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ORIGINAL PAPER

The Double Laplace Transforms and Their Properties with Applications to Functional, Integral and Partial Differential Equations

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Abstract Although a very vast and extensive literature including books and papers on the Laplace transform of a function of a single variable, its properties and applications is available, but a very little or no work is available on the double Laplace transform, its properties and applications.This paper deals with the double Laplace transforms and their properties with examples and applications to functional, integral and partial differential equations. Several simple theorems dealing with general properties of the double Laplace transform are proved. The convolution, its properties and convolution theorem with a proof are discussed in some detail. The main focus of this paper is to develop the method of the double Laplace transform to solve initial and boundary value problems in applied mathematics, and mathematical physics.

Keywords Double Laplace transform · Single Laplace transform · Convolution · Functional · Integral and partial differential equations

Mathematics Subject Classification 44A10 · 44A30 · 44A35

Introduction with Historical Comments

'What we know is not much. What we do not know is immense.' Pierre-Simon Laplace

'The greatest mathematicians like Archimedes, Newton, and Gauss have always been able to combine theory and applications into one.'

Felix Klein

In his celebrated study of probability theory and celestial mechanics, P. S. Laplace (1749–1827) introduced the idea of the Laplace transform in 1782. Laplace's classic treatise on *La*

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Théorie Analytique des Probabilities (Analytical Theory of Probability) contained some basic results of the Laplace transform which is one of the oldest and most commonly used linear integral transforms available in the mathematical literature. This has effectively been used in finding the solutions of linear differential, difference and integral equations.

On the other hand , Joseph Fourier's (1768–1830) monumental treatise on *La Théorie Analytique de la Chaleur (The Analytical Theory of Heat)* provided the modern mathematical theory of heat conduction, Fourier series, and Fourier integrals with applications. In his treatise, he discovered a double integral representation of a non-periodic function f(x) for all real x which is universally known as the *Fourier Integral Theorem* in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left[\int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi \right] dk. \tag{1}$$

The deep significance of this theorem has been recognized by mathematicians and mathematical physicists of the nineteenth and twentieth centuries. Indeed, this theorem is regarded as one of the most fundamental representation theorems of modern mathematical analysis and has widespread mathematical, physical and engineering applications. According to Lord Kelvin (1824–1907) and Peter Guthrie Tait (1831–1901) once said: "Fourier's Theorem, which is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly recondite question in modern physics $\cdots$ ″. Another remarkable fact is that the Fourier integral theorem was used by Fourier to introduce the *Fourier transform and the inverse Fourier transform.* This celebrated work of Fourier was known to Laplace, and, in fact, the Laplace transform is a special case of the Fourier transform. It was also A. L. Cauchy (1789–1857) who also used independently some of the ideas of the thoery of Fourier transforms. At the same time, S. D. Poisson (1781–1840) also independently applied the method of Fourier transforms in his research on the propagation of water waves. Although both Laplace and Fourier transforms have been discovered in the 19th century, it was the British electrical engineer, Oliver Heaviside (1850–1925) who made the Laplace transform very popular by applying it to solve ordinary differential equations of electrical circuits and systems, and then to develop modern operational calculus in less rigorous way. He first recognized the power and success of his operational method and then used it as a powerful and effective tool for the solutions of telegraph equations and the second order hyperbolic partial differential equations with constant coefficients. Subsequently, T. J. Bromwich (1875–1930) first successfully introduced the thoery of complex functions to provide formal mathematical justification of Heaviside's operational calculus. After Bromwich's work, notable contributions to rigorous formulation of operational calculus were made by J. R. Carson (1886–1940), Van der Pol, (1892–1977), G. Doetsch (1889–1959) and many others.

Both Laplace and Fourier transforms have been studied very extensively and have found to have a wide variety of applications in mathematical, physical, statistical, and engineering sciences and also in other sciences. At present, there is a very extensive literature available of the Laplace transform of a function $f(t = x)$ of one variable $t = x$ and its applications (see Sneddon [1], Churchill [2], Schiff [3], Debnath and Bhatta [4]). But there is very little or no work available on the double Laplace transforms of f(x,y) of two positive real variables x and y and their properties and applications.

So, the major objective of this paper is to study the double Laplace transform, its properties with examples and applications to functional, integral and partial differential equations. Several simple theorems dealing with general properties of the double Laplace theorem are proved. The convolution of f(x,y) and g(x,y), its properties and convolution theorem with a proof are discussed in some detail. The main focus of this paper is to develop the method of the double Laplace transform to solve initial and boundary value problems in applied mathematics, and mathematical physics.

Definition of the Double Laplace Transform and Examples

The double Laplace transform of a function $f(x, y)$ of two variables x and y defined in the first quadrant of the x–y plane is defined by the double integral in the form

$$\begin{aligned}\bar{\bar{f}}(p, q) &= \mathcal{L}_2\,[f(x, y)] = \mathcal{L}\,[\mathcal{L}\,\{f(x, y);\;\, x \to p\};\;\, y \to q] = \mathcal{L}[\bar{f}(p, y); y \to q] \\ &= \int_0^\infty \int_0^\infty f(x, y)\, e^{-(px+qy)}\, dxdy, \end{aligned} \tag{2}$$

provided the integral exists, where we follow Debnath and Bhatta [4] to denote the Laplace transform $\bar{f}(p) = \mathcal{L}\{f(x); x \to p\}$ of $f(x)$ and to define by

$$\bar{f}(p) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-px} f(x)dx, \quad Re(p) > 0, \tag{3}$$

and $\mathcal{L} \equiv \mathcal{L}_1$ is used throughout this paper. Similarly, $\mathcal{L}^{-1} \equiv \mathcal{L}_1^{-1}$ is used to denote the inverse Laplace transformation of $\bar{f}(p)$ and to define by

$$f(x) = \mathcal{L}^{-1}\{\bar{f}(p)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px}\, \bar{f}(p)dp, \quad c \geq 0. \tag{4}$$

Evidently, $\mathcal{L}_2$ is a linear integral transformation as shown below :

$$\begin{aligned}\mathcal{L}_2\,[a_1 f_1(x, y) + a_2 f_2(x, y)] &= \int_0^\infty \int_0^\infty [a_1 f_1(x, y) + a_2 f_2(x, y)]\, e^{-(px+qy)} dxdy \\ &= \int_0^\infty \int_0^\infty a_1 f_1(x, y) e^{-(px+qy)} dxdy + \int_0^\infty \int_0^\infty a_2 f_2(x, y) e^{-(px+qy)} dxdy \\ &= a_1 \int_0^\infty \int_0^\infty f_1(x, y) e^{-(px+qy)} dxdy + a_2 \int_0^\infty \int_0^\infty f_2(x, y) e^{-(px+qy)} dxdy \\ &= a_1 \mathcal{L}_2\,[f_1(x, y)] + a_2 \mathcal{L}_2\,[f_2(x, y)], \end{aligned} \tag{5}$$

where a_1 and a_2 are constants.

