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ORIGINAL ARTICLE

Approximations of Solutions for an Impulsive Fractional Differential Equation with a Deviated Argument

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Abstract In the present work, we consider an impulsive fractional differential equation with a deviated argument in an arbitrary separable Hilbert space H. We obtain an associated integral equation and then consider a sequence of approximate integral equations. The existence and uniqueness of solutions to every approximate integral equation is obtained by using analytic semigroup and Banach fixed point theorem. Next we demonstrate the convergence of the solutions of the approximate integral equations to the solution of the associated integral equation. We study the Faedo–Galerkin approximation of the solution and establish some convergence results. Finally, we consider an example to show the effectiveness of obtained theory.

Keywords Analytic semigroup · Banach fixed point theorem · Caputo derivative · Impulsive differential equation · Faedo–Galerkin approximation

Mathematics Subject Classification 34K37 · 34K30 · 35R11 · 47N20

Introduction

In recent few decades, researcher has developed great interest in fractional calculus due to its wide applicability in science and engineering. Tools of fractional calculus have been available and applicable to deal with many physical and real world problems such as anomalous diffusion process, traffic flow, nonlinear oscillation of earthquake, real system characterized by power laws, critical phenomena, scale free process, describe viscoelastic materials and

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many others. The details on the theory and its applications can be found in [1–4] and papers [5–9] and references cited therein.

On the other hand, many real world processes and phenomena which are subjected during their development to short-term external influences can be modeled as impulsive differential equation with fractional order which have been used efficiently in modelling many practical problems. Their duration is negligible compared with the total duration of the entire process and phenomena. Such process is investigated in various fields such as biology, physics, control theory, population dynamics, economics, chemical technology, medicine and so on. For the study for impulsive differential equation, we refer to monograph [10,11], and papers [12–23] and references given therein.

The purpose of this work is to establish the approximation of the solution to following differential equation with deviated argument in a separable Hilbert space $(H, \|\cdot\|, (\cdot,\cdot))$

$$
\begin{aligned}
{}^{c}D^{q}_{0+}x(t) &= -Ax(t) + f\,(t, x(t), x(a(x(t), t)))\,, \\
&\quad 0 \le t \le T_0 < \infty, \quad t \neq t_i, && (1)\\
\Delta x(t_i) &= I_i(x(t_i)), \quad i = 1, 2, \cdots, p, \quad p \in \mathbb{N} && (2)\\
x(0) &= u_0, && (3)
\end{aligned}
$$

where $0 < q < 1$, ${}^{c}D^{q}_{0+}$ is the fractional derivative in Caputo sense with single base point 0, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T_0$ are pre-fixed numbers, $\Delta x|_{t=t_i} = x(t_i^+) - x(t_i^-)$ and $x(t_i^+) = \lim_{h\to 0+} x(t_i + h)$ and $x(t_i^-) = \lim_{h\to 0-} x(t_i + h)$ denote the right and left limits of $x(t)$ at $t = t_i$, respectively. In (1), $A : D(A) \subset H \to H$ is a closed, positive definite and self adjoint linear operator with dense domain $D(A)$. We assume that $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators on H. The functions $f : [0, T_0] \times H^2 \to H$, $a : H \times [0, T_0] \to \mathbb{R}$, $I_i : H \to H$ are appropriate functions to be mentioned later. For more details of differential equation with deviating argument, we refer to papers [24–26] and references given therein.

In the present work, we investigate the Faedo–Galerkin approximations of the solutions for (1)–(3). The Faedo–Galerkin approximations of the solutions in a separable Hilbert space to the following system

$$
x'(t) + Ax(t) = M(x(t)), \quad x(0) = u_0 \tag{4}
$$

has been studied first by Miletta [27] under the assumption that $-A$ is the infinitesimal generator of an analytic semigroup and the nonlinear function M is Lipschitz continuous on a ball in $D(A^{\alpha})$, $0 < \alpha < 1$. Bahuguna and Srivastava [28] has discussed the more general cases. For a nice introduction on existence of an approximate solution and associated study of different problems are broadly talked about in the references [28–34].

The organization of the article is as follows: In Sect. 2, We provide some basic definitions, lemmas and theorems as preliminaries as these are useful for proving our results. In Sect. 3, we prove the existence and uniqueness of the approximate solutions by using analytic semigroup and Banach fixed point theorem. In Sect. 4, we show the convergence of the solution to each of the approximate integral equations with the limiting function which satisfies the associated integral equation and the convergence of the approximate Feado-Galerkin solutions will be shown in Sect. 5. In Sect. 6, we provide an example to illustrate the obtained theory.

Preliminaries and Assumptions

In this segment, some basic definitions, preliminaries, Theorems and Lemmas and assumptions which will be used to prove existence result, is stated.
Throughout the work, we assume that $(H, \| \cdot \|, < \cdot, \cdot >)$ is a separable Hilbert space. The symbol $C([0, T_0]; H)$ stands for the Banach space of all the continuous functions from $[0, T_0]$ into H equipped with the norm $\| z(t)\|_C = \sup_{t\in[0,T_0]} \| z(t)\|_H$ and $L^p((0, T_0); H)$ stands for Banach space of all Bochner-measurable functions from $(0, T_0)$ to H with the norm

$$\| z\|_{L^p} = \left(\int_{(0,T_0)} \| z(s)\|_H^p ds\right)^{1/p}.$$

Since $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{\mathcal{T}(t); t \geq 0\}$. Therefore, there exist constants $C \geq 1$ and $\delta \geq 0$ such that $\| \mathcal{T}(t)\| \leq Ce^{\delta t}$, $t \geq 0$. In addition, we note that

$$\| \frac{d^j}{dt^j}\mathcal{T}(t)\| \leq M_j, \quad t > t_0,\ t_0 > 0, \tag{5}$$

where M_j are some positive constants. Henceforth, without loss of generality, we might accept that $\mathcal{T}(t)$ is uniformly bounded by M i.e., $\| \mathcal{T}(t)\| \leq M$ and $0 \in \rho(-A)$ i.e., $-A$ is invertible. This permits us to define the positive fractional power A^α as closed linear operator with domain $D(A^\alpha) \subseteq H$ for $\alpha \in (0, 1]$. Moreover, $D(A^\alpha)$ is dense in H with the norm

$$\| y\|_\alpha = \| A^\alpha y\|. \tag{6}$$

Hence, we signify the space $D(A^\alpha)$ by H_α endowed with the α-norm $(\| \cdot \|_\alpha)$. Also, we have that $H_\kappa \hookrightarrow H_\alpha$ for $0 < \alpha < \kappa$ and therefore, the embedding is continuous. Then, we define $H_{-\alpha} = (H_\alpha)^*$, for each $\alpha > 0$. The space $H_{-\alpha}$ stands for the dual space of H_α, is a Banach space with the norm $\| z\|_{-\alpha} = \| A^{-\alpha}z\|$. For study on the fractional powers of closed linear operators, we refer to book by Pazy [35].

Lemma 2.1 *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{\mathcal{T}(t) : t \geq 0\}$ such that $\| \mathcal{T}(t)\| \leq M$, for $t \geq 0$ and $0 \in \rho(-A)$. Then,*

(i) For $0 \leq \alpha \leq 1$, H_α is a Hilbert space.
(ii) The operator $A^\alpha\mathcal{T}(t)$ is bounded for every $t > 0$ and

$$\| A\mathcal{T}(t)\| \leq Mt^{-1}, \tag{7}$$

$$\| A^\alpha\mathcal{T}(t)\| \leq M_\alpha t^{-\alpha}. \tag{8}$$

Now, we state some basic definitions and properties of fractional calculus.

Definition 2.1 The Riemann–Liouville fractional integral operator J is defined as

$$J^q_{0+}F(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s)ds, \tag{9}$$

where $F \in L^1((0, T_0); H)$ and $q > 0$ is the order of the fractional integration.

Definition 2.2 The Riemann-Liouville fractional derivative is given as

$${}^{RL}D^q_{0+}F(t) = D^m_t J^{m-q}_{0+}F(t),\ m-1 < q < m,\ m \in \mathbb{N}, \tag{10}$$

where $D^m_t = \frac{d^m}{dt^m},\ F \in L^1((0, T_0); H),\ J^{m-q}_{0+} \in W^{m,1}((0, T_0); H)$.

Definition 2.3 The Caputo fractional derivative is given as

$$ {}^{c}D_{0+}^{q}F(t)=\frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}F^{m}(s)ds,\quad m-1<q<m, \tag{11}$$

where $F\in C^{m-1}((0,T_0);H)\cap L^1((0,T_0);H)$.

