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ORIGINAL PAPER

A Novel Application of the Classical Banach Fixed Point Theorem

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Abstract Using the classical Banach fixed point theorem, we propose a novel method to obtain existence and uniqueness result pertaining to the solutions of semilinear elliptic partial differential equation of the type $\Delta u + f(x, u, Du) = 0$, in $\Omega \subset \mathbb{R}^n$ and $u|_{\partial\Omega} = 0$, in a suitable Sobolev space. Here $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is either a linear or a non-linear Lipshitz continuous function. The approach attempted here can be used as an algorithm by the numerical analysts to determine a solution to a partial differential equation of the above type.

Keywords Fixed point · Contraction map · Sobolev space · Elliptic PDE · Fundamental solution

Introduction

Elliptic partial differential equations have become a major case of study since a very long time owing to its applications in many physical and engineering problems ([1,2] and the references therein). It is also today one of the richly enhanced field of research in Mathematics. The applications of elliptic partial differential equations are almost unrestricted spreading across fields like Fluid Mechanics, Electro-magnetics, Biological systems and finance ([3–8] and the references therein). Though many a times, one would attempt analytical solutions, the real life problems may strictly demand numerical treatment. However, before investigating a numerical solution, it is customary from Mathematician's perspective to show the existence and uniqueness of solution to a given boundary value problem. This process may take one to situations where classical solutions are not supported and invites some restrictions on the regularity of the solution. Many classical theories like the Riesz representation theorem [9], Lax–Milgram theorem [10], various fixed point methods [11,12] etc. have been extensively used to establish the existence and uniqueness results pertaining to the solution of elliptic

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partial differential equations. There are also a few number of classical texts and lecture notes [13–15,18] and the references therein, which give a detailed exposition to the study of elliptic Partial differential equations. One important class of an elliptic boundary value problem is

$$\Delta u + f(x, u) = 0, x \in \mathbb{R}^n, n \geq 1$$
$$u|_{\partial\Omega} = g. \tag{1}$$

An early evidence of existence and uniqueness result pertaining to the problem in (1) was established by Picard [19], for the two-dimensional case. Such a type of problems are extensively used in the study of convection diffusion limited processes dealing with heat and mass transfer. However, we will not discuss anything about this physical phenomena in this paper. Considered in the context of partial differential equations ($n \geq 1$), the above equation has been a subject of great study. Lair and Shaker [20,21] considered the following problem,

$$\Delta u + p(x)f(u) = 0, \quad \forall x \in \Omega \subseteq \mathbb{R}^n,$$
$$u|_{\partial\Omega} = 0, \tag{2}$$

where $f(u)$ is expected to satisfy one of the following three conditions—(i) $f^{'}(s) \leq 0$, (ii) $f(s) > 0$, for $s > 0$, (iii) $\int_0^\epsilon f(s)ds < \infty$ for some $\epsilon > 0$, and showed that the problem in (2) has a unique positive solution in $H_0^{1,2}(\Omega)$, if the function p is non-trivial, non-negative, L^2 function. We shall give the description of the spaces later and please refer the same for details. The following semilinear elliptic equation was considered by Barroso [22]

$$-\Delta u + \lambda u = f(x, u, \mu), \quad \forall x \in \Omega,$$
$$u|_{\partial\Omega} = 0, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 3$ with $C^{1,1}$ boundary, λ is a parameter close to zero, f is a specific Caratheodory function and μ is a non-zero measure. The existence and uniqueness results for this depend on the behaviour of f. Barroso [22] has shown existence and uniqueness using a variant of Krasnoselskiis fixed point theorem. Oloffson [23] considered

$$-\Delta u + f(u)\mu = 0, \quad \forall x \in \Omega, \tag{4}$$

where $\Omega \subseteq \mathbb{R}^n, n \geq 2$. The general solution of the above elliptic equation was considered under relaxed regularity assumptions on Ω, μ and f.

