\rho_l}{\rho(\rho-\rho_l)}. \tag{15}$$

Furthermore, since $(\rho + \rho_l)/2 < \rho$, it follows that

$$b_l^2 > \frac{(B^2 - B_l^2)\rho_l}{2\mu\rho(\rho-\rho_l)}, \tag{16}$$

and hence from (15) and (16), we obtain

$$b_l^2 + c_l^2 > \frac{(B^2 - B_l^2)\rho_l}{2\mu\rho(\rho-\rho_l)} + \frac{(p-p_l)\rho_l}{\rho(\rho-\rho_l)}, \tag{17}$$

thereby implying that

$$\frac{\rho_l\sqrt{(p-p_l+(B^2-B_l^2)/2\mu)((1/\rho_l)-(1/\rho))}}{\rho-\rho_l} > -w. \tag{18}$$

Also, from (6)–(9) imply that $u - w < (\rho u - \rho_l u_l)/(\rho - \rho_l) = s_1$, and hence $\lambda_1(U) < s_1$.

Lastly, we show that $s_1 < \lambda_2(U)$. From (19), we have

$$\sqrt{\frac{(B^2 - B_l^2)\rho_l}{2\mu\rho(\rho-\rho_l)} + \frac{(p-p_l)\rho_l}{\rho(\rho-\rho_l)}} < w.$$

For 1-shock curve, using (8) we obtain $\frac{(u-u_l)\rho_l}{\rho-\rho_l} < w$, which implies that $s_1 < \lambda_2(U)$. Therefore 1-shock satisfies Lax conditions; proof for 2-shocks follows on similar lines.

Conversely, we want to prove that for 1-shock Lax conditions hold. It follows from (11) that for 1-shock waves, we have $s_1 < u_l - w_l$, where s_1 is the speed of the one-shock wave, which implies that $w_l < \tilde{u}_l$ and $u - w < s_1 < u + w$, and hence $|\tilde{u}| < w$. From (6), we have $\rho\tilde{u} = \rho_l\tilde{u}_l$. Since ρ and ρ_l are positive, so both $\tilde{u}$ and $\tilde{u}_l$ must have the same sign. Therefore, $\tilde{u} > 0$ and $\tilde{u}_l > 0$, the gas speed on both sides of the shock is greater than the shock speed, so particles cross the shock from the left to the right for one-shock waves. In the case of 2-shock waves, the shock inequalities give $|\tilde{u}_l| < w_l$ and $\tilde{u} < -w < 0$, which imply that the shock speed is greater than the gas speed on both sides of the shock, and so the particles cross 2-shocks from the right to the left.

For both the shock families, $\tilde{u}_l$ and $\tilde{u}$ are non-zero so that $L = \rho\tilde{u} = \rho_l\tilde{u}_l \neq 0$. Thus, for 1-shock waves, we have $\tilde{u}_l^2 > w_l^2$ and $w^2 > \tilde{u}^2$. From (7), yields

$$p + \rho\tilde{u}^2 + B^2/2\mu = p_l + \rho_l\tilde{u}_l{}^2 + B_l^2/2\mu, \tag{19}$$

which, by virtue of the fact that $\tilde{u}_l^2 > w_l^2$ and $w^2 > \tilde{u}^2$, yields

$$p + \rho w^2 + B^2/2\mu > p_l + \rho_l w_l^2 + B_l^2/2\mu$$

that is,

$$\left(1 + \frac{\gamma}{1-a\rho}\right)p + \frac{3k_2^2}{2\mu}\rho^2 > \left(1 + \frac{\gamma}{1-a\rho_l}\right)p_l + \frac{3k_2^2}{2\mu}\rho_l^2,$$

which implies that $p > p_l$ and $\rho > \rho_l$, hence $B > B_l$, $u < u_l$. Since for one-shock waves, $\tilde{u}$ and $\tilde{u}_l$ are positive. In similar way, for 2-shock waves, we can prove that $p < p_l$ and $B < B_l$, hence $\rho < \rho_l$, $u < u_l$. Therefore, both the shock waves are compressive.

Now we show that the shock curves are starlike with respect to (ρ_l, u_l).

Theorem 3 *The 1-shock and 2-shock curves are starlike with respect to* (ρ_l, u_l) *when* $p = k_1\left(\frac{\rho}{1-a\rho}\right)^{\gamma}$ *and* $B = k_2\rho$ *for values of* γ *lying in the range* $1 < \gamma < 2$.

Proof We shall show that any ray through the point (ρ_l, u_l) intersects 1-shock curve in at most one point; for this, it is sufficient to show that for any two rays through (ρ_l, u_l) and two different points (ρ_1, u_1), (ρ_2, u_2) on the 1-shock curve, their slopes are different.

The slope of the lines joining (ρ_l, u_l) with (ρ_1, u_1) and (ρ_2, u_2) are respectively $\frac{u_1-u_l}{\rho_1-\rho_l}$ and $\frac{u_2-u_l}{\rho_2-\rho_l}$. For the 1-shock curve, (8) implies that $\left(\frac{u-u_l}{\rho-\rho_l}\right)^2 = f_1(\rho)+f_2(\rho)$, where $f_1(\rho) = \frac{p-p_l}{\rho_l\rho(\rho-\rho_l)}$ and $f_2(\rho) = \frac{B^2-B_l^2}{2\mu\rho_l\rho(\rho-\rho_l)}$.

Now we show that $f_1'(\rho) < 0$ and $f_2'(\rho) < 0$. Differentiate $f_1(\rho)$ with respect to ρ, we get

$$f_1'(\rho) = \frac{\rho_l\rho(\rho-\rho_l)p' - \rho_l(p-p_l)(2\rho-\rho_l)}{\rho_l^2\rho^2(\rho-\rho_l)^2}.$$

Let $g_1(\rho) = \rho_l\rho(\rho-\rho_l)p' - \rho_l(p-p_l)(2\rho-\rho_l)$, so that $g_1(\rho_l) = 0$. Now $g_1'(\rho) = \rho_l\rho(\rho-\rho_l)p'' - 2\rho_l(p-p_l)$ and $g_1'(\rho_l) = 0$. Further, using $p = k_1\left(\frac{\rho}{1-a\rho}\right)^{\gamma}$, we get $g_1''(\rho) = -\frac{k_1\gamma\rho_l\rho^{\gamma-2}}{(1-a\rho)^{\gamma+3}}[((2-\gamma(\gamma-1))\rho+(5\gamma-3)a\rho_l\rho+(\gamma-1)^2\rho_l+4a^2\rho_l\rho^2)-4(\gamma+1)a\rho^2]$. Therefore, $g_1''(\rho) < 0$ for $1 < \gamma < 2, 0 \le a \le 0.05$ and $1 \gg a\rho \ge 0$. Across 1-shock wave, we have $\rho_l < \rho$ and $g_1'(\rho) < g_1'(\rho_l) = 0$, implying thereby that $g_1(\rho)$ is a decreasing function of ρ. Thus, $g_1(\rho) < g_1(\rho_l)$, and $f_1'(\rho) < 0$. Again, we substitute $B = k_2\rho$ in $f_2(\rho)$ and differentiate with respect to ρ obtain $f_2'(\rho) = -\frac{k_2^2}{2\mu\rho^2} < 0$. Therefore, $\left(\frac{u-u_l}{\rho-\rho_l}\right)$ is an increasing function of ρ, since $\rho > \rho_l$ and $u < u_l$ for 1-shock wave, and hence 1-shock curve is starlike with respect to (ρ_l, u_l). Similarly, we can show that 2-shock curve is also starlike with respect to (ρ_l, u_l).

Rarefaction Waves

Here we construct the rarefaction wave curves. For an i rarefaction wave ($i = 1, 2$), the two constant states V_l and V are connected through a smooth transition in i-th genuinely nonlinear characteristic field, is a solution to (5) of the form

$$V(x,t) = \begin{cases} V_l, & \frac{x}{t} \le \lambda_i(V_l) \\ V(\frac{x}{t}), & \lambda_i(V_l) \le \frac{x}{t} \le \lambda_i(V) \\ V_r, & \frac{x}{t} \ge \lambda_i(V), \end{cases} \tag{20}$$

with $\lambda_i(V_l) \le \lambda_i(V)$. If we set $\varrho = \frac{x}{t}$, then the system (5) becomes $(M - \varrho I)(\dot\rho, \dot u)^{tr} = 0$, where I is a 2×2 identity matrix and an overhead dot denotes differentiation with respect to the variable ϱ. If $(\dot\rho, \dot u)^{tr} = (0, 0)$ then ρ and u are constant; but as we are interested in non-constant solutions, we consider $(\dot\rho, \dot u)^{tr} \ne (0, 0)$ and then it follows that $(\dot\rho, \dot u)^{tr}$ is an eigenvector of the matrix M corresponding to the eigenvalue ϱ. Since the matrix M has two real and distinct eigenvalues, $\lambda_1 < \lambda_2$, there are two families of rarefaction waves, R_1 and R_2 which denote, respectively, 1-rarefaction waves and 2-rarefaction waves.