We denote by $\mathcal{C}_t^{\alpha}=PC([0,t];H_\alpha),\ t\in(0,T_0]$ the space of all H_α-valued functions on $[0,t]$ such that $x(t)$ is continuous on $t\neq t_i$, left continuous at $t=t_i$ and the right limit $x(t_i^{+})$ exists for $i=1,\cdots,p$. It is clear that $\mathcal{C}_t^{\alpha}$ is a Banach space endowed with the norm

$$\| y\|_{t,\alpha}=\sup_{s\in[0,t]}\| y(s)\|_\alpha,\quad y\in\mathcal{C}_t^{\alpha}.$$

For $0\le\alpha<1$, we define

$$\mathcal{C}_t^{\alpha-1}=\{x\in\mathcal{C}_t^{\alpha}:\| x(\tau)-x(s)\|\le\mathcal{L}|\tau-s|,\ \text{for all}\ \tau,s\in[0,t]\}, \tag{12}$$

where $\mathcal{L}>0$ is a appropriate constant to be defined later.
Now, we introduce the following assumptions on A, f, a and I_i $(i=1,\cdots,p)$:

(A1) A is a closed, densely defined, positive definite and self-adjoint linear operator from $D(A)\subset H$ into H. We assume that operator A has the pure point spectrum

$$0<\lambda_0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_m\le\cdots, \tag{13}$$

with $\lambda_m\to\infty$ as $m\to\infty$ and a corresponding complete orthonormal system of eigenfunctions $\{\phi_j\}$, i.e.,

$$A\phi_j=\lambda_j\phi_j,\quad\text{and}\quad <\phi_l,\phi_j>=\delta_{lj}, \tag{14}$$

where

$$\delta_{lj}=\begin{cases}1, & j=l,\\ 0, & \text{otherwise.}\end{cases}$$

(A2) Let $W_1\subset Dom(f)$ be an open subset of $\mathbb{R}_+\times H_\alpha\times H_{\alpha-1}$, where $\alpha\in[0,1)$. For $(\tau,x,y)\in W_1$, there is a neighborhood $U_1\subset W_1$ of (τ,x,y) and positive constants $\mathcal{L}_f=\mathcal{L}_f(\tau,x,y,U_1)$ such that

$$\| f(t,x_1,y_1)-f(s,x_2,y_2)\|\le\mathcal{L}_f[|t-s|^{\mu_1}+\| x_1-x_2\|_\alpha+\| y_1-y_2\|_{\alpha-1}], \tag{15}$$

for all $(t,x_1,y_1),(s,x_2,y_2)\in U_1$ and $0<\mu_1\le1$.

(A3) For each $(x,\tau)\in W_2$, where $W_2\subset Dom(a)$ is an open subset of $H_\alpha\times\mathbb{R}_+$, there is a neighborhood $U_2\subset W_2$ of (x,τ) and positive constant $\mathcal{L}_a=\mathcal{L}_a(x,\tau,U_2)$ such that $a(\cdot,\cdot):H_\alpha\times\mathbb{R}_+\to\mathbb{R}_+$, $a(\cdot,0)=0$,

$$|a(x_1,t_1)-(x_2,t_2)|\le\mathcal{L}_a\left[\| x_1-x_2\|_\alpha+|t_1-t_2|^{\mu_2}\right], \tag{16}$$

for all $(x_1,t_1),(x_2,t_2)\in U_2$, $0<\mu_2\le1$.

(A4) All the function $I_i:H_\alpha\to H_\alpha$, $(i=1,\cdots,p)$ are continuous function such that

(i) $\| I_i(u)\|_\alpha\le L_i$, for all $\alpha\in(0,1)$.

(ii) $\| I_i(u_1)-I_i(u_2)\|_\alpha\le N_i\| u_1-u_2\|_\alpha$, for all $u_1,u_2\in H_\alpha$.

where L_i and N_i, $i = 1, \ldots, p$ are positive constants.

From [12], we adopt the following thought of solution.

Definition 2.4 A piecewise continuous function $x : [0, T_0] \to H$ is said to be a mild solution for the system (1)–(3) if $x \in \mathcal{C}^{\alpha}_{T_0} \cap \mathcal{C}^{\alpha-1}_{T_0}$ and satisfy the following impulsive integral equation

$$x(t) = \mathcal{S}_q(t)u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s, x(s), x(a(x(s), s)))ds + \sum_{0<t_i<t} \mathcal{S}_q(t-t_i)I_i(x(t_i)), \quad \text{for all } \; t \in [0, T_0]. \tag{17}$$

The operator $\mathcal{S}_q(t)$ and $\mathcal{T}_q(t)$ are defined as follows:

$$\mathcal{S}_q(t) = \int_0^\infty \zeta_q(\xi)\mathcal{T}(t^q\xi)d\xi, \tag{18}$$

$$\mathcal{T}_q(t) = q\int_0^\infty \xi\zeta_q(\xi)\mathcal{T}(t^q\xi)d\xi, \tag{19}$$

where $\zeta_q(\xi) = \frac{1}{q}\xi^{1-1/q} \times \psi_q(\xi^{-\frac{1}{q}})$ is a a probability density function defined on $(0, \infty)$ i.e., $\zeta_q(\xi) \geq 0$, $\int_0^\infty \zeta_q(\xi)d\xi = 1$ and

$$\psi_q(\xi) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\xi^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \; \xi \in (0, \infty).$$

For more details of probability function and generalized functions, we refer to papers [36–39].

Lemma 2.2 *The operator $\mathcal{S}_q(t)$, $t \geq 0$ and $\mathcal{T}_q(t)$, $t \geq 0$ are bounded linear operators and satisfy*

(i) $\| \mathcal{S}_q(t)y\| \leq M\| y\|$, $\| \mathcal{T}_q(t)y\| \leq \frac{qM}{\Gamma(1+q)}\|y\|$ and $\| A^{\alpha}\mathcal{T}_q(t)y\| \leq \frac{qM_\alpha\Gamma(2-\alpha)t^{-q\alpha}}{\Gamma(1+q(1-\alpha))}\| y\|$, for any $y \in H$.

(ii) The families $\{\mathcal{S}_q(t) : t \geq 0\}$ and $\{\mathcal{T}_q(t) : t \geq 0\}$ are strongly continuous.

(iii) If $\mathcal{T}(t)$ is compact, then $\mathcal{S}_q(t)$ and $\mathcal{T}_q(t)$ are compact operators for any $t > 0$.

Approximate Solutions and Convergence

The existence of approximate solutions for the problem (1) is established in this section. Let $\mathcal{H}_n$ be the finite dimensional subspace of H which is spanned by $\{\phi_0, \phi_1, \cdots, \phi_n\}$ and $P^n : H \to \mathcal{H}_n$ be the corresponding projection operator for $n = 0, 1, 2, \cdots,$. We define

$$f_n : \mathbb{R}_+ \times H^2 \to H, \quad \text{and} \quad I_{i,n} : H \to H, \tag{20}$$

by

$$f_n\left(t, x(t), x(a(x(t), t))\right) = f\left(t, P^n x(t), P^n x(a(x(t), t))\right), \tag{21}$$

and

$$I_{i,n}(x) = I_i(P^n x), \; \forall\, x \in H, \; n = 0, 1, 2, \ldots, \tag{22}$$

for $i = 1, 2, \ldots, p$, respectively. We choose T, $0 < T \leq T_0$ sufficiently small such that

$$T < \left\{\frac{2R}{3}\left[\frac{(1-\alpha)\Gamma(1+q(1-\alpha))}{M_\alpha N_f \Gamma(2-\alpha)}\right]\right\}^{\frac{1}{q(1-\alpha)}}, \tag{23}$$

$$\| [\mathcal{S}_q(t) - I]u_0\|_\alpha + M\sum_{i=1}^{p} L_i < \frac{R}{3}, \tag{24}$$

$$\frac{M_\alpha \Gamma(2-\alpha)T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))}\left[\mathcal{L}_f(1+\mathcal{L}\mathcal{L}_a)\right] + M\sum_{i=1}^{p} N_i < 1. \tag{25}$$

Now, we consider

$$\mathcal{B} = \left\{ y \in \mathcal{C}_T^\alpha \cap \mathcal{C}_T^{\alpha-1} : y(0) = u_0,\ \| y - u_0\|_{T,\alpha} \le R \right\}. \tag{26}$$

By the assumptions $(A2)-(A3)$, we have that f is continuous on $[0, T]$. Therefore, there exist a constant $N_f > 0$ such that

$$N_f = \mathcal{L}_f\left[T^{\mu_1} + R(1+\mathcal{L}\mathcal{L}_a) + \mathcal{L}\mathcal{L}_a T^{\mu_2}\right] + N, \quad \text{where} \quad N = \| f(0, u_0, u_0)\|, \tag{27}$$

with

$$\| f(\tau, x(\tau), x(a(x(\tau), \tau)))\| \le N_f, \quad x \in H, \quad \tau \in [0, T]. \tag{28}$$

Now, we define the operator Q_n on $\mathcal{B}$ as follows

$$\begin{aligned} Q_n x(t) = \mathcal{S}_q(t)u_0 &+ \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s) f_n(s, x(s), x(a(x(s), s)))ds \\ &+ \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_{i,n}(x(t_i)), \end{aligned} \tag{29}$$

for $t \in [0, T]$ and $x \in \mathcal{B}$.