The inverse double Laplace transform $\mathcal{L}_2^{-1}\left[\bar{\bar{f}}(p, q)\right] = f(x, y)$ is defined by the complex double integral formula

$$\begin{aligned}\mathcal{L}_2^{-1}\left[\bar{\bar{f}}(p, q)\right] &= f(x, y) \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px}\, dp\, \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{qx}\, \bar{\bar{f}}(p, q)\, dq, \end{aligned} \tag{6}$$

where $\bar{\bar{f}}(p, q)$ must be an analytic function for all p and q in the region defined by the inequalities $Re\ p \geq c$ and $Re\ q \geq d$, where c and d are real constants to be chosen suitably.

It follows from (5) that $\mathcal{L}_2^{-1}\left[\bar{\bar{f}}(p, q)\right]$ satisfies the linear property

$$\mathcal{L}_2^{-1}[a\bar{\bar{f}}(p, q) + b\bar{\bar{g}}(p, q)] = a\, \mathcal{L}_2^{-1}[\bar{\bar{f}}(p, q)] + b\, \mathcal{L}_2^{-1}[\bar{\bar{g}}(p, q)], \tag{7}$$

where a and b are constants. This shows that $\mathcal{L}_2^{-1}$ is also a linear transformation.

Examples

(a) If f(x,y) = 1 for x > 0 and y > 0, then

$$\bar{\bar{f}}(p,q) = \mathcal{L}_2\{1\} = \int_0^\infty \int_0^\infty e^{-px} e^{-qy} dx dy$$
$$= \int_0^\infty e^{-px} dx \int_0^\infty e^{-qy} dy = \mathcal{L}\{1; x \to p\}\mathcal{L}\{1; y \to q\} = \frac{1}{pq}. \tag{8}$$

(b) If f(x,y) = exp(ax + by) for all x and y, then

$$\mathcal{L}_2\{exp(ax+by)\} = \int_0^\infty \int_0^\infty e^{-(p-a)x} e^{-(q-b)y} dx dy$$
$$= \mathcal{L}\{e^{ax}; x \to p\}\mathcal{L}\{e^{by}; y \to q\}$$
$$= \frac{1}{(p-a)(q-b)}. \tag{9}$$

(c)

$$\mathcal{L}_2\{exp[i(ax+by)]\} = \frac{1}{(p-ia)(q-ib)}$$
$$= \frac{(p+ia)(q+ib)}{(p^2+a^2)(q^2+b^2)} = \frac{(pq-ab)+i(aq+bp)}{(p^2+a^2)(q^2+b^2)}. \tag{10}$$

Consequently,

$$\mathcal{L}_2\{\cos(ax+by)\} = \frac{pq-ab}{(p^2+a^2)(q^2+b^2)}. \tag{11}$$

$$\mathcal{L}_2\{\sin(ax+by)\} = \frac{(aq+bp)}{(p^2+a^2)(q^2+b^2)}. \tag{12}$$

(d)

$$\mathcal{L}_2[\cosh(ax+by)] = \frac{1}{2}\left[\mathcal{L}_2\{e^{ax+by}\} + \mathcal{L}_2\{e^{-(ax+by)}\}\right]$$
$$= \frac{1}{2}\left[\frac{1}{(p-a)(q-b)} + \frac{1}{(p+a)(q+b)}\right]. \tag{13}$$

Similarly,

$$\mathcal{L}_2[\sinh(ax+by)] = \frac{1}{2}\left[\frac{1}{(p-a)(q-b)} - \frac{1}{(p+a)(q+b)}\right]. \tag{14}$$

(e)

$$\mathcal{L}_2\{e^{-ax-by} f(x,y)\} = \bar{\bar{f}}(p+a, q+b). \tag{15}$$

$$\mathcal{L}_2\{(xy)^n\} = \int_0^\infty e^{-px} x^n dx \int_0^\infty y^n e^{-qy} dy$$
$$= \frac{n!}{p^{n+1}} \cdot \frac{n!}{q^{n+1}} = \frac{(n!)^2}{(pq)^{n+1}}, \tag{16}$$

where n is a positive integer.

(g)

$$\mathcal{L}_2\{(x^m y^n)\} = \frac{m!n!}{p^{m+1}q^{n+1}}, \tag{17}$$

where m and n are positive integers.

(h) If a(> -1) and b(> -1) are real numbers, then

$$\mathcal{L}_2\{x^a y^b\} = \frac{\Gamma(a+1)}{p^{a+1}} \cdot \frac{\Gamma(b+1)}{q^{b+1}}. \tag{18}$$

It follows from the definition of (2) that

$$\begin{aligned}\mathcal{L}_2\{x^a y^b\} &= \int_0^\infty \int_0^\infty e^{-px-qy}\, x^a\, y^b\, dxdy \\ &= \int_0^\infty x^a e^{-px} dx \int_0^\infty y^b e^{-qy} dy\end{aligned}$$

which is, by putting px = s, and qy = t

$$\begin{aligned}&= \frac{1}{p^{a+1}} \int_0^\infty e^{-s} s^a ds \cdot \frac{1}{q^{b+1}} \int_0^\infty e^{-t} t^b dt \\ &= \frac{\Gamma(a+1)}{p^{a+1}} \cdot \frac{\Gamma(b+1)}{q^{b+1}}.\end{aligned} \tag{19}$$

where $\Gamma(a)$ is the Euler gamma function defined by the uniformly convergent integral

$$\Gamma(a) = \int_0^\infty s^{a-1} e^{-s} ds, \quad a > 0. \tag{20}$$

This example can be used to derive the celebrated integral representation of the product of *Riemann zeta functions* $\zeta(s)$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{21}$$

It follows from (19) that

$$\mathcal{L}_2\{x^{a-1} y^{b-1}\} = \frac{\Gamma(a)}{p^a} \cdot \frac{\Gamma(b)}{q^b}. \tag{22}$$

Or,

$$\frac{\Gamma(a)}{p^a} \cdot \frac{\Gamma(b)}{q^b} = \int_0^\infty \int_0^\infty e^{px} x^{a-1} e^{-qy} y^{b-1} dxdy.$$

Summing this result over p and q from one to infinity gives

$$\begin{aligned}&\Gamma(a) \sum_{p=1}^{\infty} \frac{1}{p^a} \cdot \Gamma(b) \sum_{q=1}^{\infty} \frac{1}{q^b} \\ &= \int_0^\infty x^{a-1} \sum_{p=1}^{\infty} e^{-px} dx \int_0^\infty y^{b-1} \sum_{q=1}^{\infty} e^{-qy} dy \\ &= \int_0^\infty x^{a-1} \frac{dx}{(e^x - 1)} \int_0^\infty y^{b-1} \frac{dy}{(e^y - 1)}.\end{aligned}$$

Or,

$$\Gamma(a)\zeta(a)\Gamma(b)\zeta(b) = \int_0^\infty x^{a-1}\frac{dx}{(e^x-1)}\cdot\int_0^\infty y^{b-1}\frac{dy}{(e^y-1)}$$
$$\zeta(a)\cdot\zeta(b) = \frac{1}{\Gamma(a)}\int_0^\infty x^{a-1}\frac{dx}{(e^x-1)}\cdot\frac{1}{\Gamma(b)}\int_0^\infty y^{b-1}\frac{dy}{(e^y-1)}. \tag{23}$$

This is a double integral representation for the product of two zeta functions.