First we consider 1-rarefaction waves. Since $(M - \lambda_1 I)(\dot{\rho}, \dot{u})^{tr} = 0$ with $\lambda_1 = u - w$, we have, $w\dot{\rho} + \rho\dot{u} = 0$, implying thereby that

$$\mathbf{\Gamma}_1 = u + \int^{\rho} \frac{w(\theta)}{\theta} d\theta = constant, \tag{21}$$

which represents R_1 curves with $\mathbf{\Gamma}_1$ as the 1-Riemann invariant. Similarly, 2-rarefaction wave curves are given by

$$\mathbf{\Gamma}_2 = u - \int^{\rho} \frac{w(\theta)}{\theta} d\theta = constant, \tag{22}$$

and $\mathbf{\Gamma}_2$ is the 2-Riemann invariant.

Theorem 4 *The R_1 curve is convex and monotonic decreasing while R_2 curve is concave and monotonic increasing.*

Proof The 1-rarefaction wave is given by

$$u = u_l + \int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta, \quad \text{if } \rho \leq \rho_l \tag{23}$$

which on differentiating with respect to ρ, yields $\frac{du}{d\rho} = -\frac{w}{\rho} < 0$, and subsequently,

$$\frac{d^2u}{d\rho^2} = \frac{w}{\rho^2} - \frac{w'}{\rho}. \tag{24}$$

From (24), for $1 < \gamma < 2$ yields $\frac{d^2u}{d\rho^2} = \frac{\frac{k_1\gamma\rho^{\gamma-1}}{(1-a\rho)^{\gamma+2}}[2-(\gamma-1)-4a\rho]+\frac{k_2^2\rho}{\mu}}{2\rho^2 w} > 0$ and, therefore, u is convex with respect to ρ for 1-rarefaction waves. In similar way, we can prove for 2-rarefaction waves.

Lemma 5 *Across 1-rarefaction waves (respectively, 2-rarefaction waves), $\rho < \rho_l$ and $u > u_l$ (respectively, $\rho > \rho_l$, and $u > u_l$) if and only if, the characteristic speed increases from left hand to right hand state, i.e.,*

$$\lambda_i(V_l) < \lambda_i(V), i = 1, 2. \tag{25}$$

Proof Let the characteristic speed increases from left hand to right hand state. Therefore from the inequality (25) implies that

$$w - w_l < u - u_l, \tag{26}$$

Further, since in 1-rarefaction wave region $\mathbf{\Gamma}_1$ is constant, we have $u + \int^{\rho} \frac{w(\theta)}{\theta} d\theta = u_l + \int^{\rho_l} \frac{w(\theta)}{\theta} d\theta$, which by virtue of (26) yields $u + \int^{\rho} \frac{w(\theta)}{\theta} d\theta < u_l + \int^{\rho_l} \frac{w(\theta)}{\theta} d\theta$; which implies $\rho_l > \rho$ and $u_l < u$. Similarly, for 2-rarefaction waves, we can prove that $\rho > \rho_l$ and $u > u_l$.

Conversely, for one-rarefaction wave, we assume that $\rho_l > \rho$ and $u_l < u$. Then prove that the characteristic speed increases from the left to the right, that is, $\lambda_1(V_l) \leq \lambda_1(V)$. Since $dw/d\rho = (p'' + (B')^2/\mu)/2w > 0$, w is an increasing function of ρ; this implies that for one-rarefaction waves, $w(\rho) \leq w(\rho_l)$ or equivalently $-w_l \leq -w$. From $u_l \leq u$ and $-w_l \leq -w$ imply that $\lambda_1(V_l) \leq \lambda_1(V)$. In similar way, we can prove this for the 2-rarefaction waves.

The Riemann Problem

We solve the Riemann problem (3) and (4) in the class of functions consisting of constant states, separated by either shocks or rarefaction waves. The solution of the Riemann problem consists of at most three constant states (including U_l and U_r), which are separated either by a shock or a rarefaction wave. The Riemann invariant coordinates are

$$\mathbf{\Gamma}_1 = u + \int^{\rho} \frac{w(\theta)}{\theta} d\theta,$$
$$\mathbf{\Gamma}_2 = u - \int^{\rho} \frac{w(\theta)}{\theta} d\theta. \tag{27}$$

Lemma 6 *The mapping* $(\rho, u) \longrightarrow (\mathbf{\Gamma}_1, \mathbf{\Gamma}_2)$ *is one to one and the Jacobian of this mapping is nonzero when* $\rho > 0$.

Proof From (27), we have $\frac{\partial \mathbf{\Gamma}_1}{\partial \rho} = \frac{w}{\rho}$, $\frac{\partial \mathbf{\Gamma}_1}{\partial u} = 1$, $\frac{\partial \mathbf{\Gamma}_2}{\partial \rho} = -\frac{w}{\rho}$ and $\frac{\partial \mathbf{\Gamma}_2}{\partial u} = 1$. Thus, the Jacobian of the mapping $(\rho, u) \longrightarrow (\mathbf{\Gamma}_1, \mathbf{\Gamma}_2)$ is $2w(\rho)/\rho$, which is one to one and onto when $\rho > 0$.

When U_r is sufficiently close to U_l, the existence and uniqueness of the solution of Riemann problem for system (3) in the class of elementary waves follow from the general theorem of Lax, which applies to any system of conservation laws that is strictly hyperbolic and genuinely nonlinear in each characteristic field (see [6,7]). For arbitrary data we discuss the existence of the solution of the Riemann problem for the system (3).

We consider the physical variables as coordinate system. We divide the (ρ, u)-plane into four disjoint open regions namely I, II, III and IV; separated by the curves S_1, S_2, R_1, R_2 as well as S_1^*, S_2^*, R_1^*, R_2^* for $a = 0.03$ and $a = 0.0$ respectively; these curves are drawn in Fig. 1 for a given left state U_l. Indeed, we fix U_l and allow U_r to vary; if U_r lies on any of the above four curves, then we have seen how to solve the problem. We thus assume that U_r belongs to one of the four open regions I, II, III and IV as shown in Fig. 1.

As in [9], we define, for $\overline{U} \in \mathbb{R}^+ \times \mathbb{R}$,

$$S_i(\overline{U}) = \{(\rho, u) : (\rho, u) \in S_i(\overline{U})\},$$
$$R_i(\overline{U}) = \{(\rho, u) : (\rho, u) \in R_i(\overline{U})\},$$

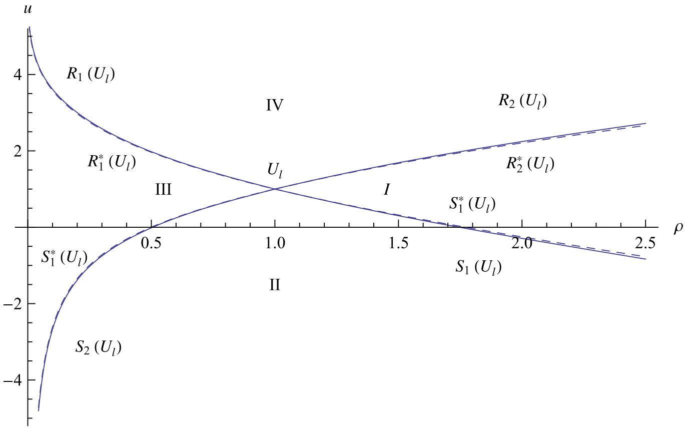

Fig. 1 Wave curves in $\rho - u$ plane for $a = 0.0$ and $a = 0.03$ as dotted and solid line respectively

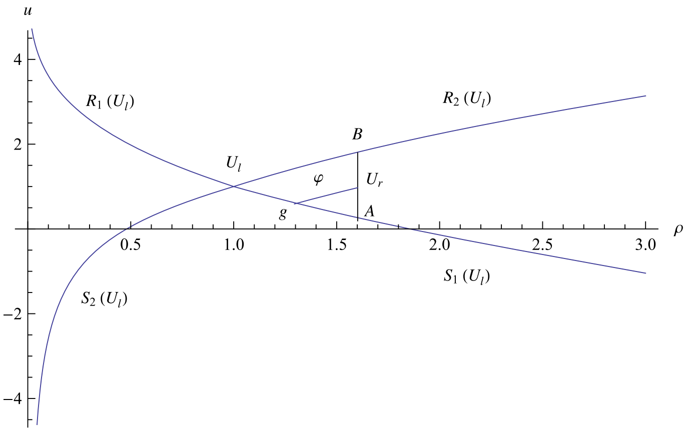

Fig. 2 U_r is in region I

and

$$T_i(\overline{U}) = S_i(\overline{U}) \cup R_i(\overline{U}), i = 1, 2.$$

For fixed $U_l \in \mathbb{R}^+ \times \mathbb{R}$, we consider the family of curves $S = \{T_2(\overline{U}) : \overline{U} \in T_1(U_l)\}$. As the (ρ, u) plane is covered univalently by the family of curves S, i.e., through each point U_r, there passes exactly one curve $T_2(\overline{U})$ of S, the solution to the Riemann problem is given as follows; we connect $\overline{U}$ to U_l on the right by a 1-wave (either shock or rarefaction wave), and then we connect U_r to $\overline{U}$ on the right by a 2-wave (either S_2 or R_2). Indeed, depending on the position of U_r we have different wave configurations.

