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ORIGINAL PAPER

EOQ Model for Time Dependent Demand and Exponentially Increasing Holding Cost Under Permissible Delay in Payment with Complete Backlogging

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Abstract In this study, economic order quantity model is considered in which demand rate is assumed to be continuous function of time and holding cost is exponentially increasing function under the condition of permissible delay in payment. The scheduling period is assumed to be a variable. Shortages are allowed which are completely backlogged. The objective of this study is to obtain the retailer's optimal replenishment policy that maximizes the total profit. Numerical examples are provided to illustrate the proposed model. Sensitivity analysis has been provided and managerial implications are discussed.

Keywords Inventory · Permissible delay · Increasing demand · Complete backlogging

Introduction

Many of the Economic Order Quantity (EOQ) models were developed under the assumption that the retailer should pay for products as soon as it is received from the supplier. However, the existing practice is that the supplier may offer a stipulated period to the retailer to settle the account. Within this stipulated period, the retailer need not have to settle the account is generally termed as permissible delay in payment.

To manage the inventory level successfully, the retailer needs to find a balance between the costs and benefits of holding stock. The costs of holding stock include the money has been spent buying the stock as well as storage. The benefits include having enough stock on hand to meet the demand of customers. Having too much stock equals extra expense for the

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retailer as it can lead to a shortfall in cash flow and incur excess storage costs. And having too little stock equals lost income in the form of lost sales, while also undermining customer confidence in retailer's ability to supply the products the retailer claims to sell. Hence keeping the right stock and being able to sell it can lead to—increased sales, new customers, increased customer confidence, improved cash flow.

The first attempt was made to describe the optimal ordering policies by Ghare and Shrader [12] in which they discussed EOQ model for an exponentially decaying inventory. Philip [42] developed an inventory model with a three-parameter Weibull distribution rate without considering shortages. In the literature referring to models with permissible delay in payments, Goyal [13] developed an EOQ model in which he ignored the difference between the selling price and the purchase cost. But Dave [10] corrected Goyal's model in which selling price exceeds the purchasing cost. Deb and Chaudhuri [11] derived inventory model with time-dependent deterioration rate. Shah [14] assumed a stochastic inventory model when delays in payments are permissible. Datta and Pal [9] considered inventory model with a linear time-dependent demand with shortages. Jamal et al. [15] generalized Aggarwal and Jaggi's [2] model to allow shortages. Under the condition of permissible delay in payments Hwang and Shinn [16] added pricing strategy to the model. Chung [17] developed an alternative approach to find EOQ under trade credit being granted. Teng [18] implemented Goyal's model by considering the difference between unit price and unit cost. Chang et al. [19] developed an EOQ model for deteriorating items under supplier credits linked to ordering quantity. Chung and Huang [20] developed an EPQ for a retailer where the supplier offers a permissible delay in payments. Huang [22] extended Goyal's model to develop an EOQ model in which the supplier offers the retailer permissible delay period. Teng and Goyal [23] addressed the shortcoming of Huang's model. Many related articles can be found in Chang and Teng [24], Chung [25], Chung and Liao [26], Goyal et al. [28], Huang [30,31], Huang and Hsu [32], Liao et al. [36], Ouyang et al. [38,39], Shinn and Hwang [46], Teng et al.[47–49], and their references. Also researchers established their inventory model under trade credit financing by assuming that the demand rate is constant. However, it is observed that the demand rate of new brand of consumer goods comes to the market, increases at the beginning of the season up to a certain moment and then remains to be constant for the rest of the time. However, they assumed that both the first derivative and second derivative of the demand rate must be greater than zero, which excluded not only a constant demand but also a linearly increasing demand which is not covered by Hsieh et al. [29]. Teng et al. [33] considered a deter inventory model when delay in payments are permissible. Musa and Sani [1] developed inventory model for deteriorating items under permissible delay in payments. Khanra et al. [34] inventory model with quadratic demand under trade credit with shortages. Chung and Cardenas-Barron [27] developed EOQ model for deteriorating items under stock-dependent demand with two-level credit policy. Ouyang et al. [40] addressed EOQ model for two-levels of trade credit policy. Cardenas-Barron [4] developed EOQ model with different back-ordering rates. Chen et al. [6] developed retailer's economic order quantity model when the supplier offers conditionally permissible delay in payments link to order quantity. Chung [7] developed EOQ model for deteriorating items under two-level trade credit policy. Wu et al. [50] developed an EOQ model with for exponentially increasing function of retailers down stream credit period. Nita et al. [37] Retailer's decision for ordering and credit policies for deteriorating items when a supplier offers order-linked credit period. Saren and Cardenas-Barron [44] developed inventory model with trade-credit policy and variable deterioration for fixed lifetime products. Chowdhury et al. [45] developed an inventory model in which demand is influenced by the selling price and the inventory level over finite planning horizon.

Khanra et al. [35] analyzed comparison between inventory followed by shortages model and shortages followed by inventory model with variable demand rate.

