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MANUSCRIPT

Incompressible Euler Limit from Boltzmann Equation with Diffuse Boundary Condition for Analytic Data

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Abstract

A rigorous derivation of the incompressible Euler equations with the no-penetration boundary condition from the Boltzmann equation with the diffuse reflection boundary condition has been a challenging open problem. We settle this open question in the affirmative when the initial data of fluid are well-prepared in a real analytic space, in 3D half space. As a key of this advance, we capture the Navier-Stokes equations of

$$\textit{viscosity} \sim \frac{\textit{Knudsen number}}{\textit{Mach number}}$$

satisfying the no-slip boundary condition, as an *intermediary* approximation of the Euler equations through a new Hilbert-type expansion of the Boltzmann equation with the diffuse reflection boundary condition. Aiming to justify the approximation we establish a novel quantitative L^p-L^∞ estimate of the Boltzmann perturbation around a local Maxwellian of such viscous approximation, along with the commutator estimates and the integrability gain of the hydrodynamic part in various spaces; we also establish direct estimates of the Navier-Stokes equations in higher regularity with the aid of the initial-boundary and boundary layer weights using a recent Green's function approach. The incompressible Euler limit follows as a byproduct of our framework.

Keyword Boltzmann equation, Incompressible Euler equation, Hilbert 6th problem, Boundary

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1 Introduction

An important and active research direction in mathematical physics/PDE is on the so-called Hilbert's sixth problem [29] seeking a unified theory of the gas dynamics including different levels of descriptions from a mathematical standpoint by connecting the behavior of solutions to equations from kinetic theory to solutions of other systems that arise in formal limits, such as the N-body problem, the Euler equations, the Navier-Stokes equations, etc. In particular, the hydrodynamic limit of the Boltzmann equation has received a great deal of attention and enthusiasm in the mathematics and physics communities since the pioneering work [30] by Hilbert, which was the first example of his sixth problem. Remarkably, all the basic fluid equations of compressible, incompressible, inviscid, or viscous fluid dynamics can be derived from the Boltzmann equation of rarefied gas dynamics upon the choice of appropriate scalings in a small mean free path limit. Though formal derivations are rather well-understood, as far as mathematical justifications go, despite great progress over the decades (for example see [1–3,13,19,23,50] and the references therein), full understanding of the hydrodynamic limit incorporating important physical applications such as boundary effects or physical phenomena is still far from being complete. The goal of this paper is to make a rigorous connection between the Boltzmann equation and the incompressible Euler equations in the presence of the boundary by bypassing the *inviscid limit* of the incompressible Navier-Stokes equations.

The dimensionless Boltzmann equation with the *Strouhal number* $\mathcal{S}t$ and the *Knudsen number* $\mathcal{K}n$ takes the form of

$$\mathcal{S}t\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathcal{K}n} Q(F, F). \tag{1.1}$$

Here the distribution function of the gas is denoted by $F(t, x, v) \geq 0$ with the time variable $t \in \mathbb{R}_+ := \{t \geq 0\}$, the space variable $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$, and the velocity variable $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The Boltzmann collision operator $Q(\cdot, \cdot)$ of the hard sphere takes the form of

$$\begin{aligned} Q(F, G) =& \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \mathrm{u}| \{F(v')G(v'_*) + G(v')F(v'_*) \\ & - F(v)G(v_*) - G(v)F(v_*)\} \mathrm{d}\mathrm{u}\mathrm{d}v_*, \end{aligned} \tag{1.2}$$

where $v' := v - ((v - v_*) \cdot \mathrm{u})\mathrm{u}$ and $v'_* := v_* + ((v - v_*) \cdot \mathrm{u})\mathrm{u}$. This operator satisfies the so-called collision invariance property: for any $F(v)$ and $G(v)$ decaying sufficiently fast as $|v| \to 0$,

$$\int_{\mathbb{R}^3} Q(F, G)(v) \Big(1, v, \frac{|v|^2 - 3}{\sqrt{6}}\Big) \mathrm{d}v = (0, 0, 0), \tag{1.3}$$

which represents the local conservation laws of mass, momentum and energy. The celebrated Boltzmann's H-theorem reveals the entropy dissipation:

$$\int_{\mathbb{R}^3} Q(F, F)(v) \ln F(v) \mathrm{d}v \leq 0, \tag{1.4}$$

for any $F(v) > 0$ decaying sufficiently fast as $|v| \to \infty$. An intrinsic equilibrium, satisfying $Q(\cdot, \cdot) = 0$, is given by a local Maxwellian associated with the density $R > 0$, the macroscopic velocity $U \in \mathbb{R}^3$ and the temperature $T > 0$

$$M_{R,U,T}(v) := \frac{R}{(2\pi T)^{\frac{3}{2}}} \exp\Big\{-\frac{|v - U|^2}{2T}\Big\}, \tag{1.5}$$

which is known as the only configuration attaining the equality in (1.4).

In addition to the Strouhal number and Knudsen number we introduce the *Mach number* $\mathcal{M}a$. By passing $\mathcal{S}t$, $\mathcal{K}n$, and $\mathcal{M}a$ to zero, one can formally derive PDEs of hydrodynamic variables for the fluctuations around the reference state $(1, 0, 1)$, which are determined as

$$\begin{aligned} &\Big(\rho(t, x), u(t, x), \theta(t, x)\Big) \\ &= \lim_{\mathcal{M}a \downarrow 0} \frac{1}{\mathcal{M}a} \int_{\mathbb{R}^3} \{F(t, x, v) - M_{1,0,1}(v)\} \Big(1, v, \frac{|v|^2 - 3}{\sqrt{6}}\Big) \mathrm{d}v. \end{aligned} \tag{1.6}$$

The famous Reynolds number appears as a ratio between the Knudsen number and Mach number through the von Karman relation:

$$\frac{1}{\mathcal{Re}} = \frac{\mathcal{Kn}}{\mathcal{Ma}}. \tag{1.7}$$

For instance, the incompressible Navier-Stokes equations with $\mathcal{Re} = 1$, namely the viscosity of order one, can be derived by setting $\mathcal{St} = \mathcal{Ma} = \mathcal{Kn} = \varepsilon$ as $\varepsilon \downarrow 0$. In this paper we are particularly interested in a scale of *large Reynolds number* as follows:

$$\mathcal{St} = \varepsilon = \mathcal{Ma} \text{ and } \mathcal{Kn} = \kappa\varepsilon \text{ with } \kappa = \kappa(\varepsilon) \downarrow 0 \text{ as } \varepsilon \downarrow 0, \tag{1.8}$$

through which we will derive the incompressible Euler equations with the no-penetration boundary condition in the limit

$$\partial_t u_E + u_E \cdot \nabla_x u_E + \nabla_x p_E = 0, \;\; \nabla_x \cdot u_E = 0 \text{ in } \Omega, \tag{1.9}$$

$$u_E \cdot n = 0 \text{ on } \partial\Omega, \tag{1.10}$$

with $\partial_t \theta + u_E \cdot \nabla_x \theta = 0$ and $\nabla_x \theta(t,x) + \nabla_x \rho(t,x) = 0$. Here $n = n(x)$ denotes an outward normal at x on the boundary $\partial\Omega$.

For the sake of simplicity we set an initial datum $\theta_0(x) = 0 = \rho_0(x)$ so that

$$\theta(t,x) = 0 = \rho(t,x) \text{ for all } t \geq 0. \tag{1.11}$$

In many important physical applications such as a turbulence theory, it would be relevant to take into account the physical boundary in the hydrodynamic limit. A boundary condition of the Boltzmann equation is determined by the interaction law of the gas with the boundary surface. One of the physical conditions is the so-called *diffuse reflection boundary condition*, which takes into account an instantaneous thermal equilibration of reflecting gas particle (see [9,12]): for $(x,v) \in \{\partial\Omega \times \mathbb{R}^3 : n(x)\cdot v < 0\}$,

$$F(t,x,v) = c_\mu M_{1,0,1}(v) \int_{n(x)\cdot\mathfrak{v}>0} F(t,x,\mathfrak{v})(n(x)\cdot\mathfrak{v}) \mathrm{d}\mathfrak{v}, \tag{1.12}$$

where we have taken an isothermal boundary with a rescaled temperature 1 for the sake of simplicity. Here, the normalization constant $c_\mu := 1/\big(\int_{n(x)\cdot\mathfrak{v}>0} M_{1,0,1}(\mathfrak{v})(n(x)\cdot\mathfrak{v})\mathrm{d}\mathfrak{v}\big)$ leads to the null flux condition $\int_{\mathbb{R}^3} F(t,x,v)(n(x)\cdot v)\mathrm{d}v = 0$ on $x \in \partial\Omega$. In particular, it is well-known that the diffuse boundary condition (1.12) is a kinetic boundary condition featuring a mismatch with the no-penetration boundary condition (1.10) of the the Euler flow under (1.8), without any small parameter with respect to $1/\mathcal{Re}$ or $\mathcal{Ma}$. One can readily see this by expanding F around a local Maxwellian $M_{1,\varepsilon u_E,1}(v)$ associated with a flow of the no-penetration boundary condition (1.10) *directly*. Unfortunately, this local Maxwellian *does not honor* the diffuse reflection boundary condition when a flow satisfies *the no-penetration boundary condition* (1.10). In fact a size of the boundary mismatch could be an order of the tangential component of the Euler flow u_E at the boundary. Therefore a uniform bound to verify the limit (1.6) in a scale of large Reynolds number (1.8) is not expected even at the formal level. This poses a major obstacle in the Euler limit from the Boltzmann equation with the diffuse reflection boundary. It is worth noting that such a mismatch does not appear at least at the formal level when the specular reflection boundary condition is imposed: $F(t, x, v) = F(t, x, R_x v)$ on $x \in \partial\Omega$ where $R_x v = v - 2n(x)(n(x) \cdot v)$; while the mismatch can possess a small factor for the so-called Maxwell boundary condition, a convex combination of the specular reflection and the diffuse reflection boundary conditions, by choosing the coefficient for diffuse reflection known as *the accommodation constant* to vanish as $\mathcal{R}e \to \infty$.

Remarkably, an analogous, better-known boundary mismatch phenomenon exists in the realm of mathematical fluid dynamics, specifically in the inviscid limit problem of the Navier-Stokes equations that addresses the validity of the Euler solutions as the leading order approximation of the Navier-Stokes solutions in the vanishing viscosity limit. The inviscid limit for the no-slip boundary condition features a boundary mismatch between two different boundary conditions for the Navier-Stokes and Euler flows. In fact, whether the solution to the Navier-Stokes equations with a $\kappa\eta_0$-viscosity (a physical constant η_0 can be computed explicitly from the Boltzmann theory as in (1.37)) satisfying the no-slip boundary condition

$$\partial_t u + u \cdot \nabla_x u - \kappa\eta_0 \Delta u + \nabla_x p = 0 \ \text{ in } \Omega, \tag{1.13}$$

$$\nabla_x \cdot u = 0 \ \text{ in } \Omega, \tag{1.14}$$

$$u = 0 \ \text{ on } \partial\Omega, \tag{1.15}$$

converges to the solution of the Euler equations satisfying the no-penetration boundary condition (1.9)-(1.10) in $\kappa = 1/\mathcal{R}e \downarrow 0$ is an outstanding problem, which is arguably the most relevant and challenging because of the mismatch of two boundary conditions between (1.15) and (1.10) resulting in the formation of boundary layers such as Prandtl layer and unbounded vorticity near the boundary. While the verification of the inviscid limit is still largely open, it holds under certain symmetry assumption on the domain and data or under the flat boundary and strong regularity such as analyticity at least near the boundary [45]. A classical way to tackle the inviscid limit problem is to study the Prandtl expansion [44,48,49]: $u(t, x_1, x_2, x_3) = u_E(t, x_1, x_2, x_3) + u_P(t, x_1, x_2, \frac{x_3}{\sqrt{\kappa}}) + O(\sqrt{\kappa})$. Recently, different frameworks that avoid the boundary layer expansion have become available [38,47,54].

The incompressible Euler limit from the Boltzmann equation turns out to be intimately tied to the inviscid limit of the incompressible Navier-Stokes equations, which accounts for the similarity of two boundary mismatches. A beautiful connection stems from the Navier-Stokes solutions of (1.13)-(1.15) in large Reynolds numbers: at least formally, not only they are approximated by the Euler equations (1.9)-(1.10) but also they approximate the Boltzmann equation (1.1) under (1.8) with (1.12), in fact better than the Euler equations (1.9)-(1.10) at each Mach number $\varepsilon > 0$, because the Navier-Stokes equations contain a high order correction term $\kappa\eta_0\Delta u$ that captures the dissipative nature of the Boltzmann collision operator (as we will see in Section 1.1). And importantly, a local Maxwellian $M_{1,\varepsilon u,1}(v)$ associated with u satisfying the no-slip boundary condition (1.15), satisfies the diffuse reflection boundary condition (1.12) without singular terms. In other words, the Navier-Stokes solutions are compatible with the diffuse reflection boundary condition. Therefore, under the scale (1.8) the Navier-Stokes solution of (1.13)-(1.15) stands in between the Boltzmann solution of (1.1), (1.12) and the Euler solution (1.9)-(1.10).

In this paper, inspired by these observations, we propose to study the Euler limit from the Boltzmann equation through the Navier-Stokes solutions that hold both features of the Euler and the Boltzmann under (1.8) at each Mach number $\varepsilon > 0$. To this end, we expand the Boltzmann solution F around a local Maxwellian associated with a Navier-Stokes flow u to (1.13)-(1.15):

$$\mu(v) := M_{1,\varepsilon u,1}(v), \tag{1.16}$$

as

$$F = \mu + \varepsilon^2 f_2\sqrt{\mu} + \varepsilon^{3/2} f_R\sqrt{\mu}, \tag{1.17}$$

and analyze (1.17) via a new Hilbert expansion presented in Section 1.1. Although the notations F^ε and f^ε may be more precise for the equation depending on ε, we will abuse the notations by dropping the superscript ε for the sake of simplicity. The next order correction f_2 can be entirely determined by the Navier-Stokes flow and it turns out that its contribution is always smaller than f_R's one in our choice of ε and κ. A choice of the range of the Mach number with respect to the Reynolds number: $\varepsilon \ll \kappa = 1/\mathscr{Re}$ in $\varepsilon \downarrow 0$ plays an important role in our analysis and the formal expansion. We will discuss the relation and its role in Section 1.2. With such a choice of the scale, uniform-in-ε estimates of the Boltzmann remainder f_R are achieved by a novel quantitative L^p-L^∞ estimate in a setting of the local Maxwellian of the Navier-Stokes approximation (1.16), along with the commutator estimates and the integrability gain of the hydrodynamic part in various spaces.

In order to establish the Euler limit by using the Navier-Stokes solutions of (1.13)-(1.15) as a reference state as $\varepsilon \downarrow 0$ in a scale of large Reynolds number (1.8), it is imperative to show the uniform-in-κ convergence of the Navier-Stokes solutions to the Euler solutions of (1.9)-(1.10), where the inviscid limit comes into play. In this paper, we take the spatial domain to be the upper-half space with periodic boundary conditions in the horizontal components and analytic data for the Navier-Stokes solutions of (1.13)-(1.15) and obtain uniform-in-κ estimates built upon a recent development on the inviscid limit problem in the half-space based on the Green's function approach using the boundary vorticity formulation [38,44,47,54].

Our main result concerns a rigorous justification of the passage from the solutions to the dimensionless Boltzmann equation (1.1) of the scale (1.8) with the diffuse reflection boundary condition (1.12) to the solution of the incompressible Euler equation (1.9) with the no-penetration boundary condition (1.10), without introducing any boundary expansion of the Boltzmann equation:

Theorem 1 (Informal statement) *We consider a half space in 3D*

$$\Omega := \mathbb{T}^2 \times \mathbb{R}_+ \ni (x_1, x_2, x_3), \quad \textit{where } \mathbb{T} \textit{ is a periodic interval of } (-\pi, \pi). \tag{1.18}$$

For some choice of ε and $\kappa(\varepsilon)$, there exists a large set of initial data u_{in}, $f_{2,in}$ and $f_{R,in}$ such that a unique solution $F(t,x,v)$ of the form (1.17) *to (1.1) and (1.12) with (1.8) exists on* $[0,T]$ *for some* $T > 0$ *and satisfies*

$$\sup_{0\le t\le T} \left\| \frac{F(t,x,v) - M_{1,\varepsilon u,1}}{\varepsilon\sqrt{M_{1,\varepsilon u,1}}} \right\|_{L^2(\Omega\times\mathbb{R}^3)} \longrightarrow 0 \ \ as \ \ \varepsilon \downarrow 0,$$

and

$$\sup_{0\le t\le T} \left\| \frac{F(t,x,v) - M_{1,\varepsilon u_E,1}}{\varepsilon(1+|v|)^2\sqrt{M_{1,0,1}}} \right\|_{L^2(\Omega\times\mathbb{R}^3)} \longrightarrow 0 \ \ as \ \ \varepsilon \downarrow 0,$$

while u and u_E denote solutions of the Navier-Stokes (1.13)-(1.15) and Euler equations (1.9)-(1.10), respectively.

The precise statement of Theorem 1 is given in Theorem 4 and Corollary 5 in Section 2.3.

Remark 1 To the best of our knowledge our result of this paper appears to be the first rigorous incompressible Euler limit result from the Boltzmann solutions with the sole diffuse reflection (therefore the accommodation constant $\sim$1) in the boundary condition! Moreover, our framework captures the inviscid limit of mathematical fluid dynamics from the Boltzmann theory.

Remark 2 Another natural choice of the scale in the study of the Euler limit might be $\varepsilon^{\mathfrak{q}} = \kappa$ with an integer $\mathfrak{q} \geq 1$. Then the second correction $\frac{1}{\kappa}Lf_2$ is shifted to the next hierarchy (see (1.27)) and as a consequence the Euler equations become the leading approximation with loss of $\kappa\eta_0\Delta u$. Without the boundary, a higher order expansion $F = \mu_E + [\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots + \varepsilon^r f_R]\sqrt{\mu_E}$ for $\mu_E = M_{1,\varepsilon u_E,1}$ has been established in [10,56]. In the presence of the boundary, on the other hand, such an expansion features a boundary mismatch. The usual approach is then drawn on a boundary layer expansion, correcting an interior Hilbert-like expansion at the boundary to satisfy the boundary conditions (for example, see [27,55]). Our approach is based on an interior expansion up to the second correction f_2 that *avoids* the boundary layer expansion under our choice of scale $\varepsilon \ll \kappa$ (see (2.11)).

Before discussing the essence of the methodology and novelty of our result, we shall briefly overview some relevant literatures on the hydrodynamic limit of the Boltzmann equation. One of the first mathematical studies of the limits at the formal level may go back to a work [30] of Hilbert, in which he introduced so-called the Hilbert expansion. Based on the truncated Hilbert expansion rigorous justifications of fluid limits have been shown as long as the solutions of corresponding fluids are bounded in some suitable spaces, for example, in the compressible fluid limits in [7,53], incompressible fluid limits in [5,10,23], diffusive limits from the Vlasov-Maxwell-Boltzmann system in [32], and relativistic fluid limits in [52]. All the derivations mentioned above did not take into account the boundary, while one of the main obstacles to study the

Boltzmann solutions with the boundary is its boundary singularity (see [20,21,35]). In [22], an L^p-L^∞ framework has been developed to construct a unique global solution of the Boltzmann equation with physical boundary conditions. Such a framework has been developed successfully in various problems of the Boltzmann theory (for example [8,14,24–27,36,37,55]). In particular, in [12,13], the authors have constructed a solution of the Boltzmann equation satisfying the diffuse reflection boundary condition and proved the validity of the hydrodynamic limit toward the incompressible Navier-Stokes-Fourier system in both steady and unsteady settings, based on a novel L^6-bound of the hydrodynamic part.

Rigorous passage from the renormalized solutions of [11] ([46] with the physical boundary) of the Boltzmann equation toward (weak) solutions of fluid equations has been also extensively explored (see [17,33,50] for the references in this direction). In particular, the program of the incompressible Navier-Stokes limit to the Leray-Hopf weak solutions has been developed successfully in [2,3,19,40,41] without the physical boundary and with the boundary in [33,42]. As for the incompressible Euler limit in terms of the entropy production, based on the relative entropy method, a dissipative solution of the incompressible Euler equations in [39] has been studied in [40,41,51] without the boundary. Notably the results have been extended to the domain with the boundary for the specular reflection boundary condition in [50], and for the Maxwell boundary condition in [4], assuming to set that the accommodation constant (a factor of diffuse reflection) vanishes as $\varepsilon \downarrow 0$.

For the rest of this section, we present the strategy and key ideas developed in the proof of our result starting with a new (formal) Hilbert expansion followed by the control of the Boltzmann remainder f_R and higher regularity of Navier-Stokes flows, for the rigorous justification of the formal expansion.

1.1 Hilbert Expansion in a Scale of Large Reynolds Number

Through a new formal Hilbert-type expansion of Boltzmann equation with the diffuse reflection boundary condition we aim to capture the Navier-Stokes equations of vanishing viscosity proportional to $\mathcal{Kn}/\mathcal{Ma}$ and satisfying the no-slip boundary condition.

It is worth pointing out that although more convenient choice of an expansion of F is seemingly the one around the global Maxwellian $\mu_0 := M_{1,0,1}$ such as $F = \mu_0 + \varepsilon(u\cdot v)\mu_0 + \varepsilon^2 \tilde{f}_2\sqrt{\mu_0} + \varepsilon\delta \tilde{f}_R\sqrt{\mu_0}$, unfortunately this choice will produce, in the Hilbert expansion (1.26)-(1.30), an unbounded term $\frac{2}{\kappa\varepsilon}\frac{1}{\sqrt{\mu_0}} Q(u\cdot v\mu_0, f_R\sqrt{\mu_0})$ even compared to the strongest control in hand, namely a dissipation term (see (1.31))! To achieve a sharper estimate, which provides weaker restriction on κ and ε, and hence weaker restriction on the initial data, we work on an expansion around the local Maxwellian μ.

It is conceptually convenient in our analysis to introduce an auxiliary parameter $\delta = \delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$, which indicates a size of the fluctuation $(F - \mu)/\varepsilon$:

$$F = \mu + \varepsilon^2 f_2\sqrt{\mu} + \varepsilon\delta f_R\sqrt{\mu}. \tag{1.19}$$

In (1.17) we have chosen $\delta = \sqrt{\varepsilon}$ and in Section 2.3 we will have the same choice such as (2.11), however in Section 2.1, Section 3 and Section 4, δ will be regarded as a free parameter and will be chosen at the last step of closing our argument (as (2.11)!).

Interior Expansion. We investigate an expansion (1.19) of the Boltzmann equation (1.1) at the local Maxwellian μ in (1.16). Let

$$Lf = \frac{-2}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f), \quad \Gamma(f, g) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} g). \tag{1.20}$$

The operators L and Γ can be read as

$$Lf(v) = \nu f(v) - Kf(v) = \nu(v) f(v) - \int_{\mathbb{R}^3} \mathbf{k}(v, v_*) f(v_*) \mathrm{d}v_*, \tag{1.21}$$

$$\begin{aligned}\Gamma(f, g)(t, v) &= \Gamma_+(f, g)(t, v) - \Gamma_-(f, g)(t, v)\\ &= \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot \mathrm{u}| \sqrt{\mu(v_*)} \big(f(t, v') g(t, v'_*) + g(t, v') f(t, v'_*) \big) \mathrm{d}\mathrm{u} \mathrm{d}v_*\\ &\quad - \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot \mathrm{u}| \sqrt{\mu(v_*)} \big(f(t, v) g(t, v_*) + g(t, v) f(t, v_*) \big) \mathrm{d}\mathrm{u} \mathrm{d}v_*,\end{aligned} \tag{1.22}$$

where the precise form of $\mathbf{k}$ is delayed to be presented in (3.19). We will demonstrate basic properties of operators L and Γ in Section 3.1. From (1.3) the null space of L, denoted by $\mathcal{N}$, is a subspace of $L^2(\mathbb{R}^3)$ spanned by orthonormal bases $\{\varphi_i \sqrt{\mu}\}_{i=0}^4$ with

$$\varphi_0 := 1, \quad \varphi_i := v_i - \varepsilon u_i \text{ for } i = 1, 2, 3, \quad \varphi_4 := (|v - \varepsilon u|^2 - 3)/\sqrt{6}. \tag{1.23}$$

We define a hydrodynamic projection $\mathbf{P}$ as an L^2_v-projection on $\mathcal{N}$ such as

$$\begin{aligned}\mathbf{P} g &:= \sum (P_j g) \varphi_j \sqrt{\mu}, \quad P_j g := \langle g, \varphi_j \sqrt{\mu} \rangle, \text{ and}\\ P g &:= (P_0 g, P_1 g, P_2 g, P_3 g, P_4 g),\end{aligned} \tag{1.24}$$

where $\langle \cdot, \cdot \rangle$ stands for an L^2_v-inner product. It is well-known that the operators enjoy $\mathbf{P} L = L \mathbf{P} = \mathbf{P} \Gamma = 0$. Importantly the linear operator L enjoys a coercivity away from the kernel $\mathcal{N}$: for $\nu(v) \geq 0$ defined in (1.21)

$$\langle Lf, f \rangle \geq \sigma_0 \| \sqrt{\nu} (\mathbf{I} - \mathbf{P}) f \|^2_{L^2(\mathbb{R}^3)} \text{ for some } \sigma_0 > 0. \tag{1.25}$$

Now we plug the expansion (1.19) into the rescaled equation (1.1) with the scale (1.8). It turns out that by relating f_2 with the flow and locating it carefully in the hierarchy we can exhibit the dissipative nature of the Boltzmann collision operator at the leading order of the fluid approximation. In particular we locate $(v-\varepsilon u)\cdot\nabla_x(\mathbf{I}-\mathbf{P})f_2$ in $\frac{1}{\delta}$-order hierarchy to capture κ-order viscosity in the fluid equation (1.13):

$$\partial_t f_R + \frac{1}{\varepsilon} v\cdot\nabla_x f_R + \frac{1}{\varepsilon^2\kappa} L f_R + \frac{(\partial_t + \varepsilon^{-1} v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}} f_R \tag{1.26}$$

$$= -\frac{1}{\varepsilon\delta}\Big\{\frac{\varepsilon^{-1}(v-\varepsilon u)\cdot\nabla_x\mu}{\sqrt{\mu}} + \frac{1}{\kappa} L f_2\Big\} \tag{1.27}$$

$$-\frac{1}{\delta}\Big\{\frac{\varepsilon^{-1}\partial_t\mu}{\sqrt{\mu}} + \frac{\varepsilon^{-1} u\cdot\nabla_x\mu}{\sqrt{\mu}} + (v-\varepsilon u)\cdot\nabla_x f_2\Big\} \tag{1.28}$$

$$-\frac{\varepsilon}{\delta}\Big\{\partial_t f_2 + u\cdot\nabla_x f_2 + \frac{(\partial_t + \varepsilon^{-1} v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}} f_2\Big\} \tag{1.29}$$

$$+\frac{2}{\kappa}\Gamma(f_2, f_R) + \frac{\varepsilon}{\delta\kappa}\Gamma(f_2, f_2) + \frac{\delta}{\varepsilon\kappa}\Gamma(f_R, f_R). \tag{1.30}$$

We can readily see an L^2-energy structure of f_R with a strong dissipation

$$\iint_{\Omega\times\mathbb{R}^3} \frac{1}{\varepsilon^2\kappa} L f_R f_R \mathrm{d}v\mathrm{d}x \gtrsim \|\varepsilon^{-1}\kappa^{-1/2}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2(\Omega\times\mathbb{R}^3)}, \tag{1.31}$$

which inherits its lower bound from the coercivity (1.25).

Let us first consider an $\frac{1}{\varepsilon\delta}$-hierarchy (1.27). For any non-vanishing term of (1.27) would cause unpleasant unboundedness, we make the term vanish entirely by solving an equation (1.27)$=0$. By the Fredholm alternative, an inverse map

$$L^{-1}: \mathcal{N}^\perp \to \mathcal{N}^\perp, \quad \text{where } \mathcal{N}^\perp \text{ stands an } L^2_v\text{-orthogonal complement of } \mathcal{N}, \tag{1.32}$$

is well-defined and hence the solvability condition is given by

$$\frac{\varepsilon^{-1}(v-\varepsilon u)\cdot\nabla_x\mu}{\sqrt{\mu}} = \sum_{\ell,m=1}^{3} \partial_\ell u_m \varphi_\ell\varphi_m\sqrt{\mu} \in \mathcal{N}^\perp. \tag{1.33}$$

This condition indeed implies the incompressible condition (1.14).

Once (1.14) holds, we have $\sum_{\ell,m=1}^3 \partial_\ell u_m\varphi_\ell\varphi_m\sqrt{\mu} = \sum_{\ell,m=1}^3 \partial_\ell u_m(\varphi_\ell\varphi_m - \frac{|v-\varepsilon u|^2}{3}\delta_{\ell m})\sqrt{\mu}$. Now we solve (1.27)$=0$ by setting

$$(\mathbf{I}-\mathbf{P})f_2 = -\kappa\sum_{\ell,m=1}^{3} A_{\ell m}\partial_\ell u_m \quad \text{with}$$

$$A_{\ell m} := L^{-1}\Big(\varphi_\ell\varphi_m\sqrt{\mu} - \frac{|v-\varepsilon u|^2}{3}\delta_{\ell m}\sqrt{\mu}\Big). \tag{1.34}$$

Then we move to an $\frac{1}{\delta}$-hierarchy (1.28). The hydrodynamic part of (1.28), unless it vanishes, would induce an unbounded term again. We expand $\delta\times$ (1.28), using (1.16) and (1.34), as

$$
\begin{aligned}
&-(v-\varepsilon u)\cdot(\partial_t u+u\cdot\nabla_x u)\sqrt{\mu}+(v-\varepsilon u)\cdot\nabla_x \mathbf{P} f_2\\
&+\kappa(v-\varepsilon u)\cdot\nabla_x\Big(\sum_{\ell,m=1}^{3} A_{\ell m}\partial_\ell u_m\Big).
\end{aligned}
\tag{1.35}
$$

The leading order term of the last term in (1.35) contributes the following to the hydrodynamic part of (1.35) as

$$
\begin{aligned}
&\kappa\sum_{\ell,m,k=1}^{3}\big\langle\varphi_i\varphi_k\sqrt{\mu},A_{\ell m}\big\rangle\partial_k\partial_\ell u_m\\
&=\kappa\sum_{\ell,m,k=1}^{3}\Big\langle\Big(\varphi_i\varphi_k-\frac{|v-\varepsilon u|^2}{3}\delta_{ik}\Big)\sqrt{\mu},A_{\ell m}\Big\rangle\partial_k\partial_\ell u_m\\
&=\kappa\sum_{\ell,m,k=1}^{3}\langle LA_{ik},A_{\ell m}\rangle\,\partial_k\partial_\ell u_m,
\end{aligned}
\tag{1.36}
$$

where we have used the fact $A_{\ell m}\in\mathcal{N}^{\perp}$ and $\frac{|v-\varepsilon u|^2}{3}\sqrt{\mu}\in\mathcal{N}$ at the first step and the definition of A_{ik} at the last step. It is well-known (e.g. Lemma 4.4 in [3]) that for some constant $\eta_0>0$

$$
\langle LA_{ik},A_{\ell m}\rangle=\eta_0(\delta_{\ell k}\delta_{mi}+\delta_{\ell i}\delta_{mk})-\frac{2}{3}\eta_0\delta_{\ell m}\delta_{ik}.
\tag{1.37}
$$

Therefore we deduce that (1.36) vanishes for $i=0,4$, and the $\kappa\eta_0$-viscosity term in (1.13) can be captured:

$$
\begin{aligned}
(1.36)&=\kappa\eta_0\sum_{\ell,m,k}\{(\delta_{\ell k}\delta_{mi}+\delta_{\ell i}\delta_{mk})-\frac{2}{3}\delta_{\ell m}\delta_{ik}\}\partial_k\partial_\ell u_m\\
&=\kappa\eta_0\{\Delta u_i-\partial_i\nabla\cdot u-\frac{2}{3}\partial_i\nabla\cdot u\}=\kappa\eta_0\Delta u_i\quad\text{for } i=1,2,3.
\end{aligned}
\tag{1.38}
$$

Here we have used the incompressible condition (1.14) at the last step. On the other hand, a leading order term of the hydrodynamic part of $(v-\varepsilon u)\cdot\nabla_x\mathbf{P}f_2$ contributes to the pressure term of (1.13) by choosing a special form of $\mathbf{P}f_2$ as in (3.1). Therefore the whole leading order terms of the hydrodynamic part in (1.28) do vanish by solving the Navier-Stokes equations (1.13) and (1.14)! For the sake of brevity we refer to Section 3 for the full expansion of (1.26)-(1.30).

Boundary Conditions. Now we consider a boundary condition of f_R. Noticeably the local Maxwellian μ becomes $M_{1,0,1}$ on the boundary from the no-slip boundary condition (1.15), and hence μ satisfies the diffuse reflection boundary condition (1.12). For the detailed study of the boundary condition of f_R we introduce the incoming and outgoing boundaries

$$\gamma_{\pm} := \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v \gtrless 0\}.$$

Since μ satisfies the diffuse reflection boundary condition (1.12) with a constant wall temperature $= 1$, by plugging (1.19) into the boundary condition, we arrive at

$$(\varepsilon^2 f_2 + \delta\varepsilon f_R)|_{\gamma_-} = c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot\mathfrak{v}>0} (\varepsilon^2 f_2 + \delta\varepsilon f_R)\sqrt{\mu(\mathfrak{v})}(n(x)\cdot\mathfrak{v})\mathrm{d}\mathfrak{v}.$$

Letting P_{γ_+} be an $L^2(\{v : n(x)\cdot v > 0\})$-projection of $\sqrt{c_\mu\mu}$, we derive that

$$\begin{aligned} f_R(t,x,v)|_{\gamma_-} &= P_{\gamma_+} f_R(t,x,v) - \frac{\varepsilon}{\delta}(1-P_{\gamma_+}) f_2(t,x,v) \\ &:= \sqrt{c_\mu\mu(v)} \int_{n(x)\cdot\mathfrak{v}>0} f_R(t,x,\mathfrak{v})\sqrt{c_\mu\mu(\mathfrak{v})}(n(x)\cdot\mathfrak{v})\mathrm{d}\mathfrak{v} \\ &\quad - \frac{\varepsilon}{\delta}(1-P_{\gamma_+}) f_2(t,x,v). \end{aligned} \tag{1.39}$$

Note that $\int_{n(x)\cdot v>0} c_\mu \mu(v)(n(x)\cdot v)\mathrm{d}v = 1$.