In general, the two methods that have been extensively used are the Schauder's fixed point theorem and the Barrier method. It was shown by Sattinger [25] that if ϕ_1, ϕ_2 be an upper and lower solution of the problem in (1), respectively, f is Hölder continuous in Ω, $\phi_1 \geq \phi_2$, then there exists a solution u such that $\phi_1(x) \geq u(x) \geq \phi_2(x)$, for every x in Ω. Other notable studies on semilinear elliptic PDEs can be found in [22–24] and the references therein. We refer here to the monograph on applications of contraction mapping principle by Brooks et. al. [26] wherein, the following general version was introduced

$$-\Delta u = f(x, u, Du), \tag{5}$$

where $u \in H_0^{1,2}(\Omega)$. This problem has been reduced to an equivalent fixed point problem in $L^2(\Omega)$, whose unique fixed point has been shown as the unique solution of the above equation. However, the corresponding estimate is $\frac{L_1}{\lambda_1} + \frac{L_2}{\sqrt{\lambda_1}} < 1$ where the constants L_1 and L_2 are such that

$$|f(x, u, Du)| \leq |f(x, 0, 0)| + L_1|u| + L_2|Du|$$

and λ_1^2 is the first and the smallest eigenvalue of $-\Delta$ on $H_0^{1,2}(\Omega)$. We derive motivation from the above [26] and consider the following problem

$$\Delta u + f(x, u, Du) = 0, \quad u|_{\partial\Omega} = 0, \tag{6}$$

Ω is a bounded domain in $\mathbb{R}^n$, $f(x, ., .)$ is a Lipshitz continuous function which satisfies the condition

$$|f(x, y_1, z_1) - f(x, y_2, z_2)| \leq K|y_1 - y_2| + L|z_1 - z_2|$$

for every pair $(x, y_1, z_1), (x, y_2, z_2) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $f(., u(.), Du(.))$ is an $L^2(\Omega)$ and $Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$.

Description of the Spaces

We begin by defining the derivative of a function in $L^p(\Omega)$ in the weak sense.
Weak derivative A function in $L^p(\Omega)$ is said to be weakly differentiable if there exists $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega u D^\beta \psi dx = (-1)^{|\beta|} \int_\Omega v \psi dx \quad \forall \psi \in C_0^\infty(\Omega).$$

In general, Sobolev spaces are defined as follows.

$$H^{k,p}(\Omega) = \{u \in L^p(\Omega) \colon D^\beta u \in L^p(\Omega), |\beta| \leq k\},$$

with norm $||u||_{k,p} = \begin{cases} \left(\sum_{|\beta|\leq k} ||D^\beta u||^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sum_{|\beta|\leq k} ||D^\beta u||_{L^\infty(\Omega)}, & p = \infty \end{cases}$ where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, $\beta_i \in \mathbb{N} \bigcup \{0\}$ for $1 \leq i \leq n$, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. Thus $D^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$. Here $D^\beta u$ are the derivatives of u in the weak sense. However, this space gives no information about the behaviour of its functions on the boundary $\partial\Omega$ which forms an integral part of the Dirichlet (Neumann and Robin) boundary value problems. Hence to tackle the boundary value problems one appeals to the trace theorem [13,16,17] to extend the functions in $H^{k,p}(\Omega)$, which is defined in Ω, to the boundary $\partial\Omega$. One important class of boundary condition is the vanishing of the function on the boundary, i.e., $u|_{\partial\Omega} = 0$. In this paper, the space in which we seek for a solution is defined as

$$H_0^{1,2}(\Omega) = \left\{u \in H^{1,2}(\Omega) \colon u|_{\partial\Omega} = 0\right\}.$$

This definition of the space preserves the completeness of the space with respect to the norm $||..||_{1,2}$ (for a proof, refer [13]). In fact $H_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Throughout the paper the derivative of u, Du, will be treated as a weak derivative.

Mathematical Approach

In this section we will prove the existence of unique solution to the problem in Eq. (6). We know from the Malgrange–Ehrenpries theorem [27,28] that there exists a *fundamental* *solution* to a linear differential operator with constant coefficients. Hence, from the considerations in the problem, there exists a fundamental solution to the operator $-\Delta$, say $F(x)$ where $F(x) \in H^{1,1}_{loc}(\mathbb{R}^n)$ (refer "Appendix"), i.e., $-\Delta F(x) = \delta_y(x)$, where δ is the Dirac *distribution*. This simple yet powerful result will be used here except that the fundamental solution will be replaced with the Green's function [13]—denoted by $G(.,.)$—satisfying the boundary condition. Using this result, we restate the problem, so as to find a fixed point to the operator defined on the right hand side of the following equation

$$\begin{aligned} u &= -\int_{\Omega} G(x,y).f(y,u(y),Du(y))dy, x \in \Omega \\ &= T(u) \text{ (say)}. \end{aligned} \tag{7}$$