Battini et al. [8] explored the integration of factors affecting the environmental impact within the traditional EOQ model. San-Jose et al. [43] developed EOQ model where the unit holding cost has two significant components: a fixed cost which represents the cost of accommodating the item in the warehouse and a variable cost given by a potential function of the length of time over which the item is held in stock in which shortages are partially backlogged. Jaggi et al. [5] developed EOQ model with allowable shortage under trade credit . Pentico et al. [21] approximated an the EOQ with partial backordering at an exponential or rational rate by a constant or linearly changing rate. Guchhait et al. [41] studied an inventory model for a deteriorating item with time dependent deterioration in imprecise environment. Taleizadeh et al. [3] developed EOQ models with incremental discounts and either full or partial backordering.

Hence in this paper, the constant demand is extended to linear non-decreasing demand function of time and holding cost is assumed to be exponentially increasing function of time containing two parameters f and d. And if d = 0 indicates that holding cost is constant. Shortages are allowed to occur which are completely backlogged. Also the necessary and sufficient conditions of the existence and uniqueness of the optimal solutions are provided. Numerical examples are presented to demonstrate the developed model and the solution procedure. Sensitivity analysis of the optimal solution with respect to major parameters of the system is carried out and their results are discussed. The paper is organized as follows: The notations and assumptions used are given in “Notations and Assumptions” section. In “Mathematical Model” section the mathematical model is developed. “Numerical Examples” section is devoted to numerical examples. Sensitivity analysis is presented in “Sensitivity Analysis” section. The paper closes with concluding remarks in “Conclusion” section.

Notations and Assumptions

The following notations and assumptions are used in this paper.

Notations

k	ordering cost per order
c	unit purchasing cost
s	unit selling price (with s>c)
δ	the backlogging parameter which is a positive constant
c_2	shortage cost per unit per order
I_e	interest earned per \$ per unit of time by the retailer
I_c	interest payable per \$ in stocks per unit of time by the supplier
I_m	the maximum inventory level for each replenishment cycle
I_b	the maximum amount of demand backlogged per cycle
M	the retailer’s trade credit period offered by supplier in years
T	inventory cycle length (decision variable)
t_1	the time at which the inventory level falls to zero (decision variable)
Q	the retailer’s order quantity
$\Pi(t_1, T)$	the retailer’s total profit function per unit of time

Assumptions

1. The demand rate $D(t)$ is given by

$$D(t) = \begin{cases} a + bt & 0 < t \le t_1 \\ -\delta(a + bt) & t_1 < t \le T \end{cases} \tag{1}$$

   where a and b are non-negative constants.
2. The holding cost is time dependent and h(t) = f exp(dt) where f and d are positive constants.
3. The replenishment rate is infinite.
4. The time horizon of the inventory model is infinite.
5. The lead time is negligible.
6. The inventory model deals with single item.
7. There is no replacement or repair of deteriorating items during the period under consideration.
8. Shortages are allowed to occur which are completely backlogged.

Mathematical Model

The model begins without shortages and ends with shortages which is depicted graphically in Fig. 1. Based on the above assumptions, the retailer orders and receives Q units of a single product from the supplier at the beginning of time t = 0. The reduction of the inventory is due to the effect of demand only in the interval [0,t_1) and the demand is backlogged in the interval [t_1,T). At time $t = t_1$ the inventory level reaches zero. Hence the change in the inventory level $I(t)$ with respect to time can be written as follows:

$$\frac{dI(t)}{dt} = -(a + bt) \quad 0 < t \le t_1 \tag{2}$$

$$\frac{dI(t)}{dt} = -\delta(a + bt) \quad t_1 < t \le T \tag{3}$$

with boundary condition $I(t_1) = 0$.

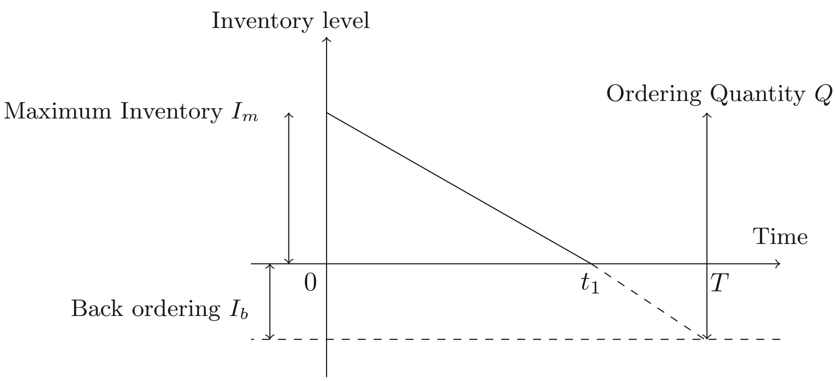

Fig. 1 Graphical representation of inventory model

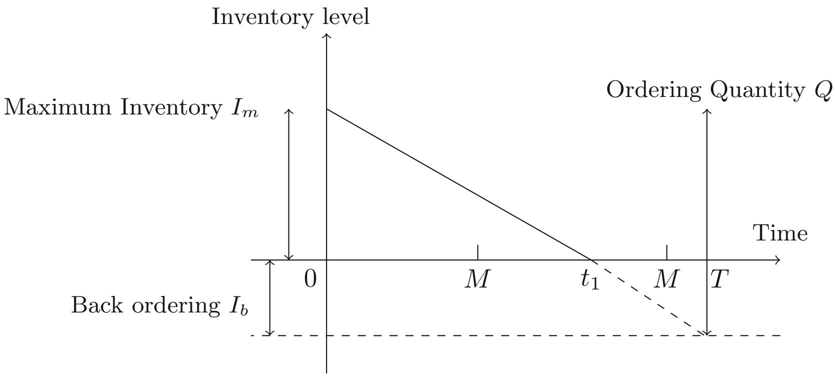

Fig. 2 Graphical representation of inventory model for various cases of M

The solutions to the above differential equations are

$$I(t) = a(t_1 - t) + \frac{b}{2}(t_1^2 - t^2) \quad 0 < t \leq t_1 \tag{4}$$

$$I(t) = \delta\Big[a(t_1 - t) + \frac{b}{2}(t_1^2 - T^2)\Big] \quad t_1 < t \leq T \tag{5}$$