Theorem 3.1 *Suppose $(A1)-(A4)$ holds and $u_0 \in D(A^\alpha)$, for $0 \le \alpha < 1$. Then, there exists a unique fixed point $x_n \in \mathcal{C}_T^\alpha \cap \mathcal{C}_T^{\alpha-1}$ of the map Q i.e., $Q_n x_n = x_n$ for each $n = 0, 1, 2, \cdots$, and x_n satisfies the following approximate integral equation*

$$\begin{aligned} x_n(t) = \mathcal{S}_q(t)u_0 &+ \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s) f_n(s, x_n(s), x_n(a(x_n(s), s)))ds \\ &+ \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_{i,n}(x_n(t_i)), \end{aligned} \tag{30}$$

for $t \in [0, T]$.

Proof To demonstrate the theorem, we first need to show that $Q_n x \in \mathcal{C}_T^\alpha \cap \mathcal{C}_T^{\alpha-1}$. It is clear that $Q_n : \mathcal{C}_T^\alpha \to \mathcal{C}_T^\alpha$. Now, it remains to show that $Q_n x \in \mathcal{C}_T^{\alpha-1}$. For $x \in \mathcal{C}_T^{\alpha-1}$, $0 < \tau < t < T$, then we have

$$\begin{aligned} &\| Q_n x(t) - Q_n x(\tau)\|_{\alpha-1} \\ &\le \| \left[\mathcal{S}_q(t) - \mathcal{S}_q(\tau)\right] u_0\|_{\alpha-1} + \int_0^\tau \| (t-s)^{q-1}\mathcal{T}_q(t-s) - (\tau-s)^{q-1}\mathcal{T}_q(\tau-s)\|_{\alpha-1} \\ &\quad \times \| f_n(s, x(s), x(a(x(s), s)))\| ds \\ &\quad + \int_\tau^t \|(t-s)^{q-1}\mathcal{T}_q(t-s)\|_{\alpha-1}\| f_n(s, x(s), x(a(x(s), s)))\| ds, \\ &\quad + \sum_{i=1}^{p} \| \left[\mathcal{S}_q(t-t_i) - \mathcal{S}_q(\tau-t_i)\right] I_{i,n}(x(t_i))\|_{\alpha-1}. \end{aligned}$$

From the first term of above inequality, we have

$$\left[\mathcal{S}_q(t) - \mathcal{S}_q(\tau)\right] A^{\alpha-1}u_0 = \int_0^\infty \zeta_q(\xi)\left[\mathcal{T}(t^q\xi) - \mathcal{T}(\tau^q\xi)\right] A^{\alpha-1}u_0 d\xi, \tag{31}$$

Also, we have that for each $z \in H$

$$\left[\mathcal{T}(t^q\xi) - \mathcal{T}(\tau^q\xi)\right] z = \int_\tau^t \frac{d}{ds}\mathcal{T}(s^q\xi)zds = \int_\tau^t q\xi s^{q-1}A\mathcal{T}(s^q\xi)zds. \tag{32}$$

Therefore, we estimate the first term as

$$\begin{aligned}
&\int_0^\infty \zeta_q(\xi)\|\mathcal{T}(t^q\xi) - \mathcal{T}(\tau^q\xi)\|\| A^{\alpha-1}u_0\|d\xi \\
&\le \int_0^\infty \zeta_q(\xi)[\int_\tau^t \|\frac{d}{ds}\mathcal{T}(s^q\xi)\|]\| u_0\|_{\alpha-1}d\xi, \\
&\le \int_0^\infty \zeta_q(\xi)[M_1(t-\tau)]\| u_0\|_{\alpha-1}d\xi, \\
&\le K_1(t-\tau)\int_0^\infty \zeta_q(\xi)d\xi, \\
&= K_1(t-\tau),
\end{aligned} \tag{33}$$

where $K_1 = M_1\| u_0\|_{\alpha-1}$. The second integrals is estimated as

$$\begin{aligned}
&\int_0^\tau \| (t-s)^{q-1}\mathcal{T}_q(t-s) - (\tau-s)^{q-1}\mathcal{T}_q(\tau-s)\|_{\alpha-1}\| f_n(s, x(s), x(a(x(s), s)))\|ds \\
&\le \int_0^\tau\int_0^\infty \zeta_q(\xi)\| \left[\frac{d}{d\varsigma}\mathcal{T}((\varsigma-s)^q\xi)|_{\varsigma=t} - \frac{d}{d\varsigma}\mathcal{T}((\varsigma-s)^q\xi)|_{\varsigma=\tau}\right] A^{\alpha-2}\| \\
&\quad \times\| f_n(s, x(s), x(a(x(s), s)))\|d\xi ds, \\
&\le \int_0^\tau\int_0^\infty \zeta_q(\xi)\left[\int_\tau^t \| A^{\alpha-2}\frac{d^2}{d\varsigma^2}\mathcal{T}((\varsigma-s)^q\xi)\|d\varsigma\right] N_f d\xi ds, \\
&\le \int_0^\tau\int_0^\infty \zeta_q(\xi)\left[\| A^{\alpha-2}\|M_2(t-\tau)\right] N_f d\xi ds, \\
&\le K_2(t-\tau),
\end{aligned} \tag{34}$$

where $K_2 = \| A^{\alpha-2}\|M_2N_fT$. The third integrals is estimated as

$$\begin{aligned}
&\int_\tau^t \|(t-s)^{q-1}\mathcal{T}_q(t-s)\|_{\alpha-1}\| f_n(s, x(s), x(a(x(s), s)))\|ds \\
&\le \int_\tau^t\int_0^\infty \zeta_q(\xi)\| [q(t-s)^{q-1}\xi A\mathcal{T}((t-s)^q\xi)]A^{\alpha-2}\| \\
&\quad \times\| f_n(s, x(s), x(a(x(s), s)))\|d\xi ds, \\
&\le \int_\tau^t\int_0^\infty \zeta_q(\xi)\| \frac{d}{d\varsigma}\mathcal{T}((\varsigma-s)^q\theta)|_{\varsigma=t}A^{\alpha-2}\|N_f d\xi ds, \\
&\le K_3(t-\tau),
\end{aligned} \tag{35}$$

where $K_3 = M_1\| A^{\alpha-2}\|N_f$. Similarly, we estimate

$$\sum_{i=1}^p \| \left[\mathcal{S}_q(t-t_i) - \mathcal{S}_q(\tau-t_i)\right] A^{\alpha-1}I_{i,n}(x(t_i))\| \le K_4(t-\tau), \tag{36}$$

where $K_4 = M_1\| A^{-1}\| \sum_{i=1}^{p} L_i$.
Thus, from the inequality (33)–(36), we obtain that

$$\| Q_n x(t) - Q_n x(\tau)\|_{\alpha-1} \leq \mathcal{L}(t-\tau), \tag{37}$$

for a positive suitable constant $\mathcal{L}$. Therefore, we conclude that $(Q_n x) \in \mathcal{C}_T^{\alpha-1}$. Hence, the $Q_n : \mathcal{C}_T^{\alpha-1} \to \mathcal{C}_T^{\alpha-1}$ is a well defined map.
Next, we prove that $Q_n : \mathcal{B} \to \mathcal{B}$. For $0 \leq t \leq T$ and $x \in \mathcal{B}$, we get that $\| (Q_n x)(t) - u_0\|_\alpha$

$$\begin{aligned}
&\leq \| \left[\mathcal{S}_q(t) - I\right] u_0\|_\alpha + \int_0^t \| (t-s)^{q-1}\mathcal{T}_q(t-s) f_n(s, x(s), x(a(x(s), s)))\|_\alpha ds \\
&\quad + \sum_{i=1}^{p} \| \mathcal{S}_q(t-t_i) I_{i,n}(x(t_i))\|_\alpha, \\
&\leq \| \left[\mathcal{S}_q(t) - I\right] u_0\|_\alpha + \frac{q M_\alpha N_f \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \int_0^t (t-s)^{q(1-\alpha)-1} ds + M \sum_{i=1}^{p} L_i, \\
&\leq \| \left[\mathcal{S}_q(t) - I\right] u_0\|_\alpha + \frac{M_\alpha N_f \Gamma(2-\alpha) T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} + M \sum_{i=1}^{p} L_i,
\end{aligned} \tag{38}$$