Observe that if there exists a solution to (7), say u, then

$$\begin{aligned} \Delta u &= -\int_{\Omega} \Delta G(x,y) f(y,u,Du)dy, \\ &= -\int_{\Omega} \delta_x(y) f(y,u,Du)dy, \\ &= -f(x,u,Du), \text{ a.e..} \end{aligned} \tag{8}$$

We use the idea due to Zhao [29] to represent a Green's function defined over an arbitrary domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$. The representation is as follows.

$$\begin{aligned} G(x,y) &= |x-y|^{2-n} \min\left\{1, \frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right\}, \\ &= |x-y|^2 \lambda(x,y) \text{ (say)}, \end{aligned} \tag{9}$$

where $d(w,\partial\Omega) = \inf\{|w-z| \colon z \in \partial\Omega\}$. It is easy to check that the function $\lambda(x,.)$ is in $C_0^\infty(\Omega)$ for a fixed $x \in \Omega$ (or a fixed $y \in \Omega$) except possibly on a set of measure zero. We will now prove that the operator T which is linear if f is linear and non-linear if f is non-linear, is a contraction map on $X = H_0^{1,2}(\Omega)$ where we will use the $H_0^{1,2}$-norm defined by $||u||_{1,2} = \left(\int_\Omega |Du|^2 dx\right)^{\frac{1}{2}}$.

$$\begin{aligned} ||Tu - Tv||_{1,2} &= \left[\int_{\Omega_x} \left|\int_{\Omega_y} D_x G(x,y)(f(y,u,Du) - f(y,v,Dv))dy\right|^2 dx\right]^{1/2}, \\ &\le \left[\int_{\Omega_x} \left(\int_{\Omega_y} |D_x G(x,y)||(f(y,u,Du) - f(y,v,Dv))|dy\right)^2 dx\right]^{1/2}, \\ &= \left[\int_{\Omega_x} \left(\int_{\Omega_y} \left||x-y|^{2-n} D_x\lambda(x,y) + (2-n)|x-y|^{1-n}\lambda(x,y)\frac{(x-y)}{|x-y|}\right|\right.\right. \\ &\qquad \left.\left. |(f(y,u,Du) - f(y,v,Dv))|dy\right)^2 dx\right]^{1/2}. \\ &\le M\left(\int_{\Omega_x}\left(\int_{\Omega_y} ||x-y| + (2-n)||x-y|^{1-n}|(f(y,u,Du)\right.\right. \\ &\qquad \left.\left. - f(y,v,Dv))|dy\right)^2 dx\right)^{1/2}, \end{aligned}$$

$$\le M\left(\int_{\Omega_x}\left(\int_{\Omega_y}(|x-y|+|(2-n)|)|x-y|^{1-n}|(f(y,u,Du)\right.\right.$$
$$\left.\left.-f(y,v,Dv))|dy\right)^2 dx\right)^{1/2},$$
$$\le MC(n,\Omega)|||\vec{x}|^{1-n}||_1||(f(.,u,Du)-f(.,v,Dv))||_2,$$
$$\le MC(n,\Omega)|||\vec{x}|^{1-n}||_1\left(\frac{K}{\lambda_1}+L\right)||u-v||_{1,2}. \tag{10}$$

where $C(n,\Omega)=|2-n|+\text{diam}(\Omega)$, $M=\max\{M_1,M_2\}>0$, $M_1=\sup_{x,y\in\Omega}\{|D_x\lambda(x,y)|\}$, $M_2=\sup_{x,y\in\Omega}\{|\lambda(x,y)|\}$. Hence

$$||Tu-Tv||_{1,2}\le MC(n,\Omega)|||\vec{x}|^{1-n}||_1\left(\frac{K}{\lambda_1}+L\right)||u-v||_{1,2} \tag{11}$$

where we have used the Lipshitz condition on f, the Young's inequality, the Cauchy-Schwartz inequality and the Poincare inequality on $H_0^{1,2}(\Omega)$ to obtain (10). The application of the Young's inequality involved the extension of the functions $|\vec{x}|^{1-n}\in L^1(\Omega)$ and $f(.,u(.),Du(.))\in L^2(\Omega)$ to $\mathbb{R}^n$ by defining it as '0' in $\mathbb{R}^n\backslash\Omega$. The mapping T will be a contraction map if $MC(n,\Omega)|||\vec{x}|^{1-n}||_1\left(\frac{K}{\lambda_1}+L\right)<1$. Hence, if this condition is met, then we can guarantee the existence of a unique fixed point to the operator T which will also satisfy (6). Thus we have the main result of this paper in the following theorem.