The maximum inventory level I_m is given by

$$I_m = I(t = 0) = at_1 + \frac{b}{2}t_1^2 \tag{6}$$

The maximum amount of demand backlogged I_b is given by

$$I_b = -I(T) = \delta\Big[a(T - t_1) + \frac{b}{2}(T^2 - t_1^2)\Big] \tag{7}$$

The retailer's order quantity Q is

$$Q = I_m + I_b = at_1 + \frac{b}{2}t_1^2 + \delta\Big[a(T - t_1) + \frac{b}{2}(T^2 - t_1^2)\Big] \tag{8}$$

The various costs associated with the retailer's profit function per cycle are listed below with interest earned and payable for $0 < M \leq t_1$ and $t_1 < M \leq T$ which is depicted in Fig. 2

a) ordering cost = k

b) holding cost (excluding interest charges)

$$\begin{aligned} C_H &= \int_0^{t_1} h(t)\, I(t)\, dt \\ &= \int_0^{t_1} f\ exp(dt)\Big[a(t_1 - t) + \frac{b}{2}(t_1^2 - T^2)\Big]\, dt \\ &= f\Big[\frac{exp(dt_1)}{d^2}\Big(a + bt_1 - \frac{b}{d}\Big) + \frac{b}{d^3} - \frac{a}{d^2} - \frac{at_1}{d} - \frac{bt_1^2}{2d}\Big] \end{aligned}$$

c) purchase cost = cQ = c$\Big[at_1 + \frac{b}{2}t_1^2 + \delta\Big[a(T - t_1) + \frac{b}{2}(T^2 - t_1^2)\Big]\Big]$

d) sales revenue = sQ = s$\left[at_1+\frac{b}{2}t_1^2+\delta\left[a(T-t_1)+\frac{b}{2}(T^2-t_1^2)\right]\right]$

e) shortage cost

$$
\begin{aligned}
c_s &= c_2\int_{t_1}^{T} -I(t)\,dt \\
&= \int_{t_1}^{T} \delta\left[a(T-t_1)+\frac{b}{2}(T^2-t_1^2)\right]dt \\
&= \delta c_2\left[\left(\frac{aT^2}{2}+\frac{bT^3}{6}\right)\right]
\end{aligned}
$$

f) **Case 1**: $0 < M \le t_1$

$$
\begin{aligned}
\text{Interest earned } &= sI_e\int_0^M D(t)\,(M-t)\,dt \\
&= sI_e\left[\frac{aM^2}{2}+\frac{bM^3}{6}\right] \\
\text{Interest payable } &= cI_c\int_M^{t_1} I(t)\,dt \\
&= cI_c\int_M^{t_1}\left[a(t_1-t)+\frac{b}{2}\left(t_1^2-T^2\right)\right]dt \\
&= cI_c\left[\frac{a(t_1-M)^2}{2}+\frac{b\left(t_1^2-Mt_1^2\right)}{2}-\frac{b(t_1^3-M^3)}{6}\right]
\end{aligned}
$$

g) **Case 2**: $t_1 < M \le T$ From 0 to t_1, the retailer sells goods and continues accumulate sales revenue to earn interest I_e.

$$
sI_e\int_0^{t_1}(a+bt)\,(t_1-t)\,dt = sI_e\left[\frac{at_1^2}{2}+\frac{bt_1^3}{6}\right]
$$

From t_1 to M, the retailer can use the sales revenue generated in [0, t_1] to earn interest. Thus the interest earned from 0 to M is

$$
sI_e\left[\frac{at_1^2}{2}+\frac{bt_1^3}{6}+(M-t_1)\left[at_1+\frac{bt_1^2}{2}\right]\right]
$$

Interest payable in this case is zero.

From the above results, the total profit per unit time can be expressed as $\Pi(t_1,T) =$ $\{$sales revenue − ordering cost − purchase cost − holding cost − interest payable + interest earned − shortage cost$\}$/T.