Theorem 7 *Let $U_l, U_r \in \mathbb{R}^+ \times \mathbb{R}$ with U_l fixed, and U_r is allowed to vary then the Riemann problem is solvable.*

Proof Suppose first that U_r lies in region I. Let the vertical line $\rho = \rho_r$ meet $R_2(U_l)$ at B, and let it meet $S_1(U_l)$ at A, as shown in Fig 2. We observe that the subfamily of curves in S, consisting of the set $\{T_2(\overline{U}) \equiv T_2(\overline{\rho}, \overline{u}) : \rho_l \leq \overline{\rho} \leq \rho_r\}$, induces a continuous mapping $g \longrightarrow \varphi(g)$ from the arc $U_l A$ to the line segment AB, see Smoller [9]; indeed, the region I is covered by curves in S. So, let us suppose that (ρ_m, u_m) is the point which is mapped to U_r. Then

$$u = u_l + \int_{\rho}^{\rho_r} \frac{w(\theta)}{\theta} d\theta - \sqrt{(p - p_l + (B^2 - B_l^2)/2\mu)((1/\rho_l) - (1/\rho))} \tag{28}$$

which on differentiation yields $\frac{du}{d\rho}|_{\rho=\rho_m} < 0$, implying thereby that (ρ_m, u_m) is unique. Similarly, we can prove that uniqueness if U_r is in region II, III and IV.

Thus if $U_r \in I$, then the solution to Riemann problem consists of 1-shock and a 2-rarefaction wave connecting U_l to U_r. Suppose U_r is in region II, then the solution consists of shocks S_1 and S_2 joining U_l to U_r. If $U_r \in III$, then the solution of Riemann problem is obtained by connecting U_l to U_r by R_1, followed by S_2. If U_r is in region IV, then the solution consists of 1-rarefaction wave and 2-rarefaction wave. Thus the set $\{T_2(\overline{U}) : \overline{U} \in T_1(U_l)\}$ covers the region I, II, III and IV in a 1-1 fashion. Therefore, the solution to the Riemann problem is solvable for arbitrary U_r lying in any of the regions I, II, III and IV.

However the vacuum state ($\rho = 0$) does occur in some cases and we have the following result:

Lemma 8 *If* $\mathbf{\Gamma}_1(\rho_l, u_l) - \mathbf{\Gamma}_2(\rho_r, u_r) \leq 0$, *then the vaccum occurs.*

Proof Across 1-rarefaction wave, 1-Riemann invariant is constant, i.e., $\mathbf{\Gamma}_1(\rho_l, u_l) = \mathbf{\Gamma}_1(\rho_m, u_m)$ and similarly across 2-rarefaction wave, 2-Riemann invariant is constant, i.e., $\mathbf{\Gamma}_2(\rho_m, u_m) = \mathbf{\Gamma}_2(\rho_r, u_r)$. So $\mathbf{\Gamma}_1(\rho_m, u_m) - \mathbf{\Gamma}_2(\rho_m, u_m) = \mathbf{\Gamma}_1(\rho_l, u_l) - \mathbf{\Gamma}_2(\rho_r, u_r) \leq 0$. But, $\mathbf{\Gamma}_1(\rho_m, u_m) - \mathbf{\Gamma}_2(\rho_m, u_m) = 2\int^{\rho_m} \frac{w(\theta)}{\theta} d\theta$, which implies that $\rho_m = 0$. Hence the proof.

Numerical Results and Discussions

For a given left state U_l and a right state U_r, we give numerical algorithm to find the unknown state U_m (see table 2) in (x, t) plane.

Case a For $\rho_l < \rho_m$ and $\rho_r \geq \rho_m$, we eliminating u_m from (8) and (22) to obtain

$$u_r - u_l + h(\rho_l, \rho_m) + \int_{\rho_r}^{\rho_m} \frac{w(\theta)}{\theta} d\theta = 0. \tag{29}$$

Case b For $\rho_l \geq \rho_m$ and $\rho_r \geq \rho_m$, we obtain from (21) and (22), that

$$u_r - u_l + \int_{\rho_l}^{\rho_m} \frac{w(\theta)}{\theta} d\theta + \int_{\rho_r}^{\rho_m} \frac{w(\theta)}{\theta} d\theta = 0. \tag{30}$$

Case c For $\rho_l \geq \rho_m$ and $\rho_r < \rho_m$, we eliminating u_m from (21) and (8), we get

$$u_r - u_l + \int_{\rho_r}^{\rho_m} \frac{w(\theta)}{\theta} d\theta + h(\rho_m, \rho_r) = 0. \tag{31}$$

Case d For $\rho_l < \rho_m$ and $\rho_r < \rho_m$, we eliminating u_m from (8),we get

$$u_r - u_l + h(\rho_l, \rho_m) + h(\rho_m, \rho_r) = 0. \tag{32}$$

Thus, for all the four possible wave patterns (29)–(32), we obtain a single nonlinear equation

$$f_r(\rho_m, U_r) + f_l(\rho_m, U_l) + u_r - u_l = 0, \tag{33}$$

where

$$f_l(\rho_m, U_l) = \begin{cases} h(\rho_l, \rho_m) & \text{if } \rho_m > \rho_l, \\ \int_{\rho_l}^{\rho_m} \frac{w(\theta)}{\theta} d\theta & \text{if } \rho_l \geq \rho_m, \end{cases}$$

and

$$f_r(\rho_m, U_r) = \begin{cases} h(\rho_m, \rho_r) & \text{if } \rho_m > \rho_r, \\ \int_{\rho_r}^{\rho_m} \frac{w(\theta)}{\theta} d\theta & \text{if } \rho_r \geq \rho_m. \end{cases}$$

We solve (33) for ρ_m by using Newton–Raphson iterative procedure with a stop criterion when the relative error is less than 10^{-8}; the initial guess for ρ_m is taken to be the average value of ρ_l and ρ_r. Once ρ_m is known, the solution for the particle velocity u_m can be obtained from (8) or (21) (respectively, from (8) or (22) depending on whether the 1-wave (respectively, 2-wave) is a shock or a rarefaction wave, the slope of the characteristic from $(0, 0)$ to (x, t) is

$$\frac{dx}{dt} = \frac{x}{t} = u - w, \tag{34}$$

Table 1 Initial data for Riemann problem

Test	ρ_l	u_l	ρ_r	u_r
1	5.99924	19.5975	5.99242	−6.19633
2	1.0	0.08	10.8	1.11
3	10.0	0.0	1.125	0.0
4	10.0	−2.0	5.40279	2.0

Table 2 Solution of the Riemann problem for $a = 0.0$, $a = 0.015$ and $a = 0.03$ with the initial data from Table 1

Test	$a = 0.0$		$a = 0.015$		$a = 0.03$	
	ρ_m	u_m	ρ_m	u_m	ρ_m	u_m
1	44.31059	6.70499	37.58569	6.70513	26.73968	6.70549
2	3.55414	−2.69535	3.61808	−2.81311	3.71766	−2.97709
3	4.16399	3.03778	4.20448	3.14363	4.27171	3.28749
4	4.04394	1.12198	4.15932	1.17591	4.31459	1.25619

where the particle velocity u and the magneto-acoustic speed w are functions of the unknown ρ. Since $\mathbf{\Gamma}_1$ is constant in 1-rarefaction wave region we have

$$u = u_l + \int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta, \tag{35}$$

which in view of (34) yields

$$u_l + \int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta - \frac{x}{t} - w = 0. \tag{36}$$

Equation (36) is solved for ρ using Newton–Raphson method and then u is found from (35). In a similar way, we find the solution inside the 2-rarefaction wave.

Four Riemann problems are selected to test the performance of numerical scheme highlighting the influence of van der Waals excluded volume, which enters into calculation through the parameter a. From the solution of the Riemann problem, we illustrate some typical wave patterns using MATLAB. Table 1 presents the data for all the four tests in terms of primitive variables. In all these cases, we consider the ratio of specific heat as $\gamma = 1.4$. The solutions of the Riemann problem with the given data (Table 1) for $k_1 = 1.0$, $k_2 = 1.0$, $\mu = 1.0$ and $a = 0.0$, $a = 0.015$, $a = 0.03$ are respectively given in Table 2.

Let us now focus on the influence of the van der Waals excluded volume through the parameter a. The main differences are related to the velocity, wave speed, position and the values of the intermediate states (region of unknown solutions) of each structure.

In test 1, the solution of the Riemann problem consists of a left and right shock waves; the solution profiles at time $t = 0.085$ are shown in Fig. 3. Consequently, as it can be observed looking at density (ρ) that the left shock wave is located at $x = 0.39832$, $x = 0.36179$ and $x = 0.25299$, and the right wave is located at $x = 0.74142$, $x = 0.77794$ and $x = 0.88671$ for $a = 0.0$, $a = 0.015$ and $a = 0.03$, respectively (see Fig. 3). It is observed that the left shock speed decreases and right shock speed increases when the van der Waals excluded volume a increases.