(i)

$$\mathcal{L}_2\left[J_0(a\sqrt{xy})\right] = \frac{4}{(4pq+a^2)}, \tag{24}$$

where $J_0(z)$ is a Bessel function of zero order. We have, by definition (2),

$$\mathcal{L}_2[J_0(a\sqrt{xy})] = \int_0^\infty\int_0^\infty e^{-(px+qy)}J_0(a\sqrt{xy})dxdy$$
$$= \int_0^\infty e^{-qy}dy\left[\int_0^\infty e^{-px}J_0(a\sqrt{xy})dx\right]$$
$$= \frac{1}{p}\int_0^\infty e^{-qy}\,exp\left(-\frac{a^2y}{4p}\right)dy = \frac{1}{p\left(q+\frac{a^2}{4p}\right)} = \frac{4}{(4pq+a^2)}.$$

Similarly,

$$\mathcal{L}_2[I_0(a\sqrt{xy})] = \frac{4}{(4pq-a^2)}, \tag{25}$$

where $I_0(z)$ is the modified Bessel function of order zero.

(j) If f(x,y) = g(x)h(y), then

$$\mathcal{L}_2[f(x,y)] = \mathcal{L}_2[g(x)\cdot h(y)]$$
$$= \int_0^\infty e^{-px}g(x)dx\int_0^\infty e^{-qy}h(y)dy$$
$$= \bar{g}(p)\,\bar{h}(q). \tag{26}$$

In particular,

$$\mathcal{L}_2\left[\frac{1}{\sqrt{xy}}\right] = \frac{\pi}{\sqrt{pq}}. \tag{27}$$

(k)

$$\mathcal{L}_2\left[erf\left(\frac{x}{2\sqrt{y}}\right)\right] = \left(\frac{1}{p\sqrt{q}}\right)\frac{1}{(p+\sqrt{q})}, \tag{28a}$$
$$\mathcal{L}_2\left[erf\left(\frac{y}{2\sqrt{x}}\right)\right] = \left(\frac{1}{q\sqrt{p}}\right)\frac{1}{(q+\sqrt{p})}. \tag{28b}$$

By definition (2),

$$\mathcal{L}_2\left[erf\left(\frac{x}{2\sqrt{y}}\right)\right] = \int_0^\infty e^{-px}dx\int_0^\infty e^{-qy}\,erf\left(\frac{x}{2\sqrt{y}}\right)dy$$

which is, using item 87 of Table B-4 of Debnath and Bhatta [4],

$$= \frac{1}{q}\int_0^\infty e^{-px}\left(1 - e^{-x\sqrt{q}}\right)dx$$
$$= \frac{1}{q}\left[\frac{1}{p} - \frac{1}{p+\sqrt{q}}\right] = \frac{1}{p\sqrt{q}(p+\sqrt{q})}.$$

Similarly, it is easy to prove (28b).
A table of double Laplace transforms can be constructed from the standard tables of Laplace transforms by using the definition (2) or directly by evaluating double integrals. The above results can be used to solve integral, functional and partial differential equations.

Existance Condition for the Double Laplace Tranform

If f(x,y) is said to be of *exponential order* a (> 0) and b(> 0) on $0 \leq x < \infty, 0 \leq y < \infty$, if there exists a positive constant K such that for all x $>$ X and y $>$ Y

$$|f(x, y)| \leq K\, e^{ax+by} \tag{29}$$

and we write

$$f(x, y) = O\,(e^{ax+by}) \quad \text{as x} \to \infty,\ \text{y} \to \infty. \tag{30}$$

Or, equivalently,

$$\lim_{x\to\infty, y\to\infty} e^{-\alpha x - \beta y}\,|f(x, y)|$$
$$= K \lim_{x\to\infty, y\to\infty} e^{-(\alpha - a)x}\, e^{-(\beta - b)y} = 0, \quad \alpha > a,\ \beta > b. \tag{31}$$

Such a function f(x,y) is simply called an *exponential* order as x $\to \infty$, y $\to \infty$, and clearly, it does not grow faster than K exp(ax + by) as x $\to \infty$, y $\to \infty$.

Theorem 2.1 *If a function f(x,y) is a continous function in every finite intervals (0,X) and (0,Y) and of exponential order exp(ax + by), then the double Laplace transform of f(x,y) exists for all p and q provided Re p $>$ a and Re q $>$ b.*

Proof We have

$$|\bar{\bar{f}}(p, q)| = |\int_0^\infty \int_0^\infty e^{-px-qy}\, f(x, y)\, dxdy|$$
$$\leq K \int_0^\infty e^{-x(p-a)}\, dx \int_0^\infty e^{-y(q-b)}\, dy$$
$$= \frac{K}{(p-a)(q-b)} \quad \text{for Re p} > a, \text{Re q} > b. \tag{32}$$

It follows from this (32) that

$$\lim_{p\to\infty, q\to\infty} |\bar{\bar{f}}(p, q)| = 0, \text{ or } \lim_{p\to\infty, q\to\infty} \bar{\bar{f}}(p, q) = 0.$$

This result can be regarded as the limiting property of the double Laplace transform. Clearly, $\bar{\bar{f}}(p, q)$ = pq or $p^2 + q^2$ is not the double Laplace transform of any function f(x,y) because $\bar{\bar{f}}(p, q)$ does not tend to zero as p $\to \infty$ and q $\to \infty$. □