On the other hand, we emphasize that, with the no-penetrate boundary condition of (1.10), the associated local Maxwellian $M_{1,\varepsilon u_E,1}$ does not satisfy the diffuse reflection boundary condition in general. Therefore the Boltzmann remainder f_R would have a singularity of an order of $1/\sqrt{\varepsilon}$ in (1.17).

1.2 Uniform Controls of the Boltzmann Remainder f_R

For a rigorous justification of the Hilbert expansion (1.19), the major task is to establish uniform-in-ε estimates of the Boltzmann remainder f_R in L^2. The equation of the Boltzmann remainder f_R in (1.26)-(1.30) with the boundary condition (1.39) features a discrepancy between the behavior of the hydrodynamic part $\mathbf{P} f_R$ and pure kinetic part $(\mathbf{I}-\mathbf{P}) f_R$: schematically an L^2-energy estimate reads

$$\begin{aligned} &\frac{d}{dt}\|f_R(t)\|_{L^2}^2 + \|\varepsilon^{-1}\kappa^{-1/2}(\mathbf{I}-\mathbf{P})f_R\|_{L^2}^2 \sim \|\nabla_x u\|_{L^\infty}\|\mathbf{P}f_R\|_{L^2}^2 \\ &\quad + \iint_{\Omega\times\mathbb{R}^3} \frac{\delta}{\varepsilon\kappa}\Gamma(\mathbf{P}f_R, \mathbf{P}f_R)(\mathbf{I}-\mathbf{P})f_R. \end{aligned}$$

A key difficulty arises from a growth of the hydrodynamic part at least as $e^{\|\nabla_x u\|_{L^\infty}}$ which might behave as an exponential of the reciprocal of some power of the viscosity κ due to the unbounded vorticity formed near the boundary, while such strong singularity of the hydrodynamic part enters the nonlinear estimate in turn. In fact such trilinear estimate can be effectively handled only by a point-wise bound of the solutions. Unfortunately as the physical boundary conditions create singularities in general

([35]), the high Sobolev estimates would not be possible. In this paper we develop a quantitative L^p-L^∞ estimate *solely* in the setting of the local Maxwellian associated with the Navier-Stokes flow, in the presence of the diffuse reflection boundary.

Thanks to a strong control of the dissipation from the spectral gap of (1.25), the nonlinear term can be bounded as

$$\delta\kappa^{-\frac{1}{2}}\|Pf_R\|_{L^\infty_t L^6_x}\|Pf_R\|_{L^2_t L^3_x}\|\varepsilon^{-1}\kappa^{-\frac{1}{2}}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}}. \tag{1.40}$$

Notably an *integrability gain* of the hydrodynamic part Pf_R should play a role; however a classical velocity average lemma $Pf_R \in H^{1/2}_x \subset L^3_x$ fails to fulfill the need in $3D$. We achieve such a higher integrability by developing a recent L^6-bound of hydrodynamic part of [13] in the setting of the local Maxwellian on the scale of large Reynolds number. We utilize the micro-macro decomposition and the equation to control $\kappa^{1/2}v\cdot\nabla_x\mathbf{P}f_R$ mainly by $\frac{1}{\varepsilon\kappa^{1/2}}L(\mathbf{I}-\mathbf{P})f_R$ and $\varepsilon\kappa^{1/2}\partial_t f_R$. We invert the operator $v\cdot\nabla_x\mathbf{P}$, employing a recent test function method of [12] in the local Maxwellian setting, to establish a crucial L^6-bound of the hydrodynamic part, which is controlled by the dissipation plus the a priori L^2-bound of $\partial_t f_R$:

$$\|\kappa^{1/2}Pf_R(t)\|_{L^6_x} \lesssim \|\varepsilon^{-1}\kappa^{-1/2}(\mathbf{I}-\mathbf{P})f_R(t)\|_{L^2_{x,v}} + \varepsilon\kappa^{1/2}\|\partial_t f_R(t)\|_{L^2_{x,v}} + l.o.t. \tag{1.41}$$

In other words we can achieve the L^6-estimate of (1.41) as "one spatial derivative gain" through the dissipation provided a temporal derivative being controlled, while the temporal derivative preserves the boundary conditions. It is a critical point in which a temporal derivative gets involved in our analysis of Boltzmann and fluids as well!

New difficulties arise as commutator estimates of $\frac{1}{\varepsilon^2\kappa}\{\partial_t Lf_R - L\partial_t f_R\}$ induce singularities even at the linear level, as well as $\partial_t(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}f_R/\sqrt{\mu}$ and the source terms in the equation of $\partial_t f_R$ possess higher temporal derivatives of the fluid with an initial layer. In fact after a careful analysis we realize such singular terms amount to

$$\frac{1}{\kappa^{\mathfrak{P}}}\int_0^t \|Pf_R(s)\|^2_{L^2_x}\mathrm{d}s,$$

while $\mathfrak{P}$ depends on the singularity of derivatives of the Navier-Stokes flow in large Reynolds numbers.

We establish a unified L^∞-estimate in the local Maxwellian setting, devising a special weight function $\mathfrak{w}_{\varrho,\mathfrak{B}}(x,v)$ in order to control an extra growth in $|v|$ from $(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}f_R/\sqrt{\mu}$ and its temporal derivative. We control f_R in $L^\infty_t L^\infty_x$ by the hydrodynamic part Pf_R in L^6 and the dissipation, studying the particle-trajectory bouncing against the diffuse reflection boundary and geometric change of variables related to the bouncing trajectories. The temporal derivative $\partial_t f_R$ needs some special attention since the source term of the equation of $\partial_t f_R$ possesses $\nabla_x\partial_t^2 u$, which turns out to have an initial-boundary layer. For that we measure $\partial_t f_R$ using a different time-space norm, namely a weighted $L^2_t L^\infty_x$, and control it by the hydrodynamic part of

$\partial_t f_R$ in $L^2_t L^3_x$ (with more singular factor-in-ε than the counterpart for f_R) and the dissipation. Although our estimate of $\partial_t f_R$ is singular than f_R due to our choice of different spaces, we are able to balance such extra singularity by the strong dissipation and careful trilinear estimates.

We establish $L^2_t L^3_x-$controls for Pf_R and $P\partial_t f_R$ via the trajectory rather than the classical average lemma. In fact a direct application of such average lemma has some subtle issue since the source terms of f_R and $\partial_t f_R$ equations are known to be bounded within a finite time interval only, while the $L^2_t L^3_x$-control enters the nonlinear estimates. In fact it is not clear whether our iteration of estimates would guarantee a nonempty finite time interval of validity. Instead we utilize the Duhamel formula along the trajectories and an extension of solutions in specially designed domains, and employ the TT^*-method developed in [14,28,31]. As a result we achieve $L^2_t L^P_x$ estimates for f_R and $\partial_t f_R$ uniformly for all $p < 3$, which gives us a sufficient bound in $L^2_t L^3_x$ by interpolating with our L^∞-estimates.

Finally upon combining all the estimates above together we are able to bound an energy by the Gronwall's inequality. The resulting bound is not uniform but growing exponentially as $e^{1/\kappa^{\mathfrak{P}}}$, in which the power depends on the higher regularity of the fluid. Luckily we are able to find a range of ε with respect to κ in a scale of large Reynolds number to absorb the Gronwall growth, and achieve a uniform bound of the Boltzmann remainder, which ensures the rigorous justification of the Hilbert expansion in Section 1.1. The main theorem of the uniform controls of the Boltzmann remainder f_R is given in Theorem 2.

1.3 Higher Regularity of Navier-Stokes Equations in the Inviscid Limit

The inviscid limit of the Navier-Stokes equations (1.13)-(1.15) is at the heart of our approach. Furthermore, in order to control f_R, as explained in the above, we need to derive quantitative higher regularity estimates of the Navier-Stokes solutions which are not directly available in the usual inviscid limit results. Before discussing new features of our analysis, we briefly discuss some prior works on the inviscid limit most relevant to our result. Due to the formation of boundary layers in the limit caused by the mismatch of boundary conditions (1.15) and (1.10), a classical way to tackle the inviscid limit problem is via the Prandtl expansion, of which rigorous justification was shown in [48,49] for well-prepared data with analytic regularity and in [44] for the initial datum with Sobolev regularity when the initial vorticity is bounded away from the boundary. In particular, the author of [44] introduced the boundary vorticity formulation of (1.13)-(1.15) (see (2.16)-(2.18)) which prompted subsequent interesting works in the field. Among others, in a recent work [47], the authors proved the inviscid limit in 2D based on the Green's function approach based on Maekawa's vorticity formulation without having to construct Prandtl boundary layer corrections but by utilizing the boundary layer weights in the norm. In [38,54], the inviscid limit was shown for initial data that is analytic only near the boundary and has finite Sobolev regularity in the complement in 2D and 3D respectively.

Our analysis of the Navier-Stokes solutions in the limit is based on the Green's function approach for the Stokes problem using the vorticity formulation (2.16)-(2.18) in the same spirit of [47]. However, the existing methods [38,47,54] do not immediately fulfill the goal of our hydrodynamic limit because the analysis of our remainder f_R requires higher regularity of Navier-Stokes solutions, more specifically L^2 and L^∞ bounds for higher order derivatives up to two temporal derivatives of $\nabla_x u$ and p and two spatial derivatives of $\partial_t u$, while the existing methods do not decipher any bounds for temporal derivatives and the boundedness of the conormal derivatives in their analytic norms does not rule out $\frac{1}{x_3}$ singularity of the normal derivative of the vorticity in the boundary layer, which may cause the loss of L^2 integrability. To get around these issues, we pursue new estimates of temporal derivatives of the vorticity ω by demanding the compatibility conditions for the initial data. With such conditions, the initial layer is absent for ω and $\partial_t\omega$; we can derive an analogous integral representation formula for $\partial_t\omega$ so that we may run the same fixed point argument for $\partial_t\omega$ as in [47] without the initial layer. For the second temporal derivative, we handle the initial-boundary layer for the horizontal part with the initial-boundary weight function. These new features allow us to attain the derivative estimates of the vorticity in the normal direction without $\frac{1}{x_3}$ singularity near the boundary at the expense of losing a power of $\sqrt{\kappa}$, which is crucial for the control of f_R. The velocity and pressure estimates are then recovered by utilizing elliptic regularity results and the Biot-Savart law in the analytic setting. The main results of Navier-Stokes solutions to (1.13)-(1.15) are given in Theorem 3.

2 Main Results

For the sake of the readers we present the precise statement of main theorems and their notations in this section. We first present the uniform controls of the Boltzmann remainder f_R of Theorem 2, and the higher regularity of the Navier-Stokes equations in the inviscid limit of Theorem 3. As a consequence of those two theorems we will show a rigorous justification of kinetic approximation of Navier-Stokes in high Reynolds numbers of Theorem 4. Then using the vorticity estimates in Theorem 3 and the famous Kato's condition in the inviscid limit, we prove a hydrodynamic limit toward the incompressible Euler equations in Corollary 5.

2.1 Uniform Controls of the Boltzmann Remainder f_R (Theorem 2)

We recall the expansion of Boltzmann solution $F = \mu + \varepsilon^2 f_2\sqrt{\mu} + \delta\varepsilon f_R\sqrt{\mu}$ in (1.19) around the local Maxwellian $\mu(v) := M_{1,\varepsilon u,1}(v)$ for any given flow (u, p) solving the incompressible Navier-Stokes equation with the no-slip boundary condition (1.13)-(1.15).

Inspired by the energy structure of the PDE and the coercivity of the linear operator L in (1.25), we define an energy and a dissipation as

$$\mathcal{E}(t) := \|f_R(t)\|^2_{L^2(\Omega\times\mathbb{R}^3)} + \|\partial_t f_R(t)\|^2_{L^2(\Omega\times\mathbb{R}^3)},$$
$$\mathcal{D}(t) := \int_0^t \|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R(s)\|^2_{L^2(\Omega\times\mathbb{R}^3)}\mathrm{d}s$$

$$
\begin{aligned}
&+\int_0^t \|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R(s)\|^2_{L^2(\Omega\times\mathbb{R}^3)}\mathrm{d}s\\
&+\int_0^t \Big(|\varepsilon^{-\frac{1}{2}} f_R(s)|^2_{L^2_\gamma}+|\varepsilon^{-\frac{1}{2}}\partial_t f_R(s)|^2_{L^2_\gamma}\Big)\mathrm{d}s. \qquad (2.1)
\end{aligned}
$$

As explained in Section 1.2, the temporal derivative gets involved mainly in order to access the L^6-bound of the hydrodynamic part $\mathbf{P}f_R$, while we will control the following auxiliary norm to be used in order to handle the nonlinearity: for $p<3$ and $t>0$

$$
\begin{aligned}
\mathcal{F}_p(t) := \sup_{0\le s\le t}\Big\{&\|\kappa^{1/2}Pf_R(s)\|^2_{L^6(\Omega)}+\|\kappa^{1/2}Pf_R\|^2_{L^2((0,s);L^p(\Omega))}\\
&+\|\kappa^{\mathfrak{P}+1/2}P\partial_t f_R\|^2_{L^2((0,s);L^p(\Omega))}+\|\varepsilon^{1/2}\kappa\mathfrak{w}_{\varrho,\text{ß}} f_R(s)\|^2_{L^\infty(\Omega\times\mathbb{R}^3)}\\
&+\|(\varepsilon\kappa)^{3/p}\kappa^{\frac{1}{2}+\mathfrak{P}}\mathfrak{w}_{\varrho',\text{ß}} f_R(s)\|^2_{L^2((0,s);L^\infty(\Omega\times\mathbb{R}^3))}\Big\}. \qquad (2.2)
\end{aligned}
$$

Here we have introduced weight functions, in order to control an extra quadratic growth in $|v|$ from $(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}f_R/\sqrt{\mu}$

$$
\mathfrak{w}_{\varrho,\text{ß}}(x,v)=\mathfrak{w}:=\exp\{\varrho|v|^2-\mathfrak{z}_{\text{ß}}(x_3)(x\cdot v)\}\ \text{ for }\ 0<\text{ß}\ll\frac{\varrho}{2\pi}\text{ and } 0<\varrho<\frac{1}{4}, \qquad (2.3)
$$

where $\mathfrak{z}_{\text{ß}}:\mathbb{R}_+\to\mathbb{R}_+$ is defined as, for $\text{ß}>0$

$$
\mathfrak{z}_{\text{ß}}(x_3)=\text{ß}\ \text{ for }\ x_3\in\Big[0,\frac{1}{\text{ß}}-1\Big],\ \text{ and }\ \mathfrak{z}_{\text{ß}}(x_3)=\frac{1}{1+x_3}\ \text{ for }\ x_3\in\Big[\frac{1}{\text{ß}}-1,\infty\Big). \qquad (2.4)
$$

We have denoted $\mathfrak{w}_{\varrho',\text{ß}}(x,v)=\mathfrak{w}'$ for $\varrho'<\varrho$. Also we have denoted the boundary norms and integral as

$$
\begin{aligned}
&|g|_{L^p_\gamma}:=\Big(\int_{\gamma_+}|g|^p+\int_{\gamma_-}|g|^p\Big)^{1/p},\ |g|_{L^p_{\gamma_\pm}}:=\Big(\int_{\gamma_\pm}|g|^p\Big)^{1/p},\\
&\int_{\gamma_\pm} f:=\int_{\partial\Omega}\int_{n(x)\cdot v\gtrless 0} f(x,v)|n(x)\cdot v|\mathrm{d}v\mathrm{d}S_x. \qquad (2.5)
\end{aligned}
$$

Next we discuss the initial data of the Boltzmann equation. We note that an initial datum of f_2 is already determined by given flow (u,p). For given initial data $f_{R,0}:=$

$f_{R,in}$, inspired by the PDE, we define

$$\begin{aligned}\partial_t f_{R,0} :=& -\frac{1}{\varepsilon} v \cdot \nabla_x f_{R,in} - \frac{1}{\varepsilon^2 \kappa} L_{in} f_{R,in} + \frac{2}{\kappa} \Gamma_{in}(f_2, f_{R,in}) \\ &+ \frac{\sqrt{\varepsilon}}{\varepsilon\kappa} \Gamma_{in}(f_{R,in}, f_{R,in}) \\ &- \frac{(\partial_t + \varepsilon^{-1} v \cdot \nabla_x)\sqrt{\mu_{in}}}{\sqrt{\mu_{in}}} f_{R,in} + (\mathbf{I}-\mathbf{P})\mathfrak{R}_1(u,p)|_{t=0} + \mathfrak{R}_2(u,p)|_{t=0},\end{aligned} \tag{2.6}$$

where $(\mathbf{I}-\mathbf{P})\mathfrak{R}_1$ and $\mathfrak{R}_2$ are defined in (3.2) with $\delta = \sqrt{\varepsilon}$ and μ_{in}, L_{in}, Γ_{in} are induced by the initial Naiver-Stokes velocity u_{in}. For the remainder f_R in (1.17), we will use the norms of the initial data:

$$\mathcal{E}(0) := \mathcal{E}(f_{R,0}) := \|f_{R,0}\|^2_{L^2(\Omega\times\mathbb{R}^3)} + \|\partial_t f_{R,0}\|^2_{L^2(\Omega\times\mathbb{R}^3)}, \tag{2.7}$$

$$\begin{aligned}\mathcal{F}_p(0) :=& \Big\{\kappa^{\frac{1}{2}} |f_{R,0}|_{L^2_\gamma} + \kappa^{\mathfrak{P}+\frac{1}{2}} |\partial_t f_{R,0}|_{L^2_\gamma} \\ &+ \varepsilon^{\frac{1}{2}}\kappa \|\mathfrak{w} f_{R,0}\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} + (\varepsilon\kappa)^{1+\frac{3}{p}} \kappa^{\mathfrak{P}} \|\mathfrak{w}' \partial_t f_{R,0}\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)}\Big\}^2.\end{aligned} \tag{2.8}$$

Theorem 2 (Uniform controls of the Boltzmann remainder f_R) *Suppose for $T > 0$ and $\mathfrak{P} \geq 1/2$*

$$\begin{aligned}&\sum_{\ell=0,1} \|\nabla_x \partial_t^\ell u\|_{L^\infty([0,T]\times\bar{\Omega})} + \frac{1}{\kappa^{1/2}} \sum_{\ell=0,1,2} \|\partial_t^\ell u\|_{L^\infty([0,T]\times\bar{\Omega})} \\ &+ \frac{1}{\kappa^{1/2}} \|p\|_{L^\infty([0,T]\times\bar{\Omega})} \lesssim \frac{1}{\kappa^{\mathfrak{P}}}.\end{aligned} \tag{2.9}$$

We further assume that, for $0 \leq \mathfrak{P}' < \mathfrak{P}$,

$$\begin{aligned}&\sum_{\ell=1,2} \|\partial_t^\ell u\|_{L^\infty([0,T];L^\infty(\bar{\Omega})\cap L^2(\Omega))} + \sum_{\substack{0\leq\ell\leq 1\\ 1\leq|\beta|\leq 2}} \|\nabla_x^\beta \partial_t^\ell u\|_{L^\infty([0,T];L^\infty(\bar{\Omega})\cap L^2(\Omega))} \\ &+ \sum_{|\beta|=1} \|\nabla_x^\beta \partial_t^2 u\|_{L^2([0,T];L^\infty(\bar{\Omega})\cap L^2(\Omega))} \\ &+ \|\partial_t^2 p\|_{L^2([0,T];L^\infty(\bar{\Omega})\cap L^2(\Omega))} + \sum_{|\beta|=0,1} \|\nabla_x^\beta \partial_t p\|_{L^\infty([0,T];L^\infty(\bar{\Omega})\cap L^2(\Omega))} \\ &\lesssim \exp\Big(\frac{1}{\kappa^{\mathfrak{P}'}}\Big).\end{aligned} \tag{2.10}$$

For given such $T > 0$, let us choose ε, δ and κ as, for some $\mathfrak{C} \gg 1$,

$$\delta = \sqrt{\varepsilon} \ \ \textit{and} \ \ \delta = \exp\Big(\frac{-\mathfrak{C}T}{\kappa^{\mathfrak{P}}}\Big). \tag{2.11}$$

Assume that an initial datum for the remainder $f_{R,in}$ satisfies, for some $p < 3$ and $|p-3| \ll 1$,

$$\sqrt{\mathcal{E}(0)} + \sqrt{\mathcal{F}_p(0)} \lesssim 1. \tag{2.12}$$

Then we construct a unique solution $f_R(t,x,v)$ of the form of

$$F = M_{1,\varepsilon u,1} + \varepsilon^2 f_2 \sqrt{M_{1,\varepsilon u,1}} + \delta\varepsilon f_R \sqrt{M_{1,\varepsilon u,1}} \ \text{ in } \ [0,T] \times \Omega \times \mathbb{R}^3,$$

which solves the Boltzmann equation (1.1) and the diffuse reflection boundary condition (1.12) with the scale of (1.8) and (2.11), and satisfies the initial condition $F|_{t=0} = M_{1,\varepsilon u_{in},1} + \varepsilon^2 f_2 \sqrt{M_{1,\varepsilon u,1}}|_{t=0} + \delta\varepsilon f_{R,in}\sqrt{M_{1,\varepsilon u,1}}|_{t=0}$, in a time interval $t \in [0,T]$. Moreover, we have

$$\delta^{\frac{1}{2}-\frac{3}{p}(1-\frac{p}{3})} \sup_{0\le t\le T} \left\{ \sqrt{\mathcal{E}(t)} + \sqrt{\mathcal{D}(t)} + \sqrt{\mathcal{F}_p(t)} \right\} \lesssim 1. \tag{2.13}$$

Remark 3 The condition (2.11) in the theorem is indeed the largest $\sqrt{\varepsilon}$ can be allowed. Any smaller $\sqrt{\varepsilon}$ than $\exp\left(\frac{-\mathfrak{C}T}{\kappa^{1/2}}\right)$ (which means $\sqrt{\varepsilon}$ decaying faster than $\exp\left(\frac{-\mathfrak{C}T}{\kappa^{1/2}}\right)$ as $\kappa \downarrow 0$) will produce the same result. In terms of (1.8) the relation (2.11) implies that the Knudsen number $\mathcal{K}n$ has to vanish only slightly faster than the Mach number $\mathcal{M}a$:

$$\mathcal{S}t = \varepsilon = \mathcal{M}a \ \text{ and } \ \sqrt{\frac{T}{\ln \varepsilon^{-1}}} \lesssim \frac{\mathcal{K}n}{\mathcal{M}a} \downarrow 0 \ \text{ as } \ \varepsilon \downarrow 0. \tag{2.14}$$

The proof of Theorem 2 will be given in Section 4.

2.2 Higher Regularity of Navier-Stokes Equations in the Inviscid Limit (Theorem 3)

For the Navier-Stokes solutions to (1.13)-(1.15), we introduce real analytic norms and function spaces, adopted from [47] and [54] for the 3D counter part with slight modifications.

In this subsection and Section 5, we will use the following notations: $x = (x_h, x_3) = (x_1, x_2, x_3) \in \mathbb{T}^2 \times \mathbb{R}_+ = \Omega$, $\nabla_x = \nabla = (\nabla_h, \partial_3) = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$; for a vector valued function $g \in \mathbb{R}^3$, $g = (g_h, g_3) = (g_1, g_2, g_3)$.

We denote the vorticity by

$$\omega = \nabla \times u, \quad u = \nabla \times (-\Delta)^{-1}\omega, \tag{2.15}$$

while the second identity is the famous Biot-Savart law. Here $(-\Delta)^{-1}$ denotes the inverse of $-\Delta$ with the zero Dirichlet boundary condition on $\partial\Omega$.

Our analysis of the Navier-Stokes solutions is based on the vorticity formulation in 3D ([43,44]):

$$\partial_t \omega - \kappa\eta_0 \Delta\omega = -u \cdot \nabla\omega + \omega \cdot \nabla u \ \text{ in } \ \Omega, \tag{2.16}$$

$$\omega|_{t=0} = \omega_{in} \ \text{ in } \ \Omega, \tag{2.17}$$

$$\kappa\eta_0(\partial_{x_3} + \sqrt{-\Delta_h})\omega_h \ = [\partial_{x_3}(-\Delta)^{-1}(-u\cdot\nabla\omega_h + \omega\cdot\nabla u_h)], \ \ \omega_3 = 0 \ \text{ on } \ \partial\Omega, \tag{2.18}$$

where $\sqrt{-\Delta_h} = |\nabla_h|$ is defined as

$$\sqrt{-\Delta_h} g(x_h, x_3) = \sum_{\xi\in\mathbb{Z}^2} |\xi| g_\xi(x_3) e^{ix_h\cdot\xi}. \tag{2.19}$$

Here, $g_\xi(x_3) = \frac{1}{(2\pi)^2}\iint_{\mathbb{T}^2} e^{-ix_h\cdot\xi} g(x_h, x_3)\mathrm{d}x_h \in \mathbb{C}$ with $\xi = (\xi_1, \xi_2) \in \mathbb{Z}^2$ denotes the Fourier transform in the horizontal variables, which satisfies $g(x_1, x_2, x_3) = \sum_{\xi\in\mathbb{Z}^2} g_\xi(x_3)e^{ix_h\cdot\xi}$. The Fourier transform can be regarded as a function $g_\xi(z)$ where z is sitting in a pencil-like complex domain: for any $\lambda > 0$,

$$\mathcal{H}_\lambda := \Big\{z \in \mathbb{C} : \operatorname{Re} z \geq 0, \ |\operatorname{Im} z| < \lambda \min\{\operatorname{Re} z, 1\}\Big\}. \tag{2.20}$$

We define analytic function spaces without the boundary layer, $\mathfrak{L}^{p,\lambda}$, for holomorphic functions with a finite norm, for $p \geq 1$,

$$\|g\|_{p,\lambda} := \sum_{\xi\in\mathbb{Z}^2} e^{\lambda|\xi|}\|g_\xi\|_{\mathcal{L}^p_\lambda} \ \text{ where } \ \|g_\xi\|_{\mathcal{L}^p_\lambda} := \sup_{0\leq\sigma\leq\lambda}\left(\int_{\partial\mathcal{H}_\sigma} |g_\xi(z)|^p|\mathrm{d}z|\right)^{1/p}. \tag{2.21}$$

Next we introduce an L^∞-based analytic boundary layer function space, for $\lambda > 0$ and $\kappa \geq 0$, that consists of holomorphic functions in $\mathcal{H}_\lambda$ with a finite norm

$$\|g\|_{\infty,\lambda,\kappa} = \sum_{\xi\in\mathbb{Z}^2} e^{\lambda|\xi|}\|g_\xi\|_{\mathcal{L}^\infty_{\lambda,\kappa}}, \tag{2.22}$$

where $\|g_\xi\|_{\mathcal{L}^\infty_{\lambda,0}} := \|e^{\bar\alpha \operatorname{Re} z} g_\xi(z)\|_{\mathcal{L}^\infty_\lambda} := \sup_{z\in\mathcal{H}_\lambda} e^{\bar\alpha\operatorname{Re} z} g_\xi(z)$ and

$$\|g_\xi\|_{\mathcal{L}^\infty_{\lambda,\kappa}} := \left\|\frac{e^{\bar\alpha\operatorname{Re} z}}{1+\phi_\kappa(z)} g_\xi(z)\right\|_{\mathcal{L}^\infty_\lambda} := \sup_{z\in\mathcal{H}_\lambda} \frac{e^{\bar\alpha\operatorname{Re} z}}{1+\phi_\kappa(z)}|g_\xi(z)|.$$

Here, a boundary layer weight function is defined as

$$\phi_\kappa(z) := \frac{1}{\sqrt{\kappa}}\phi(\frac{z}{\sqrt{\kappa}}) \ \text{ with } \ \phi(z) = \frac{1}{1+|\operatorname{Re} z|^{\mathfrak{r}}} \text{ for some } \mathfrak{r} > 1. \tag{2.23}$$

We define $\mathfrak{B}^{\lambda,\kappa}$ for holomorphic functions $g = (g_1, g_2, g_3)$ with a finite norm

$$[[g]]_{\infty,\lambda,\kappa} = \sum_{i=1,2} \|g_i\|_{\infty,\lambda,\kappa} + \|g_3\|_{\infty,\lambda,0}. \tag{2.24}$$

We note that $\mathfrak{B}^{\lambda,\kappa} \subset \mathfrak{L}^{1,\lambda}$, but $\mathfrak{B}^{\lambda,0} \subsetneq \mathfrak{L}^{\infty,\lambda}$ if $\bar{\alpha} > 0$.

Due to its singular nature of the Navier-Stokes flow in the inviscid limit, we introduce the conormal derivatives

$$D = (D_h, D_3) = (\nabla_h, \zeta(x_3)\partial_3) \ \text{ where } \ \zeta(z) = \frac{z}{1+z}. \tag{2.25}$$

With the multi-indices $\beta = (\beta_h, \beta_3) := (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3$, the higher derivatives are denoted by $D^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}D_3^{\beta_3}$ and $D_\xi^\beta = (i\xi_1)^{\beta_1}(i\xi_2)^{\beta_2}D_3^{\beta_3}$.

Now we define, for $\lambda_0 > 0$, $\gamma_0 > 0$, $\alpha > 0$, $\kappa \geq 0$, and $t \in (0, \frac{\lambda_0}{2\gamma_0})$

$$|||g|||_{\infty,\kappa} = \sup_{\lambda<\lambda_0-\gamma_0 t} \left\{ \sum_{0\leq|\beta|\leq 1} [[D^\beta g]]_{\infty,\lambda,\kappa} + \sum_{|\beta|=2} (\lambda_0 - \lambda - \gamma_0 t)^\alpha [[D^\beta g]]_{\infty,\lambda,\kappa} \right\}, \tag{2.26}$$

$$|||g|||_1 = \sup_{\lambda<\lambda_0-\gamma_0 t} \left\{ \sum_{0\leq|\beta|\leq 1} \|D^\beta(1+|\nabla_h|)g\|_{1,\lambda} + (\lambda_0 - \lambda - \gamma_0 t)^\alpha \sum_{|\beta|=2} \|D^\beta(1+|\nabla_h|)g\|_{1,\lambda} \right\}. \tag{2.27}$$

With an initial-boundary layer weight function as in [47]

$$\phi_{\kappa t}(z) = \frac{1}{\sqrt{\kappa t}}\phi\left(\frac{z}{\sqrt{\kappa t}}\right), \tag{2.28}$$

we define an initial-boundary layer function space $\mathfrak{B}^{\lambda,\kappa t}$ for holomorphic functions $g = (g_1, g_2, g_3)$ with a finite norm

$$[[g]]_{\infty,\lambda,\kappa t} = \sum_{i=1,2} \|g_i\|_{\infty,\lambda,\kappa t} + \|g_3\|_{\infty,\lambda,0}, \tag{2.29}$$

where an L^∞-based analytic norm with the initial-boundary layer is defined as

$$\|g\|_{\infty,\lambda,\kappa t} = \sum_{\xi\in\mathbb{Z}^2} e^{\lambda|\xi|}\|g_\xi\|_{\mathcal{L}^\infty_{\lambda,\kappa t}}, \quad \|g_\xi\|_{\mathcal{L}^\infty_{\lambda,\kappa t}} = \left\| \frac{e^{\bar{\alpha}\operatorname{Re} z}}{1+\phi_\kappa(z)+\phi_{\kappa t}(z)} g_\xi(z) \right\|_{\mathcal{L}^\infty_\lambda}. \tag{2.30}$$

We finally define, for $t \in (0, \frac{\lambda_0}{2\gamma_0})$,

$$|||g|||_{\infty,\kappa t} = \sup_{\lambda<\lambda_0-\gamma_0 t} \left\{ \sum_{0\leq|\beta|\leq 1} [[D^\beta g]]_{\infty,\lambda,\kappa t} + \sum_{|\beta|=2} (\lambda_0 - \lambda - \gamma_0 t)^\alpha [[D^\beta g]]_{\infty,\lambda,\kappa t} \right\}. \tag{2.31}$$

In this subsection and Section 5, α, $\bar{\alpha}$ are given positive small constants, λ_0 is a given positive constant, and γ_0 is a sufficiently large constant to be determined in Theorem 3.