Theorem 1 *Let $\Omega\subset\mathbb{R}^n$, $n\ge 3$ be a relatively compact domain with a smooth boundary and let*

$$f:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$$

be a Lipshitz continuous function satisfying

$$|f(x,y_1,z_1)-f(x,y_2,z_2)|\le K|y_1-y_2|+L|z_1-z_2|$$

for every pair $(x,y_1,z_1),(x,y_2,z_2)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$ and $f(.,u(.),Du(.))\in L^2(\Omega)$, then the elliptic boundary value problem

$$\begin{aligned}\Delta u+f(x,u,Du)&=0,\\ u|_{\partial\Omega}&=0,\end{aligned} \tag{12}$$

admits a unique solution to (10) in $H_0^{1,2}(\Omega)$, provided

$$MC(n,\Omega)|||\vec{x}|^{1-n}||_1\left(\frac{K}{\lambda_1}+L\right)<1. \tag{13}$$

Note:
If $\Omega\subset\mathbb{R}^2$ then the Green's function has the following representation

$$G(x,y)\approx\log\left(1+\frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right).$$

A Few Important Consequences and Examples of Theorem 1

Consider the semilinear elliptic partial differential equation

$$\begin{aligned} Lu + f(x, u, Du) &= 0, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{14}$$

where L is an elliptic operator in a divergence form, i.e.,

$$L = \sum_{k,j=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kj}\frac{\partial}{\partial x_j}\right), \tag{15}$$

where a_{ij} are constants and the function f having the same properties as that given in theorem. We conclude that the fundamental solution to the operator L is also in $H^{1,1}_{loc}(\mathbb{R}^n)$ (refer "Appendix"). Hence by a result due to Escauriaza [30] that $c_1 F_{\text{Laplacian}}(x, y) \leq F_{\text{semi linear}}(x, y) \leq c_2 F_{\text{Laplacian}}(x, y)$, we conclude that there exists a unique solution to (14) in $H_0^{1,2}(\Omega)$.

Another important and immediate consequence of the Theorem 1 can be seen if $f(x, 0, 0) = 0 \forall x \in \overline{\Omega}$ and f is linear in u, Du. Under these assumptions T is linear in u. From the theorem, if the multiplying factor to $||u - v||_{1,2}$ is smaller than 1, then T has a unique fixed point. But T is linear and no matter what it always fixes the '0' vector thereby allowing us to conclude that the only solution to the problem is the trivial solution—similar to '*Hopf*'s principle. We will now give few examples of partial differential equations of the type given in (6) to guarantee the existence of a solution using the condition given in (12).

Example 1 Consider the problem with dimension $n = 1$ and $\Omega = (0, 1)$

$$\begin{aligned} -u^{''} &= f(x, u, u^{'}), \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{16}$$

The Green's function corresponding to the operator $-\frac{d^2}{dx^2}$ with $u(0) = u(1) = 0$ is

$$G(x, y) = \begin{cases} |x - y| - y(1 - x), & x \leq y \\ |x - y| - x(1 - y), & x \geq y \end{cases}$$

and the corresponding first eigen value is π^2. Thus $D_x G(x, y) = y - 1$, for $x \leq y$ and is equal to y for $x \leq y$. Using similar arguments given earlier or otherwise by virtue of Theorem 1 we have the following.

$$\begin{aligned} ||Tu - Tv||_{1,2} \leq \int_0^1 &\Bigg(\Bigg|\int_0^x y(f(., u, Du) - f(., v, Dv))dy \\ &+ \int_x^1 (y-1)(f(., u, Du) - f(., v, Dv))dy\Bigg|^2 dx\Bigg)^{1/2} \\ \leq &\left(\frac{K}{\pi} + L\right)||u - v||_{1,2}. \end{aligned}$$

Thus the problem in (16) will have a unique solution if $\frac{K}{\pi} + L < 1$.