$$
\Pi(t_1,T) = \begin{cases} \Pi_1(t_1,T) & 0 < M \le t_1 \\ \Pi_2(t_1,T) & t_1 < M \le T \end{cases} \tag{9}
$$

where

$$\Pi_1(t_1,T)=\frac{1}{T}\Bigg\{(s-c)(1-\delta)\left(at_1+\frac{b}{2}t_1^2\right)-k-f\Big[\frac{exp(dt_1)}{d^2}\left(a-\frac{b}{d}\right)+exp(dt_1)\frac{bt_1}{d^2}+\frac{b}{d^3}-\frac{a}{d^2}-\frac{at_1}{d}-\frac{bt_1^2}{2d}\Big]+sI_e\Big[\frac{aM^2}{2}+\frac{bM^3}{6}\Big]-cI_c\Big[\frac{a(t_1-M)^2}{2}+\frac{b(t_1^3-t_1^2M)}{2}-\frac{b(t_1^3-M^3)}{6}\Big]-c_2\Big[\frac{a\delta t_1^2}{2}+\frac{b\delta t_1^3}{3}\Big]\Bigg\}-c_2\Big[\frac{a\delta T}{2}+\frac{b\delta T^2}{6}-a\delta t_1-\frac{b\delta t_1^2}{2}\Big] \tag{10}$$

$$\Pi_2(t_1,T)=\frac{1}{T}\Bigg\{(s-c)(1-\delta)\left(at_1+\frac{b}{2}t_1^2\right)-k-f\Big[\frac{exp(dt_1)}{d^2}\left(a-\frac{b}{d}\right)+exp(dt_1)\frac{bt_1}{d^2}+\frac{b}{d^3}-\frac{a}{d^2}-\frac{at_1}{d}-\frac{bt_1^2}{2d}\Big]-c_2\Big[\frac{a\delta t_1^2}{2}+\frac{b\delta t_1^3}{3}\Big]+sIe\Big[\frac{at_1^2}{2}+\frac{bt_1^3}{6}+(m-t_1)\left(at_1+\frac{bt_1^2}{2}\right)\Big]+(s-c)\Big[a\delta T+\frac{b\delta}{2}\Big]-c_2\Big[\frac{a\delta}{2}+\frac{b\delta T}{6}\Big]\Bigg\}+c_2\Big[a\delta t_1+\frac{b\delta t_1^2}{2}\Big] \tag{11}$$

Theoretical Results and Optimal Solutions

The necessary conditions for the total profit per unit time in Eq. (10) to be maximum at $t_1=t_1^*, T=T^*$ are $\frac{\delta\Pi_1(t_1,T)}{\delta t_1}=0$ and $\frac{\delta\Pi_2(t_1,T)}{\delta T}=0$, provided all principle minors are $\begin{vmatrix}\frac{\delta^2\Pi_1(t_1,T)}{\delta t_1^2} & \frac{\delta^2\Pi_1(t_1,T)}{\delta t_1\delta T}\\ \frac{\delta^2\Pi_1(t_1,T)}{\delta t_1\delta T} & \frac{\delta^2\Pi_1(t_1,T)}{\delta T^2}\end{vmatrix}>0$ and $\frac{\delta^2\Pi_1(t_1,T)}{\delta t_1^2}<0$, $\frac{\delta^2\Pi_1(t_1,T)}{\delta T^2}<0$

$$\frac{\delta\Pi_1(t_1,T)}{\delta t_1}=\frac{1}{T}\Bigg\{(s-c)(1-\delta)(a+bt_1)-f\Bigg[\frac{exp(dt_1)}{d}\left(a-\frac{b}{d}\right)+exp(dt_1)\frac{b}{d^2}+exp(dt_1)\frac{bt_1}{d}-\frac{a}{d}-\frac{bt_1}{d}\Bigg]-cIc\Bigg[a(t_1-M)+\frac{b}{2}(3t_1^2-2Mt_1)-\frac{bt_1^2}{2}\Bigg]-c_2\Bigg[a\delta t_1+b\delta t_1^2\Bigg]\Bigg\}+c_2\Bigg[a\delta+b\delta t_1\Bigg]=0 \tag{12}$$

$$\frac{\delta\Pi_1(t_1,T)}{\delta T}=\frac{-1}{T^2}\Bigg\{(s-c)(1-\delta)(at_1+\frac{b}{2}t_1^2)-k-f\Big[\frac{exp(dt_1)}{d^2}\left(a-\frac{b}{d}\right)+exp(dt_1)\frac{bt_1}{d^2}+\frac{b}{d^3}-\frac{a}{d^2}-\frac{at_1}{d}-\frac{bt_1^2}{2d}\Big]+sI_e\Big[\frac{aM^2}{2}+\frac{bM^3}{6}\Big]-c_2\Big[\frac{a\delta t_1^2}{2}+\frac{b\delta t_1^3}{3}\Big]\Bigg\}-c_2\Big[\frac{a\delta}{2}+\frac{b\delta T}{3}\Big]+(s-c)\frac{b\delta}{2}=0 \tag{13}$$

$$\frac{\delta^2\Pi(t_1,T)}{\delta t_1^2} = \frac{1}{T}\Big\{(s-c)b(1-\delta) - f\Big[a\,exp(dt_1) + bt_1 exp(dt_1)\Big] + \frac{bf}{d}\Big[1-exp(dt_1)\Big] - cIc\big[a+2bt_1-Mb\big] - c_2\big[a\delta+2b\delta t_1\big] + c_2 b\delta T\Big\} < 0 \tag{14}$$

$$\frac{\delta^2\Pi_1(t_1,T)}{\delta T\delta t_1} = \frac{a\delta+b\delta t_1}{T} \tag{15}$$

$$\frac{\delta^2\Pi_1(t_1,T)}{\delta T^2} = -\Big[\frac{c_2 a\delta}{T} + c_2 b\delta + \frac{cb\delta}{T}\Big] + \frac{sb\delta}{T} < 0 \tag{16}$$