Next we discuss the initial data of the velocity u_{in} and the corresponding vorticity $\omega_{in} = \nabla_x \times u_{in}$. Inspired by the PDEs, let

$$\begin{aligned} \omega_0 &:= \omega_{in}, \quad \partial_t \omega_0 := \kappa\eta_0 \Delta\omega_0 - u_0 \cdot \nabla\omega_0 + \omega_0 \cdot \nabla u_0, \\ u_0 &:= \nabla \times (-\Delta)^{-1}\omega_0, \quad \partial_t u_0 := \nabla \times (-\Delta)^{-1}\partial_t\omega_0, \\ \partial_t^2\omega_0 &:= \kappa\eta_0\Delta\partial_t\omega_0 - u_0 \cdot \nabla\partial_t\omega_0 - \partial_t u_0 \cdot \nabla\omega_0 + \omega_0 \cdot \nabla\partial_t u_0 + \partial_t\omega_0 \cdot \nabla u_0. \end{aligned} \tag{2.32}$$

Theorem 3 *Let $\lambda_0 > 0$ and $\omega_{in} \in \mathfrak{B}^{\lambda_0,\kappa}$ with (2.32) satisfy*

$$\sum_{0\le|\beta|\le 2} \|D^\beta \partial_t^\ell \omega_0\|_{1,\lambda_0} + \sum_{0\le|\beta|\le 2} \|D^\beta \partial_t^\ell \omega_0\|_{\infty,\lambda_0,\kappa} < \infty \text{ for } \ell = 0, 1, 2. \tag{2.33}$$

Further assume that $\omega_{in} = \omega_0$ and (2.32) satisfies the compatibility conditions on $\partial\Omega$

$$\begin{aligned} &\kappa\eta_0(\partial_{x_3} + \sqrt{-\Delta_h})\omega_{0,h} = [\partial_{x_3}(-\Delta)^{-1}(-u_0 \cdot \nabla\omega_{0,h} + \omega_0 \cdot \nabla u_{0,h})], \\ &\omega_{0,3} = 0, \quad \partial_t\omega_{0,3} = 0. \end{aligned} \tag{2.34}$$

Then there exists a constant $\gamma_0 > 0$ and a time $T > 0$ depending only on λ_0 and the size of the initial data such that the solution $\omega(t)$ to the vorticity formulation of the Navier-Stokes equations (2.16)–(2.18) exists in $C^1([0,T];\mathfrak{B}^{\lambda,\kappa})$ with $\partial_t^2\omega$ in $C(0,T;\mathfrak{B}^{\lambda,\kappa t})$ for $0 < \lambda < \lambda_0$ satisfying

$$\sup_{t\in[0,T]} \left[\sum_{\ell=0}^{2} \left\|\left|\partial_t^\ell \omega(t)\right|\right\|_1 + \sum_{\ell=0}^{1} \left\|\left|\partial_t^\ell \omega(t)\right|\right\|_{\infty,\kappa} + \left\|\left|\partial_t^2 \omega(t)\right|\right\|_{\infty,\kappa t} \right] < \infty. \tag{2.35}$$

Furthermore, for each $(t,x) \in [0,T] \times \Omega$,

(1) (Bounds on the vorticity and its derivatives) $\omega(t,x)$ enjoys the following bounds:

$$|\nabla_h^i \partial_t^\ell \omega_h(t,x)| \lesssim e^{-\bar{\alpha}x_3}(1 + \phi_\kappa(x_3)), \quad |\nabla_h^i \partial_t^\ell \omega_3(t,x)| \lesssim e^{-\bar{\alpha}x_3} \text{ for } i, \ell = 0, 1, \tag{2.36}$$

$$|\partial_t^2 \omega_h(t,x)| \lesssim e^{-\bar{\alpha}x_3}(1 + \phi_\kappa(x_3) + \phi_{\kappa t}(x_3)), \quad |\partial_t^2\omega_3(t,x)| \lesssim e^{-\bar{\alpha}x_3}, \tag{2.37}$$

$$|\partial_{x_3}\partial_t^\ell \omega_h(t,x)| \lesssim \kappa^{-1} e^{-\bar{\alpha}x_3}, \quad |\partial_{x_3}\partial_t^\ell\omega_3(t,x)| \lesssim e^{-\bar{\alpha}x_3}(1 + \phi_\kappa(x_3)) \text{ for } \ell = 0, 1. \tag{2.38}$$

(2) (Bounds on the velocity and its derivatives) The corresponding velocity field $u(t,x)$ satisfies the following:

$$|\partial_t^\ell u(t,x)| \lesssim 1 \text{ for } \ell = 0, 1, 2, \tag{2.39}$$

$$\sum_{1\le|\beta|\le 2} |\nabla^\beta \partial_t^\ell u(t,x)| \lesssim \big(1+\phi_\kappa(x_3)+(|\beta|-1)\kappa^{-1}\big)e^{-\min(1,\frac{\bar\alpha}{2})x_3} \quad \textit{for } \ell=0,1, \tag{2.40}$$

$$\sum_{|\beta|=1} |\nabla^\beta \partial_t^2 u(t,x)| \lesssim \big(1+\phi_\kappa(x_3)+\phi_{\kappa t}(x_3)\big)e^{-\min(1,\frac{\bar\alpha}{2})x_3}. \tag{2.41}$$

Moreover, we have the decay estimate for $\partial_t^\ell u$*:*

$$|\partial_t^\ell u| \lesssim \kappa^{-\frac12} e^{-\min(1,\frac{\bar\alpha}{2})x_3} \quad \textit{for } \ell=1,2. \tag{2.42}$$

(3) (Bounds on the pressure and its derivatives) The pressure defined in (5.73) satisfies the following:

$$|\partial_t^\ell p(t,x)| \lesssim 1 \quad \textit{for } \ell=0,1,2, \tag{2.43}$$

$$\sum_{0\le|\beta|\le 1} |\nabla^\beta \partial_t^\ell p(t,x)| \lesssim \kappa^{-\frac12} e^{-\min(1,\frac{\bar\alpha}{2})x_3} \quad \textit{for } \ell=0,1, \tag{2.44}$$

$$|\partial_t^2 p| \lesssim (\kappa^{-\frac12}+\phi_{\kappa t}(x_3))e^{-\min(1,\frac{\bar\alpha}{2})x_3}. \tag{2.45}$$

Remark 4 For simplicity of the presentation, we have taken the analytic data with the same analyticity radius in x_1, x_2 and x_3 with the exponential decay for large x_3. As shown in [38,54], more general initial data requiring the analyticity only near the boundary can be taken.

Remark 5 The horizontal vorticity ω_h and the vertical vorticity ω_3 obey different boundary conditions (2.18) which enforce different behaviors near the boundary. This is well-reflected in our L^∞ based norms in (2.24) and (2.29). As noted in [54], such incompatible behaviors of ω_h and ω_3 in 3D are dealt with the L^1 based norm (2.27) which contains one more tangential derivative $(1+|\nabla_h|)$, which is different from 2D analysis [38,47].

Remark 6 We demand the compatibility conditions in (2.34) in order to avoid singular initial-boundary layers for the temporal derivatives of the vorticity. If the first two conditions in (2.34) were not satisfied, the initial-boundary layers would occur for the first temporal derivative of the vorticity. For the second temporal derivative, we handle the initial-boundary layer for the horizontal part with the initial-boundary layer weight, while for the vertical part we further demand $\partial_t\omega_{0,3}|_{x_3=0}=0$ in order to rule out a singular initial-boundary layer caused by the Dirichlet boundary condition. This amounts to requiring the second order vanishing condition at the boundary for $\omega_{0,3}$, which is satisfied by a large class of ω_0. We remark that the first condition of (2.34) is also satisfied by a large class of ω_0. In fact, if not, by the result of [47], we can obtain a short time solution $\tilde\omega(t)$ to (2.16)–(2.18) and may reset the initial data by $\omega_0=\tilde\omega(t=t_0)$ for sufficiently small $t_0>0$.

The proof of Theorem 3 will be given in Section 5.

2.3 Main Theorem

Now we present the full statement of the main theorems of this paper:

Theorem 4 (Kinetic approximation of Navier-Stokes in large Reynolds numbers) *We consider a half space Ω in 3D as in (1.18). Suppose an initial datum of the Navier-Stokes flow u_{in} is divergence-free $\nabla_x \cdot u_{in} = 0$ in Ω and the corresponding initial vorticity $\omega_{in} = \nabla_x \times u_{in}$ belongs to the real analytic space $\mathfrak{B}^{\lambda_0,\kappa}$ of (2.24) for some $\lambda_0 > 0$ such that (2.33) holds. Further we assume that ω_{in} satisfies the compatibility conditions (2.34) on $\partial\Omega$. Then there exists a unique real analytic solution $(u(t,x), \nabla_x p(t,x))$ to* (1.13)–(1.15) *in $[0,T] \times \Omega$, while $T > 0$ only depends on λ_0 and the size of the initial data as in (2.33).*

Choosing a pressure $p(t,x)$ such that $p(t,x) \to 0$ as $x_3 \uparrow \infty$, we set the local Maxwellian and the second order correction f_2 as

$$\mu := M_{1,\varepsilon u,1} = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\left\{-\frac{|v-\varepsilon u|^2}{2}\right\},$$
$$f_2 := \mathbf{P} f_2 + (\mathbf{I} - \mathbf{P}) f_2 = p\varphi_0\sqrt{\mu} + (1.34).$$

For given such $T > 0$, let us choose ε and κ in the relation of (2.11).

Assume that an initial datum for the remainder $f_{R,in}$ satisfies (2.12) for some $p < 3$ and $|p-3| \ll 1$.

Then we construct a unique solution $f_R(t,x,v)$ of the form of

$$F = M_{1,\varepsilon u,1} + \varepsilon^2 f_2\sqrt{M_{1,\varepsilon u,1}} + \varepsilon^{3/2} f_R\sqrt{M_{1,\varepsilon u,1}} \ \ in \ \ [0,T] \times \Omega \times \mathbb{R}^3,$$

which solves the Boltzmann equation (1.1) and the diffuse reflection boundary condition (1.12) with the scale of (1.8) and (2.11), and satisfies the initial condition $F|_{t=0} = M_{1,\varepsilon u_{in},1} + \varepsilon^2 f_2\sqrt{M_{1,\varepsilon u,1}}|_{t=0} + \varepsilon^{3/2} f_{R,in}\sqrt{M_{1,\varepsilon u,1}}|_{t=0}$.

Moreover we derive that, for each ε and κ of (2.11),

$$\sup_{0\le t\le T} \left\| \frac{F(t,x,v) - M_{1,\varepsilon u(t,x),1}(v)}{\varepsilon\sqrt{M_{1,\varepsilon u(t,x),1}(v)}} \right\|_{L^2(\Omega\times\mathbb{R}^3)} \lesssim \exp\left(\frac{-\mathfrak{C}T}{2\kappa^{1/2}}\right) \ for \ \kappa \ll 1. \quad (2.46)$$

Proof The existence of the Navier-Stokes solutions follows from Theorem 3. For the remaining assertions, we note that all the estimates (2.39)–(2.42) of Theorem 3 ensure the conditions of Theorem 2 with $\mathfrak{P} = \frac{1}{2}$. Therefore the conclusion follows directly as a consequence of Theorem 2 and Theorem 3. □

The incompressible Euler limit follows as a byproduct of the main theorem:

Corollary 5 (Hydrodynamic limit toward the incompressible Euler equation) *Let $u_E(t,x)$ be a (unique) solution of the incompressible Euler equations (1.9)-(1.10) with the initial condition $u_E|_{t=0} = u_{in}$ in Ω. Then*

$$\sup_{0\le t\le T} \left\| \frac{F(t,x,v) - M_{1,\varepsilon u_E(t,x),1}(v)}{\varepsilon(1+|v|)^2\sqrt{M_{1,0,1}(v)}} \right\|_{L^2(\Omega\times\mathbb{R}^3)} \longrightarrow 0 \ \ as \ \ \varepsilon \downarrow 0.$$

Proof Note that

$$F(t,x,v) - M_{1,\varepsilon u_E(t,x),1}(v) = \big[F(t,x,v) - M_{1,\varepsilon u(t,x),1}(v)\big] \\ + \big[M_{1,\varepsilon u(t,x),1}(v) - M_{1,\varepsilon u_E(t,x),1}(v)\big].$$

The first term can be bounded as in (2.46). We bound the second term by an expansion:

$$|u(t,x) - u_E(t,x)| \int_0^{\varepsilon} |(v - \varepsilon u_E) + a(u_E - u)| e^{-\frac{|(v-\varepsilon u_E)+a(u_E-u)|^2}{2}} \mathrm{d}a.$$

Note that $\|\varepsilon u\|_{L^\infty} \ll 1$ and $\|\varepsilon u_E\|_{L^\infty} \ll 1$ from Theorem 3. Then we conclude that the second term converges to 0 as $\kappa \downarrow 0$ from Theorem 3 and the famous Kato's condition for vanishing viscosity limit in [34]. □

3 Hilbert Expansion Around a Local Maxwellian and Source Terms

In this section we complete the Hilbert expansion along with the outline of the introduction. As a result we prove

Proposition 6 *Suppose that F of (1.19), with a free parameter δ, solve (1.1) and (1.12) with (1.8) and that (u, p) solves (1.13)-(1.15). We choose a hydrodynamic part f_2 as*

$$\mathbf{P} f_2 = p\varphi_0\sqrt{\mu}, \tag{3.1}$$

with the pressure p of the Navier-Stokes flow in (1.13), and $(\mathbf{I}-\mathbf{P})f_2$ has been given in (1.34). Then f_R in (1.19) satisfies that

$$\Big[\partial_t + \frac{1}{\varepsilon} v\cdot\nabla_x + \frac{1}{\varepsilon^2\kappa} L\Big] f_R = \frac{2}{\kappa}\Gamma(f_2, f_R) + \frac{\delta}{\varepsilon\kappa}\Gamma(f_R, f_R) \\ - \frac{(\partial_t + \varepsilon^{-1} v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}} f_R + (\mathbf{I}-\mathbf{P})\mathfrak{R}_1 + \mathfrak{R}_2, \tag{3.2}$$

$$\Big[\partial_t + \frac{1}{\varepsilon} v\cdot\nabla_x + \frac{1}{\varepsilon^2\kappa} L\Big] \partial_t f_R \\ = -\frac{1}{\varepsilon^2\kappa} L_t(\mathbf{I}-\mathbf{P}) f_R + \frac{1}{\varepsilon^2\kappa} L(\mathbf{P}_t f_R) + \frac{2\delta}{\varepsilon\kappa}\Gamma(f_R, \partial_t f_R) \\ + \frac{2}{\kappa}\Gamma(f_2, \partial_t f_R) + \frac{2}{\kappa}\Gamma(\partial_t f_2, f_R) + \frac{2}{\kappa}\Gamma_t(f_2, f_R) + \frac{\delta}{\varepsilon\kappa}\Gamma_t(f_R, f_R) \\ - \frac{(\partial_t + \varepsilon^{-1} v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}} \partial_t f_R - \partial_t\Big(\frac{(\partial_t + \varepsilon^{-1} v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\Big) f_R \\ + (\mathbf{I}-\mathbf{P})\mathfrak{R}_3 + \mathfrak{R}_4, \tag{3.3}$$

where the commutators L_t, $\mathbf{P}_t$ and Γ_t are given in (3.34), while

$$e^{\varrho|v-\varepsilon u|^2}|(\mathbf{I}-\mathbf{P})\mathfrak{R}_1(t,x,v)| \lesssim \frac{1}{\delta}\kappa|\nabla_x^2 u|, \tag{3.4}$$

$$\begin{aligned} e^{\varrho|v-\varepsilon u|^2}|\mathfrak{R}_2(t,x,v)| \lesssim\ & \frac{\varepsilon}{\delta}(|p|+\kappa|\nabla_x u|)|\nabla_x u| + \frac{\varepsilon}{\delta}(|\partial_t p|+\kappa|\nabla_x u|) \\ & + \frac{\varepsilon\kappa}{\delta}(|\nabla_x\partial_t u| + |u||\nabla_x^2 u|) \\ & + \frac{\varepsilon^2}{\delta}(|p|+\kappa|\nabla_x u|)(|\partial_t u|+|u||\nabla_x u|), \end{aligned} \tag{3.5}$$

$$e^{\varrho|v-\varepsilon u|^2}|(\mathbf{I}-\mathbf{P})\mathfrak{R}_3(t,x,v)| \lesssim \frac{\kappa}{\delta}|\nabla_x^2\partial_t u|, \tag{3.6}$$

$$\begin{aligned} & e^{\varrho|v-\varepsilon u|^2}|\mathfrak{R}_4(t,x,v)| \\ & \lesssim \frac{\varepsilon}{\delta}|\partial_t^2 p| + \frac{\varepsilon\kappa}{\delta}|\nabla_x\partial_t^2 u| + \frac{\varepsilon}{\delta}|\nabla_x\partial_t p||u| + \frac{\varepsilon\kappa}{\delta}|u||\nabla_x^2\partial_t u| \\ & + \frac{\varepsilon}{\delta}\{(1+|u|)(|p|+\kappa|\nabla_x u|)+\kappa\varepsilon|\partial_t u|\}|\nabla_x\partial_t u| \\ & + \frac{\varepsilon\kappa}{\delta}(1+\varepsilon\kappa|u|)|\partial_t u||\nabla_x^2 u| + \frac{\varepsilon^2}{\delta}\{|p|+\kappa|\nabla_x u|\}|\partial_t^2 u| \\ & + \frac{\varepsilon}{\delta}\{(|u|+\varepsilon|p|+\varepsilon^2|p||u|)|\partial_t u| + (1+\varepsilon|u|)|\partial_t p|\}|\nabla_x u| \\ & \quad + \frac{\varepsilon^2\kappa}{\delta}(1+\varepsilon|u|)|\partial_t u||\nabla_x u|^2 \\ & \quad + \frac{\varepsilon}{\delta}\{|\partial_t u| + |\nabla_x p| + \varepsilon|\partial_t p| + \frac{\varepsilon}{\kappa}(|p|^2+\kappa|u||\nabla_x p| + \varepsilon\kappa|\partial_t u||p|)\}|\partial_t u|. \end{aligned} \tag{3.7}$$

At the boundary f_R and $\partial_t f_R$ satisfy

$$f_R(t,x,v)|_{\gamma_-} = P_{\gamma_+} f_R(t,x,v) - \frac{\varepsilon}{\delta}(1-P_{\gamma_+})(\mathbf{I}-\mathbf{P})f_2(t,x,v), \tag{3.8}$$

$$\begin{aligned} \partial_t f_R|_{\gamma_-} &= P_{\gamma_+}\partial_t f_R - \frac{\varepsilon}{\delta}(1-P_{\gamma_+})\partial_t(\mathbf{I}-\mathbf{P})f_2 + r_{\gamma_+}(f_R) - \frac{\varepsilon}{\delta}r_{\gamma_+}((\mathbf{I}-\mathbf{P})f_2), \\ r_{\gamma_+}(g) &:= \partial_t\sqrt{c_\mu\mu(v)}\int_{n(x)\cdot\mathfrak{v}>0} g\sqrt{c_\mu\mu(\mathfrak{v})}n(x)\cdot\mathfrak{v}d\mathfrak{v} \\ &\quad + \sqrt{c_\mu\mu(v)}\int_{n(x)\cdot\mathfrak{v}>0} g\partial_t\sqrt{c_\mu\mu(\mathfrak{v})}n(x)\cdot\mathfrak{v}d\mathfrak{v}. \end{aligned} \tag{3.9}$$

In addition,

$$e^{\varrho|v-\varepsilon u|^2}|f_2(t,x,v)| \lesssim |p(t,x)| + \kappa|\nabla_x u(t,x)|, \tag{3.10}$$

$$e^{\varrho|v-\varepsilon u|^2}|\partial_t f_2(t,x,v)| \lesssim |\partial_t p| + \kappa(|\nabla_x\partial_t u| + \varepsilon|\partial_t u||\nabla_x u|) + \varepsilon|\partial_t u||p|, \tag{3.11}$$

$$\langle v-\varepsilon u\rangle^{-2}\left|\frac{(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\right| \lesssim |\nabla_x u| + \underbrace{\varepsilon|\partial_t u| + \varepsilon|u||\nabla_x u|}_{(3.12)_*}, \tag{3.12}$$

$$\begin{aligned}&\langle v-\varepsilon u\rangle^{-2}\left|\partial_t\left(\frac{(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\right)\right| \\ &\lesssim |\nabla_x\partial_t u| + \underbrace{\varepsilon\{|\partial_t^2 u| + |u||\nabla_x\partial_t u| + |\partial_t u||\nabla_x u|\} + \varepsilon^2|\partial_t u|(|\partial_t u| + |u||\nabla_x u|)}_{(3.13)_*}.\end{aligned} \tag{3.13}$$

Remark 7 We note that due to the choice of (3.1) we remove a contribution of p^2 in $\frac{\varepsilon}{\delta\kappa}\Gamma(f_2, f_2)$. And also we remark that $\mathfrak{R}_4$ is quasi-linear for $\partial_t^2 p$ and $\nabla_x\partial_t^2 u$.

3.1 Derivatives of A_{ij} and Commutators in the Local Maxwellian Setting

First we check properties of L and Γ defined in (1.20). Recall the notation of the global Maxwellian $\mu_0 := M_{1,0,1}$. It is convenient to define

$$L_0 f(v) := \frac{-2}{\sqrt{\mu_0}}Q(\mu_0, \sqrt{\mu_0}f)(v), \quad \Gamma_0(f,g)(v) := \frac{1}{\sqrt{\mu_0}}Q(\sqrt{\mu_0}f, \sqrt{\mu_0}g)(v). \tag{3.14}$$

For a given εu, we define $\tilde{f}(\cdot) := f(\cdot + \varepsilon u)$. Then we have

$$Lf(v+\varepsilon u) = L_0\tilde{f}(v), \quad \Gamma(f,g)(v+\varepsilon u) = \Gamma_0(\tilde{f},\tilde{g})(v). \tag{3.15}$$

As in (1.23) a null space of L_0, denoted by $\mathcal{N}_0$, is a subspace of $L^2(\mathbb{R}^3)$ spanned by orthonormal bases $\{\tilde{\varphi}_i\sqrt{\mu_0}\}_{i=0}^4$ with

$$\tilde{\varphi}_0 := 1, \quad \tilde{\varphi}_i := v_i \text{ for } i = 1,2,3, \quad \tilde{\varphi}_4 := (|v|^2-3)/\sqrt{6}. \tag{3.16}$$

We denote a projection $\tilde{\mathbf{P}}$ on $\mathcal{N}_0$ as in (1.24). From standard properties of L_0 and (3.15), we can easily deduce the corresponding properties of L, namely the null space in (1.23), the spectral gap estimate in (1.25), and the existence of a unique inverse $L^{-1}: \mathcal{N}^\perp \to \mathcal{N}^\perp$ in (3.17) which is defined via $L_0^{-1}: \mathcal{N}_0^\perp \to \mathcal{N}_0^\perp$ with the identity

$$(L^{-1}f)(v) = (L_0^{-1}\tilde{f})(v-\varepsilon u). \tag{3.17}$$

The inverse enjoys the following bound which turns out useful to prove Lemma 3.

Lemma 1 *For* $0 < \varrho < \frac{1}{4}$ *and* $g \in \mathcal{N}_0^\perp$

$$\|\nu_0(v)e^{\varrho|v|^2}L_0^{-1}g(v)\|_{L_v^\infty} \lesssim \|e^{\varrho|v|^2}g(v)\|_{L_v^\infty} + \|\nu_0(v)^{-1}e^{\varrho|v|^2}g(v)\|_{L_v^2}. \tag{3.18}$$

The proof is based on the well-known decomposition of $L_0 = \nu_0 - K_0$ and the compactness of K_0: We first recall a standard decomposition

$$\begin{aligned} L_0 g(v) &= \nu_0(v) g(v) - K_0 g(v) \\ &:= \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot \mathfrak{u}| \mu_0(v_*) \mathrm{d}\mathfrak{u} \mathrm{d}v_* g(v) \\ &\quad - \frac{1}{\sqrt{\mu_0(v)}} \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot \mathfrak{u}| \big\{ \mu_0(v) \sqrt{\mu_0(v_*)} g(v_*) \\ &\quad - \mu_0(v') \sqrt{\mu_0(v'_*)} g(v'_*) - \mu_0(v'_*) \sqrt{\mu_0(v')} g(v') \big\} \mathrm{d}v_*, \end{aligned} \tag{3.19}$$

where $\langle v \rangle \lesssim \nu_0(v) \lesssim \langle v \rangle$. For (1.21) we have $\nu(v) = \nu_0(v - \varepsilon u)$ and $\mathbf{k}(v, v_*) = \mathbf{k}_0(v - \varepsilon u, v_* - \varepsilon u)$. It is well-known (see (3.50) and (3.52) in [16]) that one can write $K_0 g(v) = \int_{\mathbb{R}^3} \mathbf{k}_0(v, v_*) g(v_*) \mathrm{d}v_*$ such that for some constants $C_1, C_2 > 0$

$$\mathbf{k}_0(v, v_*) = C_1 |v - v_*| e^{-\frac{|v|^2 + |v_*|^2}{4}} - \frac{C_2}{|v - v_*|} e^{-\frac{|v - v_*|^2}{8} - \frac{1}{8} \frac{(|v|^2 - |v_*|^2)^2}{|v - v_*|^2}}. \tag{3.20}$$

It is convenient to introduce a new notation, for $\vartheta > 0$,

$$k_\vartheta(v, v_*) := \frac{1}{|v - v_*|} e^{-\vartheta |v - v_*|^2 - \vartheta \frac{(|v|^2 - |v_*|^2)^2}{|v - v_*|^2}}. \tag{3.21}$$

Clearly $|\mathbf{k}_0(v, v_*)| \lesssim k_\vartheta(v, v_*)$ for $0 < \vartheta \leq 1/8$.

Standard compactness estimates read as follows:

Lemma 2 *For $0 < \varrho < 2\vartheta$ and $C \in \mathbb{R}^3$, there exists $C_{\varrho, \vartheta} > 0$ such that*

$$\left| k_\vartheta(v, v_*) \frac{e^{\varrho |v|^2 + C \cdot v}}{e^{\varrho |v_*|^2 + C \cdot v_*}} \right| \lesssim \frac{1}{|v - v_*|} e^{-C_\varrho \frac{|v - v_*|^2}{2}} \quad \text{for } 0 < \varrho < 2\vartheta. \tag{3.22}$$

Moreover

$$\begin{aligned} &\int_{\mathbb{R}^3} (1 + |v - v_*|) k_\vartheta(v, v_*) \frac{e^{\varrho |v|^2 + C \cdot v}}{e^{\varrho |v_*|^2 + C \cdot v_*}} \mathrm{d}v_* \lesssim_{\vartheta, \varrho} \frac{1}{1 + |v|}, \\ &\int_{\mathbb{R}^3} \frac{1}{|v - v_*|} k_\vartheta(v, v_*) \frac{e^{\varrho |v|^2 + C \cdot v}}{e^{\varrho |v_*|^2 + C \cdot v_*}} \mathrm{d}v_* \lesssim_{\vartheta, \varrho} 1, \end{aligned} \tag{3.23}$$

while the same bounds replacing $|v|$ with $|v_*|$ hold for integrations over v.

The proof of (3.22) relies on a fact that the exponent has a majorant $-\vartheta |v - v_*|^2 - \vartheta \frac{(|v|^2 - |v_*|^2)^2}{|v - v_*|^2} \leq -2\vartheta (|v| + |v_*|) ||v| - |v_*||$ which is a negative definite. Note that an exponent of $\frac{e^{\varrho |v|^2}}{e^{\varrho |v_*|^2}}$ equals $\varrho (|v| + |v_*|) ||v| - |v_*||$ which can be absorbed as long as $0 < \varrho < 2\vartheta$. This yields (3.22). We refer to a proof of Lemma 5 in [20] for details to show (3.23).

Proof of Lemma 1 We consider an operator $g(v) \mapsto \nu_0^{-1} L_0 g(v) := \frac{1}{\nu_0(v)} L_0 g(v)$ on a restricted space of $\{g \in L^2(\mathbb{R}^3) : e^{\varrho|v|^2} g(v) \in L^2(\mathbb{R}^3)\}$. First we claim that

$$\begin{aligned} &\nu_0^{-1} L_0 : \{g \in L^2(\mathbb{R}^3) : e^{\varrho|v|^2} g(v) \in L^2(\mathbb{R}^3)\} \\ &\quad \to \{g \in L^2(\mathbb{R}^3) : e^{\varrho|v|^2} g(v) \in L^2(\mathbb{R}^3)\}. \end{aligned} \tag{3.24}$$

From (3.19) we have $\nu_0^{-1} L_0 g(v) = g(v) - \nu_0^{-1} e^{-\varrho|v|^2} \int_{\mathbb{R}^3} \mathbf{k}_0(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} e^{\varrho|v_*|^2} g(v_*) \mathrm{d}v_*$, and, using (3.23), for $\varrho < 2\vartheta \le 1/4$,

$$\begin{aligned} &|e^{\varrho|v|^2} \nu_0^{-1} L_0 g(v)| \\ &\quad \le |e^{\varrho|v|^2} g(v)| \\ &\qquad + \nu_0(v)^{-1} \sup_v \sqrt{\int_{\mathbb{R}^3} k_\vartheta(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} \mathrm{d}v_*} \sqrt{\int_{\mathbb{R}^3} k_\vartheta(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} |e^{\varrho|v_*|^2} g(v_*)|^2 \mathrm{d}v_*} \\ &\quad \lesssim |e^{\varrho|v|^2} g(v)| + \sqrt{\int_{\mathbb{R}^3} k_\vartheta(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} |e^{\varrho|v_*|^2} g(v_*)|^2 \mathrm{d}v_*}. \end{aligned}$$

Therefore we prove (3.24) from

$$\begin{aligned} &\|e^{\varrho|v|^2} \nu_0^{-1} L_0 g(v)\|_{L^2_v} \\ &\quad \lesssim \|e^{\varrho|v|^2} g(v)\|_{L^2_v} + \sqrt{\sup_{v_*} \int_{\mathbb{R}^3} k_\vartheta(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} \mathrm{d}v \int_{\mathbb{R}^3} |e^{\varrho|v_*|^2} g(v_*)|^2} \\ &\quad \lesssim \|e^{\varrho|v|^2} g(v)\|_{L^2_v}. \end{aligned} \tag{3.25}$$

Now we view $\{g \in L^2(\mathbb{R}^3) : e^{\varrho|v|^2} g(v) \in L^2(\mathbb{R}^3)\}$ as the Hilbert space with an inner product $\langle e^{\varrho|v|^2} \cdot, e^{\varrho|v|^2} \cdot \rangle$. Then the compactness of $\nu_0^{-1} K_0$ in this space is equivalent to the compactness of $g \mapsto \int_{\mathbb{R}^3} \mathbf{k}_0(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} g(v_*) \mathrm{d}v_*$ in a usual L^2_v. From Lemma 3.5.1 of [16], it suffices to prove that (i) $\int_{\mathbb{R}^3} \mathbf{k}_0(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} \mathrm{d}v$ is bounded in v_*, (ii) $\mathbf{k}_0(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} \in L^2(\{|v - v_*| \ge \frac{1}{n} \text{ and } |v| \le n\})$ for all $n \in \mathbb{N}$, and (iii) $\sup_v \int_{\mathbb{R}^3} \mathbf{k}_0(v, v_*) \frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}} \{\mathbf{1}_{|v-v_*| \le \frac{1}{n}} + \mathbf{1}_{|v| \ge n}\} \mathrm{d}u \to 0$ as $n \to \infty$. Both conditions (i) and (ii) come from the first bound of (3.23) directly. We prove (iii) from (3.22) and the first bound of (3.23). Now applying the Fredholm alternative to $\nu_0^{-1} L_0 = id - \nu_0^{-1} K_0$ in the Hilbert space, we obtain an inverse map $(\nu_0^{-1} L_0)^{-1}$ which is a bounded operator of the Hilbert space. Note that $L_0^{-1}(g) = (\nu_0^{-1} L_0)^{-1} (\nu_0^{-1} g)$. Hence we derive that

$$\|e^{\varrho|v|^2} L_0^{-1} g\|_{L^2_v} = \|e^{\varrho|v|^2} (\nu_0^{-1} L_0)^{-1} (\nu_0^{-1} g)\|_{L^2_v} \lesssim \|e^{\varrho|v|^2} \nu_0^{-1} g\|_{L^2_v}. \tag{3.26}$$

From the decomposition of L_0, we have $L_0^{-1}g(v) = \nu_0(v)^{-1}g(v) + \nu_0(v)^{-1}KL_0^{-1}g(v)$ for $g \in \mathcal{N}_0^{\perp}$. Then we have

$$\begin{aligned}
&|e^{\varrho|v|^2}L_0^{-1}g(v)| \\
&\le |\nu_0(v)^{-1}e^{\varrho|v|^2}g(v)| + \left|\nu_0(v)^{-1}\int_{\mathbb{R}^3}\mathbf{k}_0(v,v_*)\frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}}e^{\varrho|v_*|^2}L_0^{-1}g(v_*)\mathrm{d}v_*\right| \\
&\le \nu_0(v)^{-1}\left\{|e^{\varrho|v|^2}g(v)| + \sqrt{\int_{\mathbb{R}^3}\left|\mathbf{k}_0(v,v_*)\frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}}\right|^2\mathrm{d}v_*}\sqrt{\int_{\mathbb{R}^3}|e^{\varrho|v_*|^2}L_0^{-1}g(v_*)|^2\mathrm{d}v_*}\right\},
\end{aligned}$$

while $\left|\mathbf{k}_0(v,v_*)\frac{e^{\varrho|v|^2}}{e^{\varrho|v_*|^2}}\right|^2 \lesssim \frac{1}{|v-v_*|^2}e^{-2C_\varrho\frac{|v-v_*|^2}{2}} \in L_v^\infty L_{v_*}^1$ from (3.22). Hence we prove (3.18). $\square$

Equipped with Lemma 1 we provide bounds of A_{ij} in (1.34) and its derivatives:

Lemma 3 *For* $0 < \varrho < \frac{1}{4}$

$$\begin{aligned}
|A_{ij}(v)| &\lesssim e^{-\varrho|v-\varepsilon u|^2},\ |\nabla_x \\
A_{ij}(v)| &\lesssim \varepsilon|\nabla_x u|e^{-\varrho|v-\varepsilon u|^2},\ |\partial_t A_{ij}(v)| \lesssim \varepsilon|\partial_t u|e^{-\varrho|v-\varepsilon u|^2}, \\
|\nabla_x\partial_t A_{ij}(v)| &\lesssim \varepsilon\{|\nabla_x\partial_t u| + \varepsilon|\nabla_x u||\partial_t u|\}e^{-\varrho|v-\varepsilon u|^2}.
\end{aligned} \tag{3.27}$$

Proof It is convenient to introduce a notation, with L_0 in (3.14),

$$A_{0,ij}(v) := L_0^{-1}\Big((v_iv_j - \frac{|v|^2}{3}\delta_{ij})\sqrt{\mu_0}\Big)(v). \tag{3.28}$$

Then from (3.18) and (3.17) we can immediately prove the first bound in (3.27).