On the other hand, $f(x,y) = \exp(ax^2 + by^2)$, $a > 0$, $b > 0$ cannot have a double Laplace transform even though it is continuous but is *not* of the exponential order because

$$\lim_{x\to\infty, y\to\infty} exp(ax^2 + by^2 - px - qy) = \infty. \tag{33}$$

Basic Properties of the Double Laplace Transforms

Using Debnath and Bhatta [4], we can prove the following general properties of the double Laplace transform under suitable conditions on $f(x, y)$:

$$\text{(a)} \quad \mathcal{L}_2\left[e^{-ax-by} f(x, y)\right] = \bar{\bar{f}}(p + a, q + b), \tag{34}$$

$$\text{(b)} \quad \mathcal{L}_2\left[f(ax)\, g(by)\right] = \frac{1}{ab} \bar{f}\left(\frac{p}{a}\right) \bar{g}\left(\frac{q}{b}\right), \quad a > 0,\ b > 0, \tag{35}$$

$$\text{(c)} \quad \mathcal{L}_2\left[f(x)\right] = \frac{1}{q}\, \bar{f}(p), \qquad \mathcal{L}_2[g(y)] = \frac{1}{p}\, \bar{g}(q), \tag{36}$$

$$\text{(d)} \quad \mathcal{L}_2\left[f(x + y)\right] = \frac{1}{p - q}\left[\bar{f}(p) - \bar{f}(q)\right]. \tag{37}$$

$$\text{(e)} \quad \mathcal{L}_2\left[f(x - y)\right] = \frac{1}{p + q}\left[\bar{f}(p) + \bar{f}(q)\right], \quad \text{when } f \text{ is even.} \tag{38}$$

$$= \frac{1}{p + q}\left[\bar{f}(p) - \bar{f}(q)\right], \quad \text{when } f \text{ is odd.} \tag{39}$$

$$\text{(f)} \quad \mathcal{L}_2\left[f(x)\, H(x - y)\right] = \frac{1}{q}\left[\bar{f}(p) - \bar{f}(p + q)\right]. \tag{40}$$

$$\text{(g)} \quad \mathcal{L}_2\left[f(x)\, H(y - x)\right] = \frac{1}{q}\left[\bar{f}(p + q)\right], \tag{41}$$

$$\text{(h)} \quad \mathcal{L}_2\left[f(x)\, H(x + y)\right] = \frac{1}{q}\left[\bar{f}(p)\right], \tag{42}$$

$$(i) \quad \mathcal{L}_2\left[H(x - y)\right] = \frac{1}{p(p + q)}, \quad \text{put } f(x) = 1 \text{ in (40).} \tag{43}$$

$$\text{(j)} \quad \mathcal{L}_2\left[\frac{\partial u}{\partial x}\right] = p\, \bar{\bar{u}}(p, q) - \bar{u}_1(q), \tag{44}$$

where $\bar{\bar{u}}(p, q) = \mathcal{L}_2\left[u(x, y)\right],\ \bar{u}_1(q) = \mathcal{L}\left[u(0, y)\right]$.

$$\text{(k)} \quad \mathcal{L}_2\left[\frac{\partial u}{\partial y}\right] = q\, \bar{\bar{u}}(p, q) - \bar{u}_2(p), \quad \text{where } \bar{u}_2(p) = \mathcal{L}\left[u(x, 0)\right]. \tag{45}$$

$$\text{(l)} \quad \mathcal{L}_2\left[\frac{\partial^2 u}{\partial x^2}\right] = p^2\, \bar{\bar{u}}(p, q) - p\, \bar{u}_1(q) - \bar{u}_3(q), \tag{46}$$

where $\bar{u}_3(q) = \mathcal{L}\left[u_x(0, y)\right]$.

$$\text{(m)} \quad \mathcal{L}_2\left[\frac{\partial^2 u}{\partial y^2}\right] = q^2\, \bar{\bar{u}}(p, q) - q\, \bar{u}_2(p) - \bar{u}_4(p), \tag{47}$$

where $\bar{u}_4(p) = \mathcal{L}\left[u_y(x, 0)\right]$.

$$(n) \quad \mathcal{L}_2\left[\frac{\partial^2 u}{\partial x \partial y}\right] = pq\, \bar{\bar{u}}(p, q) - q\, \bar{u}_1(q) - p\bar{u}_2(p) + u(0, 0), \tag{48}$$

where $\mathcal{L}\left[u_x(x, 0)\right] = p\bar{u}_2(p) - u(0, 0)$ is used.

(o)

Theorem 3.1 *If* $\mathcal{L}_2[f(x,y)] = \bar{\bar{f}}(p,q)$, *then*

$$\mathcal{L}_2[f(x-\xi, y-\eta)H(x-\xi,\ y-\eta)] = e^{-\xi p-\eta q}\ \bar{\bar{f}}(p,q), \tag{49}$$

where $H(x,y)$ *is the Heaviside unit step function defined by* $H(x-a, y-b) = 1$ *when* $x > a$ *and* $y > b$ *: and* $H(x-a, y-b) = 0$ *when* $x < a$ *and* $y < b$.

Proof We have, by definition,

$$\begin{aligned}&\mathcal{L}_2[f(x-\xi, y-\eta)H(x-\xi, y-\eta)]\\&\quad= \int_0^\infty \int_0^\infty e^{-px-qy} f(x-\xi, y-\eta)H(x-\xi, y-\eta)dx\,dy\\&\quad= \int_\xi^\infty \int_\eta^\infty e^{-px-qy} f(x-\xi, y-\eta)dx\,dy\end{aligned}$$

which is, by putting $x-\xi = \tau$, $y-\eta = s$,

$$\begin{aligned}&= e^{-p\xi-q\eta} \int_0^\infty \int_0^\infty e^{-p\tau-qs} f(\tau, s)d\tau\,ds\\&= e^{-p\xi-q\eta}\,\bar{\bar{f}}(p,q).\end{aligned}$$

(p)

Theorem 3.2 *If f(x,y) is a periodic function of periods a and b, (that is,* $f(x+a, y+b) = f(x,y)$ *for all x and y), and if* $\mathcal{L}_2\{f(x,y)\}$ *exists, then*

$$\mathcal{L}_2\{f(x,y)\} = \left[1-e^{-pa-qb}\right]^{-1} \int_0^a \int_0^b e^{-px-qy} f(x,y)dx\,dy. \tag{50}$$

We have, by definition,

$$\begin{aligned}\mathcal{L}_2\{f(x,y)\} &= \int_0^\infty \int_0^\infty e^{-px-qy} f(x,y)dx\,dy\\&= \int_0^a \int_0^b e^{-px-qy} f(x,y)dx\,dy + \int_a^\infty \int_b^\infty e^{-px-qy} f(x,y)dx\,dy\end{aligned}$$

Setting $x = u+a,\ \ y = v+b$ in the second double integral, we obtain

$$\begin{aligned}\bar{\bar{f}}(p,q) &= \int_0^a \int_0^b e^{-px-qy} f(x,y)dx\,dy + \int_0^\infty \int_0^\infty e^{-pu-qv} f(u+a, v+b)du\,dv\\&= \int_0^a \int_0^b e^{-px-qy} f(x,y)dx\,dy + e^{-pa-qb} \int_0^\infty \int_0^\infty e^{-pu-qv} f(u,v)du\,dv\\&= \int_0^a \int_0^b e^{-px-qy} f(x,y)dx\,dy + e^{-pa-qb}\,\bar{\bar{f}}(p,q).\end{aligned}$$

Consequently,

$$\bar{\bar{f}}(p,q) = \left[1-e^{-pa-qb}\right]^{-1} \int_0^a \int_0^b e^{-px-qy} f(x,y)dx\,dy$$

This proves the theorem of the double Laplace transform of a periodic function.