Therefore, it gives that $Q_n(\mathcal{B}) \subset \mathcal{B}$. At long last, we will assert that Q_n is a contraction map. For $x,\ y \in \mathcal{B}$ and $0 \leq t \leq T$, we get that

$$\begin{aligned}
\| (Q_n x)(t) - (Q_n y)(t)\|_\alpha \leq & \int_0^t \| (t-s)^{q-1} A^\alpha \mathcal{T}_q(t-s)\| \\
& \times \| f_n(s, x(s), x(a(x(s), s))) - f_n(s, y(s), y(a(y(s), s)))\| ds \\
& + \sum_{i=1}^{p} \| \mathcal{S}_q(t-t_i)\| \| I_{i,n}(x(t_i)) - I_{i,n}(y(t_i))\|_\alpha.
\end{aligned} \tag{39}$$

We have the following inequalities:

$$\| f_n(s, x(s), x(a(x(s), s))) - f_n(s, y(s), y(a(y(s), s)))\| \leq \mathcal{L}_f [2 + \mathcal{L}\mathcal{L}_a] \| x - y\|_{T,\alpha}. \tag{40}$$

Similarly, we have

$$\| I_{i,n}(x(t_i)) - I_{i,n}(y(t_i))\| \leq N_i \| x - y\|_{T,\alpha}. \tag{41}$$

Using (40), (41) in (39)and obtain that

$$\begin{aligned}
\| (Q_n x)(t) - (Q_n y)(t)\| \leq & \frac{q M_\alpha \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \mathcal{L}_f \left[2 + \mathcal{L}\mathcal{L}_a\right] \| x - y\|_{T,\alpha} \\
& \times \int_0^t (t-s)^{q(1-\alpha)-1} ds + M \sum_{i=1}^{p} N_i \| x - y\|_{T,\alpha}, \\
\leq & \left[\frac{M_\alpha \Gamma(2-\alpha) T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} \mathcal{L}_f (2 + \mathcal{L}\mathcal{L}_a) + M \sum_{i=1}^{p} N_i \right] \\
& \times \| x - y\|_{T,\alpha}.
\end{aligned} \tag{42}$$

From the inequality (25), we get

$$\| (Q_n x)(t) - (Q_n y)(t)\| < \| x - y\|_{T,\alpha}. \tag{43}$$

Therefore, it implies that the map Q_n is a contraction map and has a unique fixed point $x_n \in \mathcal{B}$ i.e., $Q_n x_n = x_n$ and x_n satisfies the approximate integral equation

$$x_n(t) = \mathcal{S}_q(t)u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s) f_n(s, x_n(s), x_n(a(x_n(s), s)))ds + \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_{i,n}(x_n(t_i)),$$

for $t \in [0, T]$. □

Lemma 3.2 *Assume that hypotheses* $(A1) - (A4)$ *hold. If* $u_0 \in D(A^\alpha)$*, where* $0 < \alpha < 1$*, then* $x_n(t) \in D(A^\upsilon)$ *for all* $t \in (0, T]$ *with* $0 \le \upsilon < 1$*. Furthermore, if* $u_0 \in D(A)$ *then* $x_n(t) \in D(A^\upsilon)$ *for all* $t \in [0, T]$ *with* $0 \le \upsilon < 1$.

Proof From Theorem (3.1), we have that there exists a unique $x_n \in \mathcal{B} \subset \mathcal{C}_T^{\alpha-1}$ such that x_n satisfy the Eq. (30). Theorem 2.6.13 in Pazy [35] implies that $T(t) : H \to D(A^\upsilon)$ for $t > 0$ and $0 \le \upsilon < 1$ and for $0 \le \upsilon \le \eta < 1$, $D(A^\eta) \subseteq D(A^\upsilon)$. It is not difficult to see that Hölder continuity of x_n might be made using the similar arguments from Eqs. (33)–(36). Additionally from Theorem 1.2.4 in Pazy [35], we have that $T(t)y \in D(A)$ if $y \in D(A)$. The result follows from these facts and the fact that $D(A) \subseteq D(A^\upsilon)$ for $0 \le \upsilon \le 1$. This finishes the proof of Lemma. □

Corollary 3.1 *Suppose that the hypotheses* $(A1) - (A4)$ *hold. If* $u_0 \in D(A^\alpha)$ *with* $0 < \alpha < 1$*, then for any* $t_0 \in (0, T]$*, there exists a constant* U_{t_0} *such that*

$$\|A^\upsilon x_n(t)\| \le U_{t_0}, \quad n = 1, 2, 3, \cdots,$$

for all $t_0 \le t \le T$ *independent of* n*, where* $0 < \alpha < \upsilon < \beta$*. Furthermore, if* $u_0 \in D(A)$*, there exist a positive constant* U_0 *such that* $\|A^\upsilon x_n(t)\| \le U_0$*,* $t \in [0, T]$*,* $n = 1, 2, \cdots$*,.*

Proof Let $u_0 \in D(A^\alpha)$. Applying A^υ on the both the sides of (30) and $t_0 \le t \le T$, we get

$$\begin{aligned}
&\| A^\upsilon x_n(t)\| \\
&\le \| A^\upsilon \mathcal{S}_q(t)u_0\| + \int_0^t (t-s)^{q-1}\|A^\upsilon \mathcal{T}_q(t-s)\|\| f_n(s, x_n(s), x_n(a(x_n(s), s)))\|ds \\
&\quad + \sum_{i=1}^{p} \| \mathcal{S}_q(t-t_i)A^\upsilon I_i(x(t_i))\|, \\
&\le M_\upsilon t_0{}^{-q\upsilon}\| u_0\| + \frac{qM_\upsilon N_f \Gamma(2-\upsilon)}{\Gamma(1+q(1-\upsilon))} \int_0^t (t-s)^{q(1-\upsilon)-1}ds + M\sum_{i=1}^{p} L_i, \\
&\le M_\upsilon t_0{}^{-q\upsilon}\| u_0\| + \frac{M_\upsilon N_f \Gamma(2-\upsilon)T^{q(1-\upsilon)}}{(1-\upsilon)\Gamma(1+q(1-\upsilon))} + M\sum_{i=1}^{p} L_i, \qquad (44)\\
&\le U_{t_0}. \qquad (45)
\end{aligned}$$

Again, for $0 \le t \le T$ and $u_0 \in D(A^\alpha)$, we have

$$\| A^\upsilon x_n(t)\| \le M\| u_0\|_\upsilon + \frac{M_\upsilon N_f \Gamma(2-\upsilon)T^{q(1-\upsilon)}}{(1-\upsilon)\Gamma(1+q(1-\upsilon))} + M\sum_{i=1}^{p} L_i. \qquad (46)$$

Since, we might displace the first term in (44) by $M\| u_0\|_\upsilon$.
Moreover, if $u_0 \in D(A)$, then $u_0 \in D(A^\upsilon)$ for $0 \le \upsilon < 1$. Therefore, we can effortlessly get the required result. This finishes the proof of Lemma. □

Convergence of Solutions

The convergence of the solution $x_n \in H_\alpha$ of the approximate integral Eq. (30) to a unique solution $x(\cdot)$ of the Eq. (17) on $[0, T]$ is discussed in this section.

Theorem 4.1 *Suppose that* $(A1)-(A4)$ *are satisfied. If* $u_0 \in D(A^\alpha)$*, then*

$$\lim_{m\to\infty} \sup_{\{n\geq m, t_0\leq t\leq T\}} \| x_n(t) - x_m(t)\|_\alpha = 0, \tag{47}$$

for every $t_0 \in (0, T]$.