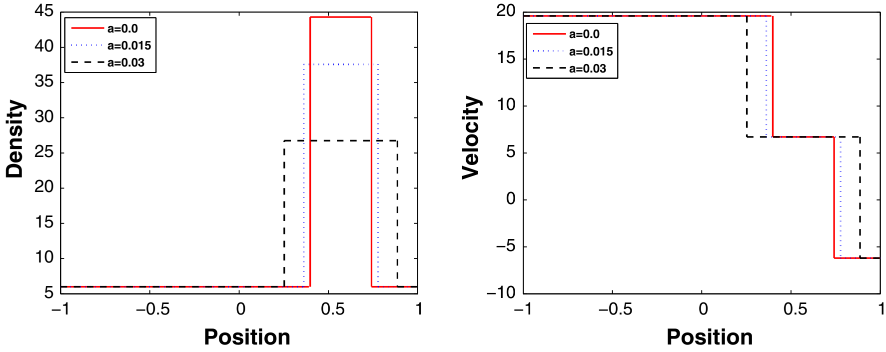

Fig. 3 Test 1: The solution for density and velocity at time $t = 0.085$ for different values of a

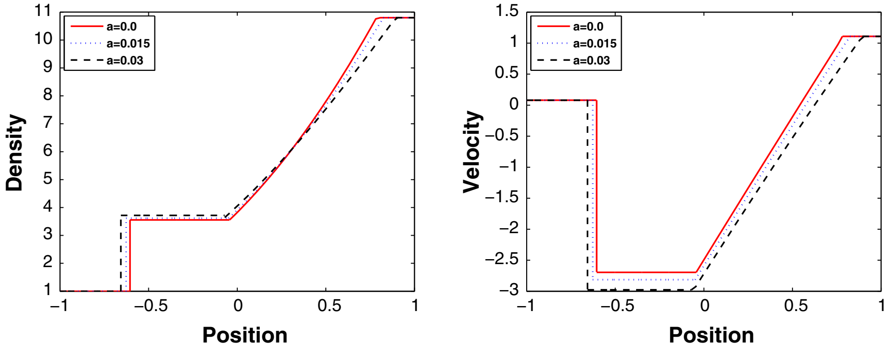

Fig. 4 Test 2: The solution for density and velocity at time $t = 0.16$ for different values of a

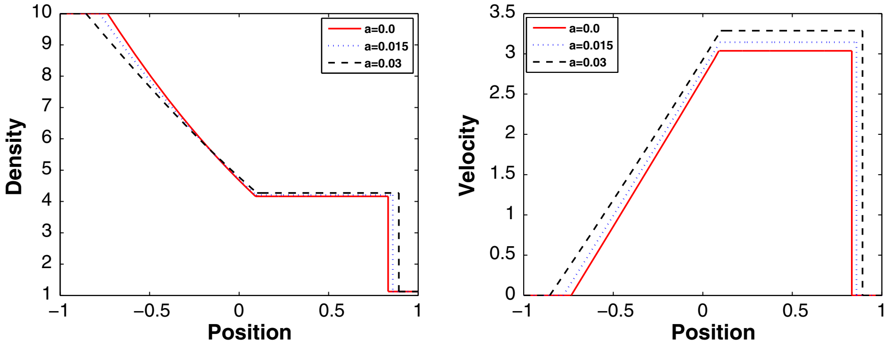

Fig. 5 Test 3: The solution for density and velocity at time $t = 0.20$ for different values of a

In test 2, the solution consists of a left shock wave and right rarefaction wave; the solution profiles at time $t = 0.16$ are shown in Fig. 4. In this case, the velocity decreases when the van der Waals excluded volume a increases. In particular, the intermediate states for density and velocity are higher when a is higher.

In test 3, the solution consists of a left rarefaction wave and a right shock wave; the solution profiles at time $t = 0.20$ are shown in Fig. 5. Let us focus on the position of head of the left rarefaction wave and the right shock wave for density profile. The head of the left rarefaction

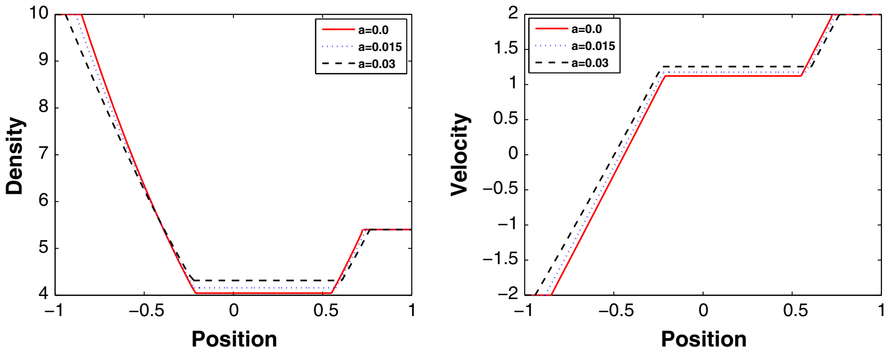

Fig. 6 Test 4: The solution for density and velocity at time $t = 0.15$ for different values of a

wave is located at $x = -0.73529$, $x = -0.77959$ and $x = -0.85504$, and the right shock wave is located at $x = 0.83247$, $x = 0.85841$ and $x = 0.89256$ for $a = 0.0$, $a = 0.015$ and $a = 0.03$. The velocity increases for the van der Waals excluded volume a increases.

In test 4, the solution consists of a left rarefaction wave and right rarefaction wave; the solution profiles at time $t = 0.15$ are shown in Fig. 6. From this test case, we noticed that the intermediate states and velocity increases for the increases of a. For the density profile, the wave speed for right rarefaction wave increases when a increases but decreases for left rarefaction wave.

Interaction of Elementary Waves

The interaction of elementary waves, obtained from the Riemann problem (4), gives rise to new emerging elementary waves. We define the initial function, with two jump discontinuities at x_1 and x_2, as follows:

$$U(x, t_0) = \begin{cases} U_l, & -\infty < x \leq x_1 \\ U_*, & x_1 < x \leq x_2 \\ U_r, & x_2 < x < \infty, \end{cases} \tag{37}$$

with an appropriate choice of U_* and U_r in terms of U_l and arbitrary x_1 and $x_2 \in \mathbb{R}$. With the above initial data, we have two Riemann problems locally. An elementary wave of the first Riemann problem may interact with an elementary wave of the second Riemann problem, and a new Riemann problem is formed at the time of interaction.

Here, we use the notation $S_2R_1 \rightarrow R_1S_2$, which means that a 2-shock wave, S_2, of first Riemann problem (connecting U_l to U_*) interacts with 1-rarefaction, R_1, of second Riemann problem (connecting U_* to U_r), and the interaction leads to a new Riemann problem (connecting U_l to U_r via U_m), the solution of which consists of 1-rarefaction, R_1, and 2-shock wave S_2 (i.e., R_1S_2). The possible interactions of elementary waves belonging to different families are R_2R_1, R_2S_1, S_2R_1 and S_2S_1 while the elementary waves belonging to same families are R_2S_2, S_2R_2, S_1R_1, R_1S_1, S_1S_1 and S_2S_2.

Interaction of Elementary Waves from Different Families

(i) Collision of two shocks (S_2S_1): We consider that U_l is connected to U_* by 2-shock, S_2, of first Riemann problem and U_* is connected to U_r by a 1-shock, S_1, of second Riemann

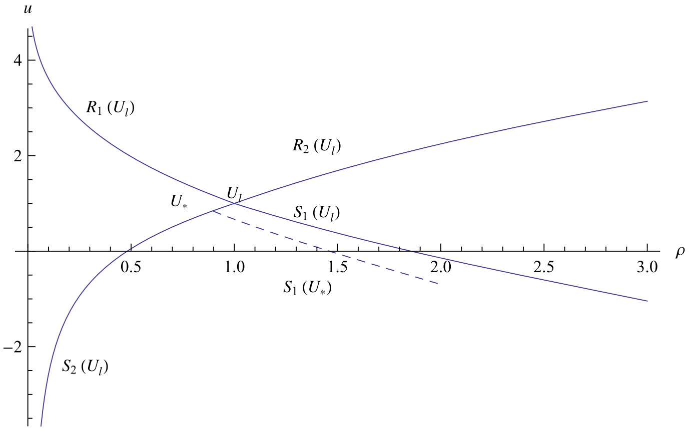

Fig. 7 Collision of S_2S_1

problem. In other words, for a given U_l, we choose U_* and U_r in such a way that $\rho_* < \rho_l, u_* = u_l - h(\rho_l, \rho_*)$ and $\rho_* < \rho_r, u_r = u_* - h(\rho_*, \rho_r)$. Since speed of 2-shock of the first Riemann problem is positive and speed of 1-shock of the second Riemann problem is negative, S_2 overtakes S_1. In order to show that for any arbitrary state U_l, the state U_r lies in the region II (see Fig. 7), it is sufficient to prove that $h(\rho_*, \rho) - h(\rho_l, \rho) + h(\rho_l, \rho_*) > 0$ for $\rho_* < \rho_l$ and $\rho_* < \rho$. Suppose $h(\rho_*, \rho) - h(\rho_l, \rho) + h(\rho_l, \rho_*) \leq 0$. Then $h^2(\rho_l, \rho_*) + h^2(\rho_*, \rho) + 2h(\rho_l, \rho_*)h(\rho_*, \rho) \leq h^2(\rho_l, \rho)$, which implies that

$$\left(p_* + \frac{B_*^2}{2\mu} - p - \frac{B^2}{2\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho_*}\right) + \left(p_* + \frac{B_*^2}{2\mu} - p_l - \frac{B_l^2}{2\mu}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_*}\right) + 2h(\rho_l, \rho_*)h(\rho_*, \rho) \leq 0. \tag{38}$$