Convolution and Convolution Theorem of the Double Laplace Transforms

The convolution of $f(x, y)$ and $g(x, y)$ is denoted by $(f * *g)(x, y)$ and defined by

$$(f * *g)(x, y) = \int_0^x \int_0^y f(x - \xi, y - \eta)\, g(\xi, \eta)\, d\xi d\eta. \tag{51}$$

The convolution is commutative, that is,

$$(f * *g)(x, y) = (g * *f)(x, y). \tag{52}$$

This follows from the definition (51). It can easily be verified that the following properties of convolution hold :

$$[f * *(g * *h)](x, y) = [(f * *g) * *h](x, y) \quad \text{(Associative)}, \tag{53}$$

$$[f * *(ag + bh)](x, y) = a(f * *g)(x, y) + b(f * *h)(x, y) \quad \text{(Distributive)}, \tag{54}$$

$$(f * *\delta)(x, y) = f(x, y) = (\delta * *f)(x, y), \quad \text{(Identity)}, \tag{55}$$

where $\delta(x, y)$ is the Dirac delta function of x and y. By virtue of these convolution properties, it is clear that the set of all double Laplace transformable functions form a commutative semigroup with respect to the convolution operation **. This set does not, in general, form a group because $f * *g^{-1}$ does not have a double Laplace transform.

Theorem 4.1 (Convolution Theorem). *If* $\mathcal{L}_2\{f(x, y)\} = \bar{\bar{f}}(p, q)$, *and* $\mathcal{L}_2\{g(x, y)\} = \bar{\bar{g}}(p, q)$, *then*

$$\mathcal{L}_2[(f * *g)(x, y)] = \mathcal{L}_2\{f(x, y)\}\mathcal{L}_2\{g(x, y)\} = \bar{\bar{f}}(p, q)\, \bar{\bar{g}}(p, q). \tag{56}$$

Or, equivalently,

$$\mathcal{L}_2^{-1}[\bar{\bar{f}}(p, q)\, \bar{\bar{g}}(p, q)] = (f * *g)(x, y), \tag{57}$$

where $(f * *g)(x, y)$ is defined by the double integral (51) which is often called the *Convolution integral (or Faltung)* of $f(x, y)$ and $g(x, y)$. Physically, $(f * *g)(x, y)$ represents the *output* of $f(x, y)$ and $g(x, y)$.

Proof We have, by definition,

$$\begin{aligned}\mathcal{L}_2[(f * *g)(x, y)] &= \int_0^\infty \int_0^\infty e^{-px-qy} (f * *g)(x, y) dx dy \\ &= \int_0^\infty \int_0^\infty e^{-px-qy} \left[\int_0^x \int_0^y f(x - \xi, y - \eta) g(\xi, \eta) d\xi d\eta\right] dx dy\end{aligned}$$

which is, using the Heaviside unit step function,

$$\begin{aligned}&= \int_0^\infty \int_0^\infty e^{-px-qy} \left[\int_0^\infty \int_0^\infty f(x - \xi, y - \eta)\, H(x - \xi, y - \eta) g(\xi, \eta) d\xi d\eta\right] dx dy \\ &= \int_0^\infty \int_0^\infty g(\xi, \eta) d\xi d\eta \left[\int_0^\infty \int_0^\infty e^{-px-qy} f(x - \xi, y - \eta)\, H(x - \xi, y - \eta)\right] dx dy\end{aligned}$$

which is, by Theorem 3.1,

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty g(\xi,\eta)\cdot e^{-p\xi-q\eta}\bar{\bar{f}}(p,q)d\xi d\eta \\
&= \bar{\bar{f}}(p,q)\int_0^\infty \int_0^\infty e^{-p\xi-q\eta} g(\xi,\eta)d\xi d\eta \\
&= \bar{\bar{f}}(p,q)\bar{\bar{g}}(p,q).
\end{aligned}
$$

This completes the proof of the convolution theorem. □

Corollary 4.1 *If* $f(x,y)=a(x)\,b(y)$ *and* $g(x,y)=c(x)\,d(y)$*, then*

$$\mathcal{L}_2[(f**g)(x,y)] = \mathcal{L}\{(a*c)(x)\}\mathcal{L}\{(b*d)(y)\}. \tag{58}$$

Or, equivalently,

$$\bar{\bar{f}}(p,q)\bar{\bar{g}}(p,q) = \bar{a}(p)\bar{b}(p)\bar{c}(q)\bar{d}(q),$$

where $$(f*g)(x) = \int_0^x f(x-\xi)g(\xi)d\xi \quad \text{and}$$

$$\mathcal{L}\{(f*g)(x)\} = \bar{f}(p)\bar{g}(p) \quad \text{(see Debnath and Bhatta [4].} \tag{59}$$

We prove (52) and (53) by means of convolution Theorem 4.1.
We apply $\mathcal{L}_2$ to the left hand side of (52) so that by Convolution Theorem 4.1

$$
\begin{aligned}
\mathcal{L}_2[(f**g)(x,y)] &= \bar{\bar{f}}(p,q)\ \bar{\bar{g}}(p,q) = \bar{\bar{g}}(p,q)\bar{\bar{f}}(p,q) \\
&= \mathcal{L}_2[(g**f)(x,y)].
\end{aligned} \tag{60}
$$

Application of $\mathcal{L}_2^{-1}$ to both sides of (60) gives

$$(f**g)(x,y) = (g**f)(x,y).$$

Similarly, we apply $\mathcal{L}_2$ to the left hand side of (53) and use the convolution theorem (4.1) so that

$$
\begin{aligned}
&\mathcal{L}_2[f**(g**h)(x,y)] \\
&\quad = \bar{\bar{f}}(p,q)\cdot \mathcal{L}_2[(g**h)(x,y)] \\
&\quad = \bar{\bar{f}}(p,q)\cdot \bar{\bar{g}}(p,q)\bar{\bar{h}}(p,q) \\
&\quad = [\bar{\bar{f}}(p,q)\cdot \bar{\bar{g}}(p,q)]\bar{\bar{h}}(p,q) \\
&\quad = \mathcal{L}_2[(f**g)(x,y)]\cdot \mathcal{L}_2[h(x,y)] \\
&\quad = \mathcal{L}_2[\{(f**g)(x,y)\}**h(x,y)] \\
&\quad = \mathcal{L}_2[\{(f**g)**h\}(x,y)].
\end{aligned} \tag{61}
$$

Application of $\mathcal{L}_2^{-1}$ to (61) proves the associative property (53).