Example 2 Consider the problem

$$\begin{aligned} -\Delta u &= f(x, u, Du), \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{17}$$

where $\Omega = (0, 1) \times (0, \pi)$.

When $\Omega \subset \mathbb{R}^2$ we use the representation, due to Zhao [29], $G(x, y) = \log\left(1 + \frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right)$. For $\epsilon << 1$ we consider,

$$\begin{aligned}
&||Tu - Tv||_{1,2}\\
&= \left(\int_\Omega \left|\int_\Omega D_x G(x, y)(f(y, u, Du) - f(y, v, Dv))dy\right|^2 dx\right)^{1/2}\\
&= \left(\int_\Omega \left|\int_{B(x,\epsilon)} D_x(2\log d(x, \partial\Omega) + 2\log|x - y|)(f(y, u, Du) - f(y, v, Dv))dy\right|^2 dx\right.\\
&\quad \left. + \int_\Omega \left|\int_{\Omega\setminus B(x,\epsilon)} D_x \log\left(1 + \frac{d(x, \partial\Omega)d(y, \partial\Omega)}{|x - y|^2}\right)(f(y, u, Du) - f(y, v, Dv))dy\right|^2 dx\right)^{1/2}
\end{aligned}$$

The metric $d(., \partial\Omega)$ is differentiable almost everywhere and hence bounded.
Let us analyze the following integral

$$\begin{aligned}
&\int_{B(x,\epsilon)} |(D_x 2\log|x - y|)(f(y, u, Du) - f(y, v, Dv))|dy\\
&\quad = \int_0^\epsilon \int_0^{2\pi} |(D_{r_x,\theta} 2\log r_x)(f(r_x, \theta, u, Du) - f(r_x, \theta, v, Dv))|r_x dr_x d\theta
\end{aligned}$$

where $r_x = |y - x|$, $D_{r_x,\theta} = \hat{e}_{r_x}\frac{\partial}{\partial r_x} + \hat{e}_\theta r_x \frac{\partial}{\partial\theta}$. Since $f(., u(.), Du(.)) \in L^2(\Omega) \subset L^1(\Omega)$ we have $0 \le \int_0^\epsilon \int_0^{2\pi} |2(f(r_x, \theta, u, Du) - f(r_x, \theta, v, Dv))|dr_x d\theta < \infty$. Hence,

$$\begin{aligned}
&\int_{B(x,\epsilon)} |(D_x 2\log|x - y|)(f(y, u, Du) - f(y, v, Dv))|dy\\
&\quad \le \epsilon \int_0^\epsilon \int_0^{2\pi} |2(f(r_x, \theta, u, Du) - f(r_x, \theta, v, Dv))|dr_x d\theta \to 0\\
&\quad \text{as } \epsilon \to 0.
\end{aligned}$$

Similarly for $x \in \Omega$ we have $|\int_{B(x,\epsilon)} D_x 2\log(d(x, \partial\Omega))(f(y, u, Du) - f(y, v, Dv))dy| \to 0$ as $\epsilon \to 0$. Thus we have

$$||Tu - Tv||_{1,2} \le M\left(\frac{K}{\lambda_1} + L\right)||u - v||_{1,2}, \tag{18}$$

where $M = \sup_{\Omega\times\Omega}\left\{|D_x \log\left(1 + \frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right)|\right\}$.
Existence of a unique solution is guaranteed if $M\left(\frac{K}{\lambda_1} + L\right) < 1$.

Example 3 Alarcón et al. [31] considered the following problem

$$\begin{aligned}
-\Delta u + \alpha u &= g(|Du|) + \lambda h(x), \quad \forall x \in \Omega,\\
u &= 0, x \in \partial\Omega,
\end{aligned} \tag{19}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with boundary $\partial\Omega$, $\alpha > 0$, g is an arbitrary C^1 function which is increasing, g is Lipshitz continuous with Lipshitz constant L, h is a non-negative function in $C^1(\overline{\Omega})$. Hence if $MC(n, \Omega)|||\vec{x}|^{1-n}||_1\left(\frac{\alpha}{\lambda_1} + L\right) < 1$, where λ_1 is as described previously, then there exists a unique solution to the problem (19) in $H_0^{1,2}(\Omega)$.