Recall the notations in (3.14) and (3.15). By taking a derivative to L_0 (3.28), it follows that, from the decomposition of $L_0A_{0,ij}(v) = \nu_0(v)A_{0,ij}(v) - \int_{\mathbb{R}^3}\mathbf{k}_0(v,v-v_*)A_{0,ij}(v-v_*)\mathrm{d}v_*$ and (3.20),

$$\begin{aligned}
L_0\partial_{v_k}A_{0,ij} &= \partial_{v_k}(v_iv_j - \frac{|v|^2}{3})\sqrt{\mu_0} + (v_iv_j - \frac{|v|^2}{3})\partial_{v_k}\sqrt{\mu_0} \\
&\quad - \left\{\partial_{v_k}\nu_0(v)A_{0,ij}(v) - \int_{\mathbb{R}^3}\partial_{v_k}[\mathbf{k}_0(v,v-v_*)]A_{0,ij}(v-v_*)\mathrm{d}v_*\right\}.
\end{aligned} \tag{3.29}$$

From (3.20) and $\nabla_v(|v|^2 - |v-v_*|^2)^2 = 4v_*(|v| + |v-v_*|)(|v - |v-v_*|)$, it follows $|\nabla_v[\mathbf{k}_0(v,v-v_*)]| \lesssim |v_*|\exp\{-\frac{|v-v_*|^2+|v*|^2}{8}\} + \frac{1}{|v_*|}\exp\{-\frac{|v_*|^2}{8} - \frac{1}{8}\frac{(|v|^2-|v-v_*|^2)^2}{|v_*|^2}\}$. From the first bound of (3.23), it follows $|\int_{\mathbb{R}^3}\partial_{v_k}[\mathbf{k}_0(v,v-v_*)]A_{0,ij}(v-v_*)\mathrm{d}v_*| \lesssim e^{-\varrho|v|^2}$ for any $0 < \varrho < 1/4$. Recall a projection $\tilde{\mathbf{P}}$ on $\mathcal{N}_0$. Then $|(\mathbf{I} - \tilde{\mathbf{P}})$ r.h.s. of (3.29)$| \lesssim e^{-\varrho|v|^2}$. Now applying (3.18) to $\partial_{v_k}A_{0,ij} = L_0^{-1}((\mathbf{I} -$

$\tilde{\mathbf{P}}$) r.h.s. of (3.29)) we derive

$$|\nabla_v A_{0,ij}(v)| \lesssim e^{-\varrho|v|^2} \text{ for any } 0 < \varrho < 1/4. \tag{3.30}$$

From (1.34) and (3.17), and the fact $\tilde{\varphi}_i = v_i$ for $i = 1, 2, 3$, and $\tilde{\varphi}_4 = \frac{|v|^2}{3}$ (the notation $\tilde{f}$ is defined in (3.15)), we have

$$A_{ij}(v) = L_0^{-1}\Big((v_i v_j - \frac{|v|^2}{3}\delta_{ij})\sqrt{\mu_0}\Big)(v - \varepsilon u) = A_{0,ij}(v - \varepsilon u). \tag{3.31}$$

Therefore we prove the second and third bounds in (3.27) using the fact that $\nabla_{x,t} A_{ij}(v) = -\varepsilon \nabla_{x,t} u \nabla_v A_{0,ij}(v - \varepsilon u)$.

Now we prove

$$|\nabla_v^2 A_{0,ij}(v)| \lesssim e^{-\varrho|v|^2}. \tag{3.32}$$

By taking one more derivative to (3.29), we derive that

$$\begin{aligned}
&L_0 \partial_{v_k} \partial_{v_\ell} A_{0,ij} \\
&= \partial_{v_k} \partial_{v_\ell} (v_i v_j - \frac{|v|^2}{3})\sqrt{\mu_0} + \partial_{v_\ell}(v_i v_j - \frac{|v|^2}{3})\partial_{v_k}\sqrt{\mu_0} \\
&\quad + (v_i v_j - \frac{|v|^2}{3})\partial_{v_\ell}\partial_{v_k}\sqrt{\mu_0} \\
&\quad - \partial_{v_k}\partial_{v_\ell}\nu_0(v) A_{0,ij}(v) - \partial_{v_k}\nu_0(v)\partial_{v_\ell} A_{0,ij}(v) \\
&\quad + \int_{\mathbb{R}^3} \partial_{v_k}\partial_{v_\ell}[\mathbf{k}_0(v, v - v_*)] A_{0,ij}(v - v_*) \\
&\quad + \partial_{v_k}[\mathbf{k}_0(v, v - v_*)]\partial_{v_\ell}[A_{0,ij}(v - v_*)] \mathrm{d}v_*.
\end{aligned}$$

The terms in the first two lines in r.h.s are easily bounded above as $e^{-\varrho|v|^2}$, recalling the fact $|\nabla_v \nu_0(v)| + |\nabla_v^2 \nu_0(v)| \lesssim 1$. We only focus on the terms in the last line. From $|\partial_{v_\ell} \nabla_v (|v|^2 - |v - v_*|^2)^2| \leq \big|4v_*\big(\frac{v_\ell}{|v|} + \frac{(v-v_*)_\ell}{|v-v_*|}\big)(|v| - |v - v_*|)\big| + 4\big|v_*(|v| + |v - v_*|)\big(\frac{v_\ell}{|v|} - \frac{(v-v_*)_\ell}{|v-v_*|}\big)\big| \lesssim |v_*|^2 + |v_*||v|$, we have $|\partial_{v_\ell} \nabla_v [\mathbf{k}_0(v, v - v_*)]| \lesssim |v_*| \exp\{-\frac{|v-v_*|^2+|v*|^2}{8}\} + \frac{1+|v|}{|v_*|^2} \exp\{-\frac{|v_*|^2}{8} - \frac{1}{8}\frac{(|v|^2-|v-v_*|^2)^2}{|v_*|^2}\}$. Using the second estimate of (3.23) with the first bound of (3.27), we have $\big|\int_{\mathbb{R}^3} \partial_{v_k}\partial_{v_\ell}[\mathbf{k}_0(v, v - v_*)] A_{0,ij}(v - v_*)\mathrm{d}v_*\big| \lesssim e^{-\varrho|v|^2}$. From (3.30) and the first bound of (3.27), it follows that $\big|\int_{\mathbb{R}^3} \partial_{v_k}[\mathbf{k}_0(v, v - v_*)]\partial_{v_\ell}[A_{0,ij}(v - v_*)]\mathrm{d}v_*\big| \lesssim e^{-\varrho|v|^2}$. Now we invert the operator L_0 and use (3.18) to conclude (3.32).

Finally from $\partial_t \nabla_x A_{ij}(v) = -\varepsilon \partial_t \nabla_x u \nabla_v A_{0,ij}(v - \varepsilon u) + \varepsilon^2 \nabla_x u \partial_t u \nabla_v^2 A_{0,ij}(v - \varepsilon u)$, (3.30), and (3.32), we conclude the last estimate of (3.27). □

For the estimates of $\partial_t f_R$ we derive the commutator estimate of $\partial_t L - L\partial_t$ and the corresponding one for Γ as follows.

Lemma 4 *Suppose $\varepsilon|u| \lesssim 1$ in the definition of μ in (1.16). For L and Γ in (1.20) and (1.21),*

$$\begin{aligned} \partial_t(Lf) &= L\partial_t f + L_t(\mathbf{I}-\mathbf{P})f - L(\mathbf{I}-\mathbf{P})(\mathbf{P}_t f), \\ \partial_t(\Gamma(f,g)) &= \{\Gamma(\partial_t f, g) + \Gamma(f, \partial_t g)\} + \Gamma_t(f,g), \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} L_t g(t,v) &:= -\varepsilon\partial_t u \cdot \nabla_v \nu_0(v-\varepsilon u) g(t,v) \\ &\quad + \varepsilon \partial_t u \cdot \int_{\mathbb{R}^3} (\nabla_v \mathbf{k}_0 + \nabla_{v_*}\mathbf{k}_0)(v-\varepsilon u, v_* - \varepsilon u) g(t,v_*) \mathrm{d}v_*, \\ (\mathbf{I}-\mathbf{P})\mathbf{P}_t g &:= -\varepsilon \sum_{j=0}^{4} (P_j g)(\mathbf{I}-\mathbf{P})\big(\partial_t u \cdot \nabla_v (\varphi_j \sqrt{\mu})\big), \\ \Gamma_t(f,g)(t,v) &:= \frac{\varepsilon}{2} \iint_{\mathbb{R}^3\times\mathbb{S}^2} |(v-v_*)\cdot \mathfrak{u}| \partial_t u \cdot (v_* - \varepsilon u)\sqrt{\mu(v_*)} \big\{ f(t,v')g(t,v_*') \\ &\quad + g(t,v')f(t,v_*') - f(t,v)g(t,v_*) - g(t,v)f(t,v_*)\big\} \mathrm{d}\mathfrak{u}\mathrm{d}v_*. \end{aligned} \tag{3.34}$$

We have

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} L_t(\mathbf{I}-\mathbf{P})f(v)g(v)\mathrm{d}v \right| \lesssim \varepsilon|\partial_t u| \|\nu^{1/2}(\mathbf{I}-\mathbf{P})f\|_{L^2_v} \|\nu^{1/2} g\|_{L^2_v}, \\ &\left| \int_{\mathbb{R}^3} L(\mathbf{P}_t f)(v) g(v) \mathrm{d}v \right| \lesssim \varepsilon |\partial_t u| |Pf| \|\nu^{1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_v}, \\ &\left| \int_{\mathbb{R}^3} \Gamma_t(f,g)(v)h(v)\mathrm{d}v \right| \\ &\quad \lesssim \varepsilon|\partial_t u| \Big(\|e^{\varrho|v|^2 + C\cdot v} g\|_{L^\infty_v} \|\nu^{1/2}(\mathbf{I}-\mathbf{P})f\|_{L^2_v} \\ &\qquad + \|e^{\varrho|v|^2+C\cdot v} f\|_{L^\infty_v} \|\nu^{1/2}(\mathbf{I}-\mathbf{P})g\|_{L^2_v} + |Pf||Pg| \Big) \|\nu^{1/2} h\|_{L^2_v}. \end{aligned} \tag{3.35}$$

Pointwise estimates are given as follows: for $0 < \varrho < 1/4$ and $C \in \mathbb{R}^3$

$$\begin{aligned} &|L_t(\mathbf{I}-\mathbf{P})f(t,v) - L(\mathbf{P}_t f)(t,v)| \\ &\quad \lesssim \varepsilon|\partial_t u| \|e^{\varrho|v|^2 + C\cdot v} f(t,v)\|_{L^\infty_v} \nu(v)^2 e^{-\varrho|v-\varepsilon u|^2}, \\ &\quad |\Gamma_t(f,g)(t,v)| \lesssim \varepsilon |\partial_t u| \|e^{\varrho|v|^2+C\cdot v} f(t,v)\|_{L^\infty_v} \|e^{\varrho|v|^2+C\cdot v} g(t,v)\|_{L^\infty_v} \frac{\nu(v)}{e^{\varrho|v|^2+C\cdot v}}, \\ &\quad |\Gamma(f,g)(v)| \lesssim \|e^{\varrho|v|^2+C\cdot v} f(v)\|_{L^\infty_v} \|e^{\varrho|v|^2+C\cdot v} g(v)\|_{L^\infty_v} \frac{\nu(v)}{e^{\varrho|v|^2+C\cdot v}}, \end{aligned} \tag{3.36}$$

and

$$|\Gamma(f,g)(v)| \lesssim \|e^{\varrho|v|^2+C\cdot v} f\|_\infty \Big(\nu(v)|g(v)| + \int_{\mathbb{R}^3} k_\vartheta(v,v_*)|g(v_*)|\mathrm{d}v_* \Big). \tag{3.37}$$

Proof The decomposition (3.33) with (3.34) comes from a direct computation to (1.21) and $\partial_t(Lf_R) = \partial_t(L(\mathbf{I}-\mathbf{P})f_R) = L(\mathbf{I}-\mathbf{P})\partial_t f_R + L_t(\mathbf{I}-\mathbf{P})f_R + L(-\mathbf{P}_t f_R)$. On the other hand, from (3.20) it is easy to check that, for any $0<\varrho<1/4$,

$$|\nabla_v \nu_0(v)| = \Big| \iint_{\mathbb{R}^3\times\mathbb{S}^2} \mathfrak{u}\frac{(v-v_*)\cdot\mathfrak{u}}{|(v-v_*)\cdot\mathfrak{u}|}\mu_0(v_*)\mathrm{d}\mathfrak{u}\mathrm{d}v_*\Big| \lesssim 1,$$
$$|\nabla_v \mathbf{k}_0(v,v_*)| + |\nabla_{v_*}\mathbf{k}_0(v,v_*)| \lesssim \Big(|v-v_*|^{-1} + \nu_0(v)^2|v-v_*|\Big)k_{\varrho/2}(v,v_*).$$

These estimates above combining with (3.23) and $\mathbf{P}L=0$ yield the first two estimates of (3.35).

We derive Γ_t as in the third identity of (3.34) from a direct computation to (1.22) and $\partial_t\sqrt{\mu(v_*)} = -\varepsilon\partial_t u\cdot\nabla_v\sqrt{\mu_0(v_*-\varepsilon u)} = \frac{1}{2}\varepsilon\partial_t u\cdot(v_*-\varepsilon u)\sqrt{\mu_0(v_*-\varepsilon u)}$. Then it is standard (see Lemma 2.13 in [13] for example) to have the last estimate in (3.35).

The first bound of (3.36) is a direct consequence of applying (3.22) and (3.23) to the first identity of (3.34). For the second bound of (3.36) we bound it as

$$\begin{aligned}
&\varepsilon|\partial_t u|\|e^{\varrho|v|^2+C\cdot v}f(t,v)\|_{L^\infty_v}\|e^{\varrho|v|^2+C\cdot v}g(t,v)\|_{L^\infty_v}\frac{1}{e^{\varrho|v|^2+C\cdot v}}\\
&\quad\iint_{\mathbb{R}^3\times\mathbb{S}^2}|(v-v_*)\cdot\mathfrak{u}|e^{-\varrho|v_*|^2-C\cdot v_*}\mathrm{d}\mathfrak{u}\mathrm{d}v_*\\
&\quad\lesssim \frac{\nu(v)}{e^{\varrho|v|^2+C\cdot v}}\varepsilon|\partial_t u|\|e^{\varrho|v|^2+C\cdot v}f(t,v)\|_{L^\infty_v}\|e^{\varrho|v|^2+C\cdot v}g(t,v)\|_{L^\infty_v}
\end{aligned}$$

For the last bound of (3.36) we recall a standard estimate (e.g. [16]) that

$$|\frac{1}{\nu_0(v)}e^{\varrho|v|^2+C\cdot v}\Gamma_0(f,g)(v)| \lesssim \|e^{\varrho|v|^2+C\cdot v}f(v)\|_{L^\infty_v}\|e^{\varrho|v|^2+C\cdot v}g(v)\|_{L^\infty_v}.$$

From the second equality of (3.15) we deduce the last bound of (3.36). A bound (3.37) is standard. □

3.2 Proof of Proposition 6

We verify two statements of Section 1.1 Hilbert Expansion. Firstly, we will show that the solvability condition (1.33) implies the incompressible condition (1.14). From (1.16), (1.23), and direct computations we verify the first identity of (1.33). Then from the oddness of the integrand with respect to the variable φ_i we derive that $\Big\langle \varphi_i\sqrt{\mu}, \frac{\varepsilon^{-1}(v-\varepsilon u)\cdot\nabla_x\mu}{\sqrt{\mu}}\Big\rangle = 0$ for $i=1,2,3$. For $i=0,4$, we compute that

$$\begin{aligned}
&\Big\langle \varphi_i\sqrt{\mu}, \frac{\varepsilon^{-1}(v-\varepsilon u)\cdot\nabla_x\mu}{\sqrt{\mu}}\Big\rangle\\
&\quad= \sum_{\ell=1}^3\langle\varphi_i\sqrt{\mu},\varphi_\ell\varphi_\ell\sqrt{\mu}\rangle\partial_\ell u_\ell = \left\{\delta_{i0}+\delta_{i4}\sqrt{\frac{2}{3}}\right\}(\nabla_x\cdot u)\ \text{ for } i=0,4.
\end{aligned}$$

This shows that (1.33) implies (1.14).

Secondly, we will verify the following statement of Section 1.1: the leading order terms of the hydrodynamic part in (1.28) vanish by solving the Navier-Stokes equations (1.13)-(1.15). Consider (1.28). We set $\mathbf{P} f_2 = \{\tilde{\rho}\varphi_0 + \sum_{\ell=1}^{3} \tilde{u}_\ell \varphi_\ell + \tilde{\theta}\varphi_4\}\sqrt{\mu}$ whose coefficients will be determined as in (3.1). Then the leading order term of (1.28)$= \frac{1}{\delta}$ (1.35) can be decomposed as

$$
\begin{aligned}
&-\frac{1}{\delta}\mathbf{P}\Big((v-\varepsilon u)\cdot(\partial_t u + u\cdot\nabla_x u)\sqrt{\mu}\\
&\quad \underbrace{+ (v-\varepsilon u)\cdot\big(\nabla_x\tilde{\rho}\varphi_0\sqrt{\mu} - \sum_{\ell=1}^{3}\nabla_x\tilde{u}_\ell\varphi_\ell\sqrt{\mu} + \nabla_x\tilde{\theta}\varphi_4\sqrt{\mu}\big)}_{(3.38)_*}\\
&\quad - \sum_{\ell,m=1}^{3}\kappa(v-\varepsilon u)\cdot A_{\ell m}\nabla_x\partial_\ell u_m\Big), \qquad (3.38)\\
&\quad -\frac{1}{\delta}(\mathbf{I}-\mathbf{P})\Big((v-\varepsilon u)\cdot\big(\nabla_x\tilde{\rho}\varphi_0\sqrt{\mu} + \sum_{\ell=1}^{3}\nabla_x\tilde{u}_\ell\varphi_\ell\sqrt{\mu} + \nabla_x\tilde{\theta}\varphi_4\sqrt{\mu}\big)\Big)\\
&\quad +\frac{1}{\delta}(\mathbf{I}-\mathbf{P})\Big(\sum_{\ell,m=1}^{3}\kappa(v-\varepsilon u)\cdot A_{\ell m}\nabla_x\partial_\ell u_m\Big), \qquad (3.39)
\end{aligned}
$$

while the lower order term consists of

$$
\begin{aligned}
&-\frac{1}{\delta}(v-\varepsilon u)\cdot\nabla_x\Big(\tilde{\rho}\varphi_0\sqrt{\mu} + \sum_{\ell=1}^{3}\tilde{u}_\ell\varphi_\ell\sqrt{\mu} + \tilde{\theta}\varphi_4\sqrt{\mu}\Big)\\
&\quad + \frac{1}{\delta}(v-\varepsilon u)\cdot\Big(\nabla_x\tilde{\rho}\varphi_0\sqrt{\mu} + \sum_{\ell=1}^{3}\nabla_x\tilde{u}_\ell\varphi_\ell\sqrt{\mu} + \nabla_x\tilde{\theta}\varphi_4\sqrt{\mu}\Big) \qquad (3.40)\\
&\quad + \frac{\kappa}{\delta}\sum_{\ell,m=1}^{3}(v-\varepsilon u)\cdot\nabla_x A_{\ell m}\partial_\ell u_m.
\end{aligned}
$$

First we focus on a leading order contribution of $(3.38)_*$ in (3.38). A direct computation yields

$$
\begin{aligned}
&\big\langle\varphi_i\sqrt{\mu}, (3.38)_*\big\rangle\\
&= \begin{cases} C_i\nabla_x\cdot\tilde{u}, \quad i=0,4\\ \partial_i\tilde{\rho}\langle\varphi_i\sqrt{\mu},\varphi_i\sqrt{\mu}\rangle + \partial_i\tilde{\theta}\langle\varphi_i\sqrt{\mu},\varphi_i\varphi_4\sqrt{\mu}\rangle = \partial_i\Big(\tilde{\rho} + \sqrt{\tfrac{2}{3}}\tilde{\theta}\Big), \quad i=1,2,3. \end{cases}
\end{aligned} \qquad (3.41)
$$

Among many other choices we make a special choice $(\tilde\rho, \tilde u, \tilde\theta) = (p, 0, 0)$ which is equivalent to (3.1). From (1.38), (3.41), and (3.1), it follows that for (u, p) solving (1.13)

$$(3.38) = \frac{1}{\delta}(v - \varepsilon u)\sqrt{\mu} \cdot \left\{\partial_t u + u \cdot \nabla_x u - \kappa\eta_0 \Delta u + \nabla_x p\right\}$$
$$= \frac{1}{\delta}(v - \varepsilon u)\sqrt{\mu} \cdot (1.13) = 0, \tag{3.42}$$

which verifies the second statement of Section 1.1.

Now we turn to proving the estimates. While the leading order terms vanish in (1.28), the rest of terms of (1.28) are bounded as follows. Upon the choice of (3.1), the first term of (3.39) vanishes and the first line of (3.40) are bounded by

$$\left|\frac{\varepsilon}{2\delta}(v - \varepsilon u) \cdot \nabla_x u \cdot (v - \varepsilon u)\mathbf{P} f_2\right| \lesssim \frac{\varepsilon}{\delta}|\nabla_x u||p|\langle v - \varepsilon u\rangle^2\sqrt{\mu}. \tag{3.43}$$

From (3.27) we deduce that the second term of (3.39) and the second line of (3.40) are bounded respectively by

$$\frac{\kappa}{\delta}|\nabla_x^2 u||v - \varepsilon u|e^{-\varrho|v-\varepsilon u|^2}, \ \frac{\varepsilon\kappa}{\delta}|\nabla_x u||v - \varepsilon u|e^{-\varrho|v-\varepsilon u|^2} \ \text{for any } 0 < \varrho < 1/4. \tag{3.44}$$

In conclusion we end up with the following result: Assume (u, p) solves (1.13)-(1.15), and both (1.34) and (3.1) hold. Then

$$|(1.28) - (3.39)| \lesssim \frac{\varepsilon}{\delta}\{|\nabla_x u||p| + \kappa|\nabla_x u|\}\langle v - \varepsilon u\rangle^2 e^{-\frac{|v-\varepsilon u|^2}{4}}, \tag{3.45}$$
$$|(\mathbf{I} - \mathbf{P})(3.39)| = |(3.39)| \lesssim \frac{1}{\delta}\kappa|\nabla_x^2 u|e^{-\varrho|v-\varepsilon u|^2}. \tag{3.46}$$

The term $\partial_t(1.28)$ can be bounded similarly. The entire leading order term of $\partial_t(1.28)$ can be decomposed as

$$-\frac{1}{\delta}\mathbf{P}\Big((v - \varepsilon u) \cdot \partial_t(\partial_t u + u \cdot \nabla_x u)\sqrt{\mu} + (v - \varepsilon u) \cdot \Big(\nabla_x \partial_t p\varphi_0\sqrt{\mu}\Big)$$
$$- \sum_{\ell,m=1}^{3} \kappa(v - \varepsilon u) \cdot A_{\ell m}\nabla_x \partial_\ell \partial_t u_m\Big), \tag{3.47}$$
$$-\frac{1}{\delta}(\mathbf{I} - \mathbf{P})\Big((v - \varepsilon u) \cdot (\nabla_x \partial_t p\varphi_0\sqrt{\mu})\Big)$$
$$+\frac{1}{\delta}(\mathbf{I} - \mathbf{P})\Big(\sum_{\ell,m=1}^{3} \kappa(v - \varepsilon u) \cdot A_{\ell m}\nabla_x \partial_\ell \partial_t u_m\Big). \tag{3.48}$$

Following the argument to get (1.38) and (3.41), we derive that

$$(3.47) = -\frac{1}{\delta}(v-\varepsilon u)\sqrt{\mu}\cdot\partial_t(1.13) = 0. \tag{3.49}$$

On the other hand,

$$|(3.48)| \lesssim \frac{1}{\delta}\kappa|\nabla_x^2\partial_t u|e^{-\varrho|v-\varepsilon u|^2}. \tag{3.50}$$

Now the lower order term $\partial_t(1.28) - (3.47) - (3.48)$ consists of

$$\begin{aligned}&\frac{\varepsilon}{\delta}\partial_t u\cdot\Big\{(\partial_t u + u\cdot\nabla_x u)\sqrt{\mu} + \nabla_x \mathbf{P} f_2 - \kappa\nabla_x\Big(\sum_{\ell,m=1}^{3} A_{\ell m}\partial_\ell u_m\Big)\Big\}\\ &\quad+\frac{\kappa}{\delta}(v-\varepsilon u)\cdot\nabla_x\Big(\sum_{\ell,m=1}^{3}\partial_t A_{\ell m}\partial_\ell u_m\Big)\\ &\quad-\frac{1}{\delta}(v-\varepsilon u)\cdot\nabla_x\partial_t\big(p\varphi_0\sqrt{\mu}\big) + \frac{1}{\delta}(v-\varepsilon u)\cdot\big(\nabla_x\partial_t p\varphi_0\sqrt{\mu}\big).\end{aligned} \tag{3.51}$$

Since the lower order term of $\partial_t(1.28)$ always contains $|\partial_t(v-\varepsilon u)| \le \varepsilon|\partial_t u|$, they can be bounded by, from (3.27) and (3.1),

$$\begin{aligned}|(3.51)| \lesssim\ & \frac{\varepsilon}{\delta}|\partial_t u|\big\{|\partial_t u| + |u||\nabla_x u| + |\nabla_x p|\\ &\qquad + \varepsilon|\nabla_x u||p| + \kappa\varepsilon|\nabla_x u|^2 + \kappa|\nabla_x^2 u|\big\}e^{-\varrho|v-\varepsilon u|^2}\\ &+\frac{\varepsilon}{\delta}\{|\nabla_x\partial_t u|(1+\kappa|\nabla_x u|) + |\partial_t p||\nabla_x u|\}e^{-\varrho|v-\varepsilon u|^2}.\end{aligned} \tag{3.52}$$

Now we consider (1.29). From (1.34), (3.1), and (3.27), we derive that

$$\begin{aligned}&|(\partial_t + u\cdot\nabla_x)\mathbf{P} f_2|\\ &\quad\lesssim \Big\{|\partial_t p| + |u||\nabla_x p| + \varepsilon|p|\{|\partial_t u| + |u||\nabla_x u|\}\Big\}\langle v-\varepsilon u\rangle^2 e^{-\frac{|v-\varepsilon u|^2}{4}},\end{aligned} \tag{3.53}$$

$$\begin{aligned}&|\partial_t(\partial_t + u\cdot\nabla_x)\mathbf{P} f_2|\\ &\quad\lesssim \Big\{|\partial_t^2 p| + |\partial_t u||\nabla_x p| + |u||\nabla_x\partial_t p| + \varepsilon|\partial_t p|\{|\partial_t u| + |u||\nabla_x u|\}\\ &\qquad + \varepsilon|p|\{|\partial_t^2 u| + |\partial_t u||\nabla_x u| + |u||\nabla_x\partial_t u|\}\\ &\qquad + \varepsilon|\partial_t u|\{\text{r.h.s. of } (3.53)\}\Big\}\langle v-\varepsilon u\rangle^2 e^{-\frac{|v-\varepsilon u|^2}{4}},\end{aligned} \tag{3.54}$$

and, for $0 < \varrho < 1/4$,

$$\begin{aligned}&|(\partial_t + u\cdot\nabla_x)(\mathbf{I}-\mathbf{P}) f_2|\\ &\quad\lesssim \kappa\Big\{\{|\nabla_x\partial_t u| + |u||\nabla_x^2 u|\} + \varepsilon\{|\partial_t u| + |u||\nabla_x u|\}|\nabla_x u|\Big\}e^{-\varrho|v-\varepsilon u|^2},\end{aligned} \tag{3.55}$$

$$
\begin{aligned}
&|\partial_t(\partial_t + u\cdot\nabla_x)(\mathbf{I}-\mathbf{P})f_2| \\
&\lesssim \kappa\Big\{\{|\nabla_x\partial_t^2 u| + |\partial_t u||\nabla_x^2 u| + |u||\nabla_x^2\partial_t u|\} + \varepsilon\{|\partial_t u| + |u||\nabla_x u|\}|\nabla_x\partial_t u| \\
&\quad +\varepsilon\{|\partial_t^2 u| + |\partial_t u||\nabla_x u| + |u||\nabla_x\partial_t u|\}|\nabla_x u| \\
&\quad +\varepsilon|\partial_t u|\{\text{r.h.s. of (3.55)}\}\Big\}e^{-\varrho|v-\varepsilon u|^2}.
\end{aligned} \tag{3.56}
$$

Next we consider the last term in (1.29). From (1.16) and (1.14)

$$
\begin{aligned}
\frac{(\partial_t + \varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}} &= \frac{1}{2}\big[(v-\varepsilon u)\cdot\nabla_x u\cdot(v-\varepsilon u) \\
&\quad + \varepsilon(\partial_t u + u\cdot\nabla_x u)\cdot(v-\varepsilon u)\big], \\
\partial_t\left(\frac{(\partial_t + \varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\right) &= \frac{1}{2}\big[\varepsilon(\partial_t^2 u + u\nabla_x\partial_t u - \partial_t u\cdot\nabla_x u)\cdot(v-\varepsilon u) \\
&\quad - \varepsilon^2\partial_t u\cdot(\partial_t u + u\cdot\nabla_x u) \\
&\quad + (v-\varepsilon u)\cdot\nabla_x\partial_t u\cdot(v-\varepsilon u)\big],
\end{aligned}
$$

and hence we derive (3.12) and (3.13).

Applying (3.27) to (1.34), it follows that, for $0<\varrho<\frac{1}{4}$,

$$
\begin{aligned}
|(\mathbf{I}-\mathbf{P})f_2| &\lesssim \kappa|\nabla_x u|e^{-\varrho|v-\varepsilon u|^2}, \\
|\partial_t(\mathbf{I}-\mathbf{P})f_2| &\lesssim \kappa\{|\partial_t\nabla_x u| + \varepsilon|\partial_t u||\nabla_x u|\}e^{-\varrho|v-\varepsilon u|^2}.
\end{aligned} \tag{3.57}
$$

From (3.1)

$$
|\mathbf{P}f_2| \lesssim |p|e^{-\frac{|v-\varepsilon u|^2}{4}}, \quad |\partial_t\mathbf{P}f_2| \lesssim |\partial_t p|e^{-\frac{|v-\varepsilon u|^2}{4}} + \varepsilon|\partial_t u||p|\langle v-\varepsilon u\rangle e^{-\frac{|v-\varepsilon u|^2}{4}}. \tag{3.58}
$$

These estimates give (3.10) and (3.11).

The last term of (1.29) is bounded as

$$
\begin{aligned}
&\frac{\varepsilon}{\delta}\left|\frac{(\partial_t + \varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}f_2\right| \\
&\quad \lesssim \frac{\varepsilon}{\delta}\{|p| + \kappa|\nabla_x u|\}\{|\nabla_x u| + \varepsilon(|\partial_t u| + |u||\nabla_x u|)\}e^{-\varrho|v-\varepsilon u|^2}, \qquad (3.59) \\
&\frac{\varepsilon}{\delta}\left|\partial_t\left(\frac{(\partial_t + \varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}f_2\right)\right| \\
&\lesssim \frac{\varepsilon}{\delta}\{|\nabla_x u| + \varepsilon(|\partial_t u| + |u||\nabla_x u|)\} \\
&\quad \times\{|\partial_t p| + \kappa|\partial_t\nabla_x u| + \varepsilon|\partial_t u|(|p| + \kappa|\nabla_x u|)\}e^{-\varrho|v-\varepsilon u|^2} \\
&\quad\quad +\frac{\varepsilon}{\delta}(|p| + \kappa|\nabla_x u|)\{|\nabla_x\partial_t u| + \varepsilon(|\partial_t^2 u| + |u||\nabla_x\partial_t u| + |\partial_t u||\nabla_x u|)
\end{aligned}
$$

$$+\varepsilon^2|\partial_t u|(|\partial_t u|+|u||\nabla_x u|)\}e^{-\varrho|v-\varepsilon u|^2}. \tag{3.60}$$

Lastly from (3.36), (3.1), and (1.34)

$$\begin{aligned}\frac{\varepsilon}{\delta\kappa}|\Gamma(f_2,f_2)(v)| &= \frac{\varepsilon}{\delta\kappa}|\Gamma(f_2,f_2)(v)-\Gamma(\mathbf{P}f_2,\mathbf{P}f_2)| \\ &\lesssim \frac{\varepsilon}{\delta}(|p|+\kappa|\nabla_x u|)|\nabla_x u|\nu(v)e^{-\varrho|v-\varepsilon u|^2}, \end{aligned}\tag{3.61}$$

$$\begin{aligned}\frac{\varepsilon}{\delta\kappa}|\partial_t\Gamma(f_2,f_2)(v)| \lesssim \frac{\varepsilon}{\delta}\{&(|p|+\kappa|\nabla_x u|)|\partial_t\nabla_x u| \\ &+(|\partial_t p|+\kappa|\nabla_x\partial_t u|)|\nabla_x u|\}\nu(v)e^{-\varrho|v-\varepsilon u|^2} \\ &+\frac{\varepsilon^2}{\delta}|\partial_t u|(|p|+\kappa|\nabla_x u|)|\nabla_x u|\nu(v)e^{-\varrho|v-\varepsilon u|^2}, \end{aligned}\tag{3.62}$$

where we have used $\Gamma(\mathbf{P}f_2,\mathbf{P}f_2)=\Gamma(p\sqrt{\mu},p\sqrt{\mu})=0$ to eliminate the contribution of p^2 in (3.61).