Applications of the Double Laplace Transforms to Functional and Partial Differential Equations

Functional Equations

Cauchy's Functional Equation

This equation has the standard form as

$$f(x+y) = f(x) + f(y), \tag{62}$$

where f is an unknown function.
We apply the double Laplace transform $\bar{\bar{f}}(p,q)$ of $f(x,y)$ to (62) combined with (36) and (37) to obtain

$$\mathcal{L}_2\left[f(x+y)\right] = \mathcal{L}_2\left[f(x)\right] + \mathcal{L}_2\left[f(y)\right]$$

or,

$$\frac{1}{p-q}\left[\bar{f}(q) - \bar{f}(p)\right] = \frac{1}{q}\,\bar{f}(p) + \frac{1}{p}\,\bar{f}(q) \tag{63}$$

that is,

$$\bar{f}(p)\left[\frac{1}{q} + \frac{1}{p-q}\right] = \bar{f}(q)\left[\frac{1}{p-q} - \frac{1}{p}\right].$$

Simplifying this equation, we get

$$p^2\,\bar{f}(p) = q^2\,\bar{f}(q), \tag{64}$$

where the left hand side is a function of p alone and right hand side is a function of q alone. This equation is true provided each side is equal to an arbitrary constant k so that

$$p^2\,\bar{f}(p) = k$$

Or,

$$\bar{f}(p) = \frac{k}{p^2}.$$

The inverse transform gives the solution of the Cauchy functional equation (62) as

$$f(x) = kx, \tag{65}$$

where k is an arbitrary constant.

The Cauchy–Abel Functional Equation

This equation for an unknown function $f(x)$ has the form

$$f(x+y) = f(x)f(y). \tag{66}$$

We apply the double Laplace transform to (66) with (35) and (37) to obtain

$$\mathcal{L}_2\left[f(x+y)\right] = \mathcal{L}_2\left[f(x)f(y)\right],$$

or,

$$\frac{1}{p-q}\left[\bar{f}(q)-\bar{f}(p)\right]=\bar{f}(p)\bar{f}(q). \tag{67}$$

Simplifying this equation leads to the separable form

$$\frac{1-p\bar{f}(p)}{\bar{f}(p)}=\frac{1-q\bar{f}(q)}{\bar{f}(q)}. \tag{68}$$

Equating each side to an arbitrary constant k, we obtain

$$\bar{f}(p)=\frac{1}{p+k}. \tag{69}$$

Thus, the inverse transform gives the solution as

$$f(x)=e^{-kx}. \tag{70}$$

Partial Differential Equations

(a) Solve the equation

$$au_x+bu_y=0, \tag{71}$$

with

$$u(x,0)=f(x),\ x>0;\ u(0,y)=0,\ y>0. \tag{72}$$

Application of the double Laplace transform $\bar{\bar{u}}$(p,q) of u(x,y) to (71) gives

$$a\ \mathcal{L}_2[u_x]+b\ \mathcal{L}_2[u_y]=0.$$

Or,

$$a[p\ \bar{\bar{u}}(p,q)-\mathcal{L}[u(0,y)]]+b(q\ \bar{\bar{u}}(p,q)-\mathcal{L}[u(x,0)])=0.$$

Or,

$$(ap+bq)\bar{\bar{u}}(p,q)=b\bar{f}(p).$$

Or,

$$\bar{\bar{u}}(p,q)=\bar{f}(p)\ \frac{1}{(q+\frac{a}{b}\ p)} \tag{73}$$

The inverse Laplace transformation wih respect to q gives

$$\bar{u}(p,y)=\bar{f}(p)\ exp\left(-\frac{ap}{b}y\right) \tag{74}$$

The inverse transformation with respect to p yields the solution

$$u(x,y)=\mathcal{L}^{-1}\left\{\bar{f}(p)\ exp\left(-\frac{ap}{b}y\right)\right\}=f(x)*\delta\left(x-\frac{ay}{b}\right),\quad \text{by the convolution theorem,}$$
$$=\int_0^x f(x-\tau)\delta\left(\tau-\frac{ay}{b}\right)d\tau=f\left(x-\frac{ay}{b}\right). \tag{75}$$

(b) Solve the first-order partial differential equation

$$u_x=u_y,\qquad u(x,0)=f(x),\ x>0,\quad u(0,y)=g(y),\ y>0. \tag{76}$$

Applying the double Laplace transform to equation (76) gives

$$\begin{aligned}\mathcal{L}_2\left[u_x\right] &= \mathcal{L}_2\left[u_y\right]\\ p\bar{\bar{u}}(p,q) - \bar{u}_1(q) &= q\bar{\bar{u}}(p,q) - \bar{u}_2(p)\\ \bar{\bar{u}}(p,q) &= \frac{\bar{u}_1(q) - \bar{u}_2(p)}{p-q},\end{aligned}$$

where

$$\begin{aligned}\bar{u}_1(q) &= \mathcal{L}\{u(0,y)\} = \mathcal{L}\{g(y)\} = \bar{g}(q),\\ \bar{u}_2(p) &= \mathcal{L}\{u(x,0)\} = \mathcal{L}\{f(x)\} = \bar{f}(p).\end{aligned}$$

Thus, inverting, we have

$$u(x,y) = \mathcal{L}_2^{-1}\left[\bar{\bar{u}}(p,q)\right] = \mathcal{L}_2^{-1}\left[\frac{\bar{u}_1(q) - \bar{u}_2(p)}{p-q}\right]. \tag{77}$$

In particular, if $u(x,0) = 1$ and $u(0,y) = 1$, so that $\bar{u}_1(q) = 1/q$ and $\bar{u}_2(p) = 1/p$, then

$$\bar{\bar{u}}(p,q) = \frac{1/q - 1/p}{p-q} = \frac{1}{pq} \tag{78}$$

Thus, the inverse of the double Laplace transform gives the solution

$$u(x,y) = \mathcal{L}_2^{-1}\left\{\frac{1}{pq}\right\} = 1. \tag{79}$$

(b) D' Alembert's Wave Equation in a Quarter Plane The standard wave equation is

$$c^2 u_{xx} = u_{tt}, \qquad x \geq 0, \quad t > 0, \tag{80}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x > 0, \tag{81}$$

$$u(0,t) = 0, \quad u_x(0,t) = 0. \tag{82}$$

We apply the double Laplace transform $\bar{\bar{u}}(p,q) = \mathcal{L}_2\left[u(x,t)\right]$ defined by

$$\bar{\bar{u}}(p,q) = \int_0^\infty \int_0^\infty u(x,t)\, e^{-px-qt}\, dx dt, \tag{83}$$

to the wave equation system (80)–(82) so that

$$\begin{aligned}&c^2\mathcal{L}_2\left[u_{xx}(x,t)\right] = \mathcal{L}_2\left[u_{tt}(x,t)\right]\\ &c^2\left[p^2\,\bar{\bar{u}}(p,q) - p\,\mathcal{L}\left\{u(0,t)\right\} - \mathcal{L}\left\{u_x(0,t)\right\}\right]\\ &\quad = q^2\,\bar{\bar{u}}(p,q) - q\,\mathcal{L}\left\{u(x,0)\right\} - \mathcal{L}\left\{u_t(x,0)\right\}\\ &\left(c^2p^2 - q^2\right)\bar{\bar{u}}(p,q) = -\left[q\bar{f}(p) + \bar{g}(p)\right].\end{aligned}$$