Clearly

$$\begin{vmatrix} \frac{\delta^2\Pi_1(t_1,T)}{\delta t_1^2} & \frac{\delta^2\Pi_1(t_1,T)}{\delta t_1\delta T} \\ \frac{\delta^2\Pi_1(t_1,T)}{\delta t_1\delta T} & \frac{\delta^2\Pi_1(t_1,T)}{\delta T^2} \end{vmatrix} = \Big\{\frac{sb\delta}{T} - \Big[\frac{c_2 a\delta}{T} + c_2 b\delta + \frac{cb\delta}{T}\Big]\Big\} \times\frac{1}{T}\Big\{(s-c)b(1-\delta) - f\Big[a\,exp(dt_1)+bt_1 exp(dt_1)\Big] + \frac{bf}{d}\Big[1-exp(dt_1)\Big] - cIc\big[a+2bt_1-Mb\big] - c_2\big[a\delta+2b\delta t_1\big] + c_2 b\delta T\Big\} - \Big\{\frac{a\delta+b\delta t_1}{T}\Big\}^2 > 0 \tag{17}$$

Hence $\Pi_1(t_1,T)$ is convex.

$$\frac{\delta\Pi_2(t_1,T)}{\delta t_1} = \frac{1}{T}\Big\{(s-c)(1-\delta)(a+bt_1) - f\Big[\frac{exp(dt_1)}{d}\Big(a-\frac{b}{d}\Big) + \frac{exp(dt_1)b}{d^2} + \frac{exp(dt_1)bt_1}{d} - \frac{a}{d} - \frac{bt_1}{d}\Big] + sIe\Big[at_1 + \frac{bt_1^2}{2} - \Big(at_1+\frac{bt_1^2}{2}\Big) + (M-t_1)(a+bt_1)\Big] - c_2\Big[a\delta t_1 + b\delta t_1^2\Big]\Big\} + c_2\Big[a\delta + b\delta t_1\Big] = 0 \tag{18}$$

$$\frac{\delta\Pi_2(t_1,T)}{\delta t_1} = \frac{-1}{T^2}\Big\{(s-c)(1-\delta)\Big(at_1+\frac{b}{2}t_1^2\Big) - k - f\Big[\frac{exp(dt_1)}{d^2}\Big(a-\frac{b}{d}\Big) + exp(dt_1)\frac{bt_1}{d^2} + \frac{b}{d^3} - \frac{a}{d^2} - \frac{at_1}{d} - \frac{bt_1^2}{2d}\Big] - c_2\Big[\frac{a\delta t_1^2}{2} + \frac{b\delta t_1^3}{3}\Big] + sIe\Big[\frac{at_1^2}{2} + \frac{bt_1^3}{6} + (m-t_1)\Big(at_1+\frac{bt_1^2}{2}\Big)\Big]\Big\} + (s-c)\frac{b\delta}{2} - c_2\Big[\frac{a\delta}{2} + \frac{b\delta T}{3}\Big] = 0 \tag{19}$$

Similarly it can be shown that $\Pi_2(t_1,T)$ is convex.

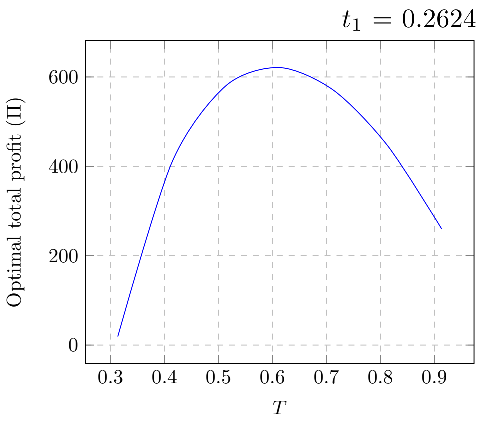

Fig. 3 Graph of T versus total profit Π

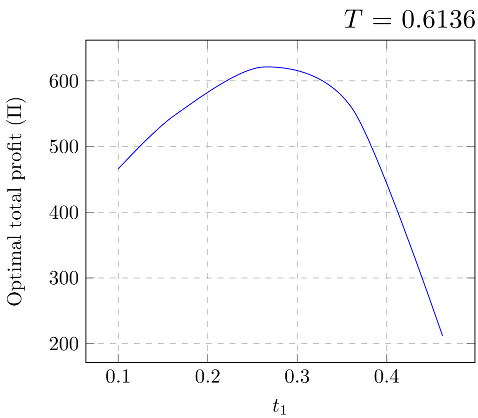

Fig. 4 Graph of t_1 versus total profit Π

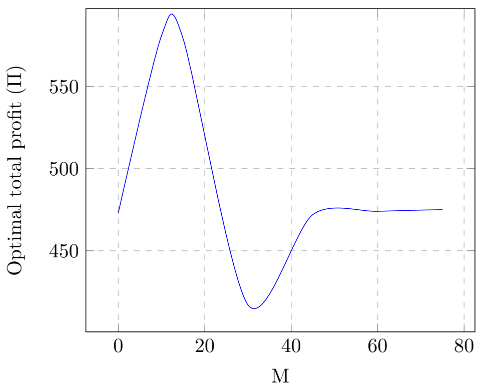

Fig. 5 Graph of M versus total profit Π

By solving Eqs. (12–13), the value of t_1, T can be obtained and with the use of this value, Eq. (10) provides the maximum total profit per unit time of the inventory model. Similarly solving Eqs. (14, 15), we can find t_1, T and Eq. (11) provides the maximum total profit per unit time.Hence, our aim is to find the optimal value of t_1 and T which maximizes the total profit $\Pi(t_1, T)$ (Figs. 3, 4, 5, 6, 7).