Finally we wrap up the estimates of the source term of (3.2) to show (3.4) and (3.5). The term $(\mathbf{I}-\mathbf{P})\mathfrak{R}_1$ consists of (3.39), which is bounded as (3.46) and hence we prove (3.4). The rest of terms form $\mathfrak{R}_2$, which can be proved to be bounded as (3.5), from (3.45), (3.53), (3.55), (3.59), and (3.61).

Now we consider the source term of (3.3). The term $(\mathbf{I}-\mathbf{P})\mathfrak{R}_3$ consists of (3.48), which is bounded as (3.50). From (3.49), (3.52), (3.54)-(3.56), (3.12), (3.13), (3.60), (3.62), and (3.36), we prove (3.7).

4 A Priori Estimates for f_R

For each $\varepsilon>0$ an existence of a unique solution in a time interval $[0,\infty)$ can be found in [13]. Thereby we only focus on a priori estimates of f_R in different spaces. For the sake of simplicity at times we will use simplified notations

$$\begin{aligned}&\|g(t,x,v)\|_{L_t^{p_1}L_x^{p_2}L_v^{p_3}} \\ &:= \Big\|\Big\|\|g(t,x,v)\|_{L_v^{p_3}(\mathbb{R}^3)}\Big\|_{L_x^{p_2}(\Omega)}\Big\|_{L_t^{p_1}([0,T])}, \quad \|g\|_{L^p_{t,x,v}} := \|g\|_{L_t^pL_x^pL_v^p}. \end{aligned}\tag{4.1}$$

Recall the boundary integral and the norms in (2.5). Also recall $\mathfrak{w}=\mathfrak{w}_{\varrho,\text{ß}}(x,v)$ in (2.3) and $\mathfrak{w}'=\mathfrak{w}_{\varrho',\text{ß}}(x,v)$ for $0<\varrho'<\varrho$.

4.1 L^2-Energy Estimate

Our starting point is a basic L^2-energy estimate for the Boltzmann remainder f_R and its temporal derivative $\partial_t f_R$ in which the dissipation (1.31) plays an important role in the nonlinear estimate.

Proposition 7 *Under the same assumptions in Proposition 6, we have*

$$\begin{aligned}&\|f_R(t)\|^2_{L^2_{x,v}} + d_2\int_0^t \|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2_{x,v}} + \int_0^t |\varepsilon^{-\frac{1}{2}}f_R|^2_{L^2_\gamma}\\ &\lesssim \|f_R(0)\|^2_{L^2_{x,v}} + (1+\|(3.12)\|_{L^\infty_{t,x}})\int_0^t \|Pf_R(s)\|^2_{L^2_x}\mathrm{d}s\\ &\quad+\frac{\delta^2}{\kappa^3}\|\kappa^{1/2}Pf_R(s)\|^2_{L^\infty_t L^6_x}\|\kappa^{1/2}Pf_R\|^2_{L^2_tL^3_x}\\ &\quad+\frac{\varepsilon\kappa^2}{\delta^2}|\nabla_x u|^2_{L^2_tL^2(\partial\Omega)} + \kappa\varepsilon^{1/8}\|\nabla_x u\|^2_{L^2_{t,x}} + \kappa\varepsilon^2\|(3.4)\|^2_{L^2_{t,x}} + \|(3.5)\|^2_{L^2_{t,x}},\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}d_2 :=\; &\frac{\sigma_0}{2} - \delta\varepsilon\|\mathfrak{w}f_R\|_{L^\infty_{t,x,v}} - (\varepsilon^{\frac{15}{16}}\|\mathfrak{w}f_R\|_{L^\infty_{t,x,v}})^2 - \varepsilon^2\|(3.10)\|_{L^\infty_{t,x}}\\ &-\frac{\varepsilon^2}{\kappa}\|(3.10)\|^2_{L^\infty_{t,x}} - \varepsilon\kappa^{1/2}\|(3.12)_*\|_{L^\infty_{t,x}}.\end{aligned} \tag{4.3}$$

Here $L^p_t = L^p_t([0,t])$ in particular. Note that a weighted L^∞-bound of f_R is involved in this energy estimate, where the weight $\mathfrak{w} = \mathfrak{w}_{\varrho,\text{ß}}(x,v)$ is defined in (2.3).

We also have

$$\begin{aligned}&\|\partial_t f_R(t)\|^2_{L^2_{x,v}} + d_{2,t}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|^2_{L^2_{t,x,v}}\\ &\quad+|\varepsilon^{-\frac{1}{2}}\partial_t f_R|^2_{L^2_tL^2_\gamma} - \varepsilon\|\partial_t u\|_{L^\infty_{t,x}}|f_R|^2_{L^2_tL^2_\gamma}\\ &\lesssim \|\partial_t f_R(0)\|^2_{L^2_{x,v}} + \kappa^{-1}\delta^2\|Pf_R\|^2_{L^\infty_tL^6_x}\{\|Pf_R\|^2_{L^2_tL^3_x} + \|P\partial_t f_R\|^2_{L^2_tL^3_x}\}\\ &\quad+\Big\{\kappa^{-1}\|\partial_t u\|^2_{L^\infty_{t,x}} + \|\nabla_x\partial_t u\|_{L^\infty_{t,x}} + \varepsilon\|\partial_t^2 u\|_{L^2_tL^\infty_x}\\ &\qquad+\varepsilon\kappa^{-1/2}(1+\|\partial_t u\|_{L^\infty_{t,x}})\|(3.10)\|_{L^\infty_{t,x}}\\ &\qquad+\|(3.12)\|_{L^\infty_{t,x}} + \|(3.13)_*\|_{L^\infty_{t,x}}\Big\}\times\int_0^t\|P\partial_t f_R(s)\|^2_{L^2_x}\mathrm{d}s\\ &\quad+\Big\{\kappa^{-1}\|\partial_t u\|^2_{L^\infty_{t,x}} + \|\nabla_x\partial_t u\|_{L^\infty_{t,x}} + \varepsilon\|\partial_t^2 u\|_{L^2_tL^\infty_x} + (\varepsilon\|(3.10)\|_{L^\infty_{t,x}})^2\\ &\qquad+(\varepsilon\kappa^{-1/2}\|(3.11)\|_{L^\infty_{t,x}})^2 + \|(3.13)_*\|_{L^\infty_{t,x}}\Big\}\times\int_0^t\|Pf_R(s)\|^2_{L^2_x}\mathrm{d}s\\ &\quad+\Big\{\varepsilon(1+\varepsilon\|(3.10)\|_{L^\infty_{t,x}})\|\partial_t u\|_{L^\infty_{t,x}} + \varepsilon\kappa\|\nabla_x\partial_t u\|_{L^\infty_{t,x}} + \varepsilon^2\kappa\|\partial_t^2 u\|_{L^2_tL^\infty_x}\\ &\qquad+(\varepsilon\kappa^{1/2}\|(3.13)_*\|_{L^\infty_{t,x}})^2 + (\varepsilon\delta\|\mathfrak{w}f_R\|_{L^\infty_{t,x,v}})^2\Big\}\times\|\varepsilon^{-1}\kappa^{-1/2}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2_{t,x,v}}\\ &\quad+e^{-\frac{\varrho}{4\varepsilon^2}}\{\|(3.12)\|^2_{L^\infty_{t,x}} + \|\nabla_x\partial_t u\|^2_{L^2_tL^\infty_x}\} + (\varepsilon\kappa^{1/2}\|(3.6)\|_{L^2_{t,x}})^2 + \|(3.7)\|^2_{L^2_{t,x}}\\ &\quad+\frac{\varepsilon\kappa^2}{\delta^2}||\partial_t\nabla_x u| + \varepsilon|\partial_t u||\nabla_x u||^2_{L^2_tL^2(\partial\Omega)} + \frac{\varepsilon^3\kappa^2}{\delta}|\nabla_x u|^2_{L^2_tL^2(\partial\Omega)}\|\partial_t u\|_{L^\infty_{t,x}},\end{aligned} \tag{4.4}$$

where $\partial_t f_R(0,x,v) := f_{R,t}(0,x,v)$ *is defined in* (2.6). *Here*

$$\begin{aligned} d_{2,t} :=& \frac{\sigma_0}{2} - \varepsilon(\kappa^{-1/2} + \varepsilon\|\partial_t u\|_{L^\infty_{t,x}})\|(3.10)\|_{L^\infty_{t,x}} - \varepsilon\kappa\|(3.12)\|_{L^\infty_{t,x}} \\ &- (\varepsilon\kappa^{1/2}\|(3.13)_*\|_{L^\infty_{t,x}})^2 \\ &- \varepsilon\kappa\|\nabla_x\partial_t u\|_{L^\infty_{t,x}} - \varepsilon^2\kappa\|\partial_t^2 u\|_{L^2_t L^\infty_x} + \varepsilon\|\partial_t u\|_{L^\infty_{t,x}}(1+\varepsilon\|\partial_t u\|_{L^\infty_{t,x}}) \\ &- \varepsilon^{-\frac{\varrho}{4\varepsilon^2}}(\varepsilon\kappa^{1/2}\|\mathfrak{w}_{\varrho',\beta}\partial_t f_R\|_{L^2_t L^\infty_{x,v}})^2 - \varepsilon\delta(1+\varepsilon\|\partial_t u\|_{L^\infty_{x,v}})\|\mathfrak{w} f_R\|_{L^\infty_{t,x}} \\ &- (\varepsilon\kappa^{1/2}\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}})^2, \end{aligned} \tag{4.5}$$

where $0 < \varrho' < \varrho$.

Remark 8 We utilize several different time-space norms to control the fluid source terms, which possess the initial-boundary and boundary layers as in Theorem 3.

The following trace theorem is useful to control the boundary terms.

Lemma 5 (Trace theorem)

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^t\int_{\gamma_+^N}|h|\mathrm{d}\gamma\mathrm{d}s \\ &\lesssim_N \iint_{\Omega\times\mathbb{R}^3}|h(0)| + \int_0^t\iint_{\Omega\times\mathbb{R}^3}|h| + \int_0^t\iint_{\Omega\times\mathbb{R}^3}|\partial_t h + \frac{1}{\varepsilon}v\cdot\nabla_x h|, \end{aligned} \tag{4.6}$$

where $\gamma_+^N := \{(x,v)\in\gamma_+ : |n(x)\cdot v| > 1/N$ *and* $1/N < |v| < N\}$.

The proof is standard (for example see Lemma 3.2 in [13] or Lemma 7 in [6]).

Proof of Proposition 7 First we prove (4.2). An energy estimate to (3.2) and (3.8) reads as

$$\frac{1}{2}\|f_R(t)\|^2_{L^2_{x,v}} - \frac{1}{2}\|f_R(0)\|^2_{L^2_{x,v}} + \frac{1}{\kappa\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{R}^3} f_R L f_R \tag{4.7}$$

$$+\frac{1}{2\varepsilon}\int_0^t\int_{\gamma_+}|f_R|^2 - \frac{1}{2\varepsilon}\int_0^t\int_{\gamma_-}|P_{\gamma_+}f_R - \frac{\varepsilon}{\delta}(1-P_{\gamma_+})(\mathbf{I}-\mathbf{P})f_2|^2 \tag{4.8}$$

$$= \frac{\delta}{\kappa\varepsilon}\int_0^t\iint_{\Omega\times\mathbb{R}^3}\Gamma(f_R,f_R)(\mathbf{I}-\mathbf{P})f_R \tag{4.9}$$

$$+\frac{2}{\kappa}\int_0^t\iint_{\Omega\times\mathbb{R}^3}\Gamma(f_2,f_R)(\mathbf{I}-\mathbf{P})f_R \tag{4.10}$$

$$+\int_0^t\iint_{\Omega\times\mathbb{R}^3}(\mathbf{I}-\mathbf{P})\mathfrak{R}_1(\mathbf{I}-\mathbf{P})f_R \tag{4.11}$$

$$+\int_0^t\iint_{\Omega\times\mathbb{R}^3}\mathfrak{R}_2 f_R \tag{4.12}$$

$$+\int_0^t\iint_{\Omega\times\mathbb{R}^3}\frac{-(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}|f_R|^2. \tag{4.13}$$

Among others two terms (4.9) and (4.13) are most problematic.

We start with (4.7). From the spectral gap estimate in (1.25), we have

$$(4.7) \geq \frac{1}{2}\|f_R(t)\|^2_{L^2_{x,v}} - \frac{1}{2}\|f_R(0)\|^2_{L^2_{x,v}} + \sigma_0\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2_{t,x,v}}. \quad (4.14)$$

Now we consider (4.9), in which we need integrability gain of $\mathbf{P}f_R$ in L^6_x of the next sections. From decomposition $f_R = \mathbf{P}f_R + (\mathbf{I}-\mathbf{P})f_R$ and $\Gamma = \Gamma_+ - \Gamma_-$ in (1.22), we derive

$$\begin{aligned}
|(4.9)| &\lesssim \frac{\delta}{\kappa\varepsilon}\sum_{i=\pm}\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\nu^{-\frac{1}{2}}\Gamma_i(|f_R|,(\mathbf{I}-\mathbf{P})f_R)||\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R| \\
&\quad + \frac{\delta}{\kappa\varepsilon}\sum_{i=\pm}\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\nu^{-\frac{1}{2}}\Gamma_i(|\mathbf{P}f_R|,|\mathbf{P}f_R|)||\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R| \\
&\lesssim \delta\varepsilon\|\mathfrak{w}f_R\|_{L^\infty_{x,v}}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2_{t,x,v}} \\
&\quad + \frac{\delta}{\kappa^{3/2}}\|\kappa^{1/2}Pf_R\|_{L^\infty_t L^6_x}\|\kappa^{1/2}Pf_R\|_{L^2_t L^3_x}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}}.
\end{aligned} \quad (4.15)$$

From (3.57) and (3.58),

$$\begin{aligned}
&|(4.10)| \\
&\leq \frac{\varepsilon}{\kappa^{1/2}}\|(3.10)\|_{L^\infty_{t,x}}\{\|Pf_R\|_{L^2_{t,x}} + \kappa^{\frac{1}{2}}\varepsilon\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}}\} \\
&\quad \times \|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}} \\
&\lesssim \{\varepsilon^2\|(3.10)\|_{L^\infty_{t,x}} + \frac{\varepsilon^2}{\kappa}\|(3.10)\|^2_{L^\infty_{t,x}}\}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2_{t,x,v}} + \|Pf_R\|^2_{L^2_t L^2_x}.
\end{aligned} \quad (4.16)$$

From (3.4) and (3.5) we derive that

$$\begin{aligned}
|(4.11)| &\lesssim \kappa^{1/2}\varepsilon\|(3.4)\|_{L^2_{t,x}}\|\kappa^{-1/2}\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}}, \\
|(4.12)| &\lesssim \|(3.5)\|^2_{L^2_{t,x,v}} + \|\mathbf{P}f_R\|^2_{L^2_{t,x,v}} \\
&\qquad + \kappa^{1/2}\varepsilon\|(3.5)\|_{L^2_{t,x,v}}\|\kappa^{-1/2}\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}}.
\end{aligned} \quad (4.17)$$

Next using (3.12) it follows that

$$\begin{aligned}
&|(4.13)| \\
&\lesssim \kappa^{1/2}\varepsilon\|\mathfrak{w}^{-1}\nu^{3/2}\nabla_x u\|_{L^2_{t,x,v}}\|\mathfrak{w}f_R\|_{L^\infty_{t,x,v}}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}} \\
&\quad + \|(3.12)\|_{L^\infty_{t,x}}\int_0^t\|Pf_R(s)\|^2_{L^2_x}\mathrm{d}s \\
&\quad + \varepsilon\kappa^{1/2}\|(3.12)_*\|_{L^\infty_{t,x}}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2_{t,x,v}}
\end{aligned}$$

$$\lesssim \big\{(\varepsilon^{\frac{15}{16}}\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}})^2 + \varepsilon\kappa^{1/2}\|(3.12)_*\|_{L^\infty_{t,x}}\big\}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2_{t,x,v}} + \|(3.12)\|_{L^\infty_{t,x}}\int_0^t \|Pf_R(s)\|^2_{L^2_x}\mathrm{d}s + (\kappa^{\frac{1}{2}}\varepsilon^{\frac{1}{16}}\|\nabla_x u\|_{L^2_{t,x}})^2. \tag{4.18}$$

Finally we control the boundary term (4.8) using a trace theorem (4.6). First we have, from (3.8),

$$\begin{aligned}
(4.8) =& \frac{1}{2\varepsilon}\int_0^t\int_{\gamma_+}\{|f_R|^2 - |P_{\gamma_+}f_R|^2\} - \frac{\varepsilon}{2\delta^2}\int_0^t\int_{\gamma_-}|(1-P_{\gamma_+})(\mathbf{I}-\mathbf{P})f_2|^2\\
&-\int_0^t\int_{\gamma_-}\frac{1}{\varepsilon^{1/2}}P_{\gamma_+}f_R\frac{\varepsilon^{1/2}}{\delta}(1-P_{\gamma_+})(\mathbf{I}-\mathbf{P})f_2\\
\geq& \frac{1}{2}|\varepsilon^{-\frac{1}{2}}(1-P_{\gamma_+})f_R|^2_{L^2_tL^2_{\gamma_+}} - \frac{1}{8C}|\varepsilon^{-\frac{1}{2}}P_{\gamma_+}f_R|^2_{L^2_tL^2_{\gamma_+}}\\
&-(\frac{\varepsilon}{2\delta^2}+2C\frac{\varepsilon}{\delta^2})\int_0^t\int_{\gamma_+}|(1-P_{\gamma_+})(\mathbf{I}-\mathbf{P})f_2|^2 \ \text{ for } C\gg 1,
\end{aligned} \tag{4.19}$$

where we have used the fact $|P_{\gamma_+}f_R|_{L^2_{\gamma_+}} = |P_{\gamma_+}f_R|_{L^2_{\gamma_-}}$ from $P_{\gamma_+}f_R(t,x,v)$ being a function of $(t,x,|v|)$ due to $u|_{\partial\Omega}=0$.

Now we estimate $P_{\gamma_+}f_R$. Since P_{γ_+} in (3.8) is a projection of $c_\mu\sqrt{\mu}$ on γ_+, it follows $\int_{\gamma_+}|P_{\gamma_+}f|^2 \leq 2\int_{\gamma_+^N}|P_{\gamma_+}f|^2$ for large enough $N>0$, where $\gamma_+^N := \{(x,v)\in\gamma_+ : |n(x)\cdot v| > 1/N \text{ and } 1/N < |v| < N\}$. Setting $h=|f|^2$ in (4.6) and using (3.2), (3.4), and (3.5) we derive

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t\int_{\gamma_+}|f_R|^2\mathrm{d}\gamma\mathrm{d}s\\
&\leq C_N\iint_{\Omega\times\mathbb{R}^3}|f_R(0)|^2 + \int_0^t\iint_{\Omega\times\mathbb{R}^3}|f_R|^2\\
&\quad+\int_0^t\iint_{\Omega\times\mathbb{R}^3}\Big|\Big[-\frac{1}{\varepsilon^2\kappa}Lf_R + \frac{1}{\kappa}\Gamma(f_2,f_R) + \frac{\delta}{\varepsilon\kappa}\Gamma(f_R,f_R)\\
&\qquad\qquad -\frac{(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}f_R + (\mathbf{I}-\mathbf{P})\mathfrak{R}_1 + \mathfrak{R}_2\Big]f_R\Big|\\
&\leq C_N\big\{\|f_R(0)\|^2_{L^2_{x,v}} + \|\mathbf{P}f_R\|^2_{L^2_{t,x,v}} + \|\varepsilon^{-1}\kappa^{-1/2}\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|^2_{L^2_{t,x,v}}\\
&\qquad + (4.15) + (4.16) + (4.17) + (4.18)\big\}.
\end{aligned} \tag{4.20}$$

Furthermore from (3.8) and (4.20)

$$\begin{aligned}
|f_R|^2_{L^2_tL^2_{\gamma_-}} &\lesssim |f_R|^2_{L^2_tL^2_{\gamma_+}} + \frac{\varepsilon^2}{\delta^2}|(1-P_{\gamma_+})(\mathbf{I}-\mathbf{P})f_2|^2_{L^2_tL^2_{\gamma_-}}\\
&= |f_R|^2_{L^2_tL^2_{\gamma_+}} + \frac{\varepsilon^2\kappa^2}{\delta^2}|\nabla_x u|^2_{L^2_tL^2(\partial\Omega)}.
\end{aligned} \tag{4.21}$$

Finally we collect the terms as

$$\begin{aligned}&\text{r.h.s of (4.14)} + (4.19) + \frac{1}{4C}|\varepsilon^{-\frac{1}{2}} P_{\gamma_+} f_R|^2_{L^2_t L^2_{\gamma_+}} + \frac{\varepsilon^{-1}}{16C}|f_R|^2_{L^2_t L^2_{\gamma_-}} \\ &\quad \le \text{r.h.s of (4.15)} + (4.16) + (4.17) + (4.18) + \frac{1}{4C} \times \text{r.h.s of (4.20)} \\ &\quad + \frac{\varepsilon^{-1}}{16C} \times \text{r.h.s of (4.21)}.\end{aligned}$$

We choose large N and then large C so that $\frac{C_N}{4C} \ll \sigma_0$. Using Young's inequality for products, and then moving contributions of $\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P}) f_R\|^2_{L^2_{t,x,v}}$ to l.h.s., we derive (4.2).

Next we prove (4.4). An energy estimate to (3.3) and (3.9) lead to (4.4)

$$\frac{1}{2}\|\partial_t f_R(t)\|_2^2 - \frac{1}{2}\|\partial_t f_R(0)\|_2^2 + \frac{1}{\kappa\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{R}^3} \partial_t f_R L \partial_t f_R \tag{4.22}$$

$$+ \frac{1}{2\varepsilon}\int_0^t\int_{\gamma_+} |\partial_t f_R|^2 - \frac{1}{2\varepsilon}\int_0^t\int_{\gamma_-} |P_{\gamma_+}\partial_t f_R$$

$$- \frac{\varepsilon}{\delta}(1-P_{\gamma_+})\partial_t(\mathbf{I}-\mathbf{P}) f_2 + r_{\gamma_+}(f_R) - \frac{\varepsilon}{\delta} r_{\gamma_+}((\mathbf{I}-\mathbf{P}) f_2)|^2 \tag{4.23}$$

$$= -\frac{1}{\varepsilon^2\kappa}\int_0^t \iint_{\Omega\times\mathbb{R}^3} L_t(\mathbf{I}-\mathbf{P}) f_R \partial_t f_R + \frac{1}{\varepsilon^2\kappa}\int_0^t \iint_{\Omega\times\mathbb{R}^3} L(\mathbf{P}_t f_R)(\mathbf{I}-\mathbf{P})\partial_t f_R \tag{4.24}$$

$$+ \frac{2\delta}{\kappa\varepsilon}\int_0^t \iint_{\Omega\times\mathbb{R}^3} \Gamma(f_R, \partial_t f_R)(\mathbf{I}-\mathbf{P})\partial_t f_R$$

$$+ \frac{2}{\kappa}\int_0^t \iint_{\Omega\times\mathbb{R}^3} \Gamma(f_2, \partial_t f_R)(\mathbf{I}-\mathbf{P})\partial_t f_R \tag{4.25}$$

$$+ \frac{2}{\kappa}\int_0^t \iint_{\Omega\times\mathbb{R}^3} \Gamma(\partial_t f_2, f_R)(\mathbf{I}-\mathbf{P})\partial_t f_R \tag{4.26}$$

$$+ \frac{2}{\kappa}\int_0^t \iint_{\Omega\times\mathbb{R}^3} \Gamma_t(f_2, f_R)\partial_t f_R + \frac{\delta}{\varepsilon\kappa}\int_0^t \iint_{\Omega\times\mathbb{R}^3} \Gamma_t(f_R, f_R)\partial_t f_R \tag{4.27}$$

$$+ \int_0^t \iint_{\Omega\times\mathbb{R}^3} (\mathbf{I}-\mathbf{P})\mathfrak{R}_3(\mathbf{I}-\mathbf{P})\partial_t f_R \tag{4.28}$$

$$+ \int_0^t \iint_{\Omega\times\mathbb{R}^3} \mathfrak{R}_4 \partial_t f_R \tag{4.29}$$

$$+ \int_0^t \iint_{\Omega\times\mathbb{R}^3} \frac{-(\partial_t + \varepsilon^{-1} v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}|\partial_t f_R|^2$$

$$+ \int_0^t \iint_{\Omega\times\mathbb{R}^3} \partial_t\Big(\frac{-(\partial_t + \varepsilon^{-1} v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\Big) f_R \partial_t f_R. \tag{4.30}$$

We consider the first term of (4.30). We decompose $\partial_t f_R = \mathbf{P}\partial_t f_R + (\mathbf{I}-\mathbf{P})\partial_t f_R$. The contribution of $\mathbf{P}\partial_t f_R$ can be bounded above as, from (3.12),

$$\|(3.12)\|_{L^\infty_{t,x}} \int_0^t \|P\partial_t f_R(s)\|^2_{L^2_x} \mathrm{d}s. \tag{4.31}$$

For the contribution of $(\mathbf{I}-\mathbf{P})\partial_t f_R$ we utilize an extra decomposition $\mathbf{1}_{|v|\leq\varepsilon^{-1}} + \mathbf{1}_{|v|\geq\varepsilon^{-1}}$ Then it is bounded as

$$\begin{aligned}
&\|(3.12)\|_\infty \Big\{ \iiint \mathbf{1}_{|v|\leq\varepsilon^{-1}} |v|\nu(v)|(\mathbf{I}-\mathbf{P})\partial_t f_R|^2 \\
&\quad + \iiint \mathbf{1}_{|v|\geq\varepsilon^{-1}} \frac{|v|^{3/2}}{\mathfrak{w}'(v)} \mathfrak{w}'(v)\partial_t f_R(v)\sqrt{\nu(v)}|(\mathbf{I}-\mathbf{P})\partial_t f_R| \Big\} \\
&\lesssim \|(3.12)\|_\infty \Big\{ \varepsilon^{-1} \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|^2_{L^2_{t,x,v}} \\
&\quad + e^{-\frac{\varrho}{4\varepsilon^2}} \|\mathfrak{w}'\partial_t f_R\|_{L^2_t L^\infty_{x,v}} \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \Big\}.
\end{aligned} \tag{4.32}$$

For the second term of (4.30) using (3.13) we bound it by

$$\begin{aligned}
&\|(3.13)_*\|_{L^\infty_{t,x}} \|\sqrt{\nu}(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}} \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \\
&\quad + e^{-\frac{\varrho}{4\varepsilon^2}} \|\nabla_x \partial_t u\|_{L^2_t L^\infty_x} \|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}} \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \\
&\quad + \{\|\nabla_x \partial_t u\|_{L^\infty_{t,x}} + \|(3.13)_*\|_{L^\infty_{t,x}}\} \Big\{ \int_0^t \|P f_R(s)\|^2_{L^2_x} \mathrm{d}s + \int_0^t \|P\partial_t f_R(s)\|^2_{L^2_x} \mathrm{d}s \Big\}.
\end{aligned} \tag{4.33}$$

Using (3.35) we bound (4.24) and (4.27) as

$$\begin{aligned}
|(4.24)| &\lesssim \kappa^{-\frac12} \|\partial_t u\|_{L^\infty_{t,x}} \|\kappa^{-\frac12}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}} \\
&\qquad \times \{\|P\partial_t f_R\|_{L^2_{t,x}} + \kappa^{\frac12}\varepsilon \|\kappa^{-\frac12}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}\} \\
&\qquad + \kappa^{-\frac12} \|\partial_t u\|_{L^\infty_{t,x}} \|\kappa^{-\frac12}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \|P f_R\|_{L^2_{t,x}}, \qquad (4.34) \\
|(4.27)| &\lesssim \kappa^{-\frac12}\varepsilon \|\partial_t u\|_{L^\infty_{t,x}} \|(3.10)\|_{L^\infty_{t,x}} \{\|\sqrt{\nu}(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}} \\
&\qquad\quad + \|P f_R\|_{L^2_{t,x}}\} \|\partial_t f_R\|_{L^2_{t,x,v}} \\
&\qquad + \delta\kappa^{-1} \|\partial_t u\|_{L^\infty_{t,x}} \{\|P\partial_t f_R\|_{L^2_{t,x}} + \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}\} \\
&\qquad \times \{\|P f_R\|_{L^\infty_t L^6_x} \|P f_R\|_{L^2_t L^3_x} \\
&\qquad\quad + \|\kappa^{-\frac12}\varepsilon^{-1}(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}} \kappa^{\frac12}\varepsilon \|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}}\}. \qquad (4.35)
\end{aligned}$$

The rest of terms can be controlled similarly as in the proof of (4.2):

$$(4.22) \geq \frac12 \|\partial_t f_R(t)\|^2_{L^2_{x,v}} - \frac12 \|\partial_t f_R(0)\|^2_{L^2_{x,v}} + \sigma_0 \|\kappa^{-\frac12}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|^2_{L^2_{t,x,v}}, \tag{4.36}$$

$$|(4.25)| \lesssim \{\delta\varepsilon\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}} + \varepsilon^2\|(3.10)\|_{L^\infty_{t,x}}\}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|^2_{L^2_{t,x,v}}$$
$$+ \frac{\delta}{\kappa^{3/2}}\|\kappa^{1/2}P f_R\|_{L^\infty_t L^6_x}\|\kappa^{1/2}P\partial_t f_R\|_{L^2_t L^3_x}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \tag{4.37}$$
$$+ \frac{\varepsilon}{\kappa^{1/2}}\|P f_2\|_{L^\infty_{t,x}}\|P\partial_t f_R\|_{L^2_{t,x}}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}, \tag{4.38}$$
$$|(4.26)| \lesssim \kappa^{-\frac{1}{2}}\varepsilon\|\sqrt{\nu}\partial_t f_2\|_{L^\infty_{t,x,v}}\{\|P f_R\|_{L^2_{t,x}}$$
$$+ \|\sqrt{\nu}(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}}\}\|\kappa^{-\frac{1}{2}}\varepsilon^{-1}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}, \tag{4.39}$$
$$|(4.28)| \lesssim \kappa^{1/2}\varepsilon\|(3.6)\|_{L^2_{t,x}}\|\kappa^{-1/2}\varepsilon^{-1}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}, \tag{4.40}$$
$$|(4.29)| \lesssim \|(3.7)\|_{L^2_{t,x}}\big\{\|P\partial_t f_R\|_{L^2_{t,x}} + \kappa^{1/2}\varepsilon\|\kappa^{-1/2}\varepsilon^{-1}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}\big\}. \tag{4.41}$$

Lastly we estimate (4.23) and the first term of (4.30). As in (4.19) we derive that (4.23) is bounded from below by

$$\begin{aligned}
&\frac{1}{2}|\varepsilon^{-\frac{1}{2}}(1-P_{\gamma_+})\partial_t f_R|^2_{L^2((0,T);L^2_{\gamma_+})} - \frac{1}{8C}|\varepsilon^{-\frac{1}{2}}P_{\gamma_+}\partial_t f_R|^2_{L^2((0,T);L^2_{\gamma_+})}\\
&\quad - C\Big\{\frac{\varepsilon}{\delta^2}|(1-P_{\gamma_+})\partial_t(\mathbf{I}-\mathbf{P})f_2|^2_{L^2((0,T);L^2_{\gamma_-})} + \varepsilon\|\partial_t u\|_\infty |f_R|^2_{L^2((0,T);L^2_{\gamma_+})}\\
&\quad + \|\partial_t u\|_\infty\frac{\varepsilon^3}{\delta}|(\mathbf{I}-\mathbf{P})f_2|^2_{L^2((0,T);L^2_{\gamma_+})}\Big\}\\
&\geq \frac{1}{2}|\varepsilon^{-\frac{1}{2}}(1-P_{\gamma_+})\partial_t f_R|^2_{L^2((0,T);L^2_{\gamma_+})} - \frac{1}{8C}|\varepsilon^{-\frac{1}{2}}P_{\gamma_+}\partial_t f_R|^2_{L^2((0,T);L^2_{\gamma_+})}\\
&\quad - C\Big\{\frac{\varepsilon\kappa^2}{\delta^2}||\partial_t\nabla_x u| + \varepsilon|\partial_t u||\nabla_x u||^2_{L^2_t L^2(\partial\Omega)} + \varepsilon\|\partial_t u\|_\infty|f_R|^2_{L^2((0,T);L^2_{\gamma_+})}\\
&\quad + \|\partial_t u\|_\infty\frac{\varepsilon^3}{\delta}\kappa^2|\nabla_x u|^2_{L^2_t L^2(\partial\Omega)}\Big\} \text{ for } C \gg 1,
\end{aligned} \tag{4.42}$$

where we have used $|r_{\gamma_+}(g)|_{L^2(\gamma_-)} \lesssim \varepsilon\|\partial_t u\|_\infty|g|_{L^2(\gamma_-)}$ from (3.9). Now we bound $P_{\gamma_+}\partial_t f_R$ using (4.6). Following the argument arriving at (4.20) and setting $h = |\partial_t f|^2$ we derive