Or,

$$\bar{\bar{u}}(p,q) = \frac{q\bar{f}(p) + \bar{g}(p)}{q^2 - c^2p^2}. \tag{84}$$

The inverse of the double Laplace transform gives

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\left[\bar{f}(p)\cosh cpt + \frac{\bar{g}(p)}{cp}\sinh cpt\right]dp \\ &= \frac{1}{2}\,\mathcal{L}^{-1}\left\{e^{cpt}+e^{-cpt}\right\} \\ &\quad + \frac{1}{2c}\,\mathcal{L}^{-1}\left\{\frac{\bar{g}(p)}{p}\left(e^{cpt}-e^{-cpt}\right)\right\} \\ &= \frac{1}{2}\left[f(x+ct)+f(x-ct)\right] \\ &\quad + \frac{1}{2c}\left[\mathcal{L}^{-1}\left\{\frac{\bar{g}(p)}{p}\,e^{cpt}\right\} + \mathcal{L}^{-1}\left\{\frac{\bar{g}(p)}{p}\,e^{-cpt}\right\}\right] \\ &= \frac{1}{2}\left[f(x+ct)+f(x-ct)\right] \\ &\quad + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)d\tau - \int_0^{x-ct} g(\tau)d\tau\right]. \end{aligned}$$

Hence,

$$u(x,t) = \frac{1}{2}\left[f(x+ct)+f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau. \tag{85}$$

This is the celebrated D' Alembert solution of the wave equation, where we have used

$$\mathcal{L}^{-1}\left\{e^{pa}\bar{f}(p)\right\} = f(x+a) \qquad \text{and } \mathcal{L}^{-1}\left\{\frac{\bar{g}(p)}{p}\right\} = \int_0^t g(\tau)\,d\tau.$$

(c) Fourier's Heat Equation in a Quarter Plane

The standard heat equation is

$$u_t = \kappa\, u_{xx}, \qquad x \ge 0, \quad t > 0, \tag{86}$$

$$u(x,0) = 0, \quad u(0,t) = 2T_0, \quad x > 0,\ t > 0, \tag{87}$$

$$u_x(0,t) = 0,\ t > 0, \quad u(x,t) \to 0 \text{ as } x \to \infty \tag{88}$$

where T_0 is a constant.

We apply the double Laplace transform defined by (83) to (86) with the conditions (87)–(88) to obtain

$$\begin{aligned} \mathcal{L}_2\left[u_t\right] &= \kappa\mathcal{L}_2\left[u_{xx}\right] \\ q\bar{\bar{u}}(p,q) - \mathcal{L}\left\{u(x,0)\right\} &= \kappa\left[p^2\bar{\bar{u}}(p,q) - p\mathcal{L}\left\{u(0,t)\right\} - \mathcal{L}\left\{u_x(0,t)\right\}\right]. \end{aligned}$$

Thus, we have

$$\bar{\bar{u}}(p,q) = \frac{2pT_0}{q}\,\frac{\kappa}{\kappa p^2 - q} \tag{89}$$

The inverse of the double Laplace transform gives

$$u(x,t) = 2T_0 \mathcal{L}_2^{-1}\left\{\frac{p}{q\left(p^2 - \frac{q}{\kappa}\right)}\right\}$$
$$= 2T_0 \mathcal{L}^{-1}\left\{\frac{1}{q}\cosh\left(\sqrt{\frac{q}{\kappa}}\right)x\right\}$$
$$= T_0 \mathcal{L}^{-1}\left\{\frac{1}{q}\left(e^{\sqrt{\frac{q}{\kappa}}x} + e^{-\sqrt{\frac{q}{\kappa}}x}\right)\right\}.$$

The first term above vanishes because of $u(x,t) \to 0$ as $x \to \infty$. Hence,

$$u(x,t) = T_0 \mathcal{L}^{-1}\left\{\frac{1}{q}\, e^{-\sqrt{\frac{q}{\kappa}}x}\right\}. \tag{90}$$

Now inversion yields the solution

$$u(x,t) = T_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{91}$$

(d) Two-dimensional heat conduction problem for the temperature distribution $u(x, y, t)$ in the region $x > 0,\ 0 < y < b$, with $t > 0$

The governing equation with the initial and boundary conditions is given by

$$\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}, \qquad x > 0,\ 0 < y < b, t > 0, \tag{92}$$
$$u(x, y, t = 0) = u_0(x, y),\ x > 0, 0 < y < b, \tag{93}$$
$$u(x = 0, y, t) = w_0(y, t),\ 0 < y < b,\ t > 0, \tag{94}$$
$$u(x, y = 0, t) = 0, \qquad x > 0, t > 0, \tag{95}$$
$$u(x, y = b, t) = 1, \qquad x > 0, t > 0, \tag{96}$$
$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = w_1(x, y) = \text{unknown}. \tag{97}$$

We apply the double Laplace transform with respect to x and t in the form

$$\bar{\bar{u}}(p, y, q) = \mathcal{L}_2\{u(x, y, t)\} = \int_0^\infty \int_0^\infty e^{-px-qt}\, u(x, y, t)dxdt, \tag{98}$$

so that the above initial and boundary value heat conduction problem becomes

$$\frac{d^2\bar{\bar{u}}}{dy^2} + \sigma^2\, \bar{\bar{u}} = \bar{w}_1(y, q),\ \sigma^2 = p^2 - \frac{q}{\kappa}, \tag{99}$$
$$\bar{\bar{u}}(p, y, q) = 0,\ y = 0;\ \bar{\bar{u}}(p, y, q) = \frac{1}{pq}, \qquad y = b. \tag{100}$$

It turns out that the solution of (99)–(100) is given by

$$\bar{\bar{u}}(p, y, q) = \frac{1}{pq}\frac{\sin\sigma y}{\sin\sigma b} - \frac{\sin\sigma y}{\sigma\sin\sigma b} - \frac{\sigma y}{\sigma\sin\sigma b}\int_y^b \sin(b-\eta)\bar{w}_1(\eta, q)d\eta$$
$$- \frac{\sin\sigma(b-y)}{\sigma\sin\sigma b}\int_0^y \sin\sigma\eta\ \bar{w}_1(\eta, q)d\eta. \tag{101}$$

Following Jaeger's method of evaluation of (101) based on the theory of residues (see Debnath and Bhatta [4]), we obtain the solution in the form

$$u(x,y,t) = \frac{y}{b} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} exp\left(-\frac{n\pi x}{b}\right)$$
$$\left[\int_0^{\infty} exp\left\{-\kappa t\left(\lambda^2 + \frac{n^2\pi^2}{b^2}\right)\frac{\sin\lambda x\, d\lambda}{\lambda\left(\lambda^2+\frac{n^2\pi^2}{b^2}\right)}\right\}\right]. \tag{102}$$

(e) Two-dimensional heat conduction initial and boundary value problem for the temperature distribution $u(x, y, t)$ in the region $x > 0,\ 0 < y < b$ with $t > 0$.