Proof For $0 < \alpha < \upsilon$, $n \geq m$ and $t \in (0, T]$, we have

$$\begin{aligned} &\| f_n(t, x_n(t), x_n(a(x_n(t), t))) - f_m(t, x_m(t), x_m(a(x_m(t), t)))\| \\ &\quad \leq \| f_n(t, x_n(t), x_n(a(x_n(t), t))) - f_n(t, x_m(t), x_m(a(x_m(t), t)))\| \\ &\qquad + \| f_n(t, x_m(t), x_m(a(x_m(t), t))) - f_m(t, x_m(t), x_m(a(x_m(t), t)))\|, \\ &\quad \leq \mathcal{L}_f\,[2 + \mathcal{L}\mathcal{L}_a]\,\| x_n(t) - x_m(t)\|_\alpha + \mathcal{L}_f\left[\| (P^n - P^m)x_m(t)\|_\alpha\right. \\ &\qquad \left. + \|A^{-1}\|\| (P^n - P^m)x_m(a(x_m(t), t))\|_\alpha\right]. \end{aligned}$$

Let $n > m$. Thus, $\mathcal{H}_m \subset \mathcal{H}_n$. Let $\mathcal{H}_m^\top$ be the orthogonal complement of $\mathcal{H}_m$ for each $m = 0, 1, \ldots,$. Thus, we have $\mathcal{H}_n^\top \subset \mathcal{H}_m^\top$. Also, we have $H = \mathcal{H}_m \oplus \mathcal{H}_m^\top = \mathcal{H}_n \oplus \mathcal{H}_n^\top$. Let $y \in H$ be an arbitrary element. Then, $y = y_m + z_m$ with $y_m \in \mathcal{H}_m$ and $z_m \in \mathcal{H}_m\top$. Therefore, we have that $y_m \in \mathcal{H}_m = P^m y$. It is easy to see that $z_m \in \mathcal{H}_m^\top \to z_m = \sum_{i=m+1}^n a_i\phi_i + z_m'$, where $z_m' \in \mathcal{H}_n^\top$. Let us take $y_m' = \sum_{i=m+1}^n a_i\phi_i$.
Therefore, $y = y_m + y_m' + z_m'$ and $P^n y = y_m + y_m'$. Thus,

$$P^n y - P^m y = y_m' = \sum_{i=m+1}^{n} a_i\phi_i.$$

If, $y = \sum_{i=1}^\infty a_i\phi_i$. Then, we get $\|y\|^2 = \sum_{i=1}^\infty |a_i|^2$.
Since, $A^{\alpha-\upsilon}\phi_i = \lambda_i^{\alpha-\upsilon}\phi$. Hence, we get

$$\begin{aligned} \|A^{\alpha-\upsilon}(P^n - P^m)y\|^2 &= < A^{\alpha-\upsilon}(P^n - P^m)y, A^{\alpha-\upsilon}(P^n - P^m)y >, \\ &= < \sum_{i=m+1}^{n} a_i A^{\alpha-\upsilon}\phi_i, \sum_{j=m+1}^{n} a_j A^{\alpha-\upsilon}\phi_j >, \\ &= < \sum_{i=m+1}^{n} a_i\lambda_i^{\alpha-\upsilon}\phi_i, \sum_{j=m+1}^{n} a_j\lambda_j^{\alpha-\upsilon}\phi_j >, \\ &= \sum_{i,j=m+1}^{n} a_i a_j \lambda_i^{\alpha-\upsilon}\lambda_j^{\alpha-\upsilon} < \phi_i, \phi_j >, \\ &\leq \lambda_{m+1}^{2(\alpha-\upsilon)}\left(\sum_{i=m+1}^{n} |a_i|^2\right), \\ &\leq \frac{1}{\lambda_m^{2(\upsilon-\alpha)}}\|y\|^2. \end{aligned} \tag{48}$$

Thus, we have the following estimation

$$\| (P^n - P^m)x_m(t)\|_\alpha \leq \| A^{\alpha-\upsilon}(P^n - P^m)A^\upsilon x_m(t)\| \leq \frac{1}{\lambda_m^{\upsilon-\alpha}}\| A^\upsilon x_m(t)\|.$$

Thus, we obtain

$$
\begin{aligned}
&\| f_n(t, x_n(t), x_n(a(x_n(t), t))) - f_m(t, x_m(t), x_m(a(x_m(t), t)))\| \\
&\quad \leq \mathcal{L}_f[2 + \mathcal{L}\mathcal{L}_a]\| x_n(t) - x_m(t)\|_\alpha + \mathcal{L}_f \left[\frac{1}{\lambda_m^{\upsilon-\alpha}} \| A^\upsilon x_m(t)\| \right. \\
&\qquad \left. + \frac{\| A^{-1}\|}{\lambda_m^{\upsilon-\alpha}} \| A^\upsilon x_m(a(x_m(t), t))\| \right].
\end{aligned} \tag{49}
$$

Similarly, we estimate

$$
\| I_{i,n}(x_n(t_i)) - I_{i,m}(x_m(t_i))\| \leq N_i \left[\| x_n(t_i) - x_m(t_i)\|_\alpha + \frac{1}{\lambda_m^{\upsilon-\alpha}} \| A^\upsilon x_m(t_i)\| \right]. \tag{50}
$$

We choose t_0' such that $0 < t_0' < t < T$, we have

$$
\begin{aligned}
&\| x_n(t) - x_m(t)\|_\alpha \\
&\leq \left(\int_0^{t_0'} + \int_{t_0'}^t \right) (t - s)^{q-1} \| A^\alpha \mathcal{T}_q(t - s)\| \\
&\times \| f_n(t, x_n(t), x_n(a(x_n(t), t))) - f_m (t, x_m(t), x_m(a(x_m(t), t))) \|ds \\
&+ \sum_{i=0}^{p} \| \mathcal{S}_q(t - t_i)\| \| I_{i,n}(x_n(t_i)) - I_{i,m}(x_m(t_i))\|_\alpha,
\end{aligned} \tag{51}
$$

we estimate the first integral as

$$
\begin{aligned}
&\int_0^{t_0'} (t - s)^{q-1} \| A^\alpha \mathcal{T}_q(t - s)\| \| f_n(t, x_n(t), x_n(a(x_n(t), t))) \\
&\quad - f_m(t, x_m(t), x_m(a(x_m(t), t)))\|ds \\
&\leq \int_0^{t_0'} (t - s)^{q-1} \| A^\alpha \mathcal{T}_q(t - s)\| 2N_f ds, \\
&\leq \frac{2N_f M_\alpha \Gamma(2 - \alpha)}{(1 - \alpha)\Gamma(1 + q(1 - \alpha))} [t^{q(1-\alpha)} - (t - t_0')^{q(1-\alpha)}], \\
&\leq \frac{2N_f M_\alpha \Gamma(2 - \alpha)}{(1 - \alpha)\Gamma(1 + q(1 - \alpha))} (t - b_1 t_0')^{q(1-\alpha)-1} t_0', \ 0 < b_1 < 1, \\
&\leq \frac{2N_f M_\alpha \Gamma(2 - \alpha)}{(1 - \alpha)\Gamma(1 + q(1 - \alpha))} (t_0 - t_0')^{q(1-\alpha)-1} t_0'.
\end{aligned} \tag{52}
$$

The second integral is estimated as

$$
\begin{aligned}
&\int_{t_0'}^t (t - s)^{q-1} \| A^\alpha \mathcal{T}_q(t - s)\| \| f_n(t, x_n(t), x_n(a(x_n(t), t))) \\
&\quad - f_m(t, x_m(t), x_m(a(x_m(t), t)))\|ds \\
&\leq \frac{q M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + q(1 - \alpha))} \int_0^t (t - s)^{q-1} \left\{ \mathcal{L}_f \left[2 + \mathcal{L}\mathcal{L}_a\right] \| x_n(s) - x_m(s)\|_\alpha \right. \\
&\quad \left. + \mathcal{L}_f \left[\frac{1}{\lambda_m^{\upsilon-\alpha}} \| A^\upsilon x_m(s)\| + \frac{\| A^{-1}\|}{\lambda_m^{\upsilon-\alpha}} \| A^\upsilon x_m(a(x_m(s), s))\| \right] \right\} ds, \\
&\leq \frac{q M_\alpha \mathcal{L}_f \Gamma(2 - \alpha)}{\Gamma(1 + q(1 - \alpha))} \left[\left(1 + \| A^{-1}\|\right) \frac{U_{t_0'} T^{q(1-\alpha)}}{q(1 - \alpha)\lambda_m^{\upsilon-\alpha}} + (2 + \mathcal{L}\mathcal{L}_a) \int_{t_0'}^t (t - s)^{q(1-\alpha)-1} \right.
\end{aligned}
$$

$$\times \| x_n(s) - x_m(s)\|_\alpha ds]\,. \tag{53}$$

Thus, we have

$$\begin{aligned}
&\| x_n(t) - x_m(t)\|_\alpha \\
&\le D_1 t_0' + \frac{D_2}{\lambda_m^{\upsilon-\alpha}} + D_3\| x_n(t) - x_m(t)\|_\alpha + D_4 \int_0^t (t-s)^{q(1-\alpha)-1} \\
&\quad \times \| x_n(s) - x_m(s)\|_\alpha ds,
\end{aligned} \tag{54}$$

where

$$D_1 = \frac{2N_f M_\alpha \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+q(1-\alpha))}(T-t_0')^{q(1-\alpha)-1}, \tag{55}$$

$$D_2 = \frac{qM_\alpha \mathcal{L}_f \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \times (1+\| A^{-1}\|)\frac{U_{t_0'} T^{q(1-\alpha)}}{q(1-\alpha)} + M\sum_{i=1}^{p} N_i, \tag{56}$$

$$D_3 = M\sum_{i=1}^{p} N_i, \tag{57}$$

$$D_4 = \frac{qM_\alpha \mathcal{L}_f \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}(2+\mathcal{L}\mathcal{L}_a), \tag{58}$$