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t\int_{\gamma_+}|\partial_t f_R|^2 \mathrm{d}\gamma\,\mathrm{d}s\\
&\quad\lesssim_N \|\partial_t f_R(0)\|_{L^2_{x,v}} + \|\partial_t f_R\|_{L^2_{t,x,v}}\\
&\qquad + \int_0^t\iint_{\Omega\times\mathbb{R}^3}\Big|\Big(-\frac{1}{\varepsilon^2\kappa}L\partial_t f_R + \text{r.h.s of (3.3)}\Big)\partial_t f_R\Big|\\
&\quad\lesssim_N \|\partial_t f_R(0)\|^2_{L^2_{x,v}} + \|P\partial_t f_R\|^2_{L^2_{t,x}} + \|\varepsilon^{-1}\kappa^{-1/2}\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|^2_{L^2_{t,x,v}}\\
&\qquad + (4.31) + \cdots + (4.35) + (4.37) + \cdots + (4.41).
\end{aligned} \tag{4.43}$$

We conclude (4.4) by collecting the terms. □

4.2 L^6_x-Integrability Gain for $\mathrm{P}f_R$

Proposition 8 *Under the same assumptions in Proposition 6, we have for all* $t \in [0, T]$

$$
\begin{aligned}
&d_6 \|Pf_R(t)\|_{L^6_x}\\
&\lesssim (\varepsilon\|(3.12)\|_{L^\infty_{t,x}} + \varepsilon\kappa^{-1}\|(3.10)\|_{L^\infty_{t,x}})\|f_R(t)\|_{L^2_{x,v}} + \varepsilon\|\partial_t f_R(t)\|_{L^2_{x,v}}\\
&\quad + o(1)(\kappa\varepsilon)^{1/2}\|\mathfrak{w} f_R(t)\|_{L^\infty_{x,v}} + \frac{\varepsilon}{\delta}|(3.10)|_{L^4(\partial\Omega)} + \varepsilon\|(3.4)\|_{L^2_{x,v}} + \varepsilon\|(3.5)\|_{L^2_{x,v}}\\
&\quad + \Big(\frac{1}{\varepsilon\kappa} + \frac{\delta}{\kappa}\|\mathfrak{w}_{\varrho,\ss} f_R(t)\|_{L^\infty_{x,v}}\Big)\Big\{\|(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}} + \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}\\
&\quad + \varepsilon\|\partial_t u\|_{L^\infty_{t,x}}\|Pf_R\|_{L^2_{t,x}}\Big\}\\
&\quad + \|\mathfrak{w}_{\varrho,\ss} f_R(t)\|^{1/2}_{L^\infty_{x,v}}\big\{|f_R|^{1/2}_{L^2_t L^2(\gamma_+)} + |\partial_t f_R|^{1/2}_{L^2_t L^2(\gamma_+)}\big\},
\end{aligned}
\tag{4.44}
$$

where

$$
d_6 := 1 - \Big[\frac{\delta}{\kappa}\|Pf_R(t)\|^{1/2}_{L^6_x}\|Pf_R(t)\|^{1/2}_{L^2_x} + \varepsilon\|u(t)\|_{L^\infty_x}\Big]^{1/6}. \tag{4.45}
$$

Proof For the sake of simplicity we use notations (4.1) throughout this subsection.
We view (3.2) as a weak formulation for a test function ψ

$$
\begin{aligned}
&\underbrace{\iint_{\Omega\times\mathbb{R}^3} f_R v\cdot\nabla_x\psi}_{(4.46)_1} - \underbrace{\int_\gamma f_R\psi}_{(4.46)_2} - \underbrace{\iint_{\Omega\times\mathbb{R}^3} \varepsilon\partial_t f_R\psi}_{(4.46)_3}\\
&= \iint_{\Omega\times\mathbb{R}^3}\psi\Big\{\frac{1}{\varepsilon\kappa}Lf_R - \frac{2\varepsilon}{\kappa}\Gamma(f_2, f_R) - \frac{\delta}{\kappa}\Gamma(f_R, f_R)\\
&\quad + \frac{(\varepsilon\partial_t + v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}} f_R - \varepsilon(\mathbf{I}-\mathbf{P})\mathfrak{R}_1 - \varepsilon\mathfrak{R}_2\Big\}.
\end{aligned}
\tag{4.46}
$$

The proof of the lemma is based on a recent test function method in the weak formulation ([12,13]). We define

$$
\tilde{\mathbf{P}} f_R := \Big\{a + b\cdot v + c\frac{|v|^2-3}{\sqrt{6}}\Big\}\sqrt{\mu_0} \text{ and } \tilde{P} f_R := (a, b, c), \tag{4.47}
$$

where $a := \langle f_R, \sqrt{\mu_0}\rangle$, $b := \langle f_R, v\sqrt{\mu_0}\rangle$, and $c := \langle f_R, \frac{|v|^2-3}{\sqrt{6}}\sqrt{\mu_0}\rangle$. We choose a family of test functions as

$$
\psi_a := (|v|^2 - \beta_a)v\sqrt{\mu_0}\cdot\nabla_x\varphi_a, \tag{4.48}
$$

$$
\psi^{i,j}_{b,1} := (v_i^2 - \beta_b)\sqrt{\mu_0}\partial_j\varphi^j_b, \quad i, j = 1, 2, 3, \tag{4.49}
$$

$$
\psi^{i,j}_{b,2} := |v|^2 v_i v_j\sqrt{\mu_0}\partial_j\varphi^i_b, \quad i \neq j, \tag{4.50}
$$

$$\psi_c := (|v|^2 - \beta_c) v \sqrt{\mu_0} \cdot \nabla_x \varphi_c, \tag{4.51}$$

where we choose $\beta_a = 10$, $\beta_b = 1$, $\beta_c = 5$ such that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} (|v|^2 - \beta_a) \frac{|v|^2 - 3}{\sqrt{6}} (v_1)^2 \mu_0(v) \mathrm{d}v = \int_{\mathbb{R}} (v_1^2 - \beta_b) \mu_0(v_1) \mathrm{d}v_1 \\ &= \int_{\mathbb{R}^3} (|v|^2 - \beta_c) v_i^2 \mu_0(v) \mathrm{d}v. \end{aligned} \tag{4.52}$$

Here,

$$-\Delta_x \varphi_a = a^5 \ \text{ with } \ \frac{\partial \varphi_a}{\partial n}\Big|_{\partial\Omega} = 0, \tag{4.53}$$

$$-\Delta_x \varphi_b^j = b_j^5 \ \text{ with } \ \varphi_b^j|_{\partial\Omega} = 0, \tag{4.54}$$

$$-\Delta_x \varphi_c = c^5 \ \text{ with } \ \varphi_c|_{\partial\Omega} = 0. \tag{4.55}$$

A unique solvability to the above Poisson equations when $(a, b, c) \in L^6(\Omega)$ and an estimate

$$\begin{aligned} &\|\nabla_x^2 \varphi_{(a,b,c)}\|_{L^{6/5}(\Omega)} + \|\nabla_x \varphi_{(a,b,c)}\|_{L^2(\Omega)} + \|\varphi_{(a,b,c)}\|_{L^6(\Omega)} \\ &\lesssim \||\tilde{P} f_R|^5\|_{L^{6/5}(\Omega)} \lesssim \|\tilde{P} f_R\|^5_{L^6(\Omega)}. \end{aligned} \tag{4.56}$$

is a direct consequence of Lax-Milgram and suitable extension (extend a^5 of (4.53) evenly in $x_3 \in \mathbb{R}$, and b^5 and c^5 of (4.54) and (4.55) oddly in $x_3 \in \mathbb{R}$, then solve the Poisson equation, and then restrict the whole space solutions to the half space $x_3 > 0$) and a standard elliptic estimate ($L^{\frac{6}{5}}(\Omega) \to \dot{W}^{2,\frac{6}{5}}(\Omega) \cap \dot{W}^{1,2}(\Omega) \cap L^6(\Omega)$).

From $M_{1,\varepsilon u,1}(v) = M_{1,0,1}(v) + O(\varepsilon)|u||v - \varepsilon u| M_{1,\varepsilon u,1}(v)$ we can easily check that

$$|\mathbf{P} f_R(t, x, v) - \tilde{\mathbf{P}} f_R(t, x, v)| \lesssim \varepsilon |u(t, x)||v - \varepsilon u| \sqrt{\mu} |f_R(t, x, v)|. \tag{4.57}$$

Therefore we have

$$\begin{aligned} \|P f_R(t)\|_{L^6_x} \lesssim& \|\mathbf{P} f_R(t)\|_{L^6_{x,v}} \lesssim \|\tilde{\mathbf{P}} f_R(t)\|_{L^6_{x,v}} \\ &+ \varepsilon \|u(t)\|_\infty \{\|P f_R(t)\|_{L^6_x} + \|(\mathbf{I} - \mathbf{P}) f_R(t)\|_{L^6_{x,v}}\} \\ \lesssim& (1 + \varepsilon \|u\|_\infty) \|\tilde{P} f_R(t)\|_{L^6_x} + \varepsilon \|u(t)\|_\infty \|(\mathbf{I} - \mathbf{P}) f_R(t)\|_{L^6_{x,v}}. \end{aligned} \tag{4.58}$$

Note that $\|(\mathbf{I} - \mathbf{P}) f_R(t)\|_{L^6_{x,v}} \leq \|(\mathbf{I} - \mathbf{P}) f_R(t)\|^{2/3}_{L^\infty_{x,v}} \|(\mathbf{I} - \mathbf{P}) f_R(t)\|^{1/3}_{L^2_{x,v}} \lesssim o(1)(\kappa\varepsilon)^{1/2} \|\mathfrak{w} f_R(t)\|_{L^\infty_{x,v}} + (\kappa\varepsilon)^{-1} \|(\mathbf{I} - \mathbf{P}) f_R(t)\|_{L^2_{x,v}}$. Hence to prove the lemma and (4.44) it suffices to prove the same bound for $\|\tilde{P} f_R\|_{L^6_{x,v}} := \|(a, b, c)\|_{L^6_x}$.

Following the direct computations in the proof of Lemma 2.12 in [13] we derive that

$$
(4.46)_1 = \begin{cases} -5\|a(t)\|_6^6 + o(1)\|\tilde{\mathbf{P}} f_R(t)\|_6^6 + O(1)\|(\mathbf{I}-\mathbf{P}) f_R(t)\|_6^6 & \text{if } \psi = \psi_a, \\ -2\int_\Omega b_i \partial_i \partial_j \varphi_b^j + o(1)\|\tilde{\mathbf{P}} f_R(t)\|_6^6 + O(1)\|(\mathbf{I}-\mathbf{P}) f_R(t)\|_6^6 & \text{if } \psi = \psi_{b,1}^{i,j}, \\ \int_\Omega b_j \partial_i \partial_j \varphi_b^i + \int_\Omega b_i \partial_j \partial_j \varphi_b^i + O(1)\|(\mathbf{I}-\mathbf{P}) f_R(t)\|_6^6 & \text{if } \psi = \psi_{b,2}^{i,j} \text{ and } i \neq j, \\ 5\|c(t)\|_6^6 + o(1)\|\tilde{\mathbf{P}} f_R(t)\|_6^6 + O(1)\|(\mathbf{I}-\mathbf{P}) f_R(t)\|_6^6 & \text{if } \psi = \psi_c. \end{cases} \tag{4.59}
$$

For $\|b_i\|_6^6$, using the second and third estimate of (4.59) we deduce that

$$
\begin{aligned}
\|b_i\|_{L^6(\Omega)}^6 &= -\int_\Omega b_i \Delta_x \varphi_b^i \mathrm{d}x = -\int_\Omega b_i \partial_i^2 \varphi_b^i \mathrm{d}x - \sum_{j(\neq i)} \int_\Omega b_i \partial_j^2 \varphi_b^i \mathrm{d}x \\
&= \frac{1}{2}\sum_j (4.46)_1|_{\psi_{b,1}^{j,i}} - \sum_{j(\neq i)} (4.46)_1|_{\psi_{b,2}^{i,j}} + o(1)\|\tilde{\mathbf{P}} f_R(t)\|_6^6 \\
&\quad + O(1)\|(\mathbf{I}-\mathbf{P}) f_R(t)\|_6^6.
\end{aligned} \tag{4.60}
$$

Now we consider the boundary term $(4.46)_2$. From (4.48)-(4.51) and (4.52)

$$
\int_\gamma \psi P_{\gamma_+} f_R = \begin{cases} \int_{\partial\Omega} \partial_n \varphi_a \int_{\mathbb{R}^3} (|v|^2 - \beta_a)(v\cdot n)^2 \mu_0 \mathrm{d}v \mathrm{d}S_x = 0 & \text{if } \psi = \psi_a, \\ 0 & \text{if } \psi = \psi_{b,1}^{i,j} \text{ or } \psi_{b,2}^{i,j}, \\ \int_{\partial\Omega} \partial_n \varphi_c \int_{\mathbb{R}^3} (|v|^2 - \beta_c)(v\cdot n)^2 \mu_0 \mathrm{d}v \mathrm{d}S_x = 0 & \text{if } \psi = \psi_c. \end{cases} \tag{4.61}
$$

Here we have used the Neumann boundary condition of (4.53) for ψ_a, and the last identity in (4.52) for ψ_c. For $\psi_{b,1}^{i,j}$ or $\psi_{b,2}^{i,j}$ we used the fact that the integrands are odd in v. From (3.8), we decompose $f|_\gamma = P_{\gamma_+} f + \mathbf{1}_{\gamma_+}(1-P_{\gamma_+})f - \mathbf{1}_{\gamma_-}\frac{\varepsilon}{\delta}(1-P_{\gamma_+})f_2$. From (4.61) together with (3.57) and (3.58) we have

$$
\begin{aligned}
|(4.46)_2| &= \Big| \cancel{\int_\gamma \psi P_{\gamma_+} f_R} + \int_\gamma \psi \{\mathbf{1}_{\gamma_+}(1-P_{\gamma_+}) f_R - \mathbf{1}_{\gamma_-}\frac{\varepsilon}{\delta}(1-P_{\gamma_+}) f_2\} \Big| \\
&\lesssim |\nabla_x \varphi|_{L^{4/3}(\partial\Omega)} \big\{ |(1-P_{\gamma_+}) f_R|_{4,\gamma_+} + \frac{\varepsilon}{\delta} |(3.10)|_{L^4(\partial\Omega)} \big\}
\end{aligned} \tag{4.62}
$$

where we have used $|\int_{\gamma_+} \psi (1-P_{\gamma_+}) f| \lesssim |\nabla_x \varphi|_{L^{4/3}(\partial\Omega)} |(1-P_{\gamma_+}) f|_{4,\gamma_+}$ at the last line. Here $\varphi \in \{\varphi_a, \varphi_b, \varphi_c\}$. For the first term of (4.62) we interpolate

$$
|(1-P_{\gamma_+}) f_R|_{4,\gamma_+} \lesssim |\varepsilon^{-\frac{1}{2}} (1-P_{\gamma_+}) f_R|_{2,\gamma_+}^{1/2} \varepsilon^{\frac{1}{4}} \|\mathfrak{w}_{\varrho,\ss} f_R\|_\infty^{1/2}. \tag{4.63}
$$

For the second term of (4.62), we use (4.56) and a trace theorem ($\dot{W}^{1,\frac{6}{5}}(\mathbb{T}^2 \times \mathbb{R}_+) \cap L^2(\mathbb{T}^2 \times \mathbb{R}_+) \to W^{1-\frac{1}{6/5},\frac{6}{5}}(\mathbb{T}^2)$), and the Sobolev embedding ($W^{\frac{1}{6},\frac{6}{5}}(\mathbb{T}^2) \to$

$L^{4/3}(\mathbb{T}^2)$) to conclude that

$$|\nabla_x\varphi|_{L^{\frac{4}{3}}(\mathbb{T}^2)} \lesssim |\nabla_x\varphi|_{W^{\frac{1}{6},\frac{6}{5}}(\mathbb{T}^2)} \lesssim \|\nabla_x\varphi\|_{\dot{W}^{1,\frac{6}{5}}(\mathbb{T}^2\times\mathbb{R}_+)\cap L^2(\Omega)} \lesssim \|\tilde{P}f_R\|^5_{L^6(\mathbb{T}^2\times\mathbb{R}_+)}. \tag{4.64}$$

Next we consider $(4.46)_3$. For ψ of (4.48)-(4.51) and φ of (4.53)-(4.55), using (4.56), it follows that

$$\begin{aligned}|(4.46)_3| &\lesssim \varepsilon\|\partial_t f_R\|_{L^2_{x,v}}\|\psi\|_{L^2_{x,v}} \lesssim \varepsilon\|\partial_t f_R\|_{L^2_{x,v}}\|\nabla_x\varphi\|_{L^2_x} \lesssim \varepsilon\|\partial_t f_R\|_{L^2_{x,v}}\|\tilde{P}f_R\|^5_{L^6_x}\\ &\le O(1)[\varepsilon\|\partial_t f_R\|_{L^2_{x,v}}]^6 + o(1)\|\tilde{P}f_R\|^6_{L^6_x}.\end{aligned} \tag{4.65}$$

Lastly we consider the right hand side of (4.46). From (1.21), (3.23), (3.37), and (4.56), it follows

$$\begin{aligned}&\Big|\iint_{\Omega\times\mathbb{R}^3}\psi\frac{1}{\varepsilon\kappa}Lf_R\Big| = \Big|\iint_{\Omega\times\mathbb{R}^3}\psi\frac{1}{\varepsilon\kappa}L(\mathbf{I}-\mathbf{P})f_R\Big|\\ &\lesssim \frac{1}{\varepsilon\kappa}\int_\Omega\int_{\mathbb{R}^3}|\nabla_x\varphi_{(a,b,c)}(x)|\mu(v)^{1/4}\Big[\nu(v)|(\mathbf{I}-\mathbf{P})f_R(x,v)|\\ &\quad+\int_{\mathbb{R}^3}k_\vartheta(v,v_*)|(\mathbf{I}-\mathbf{P})f_R(x,v_*)|\mathrm{d}v_*\Big]\mathrm{d}v\mathrm{d}x\\ &\lesssim \frac{1}{\varepsilon\kappa}\|\nabla_x\varphi_{(a,b,c)}\|_{L^2_x}\|(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{x,v}} \lesssim \frac{1}{\varepsilon\kappa}\|\tilde{P}f\|^5_{L^6_x}\|(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{x,v}}\\ &\le o(1)\|\tilde{P}f_R\|^6_{L^6_x} + \big[\varepsilon^{-1}\kappa^{-1}\|(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{x,v}}\big]^6.\end{aligned} \tag{4.66}$$

Note that, from (3.37), $|\Gamma(\frac{\varepsilon}{\kappa}f_2, f_R)| \lesssim \frac{\varepsilon}{\kappa}\|\mathfrak{w}_{\varrho,\beta}f_2\|_\infty\mathfrak{w}_{\varrho,\beta}(v)^{-1}\big[\nu(v)f_R(v) + \int_{\mathbb{R}^3}k_\vartheta(v,v_*)f_R(v_*)\mathrm{d}v_*\big]$. Then from (3.57) and (3.58)

$$\begin{aligned}\Big|\iint_{\Omega\times\mathbb{R}^3}\psi\frac{\varepsilon}{\kappa}\Gamma(f_2,f_R)\Big| &\lesssim \|\nabla_x\varphi_{(a,b,c)}\|_{L^2_x}\frac{\varepsilon}{\kappa}\|(3.10)\|_\infty\|f_R\|_{L^2_{x,v}}\\ &\le o(1)\|\tilde{P}f_R\|^6_{L^6_x} + \big[\varepsilon\kappa^{-1}\|(3.10)\|_\infty\|f_R\|_{L^2_{x,v}}\big]^6.\end{aligned} \tag{4.67}$$

For the contribution of $\Gamma(f_R, f_R)$ we decompose $f_R = \mathbf{P}f_R + (\mathbf{I}-\mathbf{P})f_R$. From (3.37) (or (3.36))

$$\begin{aligned}&|\Gamma(f_R,f_R)(v)|\\ &\lesssim |\Gamma(\mathbf{P}f_R,\mathbf{P}f_R)(v)| + |\Gamma((\mathbf{I}-\mathbf{P})f_R,(\mathbf{I}-\mathbf{P})f_R)(v)|\\ &\lesssim \nu(v)|Pf_R|^2 + \|\mathfrak{w}_{\varrho,\beta}f_R\|_\infty\Big\{\nu(v)|(\mathbf{I}-\mathbf{P})f_R)(v)|\\ &\quad+\int_{\mathbb{R}^3}k_\vartheta(v,v_*)|(\mathbf{I}-\mathbf{P})f_R)(v_*)|\mathrm{d}v_*\Big\}.\end{aligned} \tag{4.68}$$

Then from (4.48)-(4.51), (3.23), and the Hölder's inequality ($1 = 1/2 + 1/3 + 1/6$)

$$
\begin{aligned}
&\Big| \iint_{\Omega\times\mathbb{R}^3} \psi \frac{\delta}{\kappa}\Gamma(f_R, f_R)\Big| \\
&\lesssim \frac{\delta}{\kappa}\|\nabla_x \varphi_{(a,b,c)}\|_{L^2_x}\Big\{\|Pf_R\|_{L^3_x}\|Pf_R\|_{L^6_x} + \|\mathfrak{w}_{\varrho,\beta} f_R\|_{L^\infty_{x,v}}\|(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{x,v}}\Big\} \\
&\lesssim \frac{\delta}{\kappa}\|\tilde{P}f_R\|^5_{L^6_x}\|Pf_R\|^{3/2}_{L^6_x}\|Pf_R\|^{1/2}_{L^2_x} \\
&\quad + \frac{\varepsilon\delta}{\kappa^{1/2}}\|\tilde{P}f_R\|^5_{L^6_x}\|\mathfrak{w}_{\varrho,\beta} f_R\|_{L^\infty_{x,v}}\|\varepsilon^{-1}\kappa^{-1/2}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{x,v}},
\end{aligned}
\tag{4.69}
$$

where we have used an interpolation $\|Pf_R\|_{L^3} \le \|Pf_R\|^{1/2}_{L^6}\|Pf_R\|^{1/2}_{L^2}$ and (4.56) at the last step. A contribution of the rest of terms in the r.h.s of (4.46) can be easily bounded as, from (3.4) and (3.5),

$$
\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} |\psi| \left|\frac{(\varepsilon\partial_t + v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}} f_R - \varepsilon(\mathbf{I}-\mathbf{P})\mathfrak{R}_1 - \varepsilon\mathfrak{R}_2\right| \\
&\quad\lesssim \|Pf_R\|^5_{L^6_x}\Big\{\varepsilon\|(3.12)\|_\infty\|f_R\|_{L^2_{x,v}} + \varepsilon\|(3.4)+(3.5)\|_{L^2_{x,v}}\Big\}.
\end{aligned}
\tag{4.70}
$$

In conclusion, collecting the terms from (4.59) with (4.60), (4.62) with (4.63) and (4.64), (4.65), (4.66), (4.67), (4.69), (4.70), and utilizing (4.58), and two facts from (A.1):

$$
\begin{aligned}
\sup_{0\le s\le t}\|(\mathbf{I}-\mathbf{P})f_R(s)\|_{L^2_{x,v}} &\lesssim \|(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}} + \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \\
&\quad + \varepsilon\|\partial_t u\|_{L^\infty_{t,x}}\|Pf_R\|_{L^2_{t,x}}, \\
\sup_{0\le s\le t}|(1-P_{\gamma_+})f_R(s)|_{L^2(\gamma_+)} &\lesssim \sup_{0\le s\le t}|f_R(s)|_{L^2(\gamma_+)} \\
&\lesssim |f_R|_{L^2_tL^2(\gamma_+)} + |\partial_t f_R|_{L^2_tL^2(\gamma_+)},
\end{aligned}
\tag{4.71}
$$

we prove (4.44). □

4.3 Average in Velocity

We prove a version of velocity lemma when a suitable bound for source terms is only known in a finite time interval. In this section we often specify domains in which an L^p-norm is taken while the simplified notation (4.1) will be used only when the domain is $[0, T]\times\Omega\times\mathbb{R}^3$.

Proposition 9 *Assume the same assumptions in Proposition 6. Then we have, for* $2 < p < 3$,

$$\begin{aligned} d_3 \big\| P f_R \big\|_{L^2_t L^p_x} &\lesssim (1 + \varepsilon \|(3.12)\|_{L^2_t L^\infty_x}) \| f_R \|_{L^\infty_t L^2_{x,v}} \\ &+ \left\{ \frac{1}{\varepsilon\kappa} + \frac{\delta}{\kappa} \|\mathfrak{w}_{\varrho,\beta} f_R\|_{L^\infty_{t,x,v}} + \|\mathfrak{w}_{\varrho,\beta} f_R\|^{\frac{p-2}{p}}_{L^\infty_{t,x,v}} \right\} \|\sqrt{\nu}(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}} \\ &+ \| f_R(0)\|_{L^2_\gamma} + \varepsilon \|(3.6)\|_{L^2_{t,x}} + \varepsilon \|(3.7)\|_{L^2_{t,x}}, \end{aligned} \tag{4.72}$$

with

$$d_3 := 1 - O(\varepsilon)\|u\|_{L^\infty_{t,x}} - \frac{\varepsilon}{\kappa} \|(3.10)\|_{L^\infty_t L_x^{\frac{2p}{p-2}}} - \frac{\delta}{\kappa} \|P f_R\|^{\frac{3(p-2)}{p}}_{L^\infty_t L^6_x} \|\mathfrak{w}_{\varrho,\beta} f_R\|^{\frac{6-2p}{p}}_{L^\infty_{t,x,v}}, \tag{4.73}$$

and for $\varrho' < \varrho$

$$\begin{aligned} d_{3,t} \big\| P \partial_t f_R \big\|_{L^2_t L^p_x} &\lesssim \frac{1}{\kappa} \|\partial_t u\|_{L^\infty_{t,x}} \big(1 + \varepsilon^2 \|(3.10)\|_{L^\infty_{t,x}}\big) \|P f_R\|_{L^2_{t,x}} \\ &+ \frac{\delta\varepsilon}{\kappa} \|\partial_t u\|_{L^\infty_{t,x}} \|P f_R\|^{\frac{3(p-2)}{p}}_{L^\infty_t L^6_x} \|\mathfrak{w} f_R\|^{\frac{6-2p}{p}}_{L^\infty_{t,x,v}} \|P f_R\|_{L^2_t L^p_x} \\ &+ \frac{\varepsilon}{\kappa} \|\partial_t u\|_{L^\infty_{t,x}} (\delta \|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}} + \|(3.10)\|_{L^\infty_{t,x}}) \|\sqrt{\nu}(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}} \\ &+ (\kappa\varepsilon)^{\frac{2}{p-2}} \|\mathfrak{w}_{\varrho',\beta} \partial_t f_R\|_{L^2_t L^\infty_{x,v}} + \|\partial_t f_R\|_{L^\infty_t L^2_{x,v}} + \varepsilon \|(3.13)\|_{L^2_t L^\infty_x} \|f_R\|_{L^\infty_t L^2_{x,v}} \\ &+ \left\{ \frac{1}{\kappa\varepsilon} + \frac{\delta}{\kappa} \|\mathfrak{w}_{\varrho,\beta} f_R\|_{L^\infty_{t,x,v}} + \frac{\varepsilon}{\kappa} \|(3.10)\|_{L^\infty_{t,x}} \right. \\ &\left. + \varepsilon \|(3.12)\|_{L^\infty_{t,x}} \right\} \|\sqrt{\nu}(\mathbf{I}-\mathbf{P}) \partial_t f_R\|_{L^2_{t,x,v}} \\ &+ \|\partial_t f_R(0)\|_{L^2_\gamma} + \frac{\varepsilon}{\kappa} \|(3.11)\|_{L^2_{t,x,v}} \|\mathfrak{w}_{\varrho,\beta} f_R\|_{L^\infty_{t,x,v}} + \varepsilon \|(3.6)\|_{L^2_{t,x}} + \varepsilon \|(3.7)\|_{L^2_{t,x}}, \end{aligned} \tag{4.74}$$

with

$$\begin{aligned} d_{3,t} := 1 - O(\varepsilon)\|u\|_{L^\infty_{t,x}} - \frac{\varepsilon}{\kappa} \|(3.10)\|_{L^\infty_t L_x^{\frac{2p}{p-2}}} - \varepsilon \|(3.12)\|_{L^\infty_t L_x^{\frac{2p}{p-2}}} \\ - \frac{\delta}{\kappa} \|P f_R\|^{\frac{3(p-2)}{p}}_{L^\infty_t L^6_x} \|\mathfrak{w}_{\varrho,\beta} f_R\|^{\frac{6-2p}{p}}_{L^\infty_{t,x,v}}, \end{aligned} \tag{4.75}$$

where both bounds are uniform-in-p for $2 < p < 3$.

We prove the proposition by several steps.

Step 1: Extension. We define a subset

$$\tilde{\Omega} := (0, 2\pi) \times (0, 2\pi) \times (0, \infty) \subset \mathbb{R}^3. \tag{4.76}$$

We regard $\tilde{\Omega}$ as an open subset but not a periodic domain as Ω. Without loss of generality we may assume that $f_R(0, x, v)$ is defined in $\mathbb{R}^3 \times \mathbb{R}^3$ and $\| f_R(0)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} \lesssim \| f_R(0)\|_{L^p(\tilde{\Omega}\times\mathbb{R}^3)}$ for all $1 \le p \le \infty$. Then we extend a solution for whole time $t \in \mathbb{R}$ as

$$f_I(t, x, v) := \mathbf{1}_{t\ge0} f_R(t, x, v) + \mathbf{1}_{t\le0}\chi_1(t) f_R(0, x, v), \tag{4.77}$$

where a smooth non-negative function χ_1 satisfies $\chi_1(t) \equiv 1$ for $t \in [-1, 0]$, $\chi_1(t) \equiv 0$ for $t < -2$, and $0 \le \frac{d}{dt}\chi_1 \le 4$.

A closure of $\tilde{\Omega}$ is given as $cl(\tilde{\Omega}) = [0, 2\pi] \times [0, 2\pi] \times [0, \infty)$. Let us define $\tilde{t}_B(x, v) \in \mathbb{R}$ for $(x, v) \in (\mathbb{R}^3\backslash\tilde{\Omega}) \times \mathbb{R}^3$. We consider $\tilde{B}(x, v) := \{s \in \mathbb{R} : x + sv \in \mathbb{R}^3\backslash cl(\tilde{\Omega})\}$. Clearly if $\tilde{B}(x, v) \neq \emptyset$ then $\{s > 0\} \subset \tilde{B}(x, v)$ or $\{s < 0\} \subset \tilde{B}(x, v)$ exclusively. If $\{s > 0\} \subset \tilde{B}(x, v)$, let I_+ be the largest interval such that $\{s > 0\} \subset I_+ \subset \tilde{B}(x, v)$. And if $\{s < 0\} \subset \tilde{B}(x, v)$, let I_- be the largest interval such that $\{s > 0\} \subset I_- \subset \tilde{B}(x, v)$.

We define

$$\tilde{t}_B(x, v) = \begin{cases} 0 & \text{if } x \in \partial\tilde{\Omega}, \\ \inf \tilde{B}(x, v) & \text{if } x \in \mathbb{R}^3\backslash cl(\tilde{\Omega}) \text{ and } \tilde{B}(x, v) \neq \emptyset \text{ and } \{s > 0\} \subset I_+ \subset \tilde{B}(x, v), \\ \sup \tilde{B}(x, v) & \text{if } x \in \mathbb{R}^3\backslash cl(\tilde{\Omega}) \text{ and } \tilde{B}(x, v) \neq \emptyset \text{ and } \{s < 0\} \subset I_- \subset \tilde{B}(x, v), \\ -\infty & \text{if } \tilde{B}(x, v) = \emptyset \text{ and } x \notin \partial\tilde{\Omega}. \end{cases} \tag{4.78}$$

Using (4.78) we define

$$\begin{aligned} f_E(t, x, v) &:= \mathbf{1}_{(x,v)\in(\mathbb{R}^3\backslash\tilde{\Omega})\times\mathbb{R}^3} f_I(t + \varepsilon\tilde{t}_B(x, v), \tilde{x}_B(x, v), v) \\ &\text{with } \tilde{x}_B(x, v) := x + \tilde{t}_B(x, v)v. \end{aligned} \tag{4.79}$$

It is easy to see that $\varepsilon\partial_t f_E + v \cdot \nabla_x f_E = 0$ in the sense of distributions.