But the left hand side of (38) is strictly positive, which leaves us with a contradiction. Hence $h(\rho_*, \rho) - h(\rho_l, \rho) + h(\rho_l, \rho_*) > 0$, i.e., the curve $S_1(U_*)$ lies below the curve $S_1(U_l)$, and therefore U_r lies in the region II. Thus, in view of the result presented in a preceding section, it follows that the interaction result is $S_2S_1 \to S_1S_2$; the computed results illustrate this case in Fig. 7.

(ii) Collision of a shock and a rarefaction (S_2R_1): Here $U_* \in S_2(U_l)$ and $U_r \in R_1(U_*)$. That is, for a given U_l, we choose U_* and U_r such that $\rho_* < \rho_l, u_* = u_l - h(\rho_l, \rho_*)$ and $\rho_r \leq \rho_*, u_r = u_* + \int_{\rho_r}^{\rho_*} \frac{w(\theta)}{\theta} d\theta$. Since 2-shock has positive velocity and 1-rarefaction wave has negative velocity, it follows that S_2 overtakes R_1. Moreover, since for any given U_l, $\int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta - \int_{\rho}^{\rho_*} \frac{w(\theta)}{\theta} d\theta + h(\rho_l, \rho_*) > 0$ for $\rho < \rho_* < \rho_l$, it follows that the curve $R_1(U_*)$ lies below the curve $R_1(U_l)$; hence U_r lies in the region III, and subsequently $S_2R_1 \to R_1S_2$. The computed results illustrate this case in Fig. 8.

(iii) Collision of two rarefaction waves (R_2R_1): We consider $U_* \in R_2(U_l)$ and $U_r \in R_1(U_*)$. In other words, for a given U_l, we choose U_* and U_r such that $\rho_l \leq \rho_*, u_* = u_l + \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta$ and $\rho_r \leq \rho_*, u_r = u_* + \int_{\rho_r}^{\rho_*} \frac{w(\theta)}{\theta} d\theta$. Since the trailing end of 2-rarefaction wave has a positive velocity (bounded above) in (x, t)-plane and that 1-rarefaction wave has a negative velocity (bounded above), interaction will take place. Since $\rho_l < \rho_*$ and $\int_{\rho}^{\rho_*} \frac{w(\theta)}{\theta} d\theta - \int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta + \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta > 0$, it follows that the curve $R_1(U_*)$ lies above the curve $R_1(U_l)$; hence U_r lies

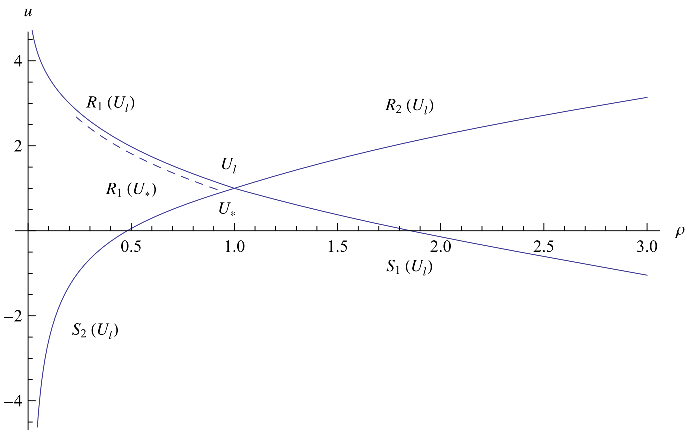

Fig. 8 Collision of $S_2 R_1$

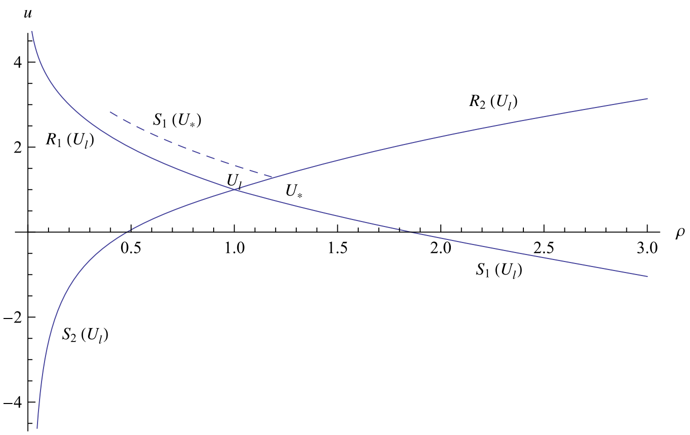

Fig. 9 Collision of $R_2 R_1$

in the region IV and the interaction result is $R_2R_1 \to R_1R_2$. The computed results illustrate this case in Fig. 9.

(iv) Collision of a rarefaction wave and a shock (R_2S_1): Here $U_* \in R_2(U_l)$ and $U_r \in S_1(U_*)$, i.e., for a given U_l, we choose U_* and U_r such that $\rho_l \le \rho_*$, $u_* = u_l + \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta$ and $\rho_* < \rho_r$, $u_r = u_* - h(\rho_*, \rho_r)$. Since 1-shock speed of second Riemann problem is less than the speed of trailing end of 2-rarefaction wave of first Riemann problem in (x, t)-plane, and therefore S_1 penetrates R_2. For any given U_l, we show that U_r lies in the region I; for this, it is enough to show that

$$\int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta + h(\rho_l, \rho) - h(\rho_*, \rho) > 0. \tag{39}$$

Since $h(\rho_l, \rho)$ is a decreasing function with respect to the first variable ρ_l, we have $h(\rho_l, \rho) > h(\rho_*, \rho)$ for $\rho_l < \rho_*$; hence, the inequality (39) follows that the curve $S_1(U_*)$ lies above the curve $S_1(U_l)$, and U_r lies in the region I. Thus the interaction result is $R_2S_1 \to S_1R_2$; the computed results illustrate this case in Fig. 10.

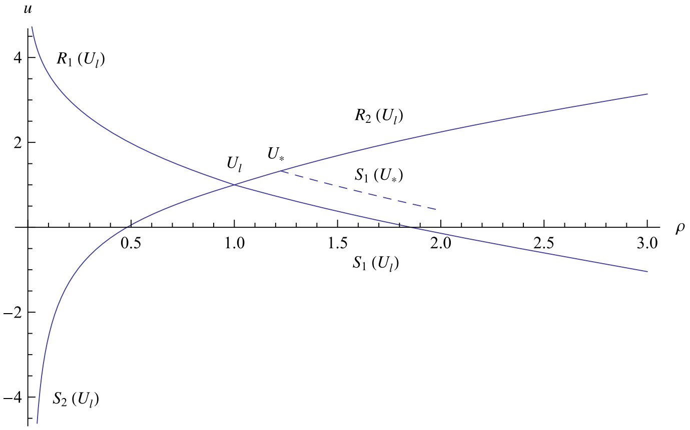

Fig. 10 Collision of R_2S_1

Interaction of Elementary Waves from Same Families

(i) 1-shock wave overtakes another 1-shock wave (S_1S_1): We consider the situation in which U_l is connected to U_* by a 1-shock of first Riemann problem and U_* is connected to U_r by a 1-shock of second Riemann problem. In other words, for a given left state U_l, the intermediate state U_*, and the right state U_r are chosen such that $\rho_l < \rho_*$ and $u_* = u_l - h(\rho_l, \rho_*)$ with Lax stability conditions

$$s_1(U_l, U_*) < \lambda_1(U_l), \quad \lambda_1(U_*) < s_1(U_l, U_*) < \lambda_2(U_*), \tag{40}$$

and $\rho_* < \rho_r$ and $u_r = u_* - h(\rho_*, \rho_r)$ with Lax stability conditions

$$s_1(U_*, U_r) < \lambda_1(U_*), \quad \lambda_1(U_r) < s_1(U_*, U_r) < \lambda_2(U_r), \tag{41}$$

