Int. J. Appl. Comput. Math (2017) 3:1557–1560
DOI 10.1007/s40819-016-0160-0

SHORT COMMUNICATION

# Amplitude-Frequency Relationship for Conservative Nonlinear Oscillators with Odd Nonlinearities

**Ji-Huan He**[1]

Published online: 4 April 2016


**Abstract** This paper studies a conservative nonlinear oscillator with odd nonlinearities, $u'' + f(u) = 0$, the square of its frequency is $f'(u_i)$, where $u_i$ is a location point. A criterion on how to choose a location point is given. Dufffing equation is used as an example to show the accuracy of the prediction.

**Keywords** Nonlinear oscillator · Frequency · Location point

## Introduction

Nonlinear vibration arises everywhere in engineering, it is of utter importance to have a fast insight into its frequency or period property, and a simple mathematical method is very much appreciated for practical applications. Though there are many analytical methods for nonlinear vibrations, among which the amplitude-frequency formulation [1] and the max-min approach [2] are widely adopted for this purpose due to shorter calculation with relatively higher accuracy. Other analytical methods for nonlinear oscillators are summarized in Refs. [3,4]. In this paper we will suggest a remarkably simple way with a relatively acceptable accuracy to conservative nonlinear oscillators with odd nonlinearities.

## Amplitude-Frequency Relationship

To illustrate the basic solution process of the new method, we first consider a linear oscillator in the form

$$u'' + ku = 0 \tag{1}$$

✉ Ji-Huan He
hejihuan@suda.edu.cn; ijnsns@aliyun.com

1 National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, 199 Ren-Ai Road, Suzhou 215123, China

where $k$ is a constant.

The square of its frequency can be easily obtained, which reads

$$\omega^2 = \frac{dg(u)}{du} = g'(u) = k \tag{2}$$

where $g(u)$ is the restoring force, $g(u) = ku$.

Now we consider a nonlinear oscillator in the form

$$u'' + f(u) = 0 \tag{3}$$

where $f(u)$ is a nonlinear restoring force, it requires $f(u)/u > 0$ and $f(0) = 0$. We extend Eq. (2) to nonlinear cases, that is

$$\omega^2 = \frac{df(u)}{du} \tag{4}$$

Equation (4) is valid only for the linear case, $df(u)/du$ is a function of $u$for nonlinear oscillators. Locating at $u = \frac{i}{N}A(i = 1, 2, 3, \ldots, N-1)$, where A is the amplitude, we have

$$\omega_i^2 = \frac{df}{du}(u = iA/N), \quad i = 1, 2, 3, \ldots, N-1 \tag{5}$$

The square of its frequency is approximately written as

$$\omega^2 = \frac{\sum_{i=1}^{N-1}\omega_i^2}{N-1} \tag{6}$$

When $N$=2, we have

$$\omega^2 = f'(u)\big|_{u=A/2} \tag{7}$$

Equation (7) can be used for a fast qualitative analysis of a nonlinear oscillator.

Consider the Duffing equation, which is

$$u'' + u + \varepsilon u^3 = 0,\ u(0) = A, u'(0) = 0 \tag{8}$$

Hereby $f(u) = u + \varepsilon u^3$. By Eq. (7), the square of its frequency can be immediately obtained, which is

$$\omega^2 = f'(u)\big|_{u=A/2} = 1 + 3\varepsilon\left(\frac{A}{2}\right)^2 = 1 + \frac{3}{4}\varepsilon A^2 \tag{9}$$

When $\varepsilon$ is small, i.e., $\varepsilon << 1$, Eq. (9) is equivalent to that obtained by the perturbation method; when $\varepsilon$ tends to infinite, the exact frequency reads [3,4]

$$\omega_{ex} = 0.9318\sqrt{\varepsilon A^2} \tag{10}$$

The accuracy of the obtained frequency by Eq. (9) reaches 7 % even when $\varepsilon \to \infty$.

If we set N = 3 and N = 4, respectively, in Eq. (6), we have

$$\omega^2 = \frac{1 + 3\varepsilon\left(\frac{A}{3}\right)^2 + 1 + 3\varepsilon\left(\frac{2A}{3}\right)^2}{2} = 1 + \frac{8}{9}\varepsilon A^2, \quad \mathrm{N} = 3 \tag{11}$$

and

$$\omega^2 = \frac{1 + 3\varepsilon\left(\frac{A}{4}\right)^2 + 1 + 3\varepsilon\left(\frac{2A}{4}\right)^2 + 1 + 3\varepsilon\left(\frac{3A}{4}\right)^2}{3} = 1 + \frac{7}{8}\varepsilon A^2, \quad \mathrm{N} = 4 \tag{12}$$

The accuracy of the frequency improves to 1.18 and 0.38 %, respectively, when $\varepsilon \to \infty$.