Next we define two cutoff functions. For any $N > 0$ we define smooth non-negative functions as

$$\begin{aligned} &\chi_2(x) \equiv 1 \text{ for } x \in [-\pi, 3\pi] \times [-\pi, 3\pi] \times [-\pi, \infty), \\ &\chi_2(x) \equiv 0 \text{ for } x \notin [-2\pi, 4\pi] \times [-2\pi, 4\pi] \times [-2\pi, \infty), \quad |\nabla_x\chi_2| \le 10, \end{aligned} \tag{4.80}$$

$$\begin{aligned} &\chi_3(v) \equiv 1 \text{ for } |v| \le N - 1, \text{ and } |v_i| \ge 2/N \text{ for all } i = 1, 2, 3, \\ &\chi_3(v) \equiv 0 \text{ for } |v| \ge N \text{ or } |v_i| \le 1/N \text{ for any } i = 1, 2, 3, \quad |\nabla_v\chi_3| \le 10. \end{aligned} \tag{4.81}$$

We denote

$$U := [-2\pi, 4\pi] \times [-2\pi, 4\pi] \times [-2\pi, \infty),$$

$$V := \{v \in \mathbb{R}^3 : |v| \leq N\} \cap \bigcap_{i=1,2,3} \{v \in \mathbb{R}^3 : |v_i| \geq 1/N\} \tag{4.82}$$

We define an extension of cut-offed solutions

$$\begin{aligned} \bar{f}_R(t,x,v) &:= \chi_2(x)\chi_3(v)\big\{\mathbf{1}_{\tilde{\Omega}}(x) f_I(t,x,v) + f_E(t,x,v)\big\} \\ &\text{for } (t,x,v) \in (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned} \tag{4.83}$$

We note that in the sense of distributions $\bar{f}_R$ solves

$$\begin{aligned} &\varepsilon \partial_t \bar{f}_R + v \cdot \nabla_x \bar{f}_R = \bar{g} \ \text{ in } (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3, \\ &\bar{g} := \frac{v \cdot \nabla_x \chi_2}{\chi_2} \bar{f}_R + \mathbf{1}_{t\geq 0} \mathbf{1}_{\tilde{\Omega}}(x)\chi_2(x)\chi_3(v)[\varepsilon\partial_t + v\cdot\nabla_x] f_R \\ &\qquad + \mathbf{1}_{t\leq 0}\{\varepsilon\partial_t \chi_1(t) f_R(0,x,v) + \chi_1(t) v \cdot \nabla_x f_R(0,x,v)\} \end{aligned} \tag{4.84}$$

Here we have used the fact that $\bar{f}_R$ in (4.84) is continuous along the characteristics across $\partial\tilde{\Omega}$ and $\{t = 0\}$. We derive that, using (4.84),

$$\bar{f}_R(t,x,v) = \frac{1}{\varepsilon}\int_{-\infty}^{t} \bar{g}(s, x - \frac{t-s}{\varepsilon} v, v)\mathrm{d}s \ \text{ for } (t,x,v) \in (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3. \tag{4.85}$$

Recall $\tilde{\varphi}_i \in \{\tilde{\varphi}_0, \cdots \tilde{\varphi}_4\}$ in (4.47). From (4.83) we note that

$$\begin{aligned} &\left\| \int_{\mathbb{R}^3} \bar{f}_R(t,x,v)\tilde{\varphi}_i(v)\sqrt{\mu_0(v)}\mathrm{d}v \right\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))} \\ &= \left\| \int_{\mathbb{R}^3} \chi_2(x)\chi_3(v) f_R(t,x,v)\tilde{\varphi}_i(v)\sqrt{\mu_0(v)}\mathrm{d}v \right\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))} \end{aligned} \tag{4.86}$$

From (1.24), we decompose

$$\begin{aligned} (4.86) \geq &\left\| \sum_j \chi_2(x)\tilde{P}_j f_R(t,x) \int_{\mathbb{R}^3} \chi_3(v)\tilde{\varphi}_j(v)\tilde{\varphi}_i(v)\mu_0(v)\mathrm{d}v \right\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))} \\ &- \left\| \int_{\mathbb{R}^3} \chi_3(v)(\mathbf{I} - \tilde{\mathbf{P}}) f_R(t,x,v)\tilde{\varphi}_i(v)\sqrt{\mu_0(v)}\mathrm{d}v \right\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))}. \end{aligned}$$

We consider the right hand side of above terms. From (4.57), $\int \tilde{\varphi}_i\tilde{\varphi}_j\mu_0 = \delta_{ij}$, and (4.81), the first term can be bounded below by $\big(1 - O(\varepsilon)\|u\|_\infty - O(\frac{1}{N})\big)\|\chi_2 P f_R\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))}$. For the second term we use (4.57), $L^2_t(0,T) \subset L^p_t(0,T)$, and $L^1(\{|v| \leq N\}) \subset L^p(\{|v| \leq N\})$ to bound it above by $C_{T,N}\|(\mathbf{I} -$

$\mathbf{P}) f_R\|_{L^p((0,T)\times\tilde{\Omega}\times\mathbb{R}^3)}+\big(O(\varepsilon)\|u\|_\infty+O(\frac{1}{N})\big)\big\|Pf_R\big\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))}$. Hence we derive

$$
\begin{aligned}
&(4.86)\\
&\geq \big(1-O(\varepsilon)\|u\|_\infty - O(\frac{1}{N})\big)\big\|Pf_R\big\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))}\\
&\quad - C_{T,N}\|(\mathbf{I}-\mathbf{P})f_R\|_{L^p((0,T)\times\tilde{\Omega}\times\mathbb{R}^3)}\\
&\geq \big(1-O(\varepsilon)\|u\|_\infty - O(\frac{1}{N})\big)\big\|Pf_R\big\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))}\\
&\quad - C_{T,N}\|\mathfrak{w}_{\varrho,\beta} f_R(t)\|^{\frac{p-2}{p}}_{L^\infty((0,T)\times\tilde{\Omega}\times\mathbb{R}^3)}\|(\mathbf{I}-\mathbf{P})f_R\|^{\frac{2}{p}}_{L^2((0,T)\times\tilde{\Omega}\times\mathbb{R}^3)}.
\end{aligned}
\tag{4.87}
$$

Step 2: Average lemma. Recall $\tilde{\varphi}_i\in\{\tilde{\varphi}_0,\cdots\tilde{\varphi}_4\}$ in (4.47). We choose $\tilde{\varphi}(v)$ such that

$$
\begin{aligned}
&\chi_3(v)|\tilde{\varphi}_i(v)|\sqrt{\mu_0(v)}\leq\tilde{\varphi}(v),\quad \tilde{\varphi}(v)\in C_c^\infty(\mathbb{R}^3)\\
&\text{and }\ \tilde{\varphi}(v)\equiv 0\ \text{ for }\ |v|\geq N\ \text{ or }\ |v_i|\leq 1/N \text{ for any } i=1,2,3.
\end{aligned}
\tag{4.88}
$$

Lemma 6 *We define*

$$
S(\bar{g})(t,x):=\frac{1}{\varepsilon}\int_{-\infty}^t\int_{\mathbb{R}^3}|\bar{g}(s,x-\frac{t-s}{\varepsilon}v,v)|\tilde{\varphi}(v)\mathrm{d}v\mathrm{d}s\ \textit{ for } (t,x)\in(-\infty,T]\times\mathbb{R}^3.
\tag{4.89}
$$

Then, for $p<3$ and $1\ll N$,

$$
\|S(\bar{g})\|_{L^2_t((0,T);L^p_x(\mathbb{T}^2\times\mathbb{R}))}\lesssim_N\|\mathbf{1}_{(t,x,v)\in\mathfrak{D}_T}\bar{g}\|_{L^2((0,T)\times(\mathbb{T}^2\times\mathbb{R})\times\{|v|\leq N\})},
\tag{4.90}
$$

where the bound (4.90) only depends on N but can be independent on $p<3$.

We remark that from (4.85) and (4.89) $\int_{\mathbb{R}^3}\bar{f}_R(t,x,v)\tilde{\varphi}_i(v)\mathrm{d}v\leq S(\bar{g})(t,x)$.

Proof of Lemma 6 We prove (4.90) by a TT^*(SS^* for our case) method. First we derive a dual of S in the following equalities:

$$
\begin{aligned}
&\int_{-\infty}^T\int_{\mathbb{R}^3}S(\bar{g})(t,x)h(t,x)\mathrm{d}x\mathrm{d}t\\
&=\int_{-\infty}^T\int_{\mathbb{R}^3}\frac{1}{\varepsilon}\int_{-\infty}^t\int_{\mathbb{R}^3}|\bar{g}(s,x-\frac{t-s}{\varepsilon}v,v)|\tilde{\varphi}(v)h(t,x)\mathrm{d}v\mathrm{d}s\mathrm{d}x\mathrm{d}t\\
&=\int_{-\infty}^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\bar{g}(s,x,v)|\left[\frac{1}{\varepsilon}\int_s^T h(t,x+\frac{t-s}{\varepsilon}v)\tilde{\varphi}(v)\mathrm{d}t\right]\mathrm{d}v\mathrm{d}x\mathrm{d}s\\
&=\int_{-\infty}^T\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\bar{g}(t,x,v)|\left[\frac{1}{\varepsilon}\int_t^T h(s,x+\frac{s-t}{\varepsilon}v)\tilde{\varphi}(v)\mathrm{d}s\right]\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&=\int_{-\infty}^T\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\bar{g}(t,x,v)|S^*(h)(t,x,v)\mathrm{d}v\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{4.91}
$$

where we have defined

$$S^*(h)(t,x,v) := \frac{1}{\varepsilon}\int_t^T h(\tau, x + \frac{\tau - t}{\varepsilon}v)\tilde{\varphi}(v)\mathrm{d}\tau. \tag{4.92}$$

Here, in the second equality of (4.91) we have used the Fubini theorem for changing order of s and t integrations, and then used a change of variables $x \mapsto x - \frac{t-s}{\varepsilon}v$. In the third equality of (4.91) we have used a change of variable $(t,s) \mapsto (s,t)$ and the fact $\mathrm{Supp}(g) \subset (-\infty, T] \times U \times V$.

On the other hand, for $1/p + 1/q = 1$, following the argument of (4.91) with $h(t,x) = \mathbf{1}_{x\in\tilde{\Omega}} h(t,x)$ we derive that

$$\begin{aligned}
&\|S(\bar{g})\|_{L^2_t((-1,T];L^p_x(\tilde{\Omega}))}\\
&= \sup_{\|h\|_{L^2_t((-1,T];L^q_x(\tilde{\Omega}))}\le 1} \int_{-1}^T \int_{\tilde{\Omega}} S(\bar{g})(t,x)h(t,x)\mathrm{d}x\mathrm{d}t\\
&= \sup_{\|h\|_{L^2_t((-1,T];L^q_x(\tilde{\Omega}))}\le 1} \int_{-1}^T \iint_{U\times V} |\bar{g}(t,x,v)|S^*(h)(t,x,v)\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.93}$$

It is important to check the integral region in space of the last term of (4.93). From (4.92), we note that if $x + \frac{\tau - t}{\varepsilon}v \notin cl(\tilde{\Omega})$ for all $\tau \in [t,T]$ then the last term would vanish since $\mathrm{supp}(h) \subset (-\infty, T] \times \tilde{\Omega}$. Therefore we can exclude (t,x,v) from the last integration in (4.93) if $L(t,x,v) \cap \tilde{\Omega} = \emptyset$ for $L(t,x,v) := \{x + \frac{\tau - t}{\varepsilon}v : \tau \in [t,T]\}$. Now we define

$$\mathfrak{D}_T := \Big\{(t,x,v) \in (-1,T] \times U \times V : L(t,x,v) \cap \tilde{\Omega} \neq \emptyset\Big\}. \tag{4.94}$$

Then we can write

$$\begin{aligned}
(4.93) &= \sup_{\|h\|_{L^2_t((-1,T];L^q_x(\tilde{\Omega}))}\le 1} \int_{-1}^T \iint_{U\times V} \mathbf{1}_{(t,x,v)\in\mathfrak{D}_T}|\bar{g}(t,x,v)|S^*(h)(t,x,v)\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\le \|\mathbf{1}_{(t,x,v)\in\mathfrak{D}_T}\bar{g}\|_{L^2((-1,T]\times U\times V)}\\
&\quad \sup_{\|h\|_{L^2_t((-1,T];L^q_x(\tilde{\Omega}))}\le 1} \big\|S^*(h)(t,x,v)\big\|_{L^2((-1,T]\times U\times V)}.
\end{aligned} \tag{4.95}$$

Therefore to prove (4.90) it suffices to show that

$$\|S^*(h)\|_{L^2((-1,T]\times U\times V)} \lesssim \|h\|_{L^2_t((-1,T];L^q_x(\tilde{\Omega}))}. \tag{4.96}$$

Note that since $\mathrm{supp}(h) \subset (-1,T] \times U$ and $\mathrm{supp}(\tilde{\varphi}) = V$ for $(x,v) \in U \times V$, we have, with $x = (x_1,x_2,x_3)$, $v = (v_1,v_2,v_3)$

$$|x_1 + \frac{\tau - t}{\varepsilon}v_1| \ge \frac{|\tau - t|}{\varepsilon}|v_1| - |x_1| \ge \frac{10\pi N\varepsilon}{\varepsilon}\frac{1}{N} - 4\pi > 4\pi \quad \text{if } \tau \ge t + 10\pi N\varepsilon.$$

Hence we can rewrite (4.92) as

$$S^*(h)(t,x,v) = \frac{1}{\varepsilon}\int_t^{\min\{T,t+10\pi N\varepsilon\}} h(\tau, x + \frac{\tau - t}{\varepsilon} v)\tilde{\varphi}(v)\mathrm{d}\tau \\ \text{for } (x,v) \in U \times V, \quad \text{if } \mathrm{supp}(h) \subset (-1,T] \times U. \tag{4.97}$$

On the other hand, from (4.91), we have for $\mathrm{supp}(h) \in (-1,T] \times \tilde{\Omega}$,

$$\begin{aligned}\|S^*(h)\|^2_{L^2((-1,T]\times U\times V)} &= \int_{-1}^{T}\iint_{U\times V} S^*(h)(t,x,v)S^*(h)(t,x,v)\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ &= \int_{-1}^{T}\iint_{U\times V} SS^*(h)(t,x)h(t,x)\mathrm{d}x\mathrm{d}t \\ &\le \|SS^*(h)\|_{L^2_t((-1,T);L^p_x(U))}\|h\|_{L^2_t((-1,T];L^q_x(\tilde{\Omega}))}.\end{aligned}$$

Therefore to show (4.96) (which will imply (4.90)) we only need to prove that, for $\mathrm{supp}(h) \subset (-1,T] \times \tilde{\Omega}$,

$$\|SS^*(h)\|_{L^2_t((-1,T];L^p_x(U))} \lesssim \|h\|_{L^2_t((-1,T];L^q_x(\tilde{\Omega}))}. \tag{4.98}$$

Now we prove (4.98). From (4.89) and (4.97), we read

$$\begin{aligned}&SS^*(h)(t,x) \\ &= \frac{1}{\varepsilon}\int_{-1}^{t}\int_{\mathbb{R}^3} S^*(h)(s, x - \frac{t-s}{\varepsilon}v, v)\tilde{\varphi}(v)\mathrm{d}v\mathrm{d}s \\ &= \frac{1}{\varepsilon^2}\int_{-1}^{t}\int_{\mathbb{R}^3}\int_s^{\min\{T,s+10\pi N\varepsilon\}} h(\tau, x - \frac{t-s}{\varepsilon}v + \frac{\tau - s}{\varepsilon}v)\mathrm{d}\tau(\tilde{\varphi}(v))^2\mathrm{d}v\mathrm{d}s \\ &= \frac{1}{\varepsilon^2}\int_{-1}^{t}\int_s^{\min\{T,s+10\pi N\varepsilon\}}\int_{\mathbb{R}^3} h(\tau, x + \frac{\tau - t}{\varepsilon}v)(\tilde{\varphi}(v))^2\mathrm{d}v\mathrm{d}\tau\mathrm{d}s.\end{aligned}$$

Now for the same reason to restrict τ-integration in (4.97) we rewrite the above expression as

$$\begin{aligned}&SS^*(h)(t,x) \\ &= \frac{1}{\varepsilon^2}\int_{\max\{-1,t-10\pi N\varepsilon\}}^{t}\int_s^{\min\{T,s+10\pi N\varepsilon\}}\int_{\mathbb{R}^3} h(\tau, x + \frac{\tau - t}{\varepsilon}v)(\tilde{\varphi}(v))^2\mathrm{d}v\mathrm{d}\tau\mathrm{d}s.\end{aligned} \tag{4.99}$$

We consider a map with the change of variables

$$v \in V \mapsto y := x + \frac{\tau - t}{\varepsilon}v \in \mathbb{R}^3, \quad \mathrm{d}v = \mathrm{d}y \Big/ \left|\frac{\partial y}{\partial v}\right| = \frac{\varepsilon^3}{|\tau - t|^3}\mathrm{d}y. \tag{4.100}$$

Now we apply (4.100) to (4.99) and derive that

$$\begin{aligned}&|SS^*(h)(t,x)|\\&\le \frac{1}{\varepsilon^2}\int_{t-10\pi N\varepsilon}^{t}\int_s^{\min\{s+10\pi N\varepsilon\}}\int_{\tilde{\Omega}}|\mathbf{1}_{\tau\in[-1,T]}h(\tau,y)|\frac{\varepsilon^3}{|\tau-t|^3}\tilde{\varphi}\\&\left(\varepsilon\frac{|y-x|}{|\tau-t|}\right)^2\mathrm{d}y\mathrm{d}\tau\mathrm{d}s.\end{aligned} \tag{4.101}$$

First using the Minkowski's inequality and the Young's inequality to a convolution in y with $1+1/p=1/q+1/(p/2)$ we have

$$\begin{aligned}&\left\|\frac{1}{\varepsilon^2}\int_{t-10\pi N\varepsilon}^{t}\int_s^{s+10\pi N\varepsilon}\int_{\tilde{\Omega}}\mathbf{1}_{\tau\in[-1,T]}|h(\tau,y)|\frac{\varepsilon^3}{|\tau-t|^3}\tilde{\varphi}\left(\varepsilon\frac{|y-x|}{|\tau-t|}\right)^2\mathrm{d}y\mathrm{d}\tau\mathrm{d}s\right\|_{L^p_x(\tilde{\Omega})}\\&\le \frac{1}{\varepsilon^2}\int_{t-10\pi N\varepsilon}^{t}\int_s^{s+10\pi N\varepsilon}\\&\quad\left\|\int_{\tilde{\Omega}}\mathbf{1}_{\tau\in[-1,T]}|h(\tau,y)|\frac{\varepsilon^3}{|\tau-t|^3}\tilde{\varphi}\left(\varepsilon\frac{|y-x|}{|\tau-t|}\right)^2\mathrm{d}y\right\|_{L^p_x(\tilde{\Omega})}\mathrm{d}\tau\mathrm{d}s\\&\le \frac{1}{\varepsilon^2}\int_{t-10\pi N\varepsilon}^{t}\int_s^{s+10\pi N\varepsilon}\|\mathbf{1}_{\tau\in[-1,T]}h(\tau,\cdot)\|_{L^q_x(\tilde{\Omega})}\\&\quad\underbrace{\left\|\frac{\varepsilon^3}{|\tau-t|^3}\tilde{\varphi}\left(\varepsilon\frac{|\cdot|}{|\tau-t|}\right)^2\right\|_{L^{p/2}_x(\tilde{\Omega})}}_{(4.102)_*}\mathrm{d}\tau\mathrm{d}s.\end{aligned} \tag{4.102}$$

From the properties of $\tilde{\varphi}\in C_c^\infty$, it follows that

$$(4.102)_*\le\frac{\varepsilon^3}{|\tau-t|^3}\left(\frac{|\tau-t|^3}{\varepsilon^3}\right)^{\frac{2}{p}}\left[\int_{\mathbb{R}^3}|\tilde{\varphi}(\tilde{y})|^p\,\mathrm{d}\tilde{y}\right]^{\frac{1}{p/2}}\lesssim\left(\frac{\varepsilon}{|\tau-t|}\right)^{3-\frac{6}{p}},$$

where $\tilde{y}=\frac{\varepsilon}{|\tau-t|}(y-x)$ with $\mathrm{d}\tilde{y}=\frac{\varepsilon^3}{|\tau-t|^3}\mathrm{d}y$. Therefore we derive that

$$\begin{aligned}&\|SS^*(h)(t,\cdot)\|_{L^p_x}\\&\lesssim\frac{1}{\varepsilon^2}\int_{t-10\pi N\varepsilon}^{t}\int_s^{s+10\pi N\varepsilon}\|\mathbf{1}_{\tau\in[-1,T]}h(\tau,\cdot)\|_{L^q_x(\tilde{\Omega})}\left(\frac{\varepsilon}{|\tau-t|}\right)^{3-\frac{6}{p}}\mathrm{d}\tau\mathrm{d}s.\end{aligned} \tag{4.103}$$

Using the Minkowski's inequality and the Young's inequality, finally we prove (4.98) as

$$\big\|\|SS^*(h)(t,\cdot)\|_{L^p_x}\big\|_{L^2_t(0,T)}$$

$$
\begin{aligned}
&\lesssim \frac{1}{\varepsilon^2}\left\|\mathbf{1}_{[t-10\pi N\varepsilon,t]}(s)\right\|_{L^1_s}\Bigg\|\sup_{s\in[t-10\pi N\varepsilon,t]}\int_s^{s+10\pi N\varepsilon} \\
&\quad \|\mathbf{1}_{\tau\in[-1,T]}h(\tau,\cdot)\|_{L^q_x(\tilde\Omega)}\left(\frac{\varepsilon}{|\tau-t|}\right)^{3-\frac{6}{p}}\mathrm{d}\tau\Bigg\|_{L^2_t} \\
&\lesssim \frac{1}{\varepsilon^2}10\pi N\varepsilon\left\|\int_{t-10\pi N\varepsilon}^{t+10\pi N\varepsilon}\|\mathbf{1}_{\tau\in[-1,T]}h(\tau,\cdot)\|_{L^q_x(\tilde\Omega)}\left(\frac{\varepsilon}{|\tau-t|}\right)^{3-\frac{6}{p}}\mathrm{d}\tau\right\|_{L^2_t} \\
&\lesssim \frac{1}{\varepsilon^2}10\pi N\varepsilon\left\|\|h(\tau,\cdot)\|_{L^q_x(\tilde\Omega)}\right\|_{L^2_\tau((-1,T])}\left\|\left(\frac{\varepsilon}{|t|}\right)^{3-\frac{6}{p}}\right\|_{L^1_t((0,10\pi N\varepsilon))} \\
&\lesssim N^{-1+\frac{6}{p}}\|h\|_{L^2_t((-1,T];L^q_x(\tilde\Omega))}.
\end{aligned}
$$

□

Step 3: Applying Lemma 6. Now we apply Lemma 6 to (4.85) and derive that

$$
\begin{aligned}
&\left\|\int_{\mathbb{R}^3}\bar f_R(t,x,v)\tilde\varphi(v)\mathrm{d}v\right\|_{L^2_t((-1,T];L^p_x(\tilde\Omega))} \\
&\lesssim \|\mathbf{1}_{(t,x,v)\in\mathfrak{D}_T}\bar g\|_{L^2((-1,T]\times U\times V)} \\
&\lesssim \|f_R(t,x,v)\|_{L^2((0,T]\times\tilde\Omega\times V)}+\|f_R(0,x,v)\|_{L^2(\tilde\Omega\times V)} \\
&\quad +\|\mathbf{1}_{(t,x,v)\in\mathfrak{D}_T}f_I(t+\varepsilon\tilde t_B(x,v),\tilde x_B(x,v),v)\|_{L^2((-1,T]\times(U\backslash\tilde\Omega)\times V)} \qquad (4.104) \\
&\quad +\|[\varepsilon\partial_t+v\cdot\nabla_x]f_R\|_{L^2((0,T]\times\tilde\Omega\times V)}, \qquad (4.105)
\end{aligned}
$$

where we have used (4.83), (4.77), (4.79), and the fact that $|v\cdot\nabla_x\chi_2(x)|\lesssim_N 1$ on $v\in V$.

First we consider (4.104). We split the cases of (4.104) according to (4.78). For $x\in\partial\tilde\Omega$, which has a zero measure in $L^2((-1,T]\times(U\backslash\tilde\Omega)\times V)$, we have $\tilde t_B(x,v)=0$ from the first line of (4.78). If $\tilde B(x,v)=\emptyset$ and $x\notin\partial\tilde\Omega$ then $\tilde t_B(x,v)=-\infty$ from the last line of (4.78) and hence $\bar f_R(-\infty)=0$ since $\chi_1(-\infty)=0$ in (4.77). Therefore we derive that

$$
\begin{aligned}
(4.104)\le\ &\|\mathbf{1}_{\{s<0\}\subset\tilde B(x,v)}\mathbf{1}_{(t,x,v)\in\mathfrak{D}_T}f_I \\
&\quad(t+\varepsilon\tilde t_B(x,v),\tilde x_B(x,v),v)\|_{L^2((-1,T]\times(U\backslash\tilde\Omega)\times V)} \qquad (4.106) \\
&+\|\mathbf{1}_{\{s>0\}\subset\tilde B(x,v)}\mathbf{1}_{(t,x,v)\in\mathfrak{D}_T}f_I \\
&\quad(t+\varepsilon\tilde t_B(x,v),\tilde x_B(x,v),v)\|_{L^2((-1,T]\times(U\backslash\tilde\Omega)\times V)}. \qquad (4.107)
\end{aligned}
$$

We need a special attention to (4.106). Since $(t,x,v)\in\mathfrak{D}_T$ we know that $\inf\{\tau\ge t: x+\frac{\tau-t}{\varepsilon}v\in cl(\tilde\Omega)\}\le T$. If $\{s<0\}\subset\tilde B(x,v)$ then, from the third line of (4.78), $\tilde t_B(x,v)=\sup\tilde B(x,v)=\sup\{s\in\mathbb{R}: x+sv\in\mathbb{R}^3\backslash cl(\tilde\Omega)\}\le(T-t)/\varepsilon$. Therefore the argument of f_I in (4.106) is confined as

$$(t+\varepsilon\tilde{t}_B(x,v),\tilde{x}_B(x,v),v)\in(-\infty,T]\times\partial\tilde{\Omega}\times V. \tag{4.108}$$

For (4.107), from the second line of (4.78), $\tilde{t}_B(x,v)=\inf\tilde{B}(x,v)=\inf\{s\in\mathbb{R}: x+sv\in\mathbb{R}^3\backslash cl(\tilde{\Omega})\}\leq 0$. Therefore $t+\varepsilon\tilde{t}_B(x,v)\leq t\leq T$ and hence the argument of f_I in (4.107) is confined as in (4.108). Now we apply the Minkowski's inequality in time, change of variables $t+\varepsilon\tilde{t}_B(x,v)\mapsto t$, and use (4.108) to derive that

$$(4.106)+(4.107)\lesssim\Big\|\|f_I(t,\tilde{x}_B(x,v),v)\|_{L^2_t((-1,T])}\Big\|_{L^2_{x,v}((U\backslash\tilde{\Omega})\times V)}. \tag{4.109}$$

Let us define an outward normal $\tilde{n}(x)$ on $\partial\tilde{\Omega}$. More precisely

$$\tilde{n}(x)=\begin{cases}(0,0,-1) & \text{if } x_3=0 \text{ and } x\in\partial\tilde{\Omega},\\ ((-1)^{\frac{x_1}{2\pi}+1},0,0) & \text{if } x_1\in\{0,2\pi\} \text{ and } x\in\partial\tilde{\Omega},\\ (0,(-1)^{\frac{x_2}{2\pi}+1},0) & \text{if } x_2\in\{0,2\pi\} \text{ and } x\in\partial\tilde{\Omega}.\end{cases} \tag{4.110}$$

From (4.82) we have therefore $(x,v)\in(U\backslash\tilde{\Omega})\times V$ then $|\tilde{n}(\tilde{x}_B(x,v))\cdot v|\geq 1/N$. We consider maps

$$\begin{aligned}&(x_1,x_3)\mapsto\tilde{x}_B(x,v)\in(0,2\pi)\times(0,2\pi)\times\{x_3=0\},\\ &\qquad\text{with }\left|\det\left(\frac{\partial(\tilde{x}_{B,1}(x,v),\tilde{x}_{B,2}(x,v))}{\partial(x_1,x_3)}\right)\right|=\left|\frac{v_2}{v\cdot\tilde{n}}\right|,\\ &(x_i,x_3)\mapsto(\tilde{x}_{B,i}(x,v),\tilde{x}_{B,3}(x,v))\in(0,2\pi)\times(0,\infty),\\ &\qquad\text{with }\left|\det\left(\frac{\partial(\tilde{x}_{B,i}(x,v),\tilde{x}_{B,3}(x,v))}{\partial(x_1,x_3)}\right)\right|=\left|\frac{v_i}{v\cdot\tilde{n}}\right|,\quad\text{for } i=1,2.\end{aligned} \tag{4.111}$$

Note that if $v\in V$ of (4.82) then $|v_i|\geq 1/N$ for all $i=1,2,3$. We define

$$\tilde{\gamma}:=\partial\tilde{\Omega}\times\mathbb{R}^3,\quad\tilde{\gamma}^N:=\partial\tilde{\Omega}\times(\mathbb{R}^3\backslash V). \tag{4.112}$$

We apply the change of variables (4.111) to (4.109):

$$\begin{aligned}(4.109)&=\left\|\left[\int_{-2\pi}^{4\pi}\int_{-2\pi}^{\infty}\int_{-2\pi}^{4\pi}\|f_I(t,\tilde{x}_B(x,v),v)\|^2_{L^2_t((-1,T])}\mathrm{d}x_1\mathrm{d}x_3\mathrm{d}x_2\right]^{1/2}\right\|_{L^2_v(V)}\\ &\leq\left\|\left[5\times 6\pi N\int_{\partial\tilde{\Omega}}\int_{-1}^{T}|f_I(t,y,v)|^2|v\cdot\tilde{n}(y)|\mathrm{d}t\mathrm{d}y\right]^{1/2}\right\|_{L^2_v(V)}\\ &\lesssim\|f_R\|_{L^2((0,T)\times\tilde{\gamma}\backslash\tilde{\gamma}^N)}+\|f_R(0)\|_{L^2(\tilde{\gamma}\backslash\tilde{\gamma}^N)}.\end{aligned} \tag{4.113}$$

We recall the trace theorem:

$$\int_0^T \int_{\tilde{\gamma}\backslash\tilde{\gamma}^{1/N}} |h| \mathrm{d}\gamma \mathrm{d}s \lesssim \sup_{t\in[0,T]} \|h(t)\|_{L^1(\tilde{\Omega}\times V)} + \int_0^T \|h(s)\|_{L^1(\tilde{\Omega}\times V)} \mathrm{d}s + \int_0^T \|[\varepsilon\partial_t + v\cdot\nabla_x]h\|_{L^1(\tilde{\Omega}\times V)} \mathrm{d}s. \tag{4.114}$$

We apply (5.33) with $h = f^2$ and derive an estimate

$$\begin{aligned} &\|f_R\|^2_{L^2((0,T)\times\tilde{\gamma}\backslash\tilde{\gamma}^N)} \\ &\lesssim \sup_{t\in[0,T]} \|f_R(t)\|^2_{L^2(\tilde{\Omega}\times V)} + \int_0^T \|f_R(s)\|^2_{L^2(\tilde{\Omega}\times V)} \mathrm{d}s \\ &\quad + \int_0^T \iint_{\tilde{\Omega}\times V} \left| f_R[\varepsilon\partial_t + v\cdot\nabla_x] f_R \right| \mathrm{d}x\mathrm{d}v\mathrm{d}s \\ &\lesssim_T \|f_R\|^2_{L^\infty([0,T];L^2(\Omega\times\mathbb{R}^3))} + \left\|[\varepsilon\partial_t + v\cdot\nabla_x]f_R\right\|_{L^2([0,T]\times\Omega\times\mathbb{R}^3)}. \end{aligned} \tag{4.115}$$

Finally we conclude a bound of (4.104) as below via (4.106), (4.107), (4.109), (4.113), and (4.115)

$$\begin{aligned} (4.104) \lesssim \|f_R(0)\|_{L^2_\gamma} + \|f_R\|_{L^\infty([0,T];L^2(\Omega\times\mathbb{R}^3))} \\ + \underbrace{\left\|[\varepsilon\partial_t + v\cdot\nabla_x]f_R\right\|_{L^2([0,T]\times\Omega\times\mathbb{R}^3)}}_{(4.116)_*}. \end{aligned} \tag{4.116}$$

Next we estimate (4.105) (and $(4.116)_*$). Using (4.84) and (3.2) we conclude that

$$\begin{aligned} &(4.105) + (4.116)_* \\ &\lesssim \Bigg\| -\frac{1}{\varepsilon\kappa} L(\mathbf{I}-\mathbf{P})f_R + \frac{\varepsilon}{\kappa}\Gamma(f_2, f_R) + \frac{\delta}{\kappa}\Gamma(f_R, f_R) \\ &\quad - \frac{(\varepsilon\partial_t + v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}} f_R + \varepsilon(\mathbf{I}-\mathbf{P})\mathfrak{R}_1 + \varepsilon\mathfrak{R}_2 \Bigg\|_{L^2((0,T]\times\Omega\times V)}. \end{aligned}$$

Following the arguments of (4.15)-(4.18), and (3.4), (3.5), we derive that

$$\begin{aligned} &(4.105) + (4.116)_* \\ &\lesssim \left\{ \frac{\varepsilon}{\kappa} \|(3.10)\|_{L^\infty_t((0,T);L_x^{\frac{2p}{p-2}}(\Omega))} + \frac{\delta}{\kappa}\|Pf_R\|_{L^\infty_t((0,T);L_x^{\frac{2p}{p-2}}(\Omega))} \right\} \\ &\|Pf_R\|_{L^2_t((0,T);L^p_x(\Omega))} \\ &\quad + \left\{ \frac{1}{\varepsilon\kappa} + \frac{\delta}{\kappa}\|\mathfrak{w}_{\varrho,\ss} f_R\|_{L^\infty_t((0,T)\times\Omega\times\mathbb{R}^3)} \right\} \|(\mathbf{I}-\mathbf{P})f_R\|_{L^2_t((0,T)\times\Omega\times\mathbb{R}^3)} \\ &\quad + \varepsilon\left\|(3.12)\right\|_{L^2_t((0,T);L^\infty_x(\Omega))} \|f_R(t)\|_{L^\infty_t((0,T);L^2(\Omega\times\mathbb{R}^3))} \end{aligned}$$

$$+\varepsilon\{\|(3.4)\|_{L^2_t((0,T);L^2_x(\Omega))}+\|(3.5)\|_{L^2_t((0,T);L^2_x(\Omega))}\}, \tag{4.117}$$

where we further bound

$$\|Pf_R\|_{L_x^{\frac{2p}{p-2}}(\Omega)}\leq \|Pf_R\|^{\frac{3(p-2)}{p}}_{L^6_x(\Omega)}\|\mathfrak{w}_{\varrho,\beta}f_R\|^{\frac{6-2p}{p}}_{L^\infty_x(\Omega)}. \tag{4.118}$$

Step 4. Proof of (4.72). First we use (4.87) and then (4.104) and (4.105). We bound (4.104) via (4.109) and (4.113), which are bounded by (4.115) and (4.117) respectively. These conclude that, for $p<3$,

$$\begin{aligned}
&\big(1-O(\varepsilon)\|u\|_\infty-O(\frac{1}{N})\big)\|Pf_R\|_{L^2_t((0,T);L^p_x(\tilde\Omega))}\\
&\quad -C_{T,N}\|\mathfrak{w}_{\varrho,\beta}f_R(t)\|^{\frac{p-2}{p}}_{L^\infty((0,T)\times\tilde\Omega\times\mathbb{R}^3)}\|(\mathbf{I}-\mathbf{P})f_R\|^{\frac{2}{p}}_{L^2((0,T)\times\tilde\Omega\times\mathbb{R}^3)}\\
&\leq \left\|\int_{\mathbb{R}^3}\bar f_R(t,x,v)\tilde\varphi_i(v)\sqrt{\mu_0(v)}\mathrm{d}v\right\|_{L^2_t((0,T);L^p_x(\tilde\Omega))} \\
&\leq \left\|\int_{\mathbb{R}^3}\bar f_R(t,x,v)\tilde\varphi(v)\mathrm{d}v\right\|_{L^2_t((0,T);L^p_x(\tilde\Omega))}\\
&\lesssim \|f_R\|_{L^\infty([0,T];L^2(\Omega\times\mathbb{R}^3))}+\|f_R(0)\|_{L^2_\gamma}+\text{r.h.s. of (4.117) with (4.118)}.
\end{aligned} \tag{4.119}$$

Then we move a contribution of $\|Pf_R\|_{L^2_t((0,T);L^p_x(\Omega))}$ to the l.h.s and use (4.118). This concludes (4.72).