where $s_1(U_l, U_*)$ is the speed of shock connecting U_l to U_*, and similarly $s_1(U_*, U_r)$ is the speed of shock connecting U_* to U_r. From (40) and (41) we obtain $s_1(U_*, U_r) < s_1(U_l, U_*)$, i.e., the 1-shock of second Riemann problem overtakes 1-shock of the first Riemann problem at a finite time, and gives rise to a new Riemann problem with data U_l and U_r. In order to solve this problem, we must determine the region in which U_r lies with respect to U_l. We claim that U_r lies in region I so that the solution of the new Riemann problem consists of S_1 and R_1. In other words, to prove our claim, we need to show that $S_1(U_r)$ lies entirely in the region I; to show this we are required to prove that for $\rho_l < \rho_* < \rho$, $h(\rho_l, \rho) - h(\rho_*, \rho) - h(\rho_l, \rho_*) > 0$. Let us assume on the contrary that $h(\rho_l, \rho) - h(\rho_*, \rho) - h(\rho_l, \rho_*) \leq 0$ for $\rho_l < \rho_* < \rho$. Then, it follows that $h^2(\rho_l, \rho) + h^2\rho_l, \rho_*) - 2h(\rho_l, \rho)h(\rho_l, \rho_*) \leq h^2(\rho_*, \rho)$, which showing thereby that

$$\left[\left(p + \frac{B^2}{2\mu} - p_l - \frac{B_l^2}{2\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho_*}\right) - \left(p_* + \frac{B_*^2}{2\mu} - p_l - \frac{B_l^2}{2\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho}\right)\right]^2 \leq 0, \tag{42}$$

which is a contradiction as the left hand side of inequality (42) is positive. Hence, $S_1S_1 \rightarrow S_1R_2$; the computed results illustrate this situation in Fig. 11.

(ii) 2-shock wave overtakes another 2-shock wave (S_2S_2): The analytical proof that U_r lies in the region III, so that $S_2S_2 \rightarrow R_1S_2$, is similar to the previous case (see Fig. 12).

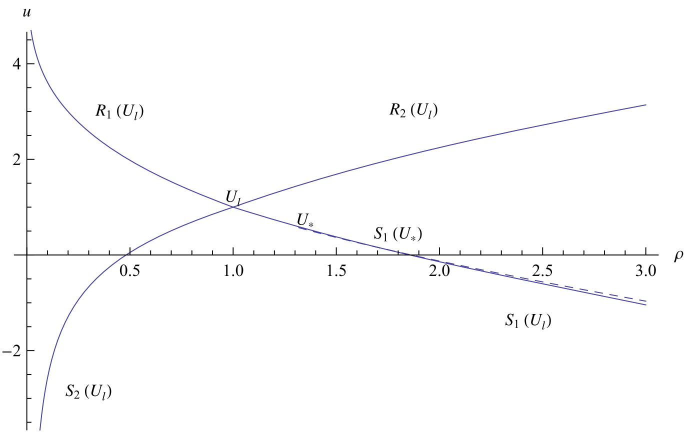

Fig. 11 S_1 overtakes S_1

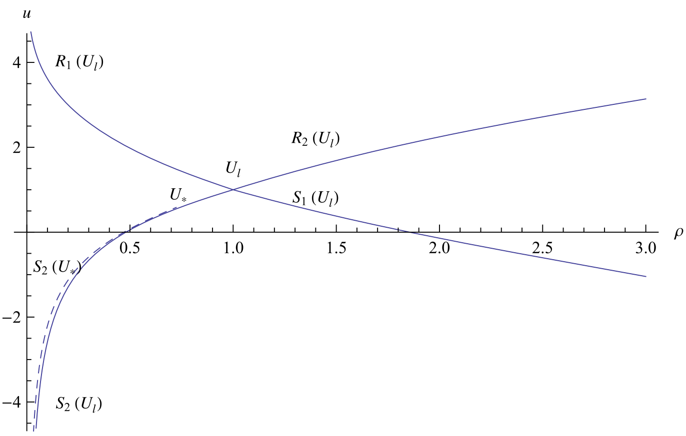

Fig. 12 S_2 overtakes S_2

(iii) 1-rarefaction wave overtakes 1-shock wave ($R_1 S_1$): In this case, U_l is connected to U_* by 1-rarefaction wave of the first Riemann problem and U_* is connected to U_r by a 1-shock of second Riemann problem. That is, for a given U_l, we choose U_* and U_r in such a way that $\rho_* \leq \rho_l, u_* = u_l + \int_{\rho_*}^{\rho_l} \frac{w(\theta)}{\theta} d\theta$ and $\rho_* < \rho_r, u_r = u_* - h(\rho_*, \rho_r)$. First we show that $S_1(U_*)$ lies below the curve $R_1(U_l)$ for $\rho_* < \rho \leq \rho_l$; in other words, for $\rho_* < \rho \leq \rho_l$

$$h(\rho_*, \rho) + \int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta - \int_{\rho_*}^{\rho_l} \frac{w(\theta)}{\theta} d\theta > 0.$$

Let us define $F_1(\rho) = h(\rho_*, \rho) + \int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta - \int_{\rho_*}^{\rho_l} \frac{w(\theta)}{\theta} d\theta$, so that $F_1(\rho_*) = 0$. Differentiating $F_1(\rho)$ with respect to ρ, we obtain $F_1'(\rho) > 0$, implying thereby that $F_1(\rho_*) < F_1(\rho)$ i.e., $F_1(\rho) > 0$; hence $S_1(U_*)$ lies below the curve $R_1(U_l)$ for $\rho_* < \rho \leq \rho_l$.

Next we prove that $S_1(U_l)$ lies above the curve $S_1(U_*)$ for $\rho_l \leq \rho$; for this it is sufficient to prove that

$$h(\rho_*, \rho) - h(\rho_l, \rho) - \int_{\rho_*}^{\rho_l} \frac{w(\theta)}{\theta} d\theta > 0,$$

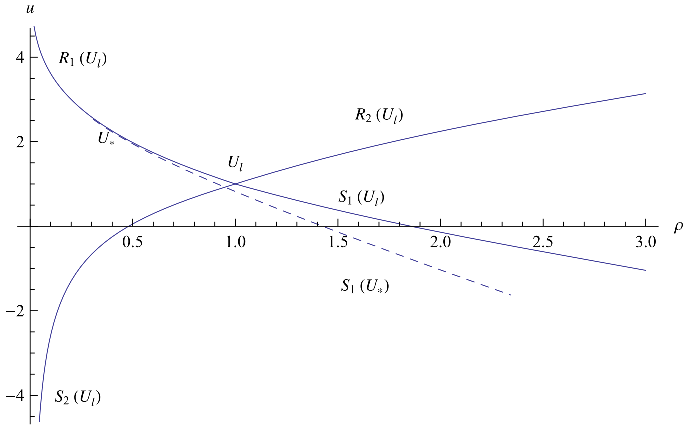

Fig. 13 R_1 overtakes S_1

for $\rho_l \le \rho$. Let us define $F_2(\rho) = h(\rho_*, \rho) - h(\rho_l, \rho) - \int_{\rho_*}^{\rho_l} \frac{w(\theta)}{\theta} d\theta$. Let us assume that $h(\rho_*, \rho) - h(\rho_*, \rho_l) \le h(\rho_l, \rho)$ for $\rho_* < \rho_l < \rho$, which implies that $h^2(\rho_*, \rho) + h^2(\rho_*, \rho_l) - 2h(\rho_*, \rho)h(\rho_*, \rho_l) \le h^2(\rho_l, \rho)$, implying thereby that $\left(p_l - p_* + \frac{B_l^2 - B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho}\right) + \left(p - p_* + \frac{B^2 - B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho_l}\right) \le 2h(\rho_*, \rho)h(\rho_*, \rho_l)$; or equivalently

$$\left[\left(p_l - p_* + \frac{B_l^2 - B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho}\right) - \left(p - p_* + \frac{B^2 - B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho_l}\right)\right]^2 \le 0. \tag{43}$$

But the left hand side of inequality (43) is positive, which leaves us with a contradiction. Hence, $h(\rho_*, \rho) - h(\rho_l, \rho) > h(\rho_*, \rho_l)$ for $\rho_* < \rho_l < \rho$, which implies that $h(\rho_*, \rho) - h(\rho_l, \rho) - \int_{\rho_*}^{\rho_l} \frac{w(\theta)}{\theta} d\theta > h(\rho_*, \rho_l) - \int_{\rho_*}^{\rho_l} \frac{w(\theta)}{\theta} d\theta = F_2(\rho_l) > 0$.

Lastly, we show that $S_2(U_l)$ and $S_1(U_*)$ intersect at some point $(\tilde{\rho}_1, \tilde{u}_1)$, where $\rho_* < \tilde{\rho}_1 < \rho_l$. To prove this, we define a new function $F_3(\rho) = h(\rho_*, \rho) - h(\rho_l, \rho) - \int_{\rho_*}^{\rho_l} \frac{w(\theta)}{\theta} d\theta$ for $\rho_* \le \rho \le \rho_l$. Since $F_3(\rho_*) < 0$, by virtue of monotonicity and intermediate value property, there exists a $\tilde{\rho}_1$ between ρ_* and ρ_l, such that $F_3(\tilde{\rho}_1) = 0$. Thus, the intersection of $S_2(U_l)$ and $S_1(U_*)$ is uniquely determined; the computed results are shown in Fig. 13. Depending on the value of ρ_r we distinguish three cases.