The governing initial and boundary value problem for the temperature distribution $u(x, y, t)$ in the region $x > 0,\ 0 < y < b$ is

$$\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}, \qquad x > 0,\ o < y < b,\ t > 0, \tag{103}$$
$$u(x, y, t = 0) = u_0(x, y) = 1,\ x > 0,\ 0 < y < b, \tag{104}$$
$$u(x, y, t) = 0 \quad when\ y = 0 \text{ and } y = b,\ x > 0 \text{ and } t > 0, \tag{105}$$
$$u(x = 0, y, t) = w_0(y, t) = 0,\ 0 < y < b,\ t > 0, \tag{106}$$
$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = w_1(y, t) = \text{unknown}. \tag{107}$$

We apply the double Laplace transform (98) so that the transformed system of (103)–(107) becomes

$$\frac{d^2\bar{\bar{u}}}{dy^2} + \sigma^2\,\bar{\bar{u}} = \bar{w}_1(y, q) - \frac{1}{kp},\ \sigma^2 = p^2 - \frac{q}{k}, \tag{108}$$
$$\bar{\bar{u}}(p, y, q) = 0 \text{ when } \quad y = 0 \text{ and } y = b. \tag{109}$$

Thus, the solution of (108)–(109) is given by

$$\bar{\bar{u}}(p, y, q) = \frac{\sin\sigma y - \sin\sigma b + \sin\sigma(b-y)}{\kappa p\sigma^2\sin\sigma b} - \frac{\sin\sigma y}{\sigma\sin\sigma b}\int_y^b \sin\sigma(b-\eta)\ \bar{w}_1(\eta, q)d\eta$$
$$- \frac{\sin\sigma(b-y)}{\sigma\sin\sigma b}\int_0^y \sin\sigma\eta\ \bar{w}_1(\eta, q)d\eta. \tag{110}$$

We follow Jaeger's method of evaluation based on the theory of residues to obtain the solution

$$u(x, y, t) = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left\{\left(\frac{2n+1}{b}\right)\pi y\right\}$$
$$\int_0^{\infty}\frac{\sin\lambda x}{\lambda} exp\left[-\kappa t\left\{\lambda^2 + \frac{(2n+1)^2\pi^2}{b^2}\right\}\right]d\lambda. \tag{111}$$

In particular, when $u_0(x, y) = x,\ x > 0,\ 0 < y < b$, we can obtain the solution for $\bar{\bar{u}}(p, y, q)$, and then the solution $u(x, y, t)$ can be obtained by inversion in the exact form

$$u(x, y, t) = \frac{4x}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin(2n+1)\left(\frac{\pi y}{b}\right) exp\left[-\kappa t\left\{\frac{(2n+1)^2\pi^2}{b^2}\right\}\right]. \tag{112}$$

Solution of Integral Equations by the Double Laplace Transform

We consider a double integral equation of the form

$$f(x, y) = h(x, y) + \lambda \int_0^x \int_0^y f(x - \xi, y - \eta) g(\xi, \eta) d\xi d\eta, \tag{113}$$

where $f(\cdot, \cdot)$ is an unknown function, λ is a given constant parameter, $h(x, y)$ and $g(x, y)$ are known functions.

We apply the double Laplace transform $\bar{\bar{f}}(p, q) = \mathcal{L}_2[f(x, y)]$ defined by (2) so that the convolution integral equation reduces to the form

$$\bar{\bar{f}}(p, q) = \bar{\bar{h}}(p, q) + \lambda\, \mathcal{L}_2[(f * * g)(x, y)]$$

which is, by the convolution Theorem 4.1,

$$\bar{\bar{f}}(p, q) = \bar{\bar{h}}(p, q) + \lambda \bar{\bar{f}}(p, q) \bar{\bar{g}}(p, q). \tag{114}$$

Consequently,

$$\bar{\bar{f}}(p, q) = \frac{\bar{\bar{h}}(p, q)}{1 - \lambda \bar{\bar{g}}(p, q)}. \tag{115}$$

The inversion of the double Laplace transform gives the solution of (114) in the form

$$f(x, y) = \mathcal{L}_2^{-1}\left[\frac{\bar{\bar{h}}(p, q)}{1 - \lambda \bar{\bar{g}}(p, q)}\right] = \mathcal{L}_2^{-1}[\bar{\bar{h}}(p, q) \cdot \bar{\bar{m}}(p, q)], \tag{116}$$

$$= \int_0^x \int_0^y h(x - \xi, y - \eta) m(\xi, \eta) d\xi d\eta, \tag{117}$$

where $\bar{\bar{m}}(p, q) = \frac{1}{1 - \lambda \bar{\bar{g}}(p, q)}$.

Thus, we obtain the formal solution of the original integral equation (113). It is necessary to obtain the inverse of the double Laplace transform for the explicit representation of the solution (116).

We illustrate the above method by simple examples.

(a) Solve the integral equation

$$f(x, y) = a - \lambda \int_0^x \int_0^y f(\xi, \eta) d\xi d\eta, \tag{118}$$

where a and λ are constant.

We apply the double Laplace transform to (118) so that

$$\bar{\bar{f}}(p, q) = \mathcal{L}_2\{a\} - \lambda \mathcal{L}_2[1 * f(x, y)]$$

or,

$$\bar{\bar{f}}(p, q) = \frac{a}{pq} - \frac{\lambda}{pq} \bar{\bar{f}}(pq).$$

Thus,

$$\bar{\bar{f}}(p, q) = \frac{a}{pq + \lambda}. \tag{119}$$

The inverse double Laplace transform of (119) gives the solution

$$f(x, y) = a\mathcal{L}_2^{-1}\left\{\frac{1}{pq+\lambda}\right\} = aJ_0(2\sqrt{\lambda xy}). \tag{120}$$

(b) Solve the integral equation

$$\int_0^x \int_0^y f(x-\xi,\ y-\eta) f(\xi,\ \eta) d\xi\, d\eta = a^2, \tag{121}$$

where a is a constant.
Application of the double Laplace transform to (121) gives

$$\bar{\bar{f}}^2(p, q) = \frac{a^2}{pq}.$$

Or,

$$\bar{\bar{f}}(p, q) = a \cdot \frac{1}{\sqrt{pq}}.$$

Using the inverse of the double Laplace transform yields the solution of (121)

$$f(x, y) = a\mathcal{L}_2^{-1}\left\{\frac{1}{\sqrt{pq}}\right\} = \frac{a}{\pi}\,\frac{1}{\sqrt{xy}}. \tag{122}$$

Concluding Remarks

Some simple examples and applications of the double Laplace transform are discussed in this paper. Some advanced problems in fluid dynamics and elasticity dealing with integral and partial differential equations will be discussed in a subsequent paper.

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