Since $1 - M\sum_{i=1}^{p} N_i > 0$, we have

$$\begin{aligned}
&\| x_n(t) - x_m(t)\|_\alpha \\
&\le \frac{1}{1-D_3}\left[D_1 t_0' + \frac{D_2}{\lambda_m^{\upsilon-\alpha}} + D_4 \int_0^t (t-s)^{q(1-\alpha)-1}\| x_n(s) - x_m(s)\|_\alpha ds\right].
\end{aligned}$$

By Lemma 5.6.7 in [35], we have that there exists a constant $\mathcal{K}$ such that

$$\| x_n(t) - x_m(t)\|_\alpha \le \frac{1}{1-D_3}[D_1 t_0' + \frac{D_2}{\lambda_m^{\upsilon-\alpha}}]\mathcal{K}, \tag{59}$$

taking supremum over $[t_0, T]$ and letting $m \to \infty$, we obtain

$$\lim_{m\to\infty} \sup_{\{n\ge m, t_0\le t\le T\}} \| x_n(t) - x_m(t)\|_\alpha \le \frac{D_1}{(1-D_3)} t_0' \mathcal{K}. \tag{60}$$

As t_0' is arbitrary, therefore the right hand side may be made as small as desired by taking t_0' sufficiently small. This completes the proof of the Theorem. □

Proposition 4.2 *If $u_0 \in D(A)$, then there exist a Cauchy sequence $x_n \in \mathcal{B}$ on* $[0, T]$ *i.e.,*

$$\| x_n - x_m\|_{T,\alpha} \to 0, \tag{61}$$

as $m, n \to \infty$.

Proof Taking $t_0 = 0$ in the proof of Theorem 4.1, we replace the term $(t_0 - t_0')^{q(1-\alpha)-1} t_0'$ by $(1-b_1)^{q(1-\alpha)-1} {t_0'}^{1-\alpha}$ in Eq. 52 and the constant $U_{t_0'}$ by the constant U_0 from the Lemma 3.1 and Corollary 3.1. □

Theorem 4.3 *Suppose that (A1)–(A4) are satisfied and $u_0 \in D(A^\alpha)$. Then, there exists a unique $x_n \in \mathcal{B}$, satisfying*

$$x_n(t) = \mathcal{S}_q(t)u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f_n(s, x_n(s), x_n(a(x_n(s), s)))ds$$
$$+\sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_{i,n}(x_n(t_i)),\ t \in [0, T],$$

and $x \in \mathcal{B}$, satisfying

$$x(t) = \mathcal{S}_q(t)u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s, x(s), x(a(x(s), s)))ds$$
$$+\sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_i(x(t_i)),\ t \in [0, T],$$

such that x_n converges to x in $\mathcal{B}$ i.e., $x_n \to x$ as $n \to \infty$.

Proof Let $u_0 \in D(A^\alpha)$. For $0 < t \leq T$, $A^\alpha x_n(t) \to A^\alpha x(t)$ as $n \to \infty$ and $x(0) = x_n(0) = u_0$ for all n. Also, for $t \in [0, T]$, we have $A^\alpha x_n(t) \to A^\alpha x(t)$ as $n \to \infty$ in H. Since $x_n \in \mathcal{B}$, therefore it follows that $x \in \mathcal{B}$ and

$$\lim_{n\to\infty} \sup_{t_0 \leq t \leq T} \| x_n(t) - x(t)\|_\alpha = 0, \quad \text{for any } t_0 \in (0, T]. \tag{62}$$

Also, we have

$$\begin{aligned}&\| f_n(t, x_n(t), x_n(a(x_n(t), t))) - f(t, x(t), x(a(x(t), t)))\| \\ &\leq \mathcal{L}_f\,[2 + \mathcal{L}\mathcal{L}_a]\,\| x_n(t) - x(t)\|_\alpha + \mathcal{L}_f \left[\| (P^n - I)x(t)\|_\alpha \right.\\ &\quad \left. +\| A^{-1}\|\, \| (P^n - I)x(a(x(t), t))\|_\alpha\right] \to 0,\end{aligned} \tag{63}$$

as $n \to \infty$. For $0 < t_0 < t$, we rewrite 30 as

$$x_n(t) = \mathcal{S}_q(t)u_0 + (\int_0^{t_0} + \int_{t_0}^t)(t-s)^{q-1}\mathcal{T}_q(t-s)f_n(s, x_n(s), x_n(a(x_n(s), s)))ds$$
$$+\sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_{i,n}(x_n(t_i)).$$

We may estimate the first integral as

$$\| \int_0^{t_0} (t-s)^{q-1}\mathcal{T}_q(t-s)f_n(s, x_n(s), x_n(a(x_n(s), s)))ds\| \leq \frac{qMN_f}{\Gamma(1+q)}T^{q-1}t_0, \tag{64}$$

Thus, we deduce that

$$\begin{aligned}&\| x_n(t) - \mathcal{S}_q(t)u_0 - \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_{i,n}(x_n(t_i)) - \int_{t_0}^t (t-s)^{q-1}\mathcal{T}_q(t-s) \\ &\times f_n(s, x_n(s), x_n(a(x_n(s), s)))ds\| \leq \left[\frac{qMN_f}{\Gamma(1+q)}T^{q-1}\right]t_0.\end{aligned} \tag{65}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned}&\| x(t)-\mathcal{S}_q(t)u_0-\sum_{i=1}^{p}\mathcal{S}_q(t-t_i)I_i(x(t_i))-\int_{t_0}^{t}(t-s)^{q-1}\mathcal{T}_q(t-s)\\&\quad\times f(s,x(s),x(a(x(s),s)))ds\|\\&\quad\le\left[\frac{qMN_f}{\Gamma(1+q)}T^{q-1}\right]t_0. \end{aligned}\tag{66}$$

Since t_0 is arbitrary, we deduce that x satisfies the integral Eq. (17).
Now, we shall show the uniqueness of the solution to Eq. (17). Let x_1 and x_2 be the two solutions of the (17). We have

$$\begin{aligned}\| x_1(t)-x_2(t)\|_\alpha &\le\int_0^t(t-s)^{q-1}\| A^\alpha\mathcal{T}_q(t-s)\|\| f(s,x_1(s),x_1(a(x_1(s,)s)))\\&\quad-f(s,x_2(s),x_2(a(x_2(s,)s)))\|ds\\&\quad+\sum_{i=1}^{p}\|\mathcal{S}_q(t-t_i)\|\| I_i(x_1(t_i))-I_i(x_2(t_i))\|,\\&\le\frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\mathcal{L}_f(2+\mathcal{L}\mathcal{L}_a)\| x_1-x_2\|_{T,\alpha}\\&\quad\int_0^t(t-s)^{q(1-\alpha)-1}ds+M\sum_{i=1}^{p}N_i\| x_1-x_2\|_{T,\alpha},\\&\le\left[\frac{M_\alpha\Gamma(2-\alpha)T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))}\mathcal{L}_f(2+\mathcal{L}\mathcal{L}_a)+M\sum_{i=1}^{p}N_i\right]\\&\quad\times\| x_1-x_2\|_{T,\alpha},\end{aligned}\tag{67}$$

By Lemma 5.6.7 in Pazy [35], we obtain that

$$\| x_1(t)-x_2(t)\|=0.\tag{68}$$

Also, we have that

$$\| x_1(t)-x_2(t)\|\le\frac{1}{\lambda_0^\alpha}\| x_1(t)-x_2(t)\|_\alpha,\tag{69}$$

From (68) and (69), we deduce that $u_1=u_2$ on $[0,T]$. Hence, the theorem is proved. □

Faedo–Galerkin Approximations

In this section, we consider the Faedo–Galerkin Approximation of a solution and show the convergence results for such an approximation.

We know that for any $0<T<T_0$, we have a unique $x\in\mathcal{C}_T^\alpha$ satisfying the following integral equation

$$\begin{aligned}x(t)&=\mathcal{S}_q(t)u_0+\int_0^t(t-s)^{q-1}\mathcal{T}_q(t-s)f(s,x(s),x(a(x(s),s)))ds\\&\quad+\sum_{i=1}^{p}\mathcal{S}_q(t-t_i)I_i(x(t_i)),\end{aligned}\tag{70}$$

for $0<t<T_0$.