(a) When $\rho_r < \tilde{\rho}_1, U_r \in III$ and the interaction result is $R_1S_1 \to R_1S_2$; indeed, 1-shock is weak compared to 1-rarefaction wave.
(b) When $\rho_r = \tilde{\rho}_1, U_r$ lies on $S_2(U_l)$ and the interaction result is $R_1S_1 \to S_2$; indeed, when two waves of first family interact, they annihilate each other, and give rise to a wave of second family.
(c) When $\rho_r > \tilde{\rho}_1, U_r \in II$ and the interaction result is $R_1S_1 \to S_1S_2$; indeed, the 1-shock of second Riemann problem, which is strong compared to the 1-rarefaction of first Riemann problem, overtakes the trailing end of 1-rarefaction wave, and a reflected shock $S_2(U_m, U_r)$, connecting a new constant state U_m on the left to U_r on the right, is produced. The transmitted wave, after interaction, is the 1-shock that joins U_l on the left and U_m on the right.

(IV) 1-shock wave overtakes 1-rarefaction wave (S_1R_1): Here $U_* \in S_1(U_l)$ and $U_r \in R_1(U_*)$. That is, for a given U_l, we choose U_* and U_r in such a way that $\rho_l < \rho_*, u_* =$

$u_l - h(\rho_l, \rho_*)$ and $\rho_r \leq \rho_*, u_r = u_* + \int_{\rho_r}^{\rho_*} \frac{w(\theta)}{\theta} d\theta$. In the (x, t) plane the speed of trailing end of 1-rarefaction wave, $\lambda_1(U_*)$, is less than 1-shock speed $s_1(U_l, U_*)$ and therefore 1-shock from left overtakes 1-rarefaction wave from right after a finite time. First we show that $R_1(U_*)$ lies below the curve $S_1(U_l)$ for $\rho_l \leq \rho \leq \rho_*$; for this we need to show $h(\rho_l, \rho_*) - h(\rho_l, \rho) - \int_{\rho}^{\rho_*} \frac{w(\theta)}{\theta} d\theta > 0$ for $\rho_l \leq \rho < \rho_*$. To prove this, we define a new function $G_1(\rho) = h(\rho_l, \rho_*) - h(\rho_l, \rho) - \int_{\rho}^{\rho_*} \frac{w(\theta)}{\theta} d\theta > 0$ for $\rho_l \leq \rho < \rho_*$. This, in view of the expression for $\psi(\theta)$ and $h(\rho_l, \rho)$, yields

$$G_1'(\rho) = -\frac{\left[\left(p - p_l + \frac{B^2 - B_l^2}{2\mu}\right)\frac{1}{\rho^2} - \left(p' + \frac{BB'}{\mu}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho}\right)\right]^2}{2h(\rho_l, \rho)} < 0,$$

which implies that $G_1(\rho) > G_1(\rho_*)$; but since $G_1(\rho_*) = 0$, we have $G_1(\rho) > 0$.

Next we show that $R_1(U_l)$ lies above the curve $R_1(U_*)$ for $\rho \leq \rho_l < \rho_*$, i.e., $g(\rho_l, \rho_*) + \int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta - \int_{\rho}^{\rho_*} \frac{w(\theta)}{\theta} d\theta > 0$ for $\rho_l \leq \rho < \rho_*$.

Since the left hand side of this inequality, $\rho \leq \rho_l < \rho_*$, turns out to be $G_1(\rho_l)$, which has already been shown to be positive, the conclusion follows.

Lastly, we show that $R_1(U_*)$ and $S_2(U_l)$ intersect uniquely at some point, say, $(\tilde{\rho_2}, \tilde{u_2})$; for this, it is enough to show that the equation $h(\rho_l, \rho_*) - h(\rho_l, \rho) - \int_{\rho}^{\rho_*} \frac{w(\theta)}{\theta} d\theta = 0$ has unique root $\tilde{\rho_2}$ such that $\tilde{\rho_2} < \rho_l$. To establish this, we define a new function $G_2(\rho) = h(\rho_l, \rho_*) - h(\rho_l, \rho) - \int_{\rho}^{\rho_*} \frac{w(\theta)}{\theta} d\theta$; since $G_2(\rho_l) > 0$, and $G_2(\rho)$ takes negative value as ρ is close to zero, in view of monotonicity and intermediate value property, it follows that the curves $R_1(U_*)$ and $S_2(U_l)$ intersect uniquely; here again we distinguish three cases depending on the value of ρ_r.

(a) When $\rho_r > \tilde{\rho_2}, U_r \in II$ and the interaction result is $S_1R_1 \to S_1S_2$; indeed, the 1-shock is sufficiently strong compared to 1-rarefaction wave which, after interaction, produces a new elementary wave.
(b) When $\rho_r = \tilde{\rho_2}, U_r \in S_2(U_l)$ and the interaction result is $S_1R_1 \to S_2$. The interaction of elementary waves of first family gives rise to a new elementary wave of second family.
(c) When $\rho_r < \tilde{\rho_2}, U_r \in III$ and the interaction result is $S_1R_1 \to R_1S_2$. The computed results shown in Fig. 14.

(V) 2-shock wave overtakes 2-rarefaction wave (S_2R_2): The S_2R_2 interaction takes place when $U_* \in S_2(U_l)$ and $U_r \in R_2(U_*)$. In other words, for a given U_l, we choose U_* and U_r in such a way that $\rho_* < \rho_l, u_* = u_l - h(\rho_l, \rho_*)$ and $\rho_* \leq \rho_r, u_r = u_* + \int_{\rho_*}^{\rho_r} \frac{w(\theta)}{\theta} d\theta$.

First we show that for $\rho_* < \rho \leq \rho_l$, $S_2(U_l)$ lies above $R_2(U_*)$, i.e.,

$$h(\rho_l, \rho_*) - h(\rho_l, \rho) - \int_{\rho_*}^{\rho} \frac{w(\theta)}{\theta} d\theta > 0,$$

for $\rho_* \leq \rho_l$. To prove this we define a new function $M_1(\rho) = h(\rho_l, \rho_*) - h(\rho_l, \rho) - \int_{\rho_*}^{\rho} \frac{w(\theta)}{\theta} d\theta > 0$. Since $M_1'(\rho) > 0$, we have $M_1(\rho) > M_1(\rho_*)$; further since $M_1(\rho_*) = 0$, it follows that $M_1(\rho) > 0$, implies thereby that $S_2(U_l)$ lies above $R_2(U_*)$ for $\rho_* < \rho \leq \rho_l$.

Next we show that the curve $R_2(U_l)$ lies above the curve $R_2(U_*)$ for $\rho_* < \rho_l \leq \rho$; for this it is enough to prove $h(\rho_l, \rho_*) - \int_{\rho_*}^{\rho} \frac{w(\theta)}{\theta} d\theta + \int_{\rho_l}^{\rho} \frac{w(\theta)}{\theta} d\theta > 0$ for $\rho_* < \rho_l \leq \rho$. We notice that the left hand side of this inequality is $M_1(\rho_l)$ which has already been shown to be positive, and hence the curve $R_2(U_l)$ lies above $R_2(U_*)$ for $\rho_* < \rho_l \leq \rho$.

Lastly, we show that $R_2(U_*)$ and $S_1(U_l)$ intersect uniquely, say, at $(\tilde{\rho_3}, \tilde{u_3})$ for $\rho_* < \rho_l < \tilde{\rho_3}$.