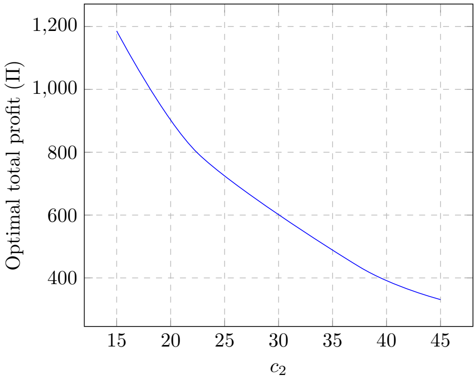

Fig. 6 Graph of c_2 versus total profit Π

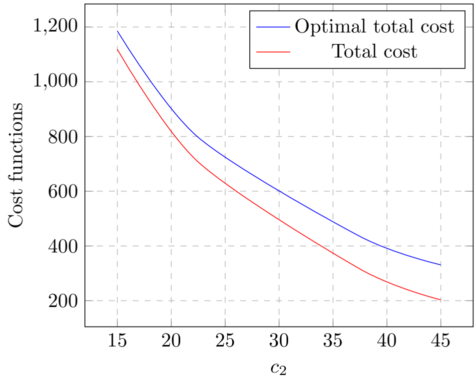

Fig. 7 Graph of c_2 versus cost functions

Numerical Examples

Numerical examples are carried out using SCILAB(5.5.0).

In each table, the column TC that corresponds to the total cost when permissible delay period M = 0.

Example 1 a = 10, b = 50, f = 15, d = 8, k = 200, c = 50, s = 60, $i_e = 0.11$, $i_c = 0.04$, M = 35/365, $\delta = 8$, $c_2 = 30$ with suitable units. The results are tabulated below.

t_1	T	Π_1	t_1	T	Π_2	Q^*	TC	Q
0.2746	0.6227	581.2332	0.3089	0.6663	516.3449	94.9492	472.8069	103.6823

Example 2 a = 10, b = 50, f = 15, d = 8, k = 140, c = 50, s = 60, $i_e = 0.12$, $i_c = 0.13$, M = 48/365, $\delta = 8$, $c_2 = 30$ with suitable units. The results are tabulated below.

t_1	T	Π_1	t_1	T	Π_2	Q^*	TC	Q
0.2624	0.6136	620.978	0.2618	0.6130	609.8865	93.9553	566.6639	94.8394

Example 3 a = 10, b = 50, f = 15, d = 8, k = 450, c = 40, s = 50, $i_e = 0.11$, $i_c = 0.04$, M = 14/365, $\delta = 8$, $c_2 = 30$ with suitable units. The results are tabulated below.

t_1	T	Π_1	t_1	T	Π_2	Q^*	TC	Q
0.3426	0.7431	201.9775	0.3567	0.7613	159.9815	125.3839	121.77	130.3446

Example 4 a = 10, b = 50, f = 15, d = 8, k = 60, c = 45, s = 50, $i_e = 0.11$, $i_c = 0.13$, M = 14/365, $\delta = 8$, $c_2 = 30$ with suitable units. The results are tabulated below.

t_1	T	Π_1	t_1	T	Π_2	Q^*	TC	Q
0.2083	0.3918	116.0514	0.2398	0.4136	128.7446	40.4518	86.3469	40.4443

Example 5 a = 10, b = 50, f = 15, d = 8, k = 200, c = 50, s = 60, $i_e = 0.11$, $i_c = 0.04$, $\delta = 8$, $c_2 = 30$ with suitable units. To study the effects of change in the parametric values of M, δ and c_2, we use the data of Example 1 and the results are presented in the following Tables 1, 2 and 3 respectively.

From the above example, the following observations are made:

Table 1 Effects of changes in parameter M of the model

M	t_1	T	Π_1	t_1	T	Π_2	Q^*	TC
10/365	0.2746	0.6227	581.2332	0.2841	0.6508	513.876	94.9492	472.8069
15/365	0.2912	0.6557	579.0719	0.2845	0.6511	466.6263	103.3927	472.8069
30/365	0.2904	0.6559	471.7846	0.2856	0.6518	468.8464	103.4611	472.8069
45/365	0.2907	0.656	472.4096	0.2867	0.6525	470.7000	103.4096	472.8069
60/365	0.2910	0.656	473.9807	0.2879	0.6533	472.7513	103.358	472.8069
75/365	0.2911	0.656	475.3911	0.309	0.6663	473.756	103.3408	472.8069

Table 2 Effects of changes in parameter δ of the model

δ	t_1	T	Π_1	t_1	T	Π_2	Q^*	TC
4	0.2972	0.7032	200.6311	0.3154	0.7069	143.0935	62.6364	101.1915
6	0.2946	0.6741	389.1862	0.3126	0.6818	328.7022	83.0289	285.5886
10	0.2857	0.6422	769.6512	0.3047	0.6543	704.9706	123.2467	661.3899
12	0.2596	0.5974	962.6652	0.3000	0.6450	894.2057	131.6652	850.8132