Also, we have a unique solution $x_n \in \mathcal{C}_T^\alpha$ of the approximate integral equation

$$x_n(t) = \mathcal{S}_q(t)u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f_n(s, x_n(s), x_n(a(x_n(s), s)))ds + \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_{i,n}(x_n(t_i)), \tag{71}$$

Applying the projection on above equation, then Faedo–Galerkin approximation is given by $v_n(t) = P^n x_n(t)$ satisfying

$$\begin{aligned} P^n x_n(t) &= v_n(t) \\ &= \mathcal{S}_q(t)P^n u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)P^n f_n(s, x_n(s), x_n(a(x_n(s), s)))ds \\ &\quad + \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)P^n I_{i,n}(x_n(t_i)), \\ &= \mathcal{S}_q(t)P^n u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)P^n f(s, v_n(s), v_n(a(v_n(s), s)))ds \\ &\quad + \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)P^n I_i(v_n(t_i)). \end{aligned} \tag{72}$$

Let solution $x(\cdot)$ of (70) and $v_n(\cdot)$ of (72), have the representation

$$x(t) = \sum_{i=0}^{\infty} \alpha_i(t)]\phi_i, \quad \alpha_i(t) = (x(t), \phi_i), \ \ i = 0, 1, 2 \ldots, \tag{73}$$

$$v_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)\phi_i, \quad \alpha_i^n(t) = (v_n(t), \phi_i), \ \ i = 0, 1, 2 \ldots, \tag{74}$$

Using (74) in (72) and taking inner product with ϕ_i, we obtain a system of fractional order integro-differential equation of the form

$$\frac{d^q}{dt^q}\alpha_i^n(t) + \lambda_i \alpha_i^n(t) = F_i^n(t, \alpha_0^n(t), \alpha_1^n(t) \ldots, \alpha_n^n(t)), \tag{75}$$

$$\Delta\alpha_i^n(t_k) = I_i^n(\alpha_i^n(t_k)), \ \ k = 1, \ldots, p, \tag{76}$$

$$\alpha_i^n(0) = \varphi_i, \tag{77}$$

where

$$F_i^n = \left(f\left(t, \sum_{i=0}^{n} \alpha_i^n \phi_i, \sum_{i=0}^{n} \tau_i^n \phi_i\right), \phi_i \right), \tag{78}$$

$$\tau_i^n = \alpha_i^n\left(a\left(\alpha_0^n, \alpha_1^n, \ldots, \alpha_n^n(t)\right)\right), \tag{79}$$

$$I_i^n = \left(I_k\left(\sum_{k=1}^{p} \sum_{i=1}^{n} \alpha_i^n(t_k)\phi_i\right), \phi_i \right), \tag{80}$$

$$\varphi_i = (u_0, \phi_i), \quad \text{for} \ \ i = 1, 2, \ldots, n. \tag{81}$$

For the convergence of α_i^n to α_i, we have the following convergence theorem.

Theorem 5.1 *Let us assume that* $(A1)--(A4)$ *are satisfied and* $u_0 \in D(A^\alpha)$. *Then there exist a unique function* $v_n \in \mathcal{B}$ *given as*

$$v_n(t) = \mathcal{S}_q(t)P^n u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f_n(s, v_n(s), v_n(a(v_n(s), s)))ds + \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)P^n I_i(v_n(t_i)), \tag{82}$$

and $x \in \mathcal{B}$ *satisfying*

$$x(t) = \mathcal{S}_q(t)u_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s, x(s), x(a(x(s), s)))ds + \sum_{i=1}^{p} \mathcal{S}_q(t-t_i)I_i(x(t_i)), \tag{83}$$

for $t \in [0, T_0]$, *such that* $v_n \to x$ *as* $n \to \infty$ *in* $\mathcal{B}$ *and* x *satisfies the Eq. (17) on* $[0, T_0]$.

The system (75)–(77) determines the α_i^n's. It can easily be investigated that

$$A^\alpha[x(t) - v(t)] = A^\alpha\left[\sum_{i=0}^{\infty}(\alpha_i(t) - \alpha_i^n(t))\phi_i\right] = \sum_{i=0}^{\infty}\lambda_i^\alpha(\alpha_i(t) - \alpha_i^n)\phi_i, \tag{84}$$

Thus, we conclude that

$$\| A^\alpha[x(t) - v(t)]\|^2 \geq \sum_{i=0}^{n}\lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2. \tag{85}$$

Theorem 5.2 *Let us assume that* $(A1)--(A4)$ *are satisfied. Then, we have the following results*

(a) If $u_0 \in D(A^\alpha)$, *then*

$$\lim_{n\to\infty}\sup_{t\in[t_0,T_0]}\left[\sum_{i=0}^{n}\lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2\right] = 0, \tag{86}$$

for any $0 < t_0 \leq T_0$.

(b) If $u_0 \in D(A)$, *then*

$$\lim_{n\to\infty}\sup_{t\in[0,T_0]}\left[\sum_{i=0}^{n}\lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2\right] = 0, \tag{87}$$

for any $0 \leq t \leq T_0$.

The statement of this hypothesis takes after from the facts specified above and the following results.

Corollary 5.1 *Assume that* $(A1)--(A4)$ *are satisfied. Then*

(a) If $u_0 \in D(A^\alpha)$, *then*

$$\sup_{t\in[t_0,T_0]} \| v_n(t) - v_m(t)\|_\alpha \to 0, \quad as \;\; m, n \to \infty, \tag{88}$$

for any $0 < t_0 \leq T_0 < T_{\max}$.

(b) If $u_0 \in D(A)$*, then*

$$\sup_{t\in[0,T_0]} \| v_n(t) - v_m(t)\|_\alpha \to 0, \quad as \;\; m, n \to \infty. \tag{89}$$

Proof For $n \geq m$ and $0 \leq \alpha < \upsilon$, we get

$$\begin{aligned} \| v_n(t) - v_m(t)\|_\alpha &= \| P^n x_n(t) - P^m x_n(t)\|_\alpha, \\ &\leq \| P^n[x_n(t) - x_m(t)]\|_\alpha + \| (P^n - P^m)x_m(t)\|_\alpha, \\ &\leq \| x_n(t) - x_m(t)\|_\alpha + \frac{1}{\lambda_m^{\upsilon-\alpha}} \| A^\upsilon x_m(t)\|. \end{aligned} \tag{90}$$

If $u_0 \in D(A^\alpha)$ then the result in (a) follows from Theorem 4.1, If $u_0 \in D(A)$, (b) follows from Proposition 4.2. □

Application

In this section, we present an example to show the feasibility of our abstract result.
Let us consider following fractional differential equation with impulsive conditions in the separable Hilbert space H

$$\frac{\partial^q w}{\partial t^q} = \frac{\partial^2 w}{\partial u^2} + \widetilde{P}(u, w(u,t)) + \widetilde{H}(t, u, w(u,t)), \;\; (u,t) \in (0,1) \times \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right), \tag{91}$$

$$\delta w|_{t=\frac{1}{2}} = \frac{2w(\frac{1}{2})^-}{2 + w(\frac{1}{2})^-}, \tag{92}$$

$$w(0,t) = w(1,t) = 0, \tag{93}$$

$$w(x, 0) = w_0(u), \quad u \in (0,1), \tag{94}$$

where $0 < q < 1$, $\widetilde{H} : \mathbb{R}^+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function which is measurable in u, locally Hölder continuous in first argument t, locally Lipschitz continuous in w and uniformly in u. The function $\widetilde{P}$ is given as

$$\widetilde{P}(u, w(u,t)) = \int_0^u (G)(u,y) w(y, h(t)|w(y,t)|) dy, \tag{95}$$

here, $h : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be locally Hölder continuous in t with $h(0) = 0$ and $\mathcal{G} \in C^1([0,1] \times [0,1], \mathbb{R})$.
Now, we take $H = L^2((0,1), \mathbb{R})$ and operator A as $Aw = d^2w/dx^2$ with domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$. Let $\alpha = 1/2$, then $H_{1/2} = D(A^{1/2}) = H_0^1(0,1)$ is a Banach space with norm $\|w\|_{1/2} := \|A^{1/2}w\|$, for $w \in D(A^{1/2})$ and $H_{-1/2} = (H_0^1(0,1))^* = H^{-1}(0,1) \equiv H^1(0,1)$ is dual space of the space $H_{1/2}$.
Now, for each $u \in (0,1)$, we may consider the function $f : \mathbb{R}^+ \times H_{1/2} \times H_{-1/2} \to H$ defined as