Now we define $M_2(\rho) = h(\rho_l, \rho) - h(\rho_l, \rho_*) + \int_{\rho_*}^{\rho} \frac{w(\theta)}{\theta} d\theta$ for $\rho_* < \rho_l \leq \rho$ so that $M_2(\rho_l) < 0$, and we can choose a constant $k > 0$ such that $M_2(\rho) > 0$ for all $\rho > k$. Then,

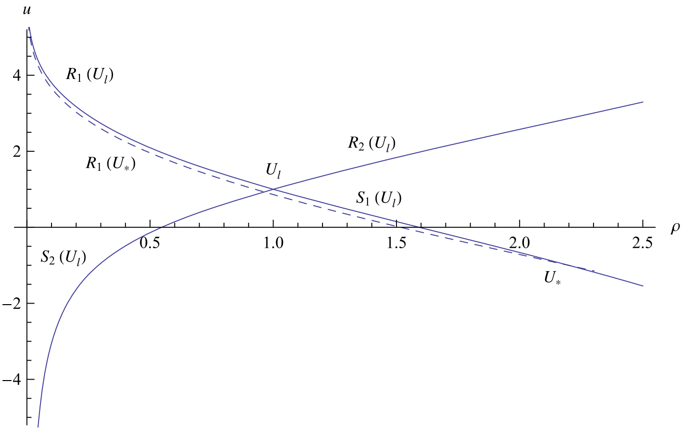

Fig. 14 S_1 overtakes R_1

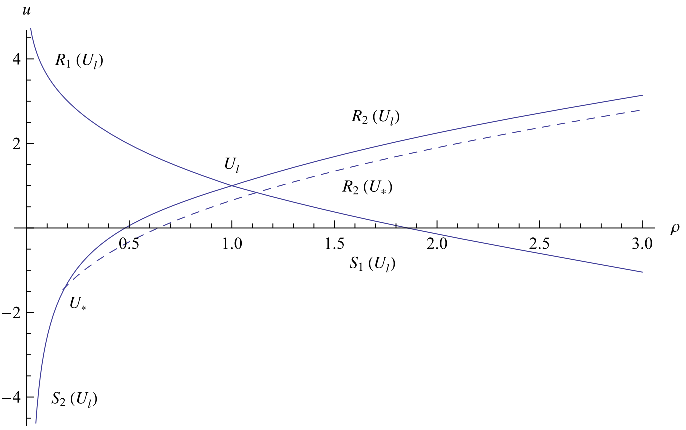

Fig. 15 S_2 overtakes R_2

there exits a $\tilde{\rho_3}$ such that $M_2(\tilde{\rho_3}) = 0$. Thus $R_2(U_*)$ and $S_1(U_l)$ intersect uniquely at $(\tilde{\rho_3}, \tilde{u_3})$ as $R_2(U_*)$ and $S_1(U_l)$ are monotone; the computed results are shown in Fig. 15. Here again the following cases arise.

(a) When $\rho_r < \tilde{\rho_3}$, $U_r \in II$ and the interaction result is $S_2R_2 \to S_1S_2$; indeed, the strength of R_2 is small computed to the elementary wave S_2, and S_2 annihilates R_2 in a finite time. The strength of reflected S_1 wave is small compared to the incident waves S_2 and R_2.
(b) When $\rho_r = \tilde{\rho_3}$, U_r lies on $S_1(U_l)$ and the interaction result is $S_2R_2 \to S_1$. The reflected shock S_1 is weak computed to incident waves S_2 and R_2.
(c) When $\rho_r > \tilde{\rho_3}$, $U_r \in I$ and the interaction result is $S_2R_2 \to S_1R_2$; indeed, R_2 is stronger than S_2.

(vi) 2-rarefaction wave overtakes 2-shock wave (R_2S_2): Here $U_* \in R_2(U_l)$ and $U_r \in S_2(U_*)$. That is, for a given U_l, we choose U_* and U_r such that $\rho_l \leq \rho_*$, $u_* = u_l + \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta$ and $\rho_r < \rho_*$, $u_r = u_* - h(\rho_*, \rho_r)$. Now we show that $R_2(U_l)$ lies above $S_2(U_*)$ for $\rho_l \leq \rho < \rho_*$, i.e.,

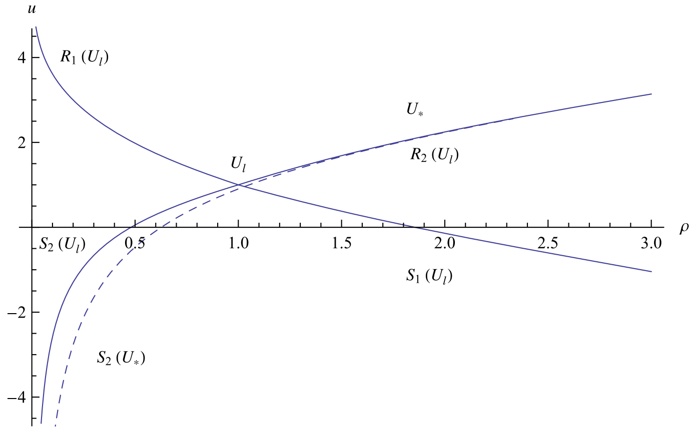

Fig. 16 R_2 overtakes S_2

$$h(\rho_*, \rho) + \int_{\rho_l}^{\rho} \frac{w(\theta)}{\theta} d\theta - \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta > 0,$$

for $\rho_l \leq \rho < \rho_*$. To prove this we define a new function $N_1(\rho) = h(\rho_*, \rho) + \int_{\rho_l}^{\rho} \frac{w(\theta)}{\theta} d\theta - \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta$ for $\rho_l \leq \rho \leq \rho_*$ so that $N_1(\rho_*) = 0$. This, in view of the expressions for $w(\theta)$ and $h(\rho_*, \rho)$, yields

$$N_1'(\rho) = -\frac{\left[\left(p' + \frac{BB'}{\mu}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_*}\right) - \left(p_* - p + \frac{B_*^2 - B^2}{2\mu}\right)\frac{1}{\rho^2}\right]^2}{2h(\rho_*, \rho)} < 0,$$

implying thereby that $N_1(\rho) > N_1(\rho_*) = 0$. Hence, the result.

Next we show that $S_2(U_*)$ lies below the curve $S_2(U_l)$ for $\rho \leq \rho_l < \rho_*$; for this it is sufficient to prove $h(\rho_*, \rho) - h(\rho_l, \rho) - \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta > 0$ for $\rho \leq \rho_l < \rho_*$. If $h(\rho_*, \rho) - h(\rho_l, \rho) > h(\rho_*, \rho_l)$ then $h(\rho_*, \rho) - h(\rho_l, \rho) - \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta > h(\rho_*, \rho_l) - \int_{\rho_l}^{\rho_*} \frac{w(\theta)}{\theta} d\theta = N_1(\rho_l) > 0$.

Let us assume on the contrary that $h(\rho_*, \rho) - h(\rho_l, \rho) \leq h(\rho_*, \rho_l)$. Then, it follows that $h(\rho_*, \rho) - h(\rho_*, \rho_l) \leq h(\rho_l, \rho)$, implying thereby that $h^2(\rho_*, \rho) + h^2(\rho_*, \rho_l) - 2h(\rho_*, \rho)h(\rho_*, \rho_l) \leq h^2(\rho_l, \rho)$; this, in view of the expressions for $h(\rho_*, \rho), h(\rho_*, \rho_l)$ and $h(\rho_l, \rho)$ yields $\left(p_l - p_* + \frac{B_l^2 - B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho}\right) + \left(p - p_* + \frac{B^2 - B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho_l}\right) \leq 2h(\rho_*, \rho)h(\rho_*, \rho_l)$, or equivalently

$$\left[\left(p_l - p_* + \frac{B_l^2 - B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho}\right) - \left(p - p_* + \frac{B^2 - B_*^2}{2\mu}\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho_l}\right)\right]^2 \leq 0. \tag{44}$$

But the left hand side of (44) is positive for $\rho \leq \rho_l < \rho_*$, which leaves us with a contradiction. Hence, $h(\rho_*, \rho) - h(\rho_l, \rho) > h(\rho_*, \rho_l)$ for $\rho \leq \rho_l < \rho_*$.

Lastly, we show that $S_2(U_*)$ and $S_1(U_l)$ intersect uniquely at a point, $(\tilde{\rho_4}, \tilde{u_4})$ for $\rho_l < \tilde{\rho_4} < \rho_*$. The proof for this follows on similar lines as discussed earlier; here also we encounter three possibilities.

(a) When $\rho_r > \tilde{\rho_4}$, $U_r \in I$ and the interaction result is $R_2 S_2 \to S_1 R_2$; indeed, R_2 is strong compared to the elementary wave S_2, and the strength of reflected S_1 is small compared to the incident waves S_2 and R_2.
(b) When $\rho_r = \tilde{\rho_4}$, $U_r \in S_1(U_l)$ and the interaction result is $R_2 S_2 \to S_1$.
(c) When $\rho_r < \tilde{\rho_4}$, $U_r \in II$ and the interaction result is $R_2 S_2 \to S_1 S_2$; indeed,the elementary wave S_2 is strong compared to R_2. The computed results illustrate this case in Fig. 16.

Conclusion

The Riemann problem for the one-dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field, is solved, at any point (x, t) in the relevant domain of interest $x_l < x < x_r; t > 0$, with $x_l < 0$ and $x_r > 0$ under the influence of van der Waals excluded volume through the parameter a. It has been observed that the position, velocity, shock speed and the values of the intermediate states of each structure in the flow, i.e., rarefaction wave and shock wave, is strongly influenced by the van der Waals excluded volume a. It is observed that the left shock speed decreases and right shock speed increases when the van der Waals excluded volume a increases in test 1. In test 2, the velocity decreases when the van der Waals excluded volume a increases and but the velocity increases for test 3. It is noticed that for the density profile of test 4, the wave speed for right rarefaction wave increases when a increases but decreases for left rarefaction wave. We have discussed all possible wave interactions of the Riemann problem. We will extend this analytical procedure to construct the exact solution and wave interactions to the case of non-isentropic flows in future.

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