## Discussion and conclusions

Equation (6) is such constructed only for simple calculation, there are many alternative determinations of the square of frequency, for example

$$\omega^2 = \frac{\sum_{i=1}^{N} \omega_i^2}{N} \tag{13}$$

where $\omega_i$ is defined by Eq. (5), or in a more general form

$$\omega^2 = \frac{\sum_{i=1}^{N} f'(u_i)}{N} \tag{14}$$

where $u_i (i = 1 \sim N)$ are location points, $0 < u_i < A$. For Duffing equation, we set N = 2 and locate two points:$u_1 = 0.5A$ and $u_2 = 0.6A$, from Eq. (14) we have

$$\omega^2 = \frac{f'(0.5A) + f'(0.6A)}{2} = 1 + 0.915\varepsilon A^2 \tag{15}$$

The accuracy of the frequency is 2.65 %.

The most simple calculation is

$$\omega^2 = f'(u_i), \quad 0 < u_i < A \tag{16}$$

The accuracy, however, depends greatly upon the location point. Hereby we give a criterion for choosing a suitable location point, see Table 1.

**Table 1** Criterion for choosing a location point

| Conditions | Location point for Eq. (16) |
|---|---|
| $uf''(u) > 0$ | $A/2 < u_i < A$ |
| $uf''(u) < 0$ | $0 < u_i < A/2$ |

For Duffing equation, we have $uf''(u) = 6\varepsilon u^2 > 0$, we choose $u_i = 0.51A$:

$$\omega^2 = \frac{df}{du}(u = 0.51A) = 1 + 0.7803\varepsilon A^2 \tag{17}$$

The accuracy of the obtained frequency improves from 7 % for $u = 0.5A$ to 5.2 % for $u = 0.51A$.

Consider another example in the form

$$u'' + u^{1/3} = 0, \ u(0) = A, u'(0) = 0 \tag{18}$$

Hereby $f(u) = u^{1/3}$, which satisfies the condition: $uf''(u) < 0$, therefore the location point should be $0 < u_i < A/2$. We choose two location points $u = 0.5A$ and $u = 0.2A$ for comparison.

By Eq. (16), we have

$$\omega^2 = \frac{1}{3}(0.5A)^{-2/3} = 0.5291A^{-2/3} \tag{19}$$

$$\omega^2 = \frac{1}{3}(0.2A)^{-2/3} = 0.9746A^{-2/3} \tag{20}$$

The exact frequency for Eq. (18) is

$$\omega_{ex} = 1.070451A^{-1/3} \tag{21}$$

It is obvious that the accuracy improves from 32.05 % for $u = 0.5A$ to 7.77 % for $u = 0.2A$, showing that the criterion given in Table 1 is practicable.

We conclude that this paper might give the most simple and direct way to outline the general solution property of a nonlinear oscillator, while the accuracy is always remarkable contrast to those obtained by the perturbation method. The error by the perturbation method tends to infinity when $\varepsilon \to \infty$ for Duffing equation [4], while all predictions in this paper are relatively acceptable even when $\varepsilon \to \infty$. The utmost simplicity of the solution process makes the method much attractive for practical applications. The examples given in this paper can be used as a paradigm for many other applications.

**Acknowledgements** The work is supported by PAPD (A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions), "Six Talent Peak" of Jiangsu Province (ZBZZ-035).

## References

1. He, J.H.: An improved amplitude-frequency formulation for nonlinear oscillators. Int. J. Nonl. Sci. Num. **9**(2), 211–212 (2008)
2. He, J.H.: Max-min approach to nonlinear oscillators. Int. J. Nonl. Sci. Num. **9**(2), 207–210 (2008)
3. He, J.H.: Some asymptotic methods for strongly nonlinear equations. Int. J. Mod. Phys. B **20**, 1141–1199 (2006)
4. He, J.H.: Nonperturbative methods for strongly nonlinear problems. dissertation.de-Verlag im Internet GmbH (2006)