$$f(t, w, z)(u) = \widetilde{P}(u, z) + \widetilde{H}(t, u, w), \tag{96}$$

with $\widetilde{P} : [0,1] \times H_{-1/2} \to H$ which is defined by

$$\widetilde{P}(u, z) = \int_0^u \mathcal{G}(u,y) z(y) dy, \tag{97}$$

and $\mathcal{H} : \mathbb{R}^+ \times [0,1] \times H_{1/2} \to H$ fulfills following conditions

$$\|\mathcal{H}(t,u,w)\| \leq \mathcal{Q}(u,t)(1+\|w\|_{1/2}), \tag{98}$$

where $\mathcal{Q}(\cdot,t) \in H$ and $\mathcal{Q}$ continuous in its second arguments.
For $w \in D(A)$ and $\lambda \in \mathbb{R}$ with $-Aw = \lambda w$, we obtain

$$< -Aw, w >=< \lambda w, w >,$$

and $\|w'\|_{L^2} = \lambda \|w\|_{L^2}$. This gives that $\lambda > 0$. Let $w(u) = C_1 \sin(\sqrt{\lambda} u) + C_2 \cos(\sqrt{\lambda} u)$ be the solution of the equation $-Aw = w'' = \lambda w$. We use the boundary condition and get $C_2 = 0$ and $\lambda = \lambda_n = n^2\pi^2$ for each $n \in \mathbb{N}$. Therefore, we get

$$w_n(u) = C_1 \sin(\sqrt{\lambda_n} u), \quad \textit{text for each} \ \ n \in \mathbb{N}, \tag{99}$$

and $< w_n, w_m >= 0, \ \ m \neq n, < w_n, w_n >= 1$.
For $w \in D(A)$, there exists a sequence of real numbers $\{\beta_n\}$ such that

$$w(u) = \sum_{n \in \mathbb{N}} \beta_n w_n(u), \ \ \sum_{n \in \mathbb{N}} (\beta_n)^2 < +\infty, \ \ \sum_{n \in \mathbb{N}} (\lambda_n)^2 (\beta_n)^2 < +\infty.$$

We also have

$$A^{1/2} w(u) = \sum_{n \in \mathbb{N}} \sqrt{\lambda_n} \beta_n w_n(u), \quad w \in D(A^{1/2}) \tag{100}$$

with $\sum_{n \in \mathbb{N}} \lambda_n (\beta_n)^2 < +\infty$. The semigroup $\mathcal{S}(t)$ have the following expression as

$$\mathcal{S}(t)w = \sum_{n=1}^{\infty} \exp(n^2 t) < w, w_n > w_n, \tag{101}$$

here, $\{w_n\}, n = 1, 2, \cdots$ denotes the orthogonal set of eigenfunctions of A defined by the (99). Now, we will show that $(A2)$-$(A3)$ are verified. For $(A2)$, we have that $\widetilde{P} : [0,1] \times H_{-1/2} \to H$ defined by

$$\widetilde{P}(u, z(u,t)) = \int_0^u \mathcal{G}(u,y) z(y,t) dy,$$

and $z(u,t) = z(y, h(t)|z(y,t)|)$. Thus, for each $u \in [0,1]$, we obtain

$$\begin{aligned} |\widetilde{P}(u, z_1(u,)) - \widetilde{P}(u, z_2(u,))| &\leq \int_0^u |\mathcal{G}(u,y)| \cdot |(z_1 - z_2)(y,\cdot)| dy, \\ &\leq \|\mathcal{G}\|_\infty \int_0^u |(z_1 - z_2)(y,\cdot)| dy. \end{aligned} \tag{102}$$

Since $z_1, z_2 \in H^1(0,1)$. Therefore, applying the Minkowski's integral inequality and getting

$$\begin{aligned} |\widetilde{P}(u, z_1(u,)) - \widetilde{P}(u, z_2(u,))|^2_{L^2(0,1)} &\leq \|\mathcal{G}\|^2_\infty \int_0^1 \int_0^y |(z_1 - z_2)(y,\cdot)|^2 dx dy, \\ &\leq \|\mathcal{G}\|^2_\infty \int_0^1 y |(z_1 - z_2)(y,\cdot)|^2 dy, \\ &\leq \|\mathcal{G}\|^2_\infty \|z_1 - z_2\|^2_{L^2(0,1)}. \end{aligned} \tag{103}$$

Since we have

$$\frac{\partial}{\partial u} \widetilde{P}(u, z(u,\cdot)) = \mathcal{G}(u,u) z(u,\cdot) + \int_0^u \frac{\partial \mathcal{G}}{\partial u}(u,u) z(y,\cdot) dy. \tag{104}$$

Thus, we estimate

$$\|\frac{\partial}{\partial u}\widetilde{P}(u,z_1(u,\cdot))-\frac{\partial}{\partial u}\widetilde{P}(u,z_2(u,\cdot))\|_{L^2(0,1)}\leq\left(\|\mathcal{G}\|_\infty+\|\frac{\partial\mathcal{G}}{\partial u}\|_\infty\right)\|z_1-z_2\|_{L^2(0,1)}.$$

Therefore,

$$\begin{aligned}\|\widetilde{P}(u,z_1(u,\cdot))-\widetilde{P}(u,z_2(u,\cdot))\|_{H^1(0,1)}&\leq\left(2\|\mathcal{G}\|_\infty+\|\frac{\partial\mathcal{G}}{\partial u}\|_\infty\right)\|z_1-z_2\|_{L^2(0,1)},\\&\leq\left(2\|\mathcal{G}\|_\infty+\|\frac{\partial\mathcal{G}}{\partial u}\|_\infty\right)\|z_1-z_2\|_{H^1(0,1)},\end{aligned}$$

The assumption on $\widetilde{H}$ gives that there exist constants $B_2>0$ and $\mu\in(0,1]$ such that

$$\|\widetilde{H}(t,u,w_1)-\widetilde{H}(s,u,w_2)\|_{H_0^1(0,1)}\leq B_2(|t-s|^\mu+\|w_1-w_2\|_{H_0^1(0,1)}), \tag{105}$$

for all $t,s\in[0,1]$, $u\in(0,1)$ and $w_1,\ w_2\in H_0^1(0,1)$. Therefore, $f:[0,1]\times H_0^1(0,1)\times H^1(0,1)\to L^2(0,1)$ defined by $f=\widetilde{P}+\widetilde{H}$ fulfills the assumption $(A2)$.
Next, we will show that $a:H_0^1(0,1)\times\mathbb{R}^+\to\mathbb{R}^+$ which is defined as $a(w(u,t),t)=h(t)|w(u,t)|$, fulfill the assumption $(A3)$. For $t\in[0,1]$

$$\begin{aligned}|a(w,t)|&=|h(t)|w(u,t)||,\\&\leq\|h\|_\infty\times\|w\|_\infty\leq\|h\|_\infty\|w\|_{H_0^1(0,1)},\end{aligned} \tag{106}$$

In the above inequality, we have used the following embedding $H_0^1(0,1)\subset C[0,1]$. By the Hölder continuity of h, we have that there exist $L_h>0$ and $\theta_1\in(0,1]$ such that

$$|h(t)-h(s)|\leq L_h|t-s|,\quad t,s\in[0,1]. \tag{107}$$

Furthermore, for $w_1,w_2\in H_0^1(0,1)$, we have

$$\begin{aligned}|a(w_1,t)-a(w_2,s)|&=|h(t)\,[|w_1(u,t)|-|w_2(u,s)|]+(h(t)-h(s))w_2(u,s)|,\\&\leq\|h\|_\infty\|w_1-w_2\|_\infty+L_h|t-s|^\theta\|w_2\|_\infty,\\&\leq\|h\|_\infty\|w_1-w_2\|_{H_0^1(0,1)}+L_h|t-s|^\theta\|w_2\|_\infty,\\&\leq\max\{\|h\|_\infty,L_h\|w_2\|_\infty\}\left(\|w_1-w_2\|_{H_0^1(0,1)}+|t-s|^\theta\right).\end{aligned}$$

For $w_1,w_2\in D(A^{1/2})$, we have

$$\|I_i(w_1)-I_i(w_2)\|_{1/2}\leq\frac{2\|w_1-w_2\|_{1/2}}{\|(2+w_1)(2+w_2)\|_{1/2}}\leq\frac{1}{2}\|w_1-w_2\|_{1/2}.$$

Thus, all the results of this section to obtain the main results can be applied.
For the particular case, we can take following example

$$\begin{aligned}f(t,w(t),w(a(w(t),t)))&=\frac{3}{\sin(w(\frac{1}{2}w(t)))+4},\quad t\in[0,1],\\I_i(w(t_i))&=\frac{|w(t_i^-)|}{9+|w(t_i^-)|}\end{aligned}$$

where $L_a=\frac{1}{2}\ L_F=3/16$ and $L_I=1/9$.

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