Conclusions

An existence and uniqueness result of a solution to the partial differential equation

$$\Delta u + f(x, u, Du) = 0, x \in \Omega \subset \mathbb{R}^n$$
$$u|_{\partial\Omega} = 0,$$

has been established using the classical Banach fixed point theorem. The result was then demonstrated by considering few important examples. The method introduced here can be used by numerical analysts to determine a weak solution to the partial differential equation of the above kind.

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Appendix

Consider

$$-\Delta F(x) = \delta(x). \tag{20}$$

On applying Fourier transform to (16), we obtain

$$\hat{F}(\xi) = \frac{1}{|\xi|^2}. \tag{21}$$

Inverse Fourier transform helps us to recover the fundamental solution of the Laplacian Δ which is as follows

$$F(x) = -\int_{\mathbb{R}^n} \frac{e^{i\langle x,\xi\rangle}}{|\xi|^2} d\xi. \tag{22}$$

It is a well known result that the fundamental solution of the Laplacian is represented in a closed form as

$$\begin{aligned} F(x) &= \frac{1}{|\omega_n|} \frac{1}{|x|^{n-2}}, \quad n \geq 3, \\ &= \log(|x|), \quad n = 2 \end{aligned} \tag{23}$$

where ω_n is the surface area of a sphere in n dimensions, S^{n-1}. It can be seen that for $n \geq 3$

$$\frac{\partial F}{\partial x_l} = \frac{(2-n)}{\omega_n} \frac{x_l}{|x|^n}, \tag{24}$$

for $l = 1, 2, \ldots, n$ and hence $|\nabla F| = \frac{1}{\omega_n} \frac{(2-n)}{|x|^{n-1}}$. This shows that $|\nabla F| \in L^1_{loc}(\mathbb{R}^n)$. A similar argument can be used for $n = 2$ as well with $\mathbf{K}$ being an arbitrary compact set in $\mathbb{R}^n$. Hence if $L = \sum_{k=1}^n \sum_{j=1}^n a_{kj} \frac{\partial^2}{\partial x_k \partial x_j}$ is an elliptic operator satisfying

$$\sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} \xi_k \xi_j > \gamma |\xi|^2, \tag{25}$$

then the fundamental solution corresponding to L, say F, also belongs to $H^{1,1}_{loc}(\mathbb{R}^n)$. This is because

$$\left|\frac{\partial}{\partial x_l}F_{laplacian}\right| = \left(\left|\int_{\mathbb{R}^n}\frac{\xi_l\cos<x,\xi>}{|\xi|^2}\right|^2+\left|\int_{\mathbb{R}^n}\frac{\xi_l\sin<x,\xi>}{|\xi|^2}\right|^2\right)^{\frac{1}{2}} \tag{26}$$

and hence

$$\begin{aligned}\int_K\left|\frac{\partial F}{\partial x_l}\right|dx &= \int_K\left|\int_{\mathbb{R}^n}\frac{i\xi_l e^{i<x,\xi>}}{\sum_{k,j}a_{kj}\xi_k\xi_j}d\xi\right|dx\\ &= \int_K\left(\left|\int_{\mathbb{R}^n}\frac{\xi_l\cos<x,\xi>}{\sum_{k,j}a_{kj}\xi_k\xi_j}d\xi\right|^2+\left|\int_{\mathbb{R}^n}\frac{\xi_l\sin<x,\xi>}{\sum_{k,j}a_{kj}\xi_k\xi_j}d\xi\right|^2\right)^{\frac{1}{2}}dx\\ &\le \frac{1}{\gamma}\int_K\left(\left|\int_{\mathbb{R}^n}\frac{\xi_l\cos<x,\xi>}{|\xi|^2}\right|^2+\left|\int_{\mathbb{R}^n}\frac{\xi_l\sin<x,\xi>}{|\xi|^2}\right|^2\right)^{\frac{1}{2}}dx, \text{ holds from (30)}\\ &= \frac{1}{\gamma}\int_K\left|\frac{\partial}{\partial x_l}F_{laplacian}\right|dx\\ &< \infty,\end{aligned} \tag{27}$$

for every $l = 1, 2, \ldots, m$. Note that we have used the condition of L being uniformly elliptic.

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