Table 3 Effects of changes in parameter c_2 of the model

c_2	t_1	T	Π_1	t_1	T	Π_2	Q^*	TC
15	0.2936	1.0282	1186.6004	0.3147	1.0385	1147.09	258.0579	1118.7151
22.5	0.2874	0.7717	797.2853	0.3077	0.7829	743.8332	146.2674	707.1147
37.5	0.2959	0.5911	434.7165	0.3124	0.6004	365.6455	81.1324	316.4963
45	0.3013	0.5497	330.5489	0.3162	0.5582	256.8753	67.4322	203.0808

1. If the permissible delay period M increases, the optimal total cost decreases first and then increases for a fixed value of δ. When M is greater than 15 days, the total profit increases. Thus the retailer should place optimum order quantity accordingly and carefully in order to gain maximum profit. If the permissible delay period set by the supplier is too short, the retailer may decide to order a quantity so as to gain profit.
   The managerial insight is as follows: Hence the retailer can maximize the total profit if the retailer gets shorter permissible delay period from the supplier.
2. If the values of the backlogging parameter δ increase, the optimum total profit and optimum order quantity Q^* increases but the optimum order cycle T^* decreases. Thus we see that the optimum total profit increases to a greater extent for large values of δ.
   The managerial insight is as follows: In order to gain maximum total profit the retailer ought to improve the backlogging rate.
3. If shortage cost c_2 increases for a fixed value of δ, the optimum total profit, optimum order quantity and the optimum order cycle all decreases. Thus, changes in c_2 result negative change in the total cost, order quantity and order cycle.
   The managerial insight is as follows: The retailer has to adopt for planned back order level strategically so as to gain maximum profit.

Sensitivity Analysis

To study the effects of change in the parametric values a, b, d, s, c, k, i_e, i_c, δ, M, c_2 on the optimal total cost, we use the data of Example 1. The sensitive analysis is performed by changing each of the parameter by −50, −25, 25 and 50 %, taking one parameter at a time and keeping the remaining parameters unchanged. Based on the computational results we obtained the numerical results which is presented in the Table 4.

Based on the computational results, we obtained the following managerial phenomena:

1. If the values of a and b of the model increase, the optimal order quantity Q^* increases. Hence, the optimal total profit of the model increases.
2. The optimal total profit alternatively decreases and increases with the increase in the values of the small variation in the values of f. But the optimal order quantity Q^* gradually decreases with the increasing values of f.
3. If we increase the value of d, then the optimal total profit and the optimal order quantity Q^* decreases. Thus the retailer must set up the holding cost strategically so as to gain maximum total profit.
4. The optimal total profit decreases with the increase in the value of ordering cost K. Also the optimal order quantity Q^* increases with the increase in the value of k. The economic interpretation is that the higher the ordering cost k is, the larger the replenishment cycle becomes. Hence, it turns out a smaller profit.
5. If the value of purchase cost c increases then both the optimal total profit and the optimal order quantity Q^* decreases.
6. Both optimal total profit and optimal order quantity Q^* increases for the increasing values of selling price c. Thus, changes in the optimal total profit indicate that the model is highly sensitive to the changes on s.
7. The changes in the optimal total profit indicate that the model is moderately sensitive to the changes on ie while it is low sensitive to the optimal order quantity Q^*.
8. The optimal total profit and Q^* decreases with the increase in the value of ic.

Table 4 Effects of changes in parameter of the model

Parameters	Changes (%)	t_1	T	Π_1	t_1	T	Π_2	Q^*	TC
a	−50	0.3223	0.7109	448.5032	0.3369	0.7183	401.263	100.0527	368.8759
	−25	0.3068	0.6834	511.7938	0.3231	0.692	457.3262	101.8320	419.2946
	25	0.2732	0.628	650.4608	0.2943	0.6409	518.2293	104.7101	529.4063
	50	0.2543	0.5992	726.2921	0.2789	0.6156	642.9354	105.6936	589.1643
b	−50	0.2918	0.6579	268.7578	0.3091	0.6665	229.0268	68.0389	194.4927
	−25	0.2911	0.6567	423.914	0.309	0.6664	372.6859	85.7252	333.648
	25	0.2902	0.6553	734.2212	0.3089	0.6663	660.0041	121.0423	611.9679
	50	0.2899	0.6549	889.3644	0.3089	0.6663	803.6633	138.7062	751.1301
f	−50	0.3549	0.7116	610.4741	0.3743	0.7229	560.1274	111.3179	509.3523
	−25	0.3168	0.6784	486.8331	0.3357	0.6892	481.097	106.5779	487.9291
	25	0.2707	0.6391	569.1654	0.2886	0.6492	502.2622	101.045	461.1848
	50	0.2547	0.6259	561.1821	0.2723	0.6357	489.6308	99.2405	451.7943
d	−50	0.3963	0.7511	610.7111	0.4265	0.7728	568.1806	117.6929	512.9638
	−25	0.3338	0.6942	592.7170	0.3568	0.7089	538.8164	109.0538	490.1434
	25	0.2585	0.6282	568.2147	0.2738	0.6359	498.4429	99.3942	459.0215
	50	0.2336	0.6072	559.3092	0.2467	0.6129	483.7643	96.4128	447.7149
k	−50	0.2429	0.5826	749.8396	0.2711	0.6032	673.4113	87.1645	633.0102
	−25	0.2709	0.6244	661.1419	0.2927	0.6384	592.9493	96.1214	550.6113
	25	0.3059	0.6818	501.2925	0.3221	0.6900	442.6418	109.7257	398.2856
	50	0.3185	0.7041	426.6948	0.3330	0.7107	371.2648	115.4319	326.3201

Table 4 continued

Parameters	Changes (%)	t_1	T	Π_1	t_1	T	Π_2	Q^*	TC
c	−50	0.4408	1.7172	7250.6566	0.4473	1.7251	7229.8104	662.2719	7203.2573
	−25	0.3549	1.1331	3183.0687	0.3703	1.1526	3147.9653	300.5462	3116.3621
s	25	0.3742	1.2463	3890.4663	0.3892	1.2653	3853.0201	359.6583	3814.8669
	50	0.4678	1.9521	9320.3534	0.4746	1.9601	9296.4343	847.2644	9258.9656
i_e	−50	0.2921	0.6509	512.1848	0.3012	0.6624	494.2918	101.4277	472.8069
	−25	0.2906	0.6559	545.5252	0.3052	0.6645	505.2503	103.3925	472.8069
	25	0.2905	0.6559	612.6018	0.3126	0.6681	527.5709	103.4097	472.8069
	50	0.2905	0.6558	646.1700	0.3161	0.6696	538.9232	103.3754	472.8069
i_c	−50	0.2915	0.6567	592.9000	0.3089	0.6663	516.3449	103.5118	472.8069
	−25	0.2911	0.6563	585.9968	0.3089	0.6663	516.3449	103.4436	472.8069
	25	0.2791	0.6284	573.7323	0.3089	0.6663	516.3449	96.0804	472.8069
	50	0.2834	0.6338	566.3573	0.3089	0.6663	516.3449	97.1513	472.8069
M	−50	0.2746	0.6227	581.2332	0.3089	0.6663	516.3449	94.9492	472.8069
	−25	0.2904	0.6558	579.0696	0.3089	0.6663	516.3449	103.3926	472.8069
	25	0.2907	0.6559	579.0735	0.3089	0.2927	516.3449	103.3753	472.8069
	50	0.2909	0.6559	579.0864	0.3089	0.6663	516.3449	103.3409	472.8069
δ	−50	0.2972	0.7032	200.6311	0.3154	0.7069	143.0935	62.6364	101.1915
	−25	0.2946	0.6741	389.1862	0.3126	0.6818	328.7022	83.0289	285.5886
	25	0.2857	0.6422	769.6512	0.3047	0.6543	704.9706	123.2467	661.3899
	50	0.2596	0.5974	962.6652	0.3000	0.6450	894.2057	131.6652	850.8132
c_2	−50	0.2936	1.0282	1186.6004	0.3147	1.0385	1147.09	258.0579	1118.7151
	−25	0.2874	0.7717	797.2853	0.3077	0.7829	743.8332	146.2674	707.1147
	25	0.2959	0.5911	434.7165	0.3124	0.6004	365.6455	81.1324	316.4963
	50	0.3013	0.5497	330.5489	0.3162	0.5582	256.8753	67.4322	203.0808

9. The changes of permissible delay M by a small amount have no impact on the optimal order quantity Q^*. Also the optimal total profit of the model is low sensitive to small variation in M.
10. If the values of the backlogging parameter δ increase, the optimum total profit and optimum order quantity Q^* increases. Thus we see that the optimum total profit increases to a greater extent for large values of δ. In order to gain maximum total profit the retailer ought to improve the backlogging rate.
11. If shortage cost c_2 increases for a fixed value of δ, the optimum total profit, optimum order quantity and the optimum order cycle all decreases. Thus, changes in c_2 result negative change in the total cost, order quantity and order cycle. Hence, the retailer has to adopt for planned back order level strategically so as to gain maximum profit.

Conclusion

In this paper EOQ model for time varying demand and variable holding cost under permissible delay with shortages is studied. In this model shortages are allowed which are completely backlogged. We have considered two different cases to obtain optimal total profit, optimal order quantity and optimal replenishment policy. Based on sensitive analysis, this paper revealed the following observations: (1) The optimal total profit of the model is low sensitive to small variation in M. Hence, the retailer can maximize the total profit if the retailer gets shorter permissible delay period from the supplier. (2) The changes in the optimal total profit indicate that the model is highly sensitive to the changes on s. (3) In order to gain maximum total profit the retailer ought to improve the backlogging rate. (4) The retailer has to adopt for planned back order level strategically so as to gain maximum profit. (5) The optimal total profit is highly sensitive on the parameters c, s and c_2 of this model. Some numerical examples are given to illustrate the proposed model. Graphical representation is provided to establish the total profit function attains it's optimum value. The sensitivity of the solution to changes in the values of the parameters between the range −50 to +50 % has also been discussed. To author's best knowledge, such type of EOQ model with variable holding cost inventory model has not yet been discussed.

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