Step 5: Sketch of proof for (4.74). We follow the same argument for (4.72). Thereby we only pin point the difference of the proof of (4.74). Recall $\partial_t f_R(0,x,v)=f_{R,t}(0,x,v)$ from (2.6). We regard $\tilde\Omega$ as an open subset but not a periodic domain as Ω. Without loss of generality we may assume that $f_{R,t}(0,x,v)$ is defined in $\mathbb{R}^3\times\mathbb{R}^3$ and $\|f_{R,t}(0)\|_{L^p(\mathbb{R}^3)\times\mathbb{R}^3}\lesssim\|f_{R,t}(0)\|_{L^p(\tilde\Omega)\times\mathbb{R}^3}$ for all $1\leq p\leq\infty$. Then we extend a solution for whole time $t\in\mathbb{R}$ as

$$f_{I,t}(t,x,v):=\mathbf{1}_{t\geq 0}\partial_t f_R(t,x,v)+\mathbf{1}_{t\leq 0}\chi_1(t)f_{R,t}(0,x,v). \tag{4.120}$$

Using $\tilde t_B(x,v)$ in (4.78) we define

$$f_{E,t}(t,x,v):=\mathbf{1}_{(x,v)\in(\mathbb{R}^3\backslash\tilde\Omega)\times\mathbb{R}^3}f_{I,t}(t+\varepsilon\tilde t_B(x,v),\tilde x_B(x,v),v). \tag{4.121}$$

We define an extension of cut-offed solutions

$$\begin{aligned}
&\bar f_{R,t}(t,x,v):=\chi_2(x)\chi_3(v)\big\{\mathbf{1}_{\tilde\Omega}(x)f_{I,t}(t,x,v)+f_{E,t}(t,x,v)\big\}\\
&\quad\text{for } (t,x,v)\in(-\infty,T]\times\mathbb{R}^3\times\mathbb{R}^3.
\end{aligned} \tag{4.122}$$

We note that in the sense of distributions $\bar{f}_{R,t}$ solves

$$\begin{aligned}
&\varepsilon\partial_t \bar{f}_{R,t} + v\cdot\nabla_x \bar{f}_{R,t} = \bar{g}_t \ \text{ in } (-\infty, T]\times\mathbb{R}^3\times\mathbb{R}^3, \quad \text{where}\\
&\bar{g}_t := \frac{v\cdot\nabla_x\chi_2}{\chi_2}\bar{f}_{R,t} + \mathbf{1}_{t\geq 0}\mathbf{1}_{\tilde{\Omega}}(x)\chi_2(x)\chi_3(v)[\varepsilon\partial_t + v\cdot\nabla_x]\partial_t f_R \\
&\quad + \mathbf{1}_{t\leq 0}\chi_2(x)\chi_3(v)\{\varepsilon\partial_t\chi_1(t) f_{R,t}(0,x,v) + \chi_1(t) v\cdot\nabla_x f_{R,t}(0,x,v)\}.
\end{aligned} \tag{4.123}$$

Here we have used the fact that $\bar{f}_{R,t}$ in (4.123) is continuous along the characteristics across $\partial\tilde{\Omega}$ and $\{t=0\}$. We derive that, using (4.123),

$$\bar{f}_{R,t}(t,x,v) = \frac{1}{\varepsilon}\int_{-\infty}^{t} \bar{g}_t(s, x - \frac{t-s}{\varepsilon}v, v)\mathrm{d}s \ \text{ for } (t,x,v)\in(-\infty,T]\times\mathbb{R}^3\times\mathbb{R}^3. \tag{4.124}$$

Now we apply Lemma 6 to (4.124) and derive that, for $p<3$,

$$\begin{aligned}
&\|S(\bar{g}_t)\|_{L^2_t((0,T);L^p_x(\mathbb{T}^2\times\mathbb{R}))}\\
&\lesssim \|\mathbf{1}_{(t,x,v)\in\mathfrak{D}_T}\bar{g}_t\|_{L^2((0,T)\times(\mathbb{T}^2\times\mathbb{R})\times\{|v|\leq N\})}\\
&\lesssim \|f_{R,t}(0)\|_{L^2(\Omega\times\mathbb{R}^3)} + \|\varepsilon\partial_t f_{R,t} + v\cdot\nabla_x f_{R,t}\|_{L^2((0,T)\times\tilde{\Omega}\times V)}\\
&\quad + \|\mathbf{1}_{(t,x,v)\in\mathfrak{D}_T} f_{I,t}(t+\varepsilon\tilde{t}_B(x,v), \tilde{x}_B(x,v), v)\|_{L^2((-1,T]\times(U\backslash\tilde{\Omega})\times V)}.
\end{aligned} \tag{4.125}$$

Following the same argument of (4.116)-(4.117) we deduce that

$$\begin{aligned}
(4.125) \lesssim &\|\partial_t f_R\|_{L^\infty([0,T];L^2(\Omega\times\mathbb{R}^3))} + \|\partial_t f_R(0)\|_{L^2_\gamma}\\
&+ \Big\| -\frac{1}{\varepsilon\kappa}L(\mathbf{I}-\mathbf{P})\partial_t f_R + \varepsilon\times \text{r.h.s. of } (3.3)\Big\|_{L^2((0,T]\times\Omega\times V)}.
\end{aligned} \tag{4.126}$$

From (4.31)-(4.33), the last term of (4.126) is bounded above by

$$\begin{aligned}
&\Big\{\frac{1}{\kappa}\|\partial_t u\|_{L^\infty_{t,x}}\Big(1+\delta\varepsilon\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}}\Big) + \frac{\varepsilon^2}{\kappa}\|(3.10)\|_{L^\infty_{t,x}}\Big\}\Big\{\|Pf_R\|_{L^2_{t,x}}\\
&+ \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}}\Big\}\\
&+\Big\{\frac{1}{\kappa\varepsilon} + \frac{\delta}{\kappa}\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}} + \frac{\varepsilon}{\kappa}\|(3.10)\|_{L^\infty_{t,x}} + \varepsilon\|(3.12)\|_{L^\infty_{t,x}}\Big\}\|\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}\\
&+\Big\{\frac{\delta}{\kappa}\|Pf_R\|^{\frac{3(p-2)}{p}}_{L^\infty_t L^6_x}\|\mathfrak{w} f_R\|^{\frac{6-2p}{p}}_{L^\infty_{t,x,v}} + \frac{\varepsilon}{\kappa}\|(3.10)\|_{L^\infty_t L_x^{\frac{2p}{p-2}}}\\
&+\varepsilon\|(3.12)\|_{L^\infty_t L_x^{\frac{2p}{p-2}}}\Big\}\|P\partial_t f_R\|_{L^2_t L^p_x}\\
&+\frac{\varepsilon}{\kappa}\|(3.11)\|_{L^2_{t,x,v}}\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}} + \varepsilon\|(3.13)\|_{L^2_t L^\infty_x}\|f_R\|_{L^\infty_t L^2_{x,v}}\\
&+\varepsilon\{\|(3.6)\|_{L^2_{t,x}} + \|(3.7)\|_{L^2_{t,x}}\}.
\end{aligned} \tag{4.127}$$

Here the most singular term comes from $\frac{1}{\varepsilon^2\kappa}L(\mathbf{P}_t f_R)$ in the r.h.s. of (3.3) .

On the other hand from (4.122) and the argument of (4.86) we derive

$$\begin{aligned}\|S(\bar{g}_t)\|_{L^2_t((0,T);L^p_x(\mathbb{T}^2\times\mathbb{R}))} &\gtrsim \Big\|\int_{\mathbb{R}^3}\bar{f}_{R,t}(t,x,v)\tilde{\varphi}_i(v)\sqrt{\mu_0(v)}\mathrm{d}v\Big\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))}\\ &\gtrsim \big(1-O(\varepsilon)\|u\|_\infty - O(\frac{1}{N})\big)\big\|P\partial_t f_R\big\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))}\\ &\quad -(\kappa\varepsilon)^{\frac{2}{p-2}}\|\mathfrak{w}'\partial_t f_R\|_{L^2_t((0,T);L^\infty_{x,v}(\Omega\times\mathbb{R}^3))} - \frac{1}{\kappa\varepsilon}\|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}.\end{aligned} \tag{4.128}$$

Here we have used

$$\begin{aligned}&\Big\|\int_{\mathbb{R}^3}\chi_2(x)\chi_3(v)(\mathbf{I}-\mathbf{P})\partial_t f_R(t,x,v)\tilde{\varphi}_i(v)\sqrt{\mu_0(v)}\mathrm{d}v\Big\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))}\\ &\leq \|(\mathbf{I}-\mathbf{P})\partial_t f_R(t,x,v)\|_{L^2_t((0,T);L^p_{x,v}(\tilde{\Omega}\times\mathbb{R}^3))}\\ &\lesssim \Big\|\|\mathfrak{w}'\partial_t f_R\|^{\frac{p-2}{p}}_{L^\infty_{x,v}(\Omega\times\mathbb{R}^3)}\|(\mathbf{I}-\mathbf{P})\partial_t f_R\|^{\frac{2}{p}}_{L^2_{x,v}(\Omega\times\mathbb{R}^3)}\Big\|_{L^2_t((0,T))}\\ &\lesssim \Big\|\|\mathfrak{w}'\partial_t f_R\|^{\frac{p-2}{p}}_{L^\infty_{x,v}(\Omega\times\mathbb{R}^3)}\Big\|_{L_t^{\frac{2p}{p-2}}((0,T))}\Big\|\|(\mathbf{I}-\mathbf{P})\partial_t f_R\|^{\frac{2}{p}}_{L^2_{x,v}(\Omega\times\mathbb{R}^3)}\Big\|_{L^p_t((0,T))}\\ &\lesssim (\kappa\varepsilon)^{\frac{2}{p}}\|\mathfrak{w}'\partial_t f_R\|^{\frac{p-2}{p}}_{L^2_t((0,T);L^\infty_{x,v}(\Omega\times\mathbb{R}^3))}(\kappa\varepsilon)^{-\frac{2}{p}}\|(\mathbf{I}-\mathbf{P})\partial_t f_R\|^{\frac{2}{p}}_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}\\ &\lesssim (\kappa\varepsilon)^{\frac{2}{p-2}}\|\mathfrak{w}'\partial_t f_R\|_{L^2_t((0,T);L^\infty_{x,v}(\Omega\times\mathbb{R}^3))} + (\kappa\varepsilon)^{-1}\|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}.\end{aligned} \tag{4.129}$$

Combining (4.128), (4.125), (4.126), and (4.127) and choosing $N \gg 1$ we conclude (4.74).

4.4 L^∞-Estimate

In this section we develop a unified L^∞-estimate in the local Maxwellian setting. We devise the weight functions to control an extra growth in $|v|$ comes from $\frac{(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}$ and its temporal derivative:

$$\mathfrak{w}_{\varrho,\text{ß}}(x,v)=\mathfrak{w}:=\exp\{\varrho|v|^2-\mathfrak{z}_{\text{ß}}(x_3)(x\cdot v)\}\ \text{ for }\ 0<\text{ß}\ll\frac{\varrho}{2\pi}\ \text{and}\ 0<\varrho<\frac{1}{4},$$

where $\mathfrak{z}_{\text{ß}}:\mathbb{R}_+\to\mathbb{R}_+$ is defined as, for ß >0

$$\mathfrak{z}_{\text{ß}}(x_3)=\text{ß}\ \text{ for }\ x_3\in\big[0,\frac{1}{\text{ß}}-1\big],\ \text{ and }\ \mathfrak{z}_{\text{ß}}(x_3)=\frac{1}{1+x_3}\ \text{ for }\ x_3\in\big[\frac{1}{\text{ß}}-1,\infty\big).$$

We often abuse the notation of $\mathfrak{w}_{\varrho,ß}$ and $\mathfrak{w}$. We compute to have

$$\begin{aligned}
&\frac{v\cdot\nabla_x \mathfrak{w}_{\varrho,ß}(x,v)}{\mathfrak{w}_{\varrho,ß}(x,v)}\\
&= -\mathfrak{z}_ß(x_3)|v|^2 - v_3\partial_{x_3}\mathfrak{z}_ß(x_3)(x_1v_1+x_2v_2+x_3v_3)\\
&= -\mathfrak{z}_ß(x_3)|v_3|^2 - x_3\partial_{x_3}\mathfrak{z}_ß(x_3)|v_3|^2 - \mathfrak{z}_ß(x_3)(|v_1|^2+|v_2|^2)\\
&\quad - \partial_{x_3}\mathfrak{z}_ß(x_3)(x_1v_1+x_2v_2)v_3\\
&= -ß\mathbf{1}_{[0,ß^{-1}-1]}(x_3)|v|^2 - \mathbf{1}_{[ß^{-1}-1,\infty)}(x_3)(1+x_3)^{-2}|v_3|^2\\
&\quad - \mathbf{1}_{[ß^{-1}-1,\infty)}(x_3)\frac{1}{1+x_3}(|v_1|^2+|v_2|^2)\\
&\quad - \partial_{x_3}\mathfrak{z}_ß(x_3)(x_1v_1+x_2v_2)v_3,
\end{aligned}$$

where we have used $\partial_{x_3}\mathfrak{z}_ß(x_3) = \mathbf{1}_{[ß^{-1}-1,\infty)}(x_3)\frac{-1}{(1+x_3)^2}$. The last term, the sole term without a sign, can be bounded as

$$\begin{aligned}
&|-\partial_{x_3}\mathfrak{z}_ß(x_3)(x_1v_1+x_2v_2)v_3|\\
&\quad\le 2\sqrt{2}\pi\mathbf{1}_{[ß^{-1}-1,\infty)}(x_3)(1+x_3)^{-2}(|v_1|^2+|v_2|^2)^{1/2}|v_3|\\
&\quad\le 4\pi^2\mathbf{1}_{[ß^{-1}-1,\infty)}(x_3)(1+x_3)^{-2}(|v_1|^2+|v_2|^2)\\
&\qquad + \frac{1}{2}\mathbf{1}_{[ß^{-1}-1,\infty)}(x_3)(1+x_3)^{-2}|v_3|^2.
\end{aligned}$$

Therefore we conclude that

$$\begin{aligned}
-v\cdot\nabla_x\mathfrak{w}_{\varrho,ß}(x,v) &\ge \Big\{ß\mathbf{1}_{[0,ß^{-1}-1]}(x_3)|v|^2 + \frac{1}{2}\mathbf{1}_{[ß^{-1}-1,\infty)}(x_3)(1+x_3)^{-2}|v_3|^2\\
&\quad + (1-4\pi^2ß)\mathbf{1}_{[ß^{-1}-1,\infty)}(x_3)\frac{1}{1+x_3}(|v_1|^2+|v_2|^2)\Big\}\mathfrak{w}_{\varrho,ß}(x,v)\\
&\ge \frac{\mathfrak{z}_ß(x_3)}{2}|v|^2\mathfrak{w}_{\varrho,ß}(x,v).
\end{aligned} \tag{4.130}$$

We consider

$$h(t,x,v) = \mathfrak{w}_{\varrho,ß}(x,v)f_R(t,x,v). \tag{4.131}$$

An equation for h can be written from (3.2) and (3.8) as

$$\partial_t h + \frac{1}{\varepsilon}v\cdot\nabla_x h + \frac{\nu_ß}{\varepsilon^2\kappa}h = \frac{1}{\varepsilon^2\kappa}K_{\mathfrak{w}}h + \mathcal{S}_h, \tag{4.132}$$

$$h|_{\gamma_-} = \mathfrak{w}P_{\gamma_+}\Big(\frac{h}{\mathfrak{w}}\Big) + r. \tag{4.133}$$

For (4.131), we have $r = -\frac{\varepsilon}{\delta}\mathfrak{w}(1-P_{\gamma_+})f_2$ and $\mathcal{S}_h := \frac{\delta}{\kappa\varepsilon}\Gamma_{\mathfrak{w}}(h,h) + \frac{2}{\kappa}\Gamma_{\mathfrak{w}}(\mathfrak{w}f_2, h) + \mathfrak{w}(\mathbf{I}-\mathbf{P})\mathfrak{R}_1 + \mathfrak{w}\mathbf{R}_2$, and

$$\nu_{ß} := \nu(v) - \varepsilon\kappa\frac{v\cdot\nabla_x \mathfrak{w}_{\varrho,ß}}{\mathfrak{w}_{\varrho,ß}} + \varepsilon^2\kappa\frac{(\partial_t + \varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}, \tag{4.134}$$

where we denote $\Gamma_{\mathfrak{w}}(\cdot,\cdot)(v) := \mathfrak{w}(v)\Gamma(\frac{\cdot}{\mathfrak{w}},\frac{\cdot}{\mathfrak{w}})(v)$ and $K_{\mathfrak{w}}(\cdot) := \mathfrak{w}K(\frac{\cdot}{\mathfrak{w}})$.

If we have

$$\varepsilon^{5/2}\kappa|\partial_t u| + \varepsilon^{1/2}\sup_{x\in\Omega}(1+x_3)|\nabla_x u(t,x)| < \infty, \tag{4.135}$$

then for sufficiently small $\varepsilon, \kappa > 0$, from (4.130),

$$\begin{aligned}\nu_{ß} &\geq \nu(v) + \frac{\varepsilon\kappa}{2}\mathfrak{z}_{ß}(x_3)|v|^2 - \varepsilon^2\kappa\{\varepsilon|\partial_t u| + |\nabla_x u||v|\}|v-\varepsilon u|\\ &\geq \frac{\nu(v)}{2} + \frac{\varepsilon\kappa}{4}\mathfrak{z}_{ß}(x_3)|v|^2.\end{aligned} \tag{4.136}$$

From (1.20), (1.22), and (2.3)

$$\begin{aligned}&|\mathfrak{w}(v)\Gamma(\frac{h}{\mathfrak{w}},\frac{h}{\mathfrak{w}})(v)|\\ &\leq \iint_{\mathbb{R}^3\times\mathbb{S}^2}|(v-v_*)\cdot\mathfrak{u}|\sqrt{\mu(v_*)}e^{-\varrho|v_*|^2+\frac{\varrho}{2}|v_*|}\\ &\qquad\times\left\{|h(v')||h(v_*')| + |h(v)||h(v_*)|\right\}\mathrm{d}\mathfrak{u}\mathrm{d}v_*\\ &\lesssim_{\varrho} \nu(v)\|h\|^2_{L^\infty_v}.\end{aligned} \tag{4.137}$$

From (3.20) clearly we have

$$\mathbf{k}(v,v_*)\frac{\mathfrak{w}_{\varrho,ß}(v)}{\mathfrak{w}_{\varrho,ß}(v_*)} \leq \mathbf{k}_{\mathfrak{w}}(v,v_*) := \frac{2C_2}{|v-v_*|}e^{-\frac{|v-v_*|^2}{8}-\frac{1}{8}\frac{(|v-\varepsilon u|^2-|v_*-\varepsilon u|^2)^2}{|v-v_*|^2}}\frac{\mathfrak{w}_{\varrho,ß}(v)}{\mathfrak{w}_{\varrho,ß}(v_*)}. \tag{4.138}$$

As in (3.23) we derive

$$\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}}(v,v_*)\mathrm{d}v_* \lesssim \frac{1}{1+|v|}. \tag{4.139}$$

Proposition 10 *Recall $\mathfrak{w}_{\varrho,ß}$ in (2.3). Assume the same assumptions in Proposition 6. In addition we assume (4.135), and the conditions of ϱ and ß in (2.3). Then*

$$\begin{aligned} &d_\infty \|\mathfrak{w}_{\varrho,ß} f_R\|_{L^\infty_{t,x,v}} \\ &\lesssim \|\mathfrak{w}_{\varrho,ß} f(0)\|_{L^\infty_{x,v}} + \frac{\varepsilon}{\delta}\|(3.10)\|_{L^\infty_{t,x}} + \varepsilon^2\kappa(\|(3.4)\|_{L^\infty_{t,x}} + \|(3.5)\|_{L^\infty_{t,x}}) \\ &+ \frac{1}{\varepsilon^{1/2}\kappa^{1/2}}\|Pf_R\|_{L^\infty_t L^6_x} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}}\Big\{\|\sqrt{\nu}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}} \\ &+ \|\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}}\Big\} \\ &+ \frac{1}{\varepsilon^{1/2}\kappa^{3/2}}\|\partial_t u\|_{L^\infty_{t,x}}\|Pf_R\|_{L^2_{t,x}}, \end{aligned} \tag{4.140}$$

where

$$d_\infty := 1 - \varepsilon^2\|(3.10)\|_{L^\infty_{t,x}} - \varepsilon\delta\|\mathfrak{w}_{\varrho,ß} f_R\|_{L^\infty_{t,x,v}}. \tag{4.141}$$

Proposition 11 *Assume the same assumptions of Proposition 10. We denote*

$$\mathfrak{w}'(x,v) := \mathfrak{w}_{\varrho',ß}(x,v) \ \textit{for}\ \varrho' < \varrho. \tag{4.142}$$

Let $p < 3$. Then

$$\begin{aligned} &d_{\infty,t}\|\mathfrak{w}'\partial_t f_R\|_{L^2_t((0,T);L^\infty_{x,v}(\Omega\times\mathbb{R}^3))} \\ &\lesssim \varepsilon\kappa^{1/2}\|\mathfrak{w}'\partial_t f_R(0)\|_{L^\infty_{x,v}} + \frac{1}{\varepsilon^{3/p}\kappa^{3/p}}\|P\partial_t f\|_{L^2_t L^p_x} \\ &+ \frac{1}{\varepsilon^{3/2}\kappa^{3/2}}\|\sqrt{\nu}(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2_{t,x,v}} \\ &+ \frac{\varepsilon}{\delta}\|(3.11)\|_{L^\infty_{x,v}} + \frac{\varepsilon^2}{\delta}\|\partial_t u\|_{L^\infty_{x,v}}\|(3.10)\|_{L^\infty_{x,v}} \\ &+ \varepsilon^2\kappa\|(3.6)\|_{L^\infty_{x,v}} + \varepsilon^2\kappa\|(3.7)\|_{L^2_t L^\infty_x} \\ &+ \varepsilon\big(\|\partial_t u\|_{L^\infty_{t,x}} + \varepsilon\|(3.11)\|_{L^\infty_{t,x}} + \varepsilon\kappa\|(3.13)\|_{L^2_t L^\infty_x}\big)\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}} \\ &+ \varepsilon\Big(\varepsilon\|(3.11)\|_{L^\infty_{t,x}} + \varepsilon\kappa\|(3.13)\|_{L^2_t L^\infty_x} \\ &+ \|\partial_t u\|_{L^\infty_{t,x}}\big(1 + \varepsilon^2\|(3.10)\|_{L^\infty_{t,x}} + \varepsilon\delta\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}}\big)\Big)\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}}, \end{aligned} \tag{4.143}$$

with

$$d_{\infty,t} := 1 - \varepsilon^2\|(3.10)\|_{L^\infty_{t,x}} - \varepsilon\delta\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}}. \tag{4.144}$$

In the proof of propositions, for simplicity, we often use $\|\cdot\|_\infty$ for $\|\cdot\|_{L^\infty_{t,x,v}}$, $\|\cdot\|_{L^\infty_{x,v}}$ or $\|\cdot\|_{L^\infty_x}$ if there would be no confusion.

Proof of Proposition 10 We define backward exit time and position as

$$t_{\mathbf{b}}(x,v) := \varepsilon \frac{x_3}{v_3}, \quad x_{\mathbf{b}}(x,v) := x - \frac{x_3}{v_3} v \ \text{ for } \ (x,v) \in \Omega \times \mathbb{R}^3. \tag{4.145}$$

Since the characteristics for (4.132) are given by $(x - \frac{t-s}{\varepsilon} v, v)$, we have, for $0 \le t - s < t_{\mathbf{b}}(x,v)$,

$$\frac{d}{ds}\Big\{e^{-\int_s^t \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} h(s, x - \frac{t-s}{\varepsilon} v, v)\Big\} = e^{-\int_s^t \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} \Big\{\frac{1}{\varepsilon^2\kappa} K_{\mathfrak{w}} h + \mathcal{S}_h\Big\}(s, x - \frac{t-s}{\varepsilon} v, v). \tag{4.146}$$

Here $e^{-\int_s^t \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} = e^{-\int_s^t \frac{\nu_{\text{ß}}(\tau, x - \frac{t-\tau}{\varepsilon} v, v)}{\varepsilon^2\kappa} d\tau}$. We regard $(x_1 - \frac{t-s}{\varepsilon} v_1, x_2 - \frac{t-s}{\varepsilon} v_2) \in \mathbb{R}^2$ belongs to $\mathbb{T}^2$ without redefining them in $[-\pi,\pi]^2$.

Now we represent h using (4.146) and (4.133) as

$$\begin{aligned} h(t,x,v) =& \mathbf{1}_{t - t_{\mathbf{b}}(x,v) < 0} e^{-\int_0^t \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} h(0, x - \frac{t}{\varepsilon} v, v) \\ &+ \int_{\max\{0, t - t_{\mathbf{b}}(x,v)\}}^t e^{-\int_s^t \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} \frac{1}{\varepsilon^2\kappa} K_{\mathfrak{w}} h(s, x - \frac{t-s}{\varepsilon} v, v) ds \qquad (4.147) \\ &+ \int_{\max\{0, t - t_{\mathbf{b}}(x,v)\}}^t e^{-\int_s^t \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} \mathcal{S}_h(s, x - \frac{t-s}{\varepsilon} v, v) ds \\ &+ \mathbf{1}_{t - t_{\mathbf{b}}(x,v) \ge 0} e^{-\int_{t - t_{\mathbf{b}}(x,v)}^t \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} h(t - t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v). \qquad (4.148) \end{aligned}$$

Since the integrand of (4.148) reads on the boundary, using the boundary condition (4.133) and (4.146) again, we represent it as

$$\begin{aligned} & h(t - t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v) \\ &= \mathfrak{w}(x_{\mathbf{b}}(x,v), v) c_\mu \sqrt{\mu(v)} \int_{\mathfrak{v}_3 < 0} h(t - t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), \mathfrak{v}) \frac{\sqrt{\mu(\mathfrak{v})}|\mathfrak{v}_3|}{\mathfrak{w}(x_{\mathbf{b}}(x,v), \mathfrak{v})} d\mathfrak{v} \\ &\quad + r(t - t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v) \\ &= \mathfrak{w}(x_{\mathbf{b}}(x,v), v) c_\mu \sqrt{\mu(v)} \int_{\mathfrak{v}_3 < 0} e^{-\int_0^{t - t_{\mathbf{b}}(x,v)} \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} \\ &\quad \times h(0, x_{\mathbf{b}}(x,v) - \frac{t - t_{\mathbf{b}}(x,v)}{\varepsilon} \mathfrak{v}, \mathfrak{v}) \frac{\sqrt{\mu(\mathfrak{v})}|\mathfrak{v}_3|}{\mathfrak{w}(x_{\mathbf{b}}(x,v), \mathfrak{v})} d\mathfrak{v} \\ &\quad + \mathfrak{w}(x_{\mathbf{b}}(x,v), v) c_\mu \sqrt{\mu(v)} \int_{\mathfrak{v}_3 < 0} \int_0^{t - t_{\mathbf{b}}(x,v)} e^{-\int_s^{t - t_{\mathbf{b}}(x,v)} \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} \\ &\quad \times \frac{1}{\varepsilon^2\kappa} K_{\mathfrak{w}} h(s, x_{\mathbf{b}}(x,v) - \frac{t - t_{\mathbf{b}}(x,v) - s}{\varepsilon} \mathfrak{v}, \mathfrak{v}) \frac{\sqrt{\mu(\mathfrak{v})}|\mathfrak{v}_3|}{\mathfrak{w}(x_{\mathbf{b}}(x,v), \mathfrak{v})} ds d\mathfrak{v} \\ &\quad + \mathfrak{w}(x_{\mathbf{b}}(x,v), v) c_\mu \sqrt{\mu(v)} \int_{\mathfrak{v}_3 < 0} \int_0^{t - t_{\mathbf{b}}(x,v)} e^{-\int_s^{t - t_{\mathbf{b}}(x,v)} \frac{\nu_{\text{ß}}}{\varepsilon^2\kappa}} \end{aligned}$$

$$\times \mathcal{S}_h(s, x_{\mathbf{b}}(x,v) - \frac{t - t_{\mathbf{b}}(x,v) - s}{\varepsilon}\mathfrak{v}, \mathfrak{v})\frac{\sqrt{\mu(\mathfrak{v})}|\mathfrak{v}_3|}{\mathfrak{w}(x_{\mathbf{b}}(x,v), \mathfrak{v})}\mathrm{d}s\mathrm{d}\mathfrak{v}$$
$$+ r(t - t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v), \tag{4.149}$$

where $r = -\frac{\varepsilon}{\delta}\mathfrak{w}(1 - P_{\gamma_+})f_2$ and $e^{-\int_0^{t-t_{\mathbf{b}}(x,v)}\frac{\nu_{\mathrm{B}}}{\varepsilon^2\kappa}} := e^{-\int_0^{t-t_{\mathbf{b}}(x,v)}\frac{1}{\varepsilon^2\kappa}\nu_{\mathrm{B}}(\tau, x - \frac{t_{\mathbf{b}}(x,v)}{\varepsilon}v - \frac{t - t_{\mathbf{b}}(x,v) - s}{\varepsilon}\mathfrak{v}, \mathfrak{v})\mathrm{d}\tau}$.

Note that, from (3.4), (3.5), (3.57), (3.58), and (4.137),

$$\begin{aligned}|\mathcal{S}_h(s, x - \frac{t-s}{\varepsilon}v, v)| &\lesssim \nu(v)\frac{\delta}{\kappa\varepsilon}\|h\|_\infty^2 + \frac{\nu(v)}{\kappa}\|(3.10)\|_\infty\|h\|_\infty \\ &\quad + \|(3.4)\|_\infty + \|(3.5)\|_\infty, \\ |\mathfrak{w}(1 - P_{\gamma_+})f_2| &\lesssim \|(3.10)\|_\infty.\end{aligned} \tag{4.150}$$

We derive a preliminary estimate as

$$|h(t,x,v)| \lesssim e^{-\frac{\nu}{2\varepsilon^2\kappa}t}\|h(0)\|_\infty$$
$$+ \varepsilon\delta \sup_{0\le s\le t}\|h(s)\|_\infty^2 + \varepsilon^2 \sup_{0\le s\le t}\|(3.10)\|_\infty\|h(s)\|_\infty$$
$$+ \frac{\varepsilon}{\delta}\sup_{0\le s\le t}\|(3.10)\|_\infty + \varepsilon^2\kappa(\|(3.4)\|_\infty + \|(3.5)\|_\infty) \tag{4.151}$$
$$+ \int_0^t \frac{e^{-\frac{\nu}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}}(v, v_*)|h(s, x - \frac{t-s}{\varepsilon}, v_*)|\mathrm{d}v_*\mathrm{d}s \tag{4.152}$$
$$+ \mathfrak{w}(x_{\mathbf{b}}(x,v), v)c_\mu\sqrt{\mu(v)}\int_{\mathfrak{v}_3<0}\int_0^{t-t_{\mathbf{b}}(x,v)}\frac{e^{-\frac{\nu}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}$$
$$\times \int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}}(\mathfrak{v}, v_*)|h(s, x_{\mathbf{b}}(x,v) - \frac{t - t_{\mathbf{b}}(x,v) - s}{\varepsilon}\mathfrak{v}, v_*)|\mathrm{d}v_*\mathrm{d}s\frac{\sqrt{\mu(\mathfrak{v})}|\mathfrak{v}_3|}{\mathfrak{w}(x_{\mathbf{b}}(x,v), \mathfrak{v})}\mathrm{d}\mathfrak{v}. \tag{4.153}$$

We note that $|h(s, x - \frac{t-s}{\varepsilon}, v_*)|$ has the same upper bound. Then we bound (4.152) by a summation of (4.151) and

$$\begin{aligned}&\sup_{\substack{(x_{\mathbf{b}}, v)\in\partial\Omega\times\mathbb{R}^3 \\ t - t_{\mathbf{b}}\ge 0}}\mathfrak{w}(x_{\mathbf{b}}, v)c_\mu\sqrt{\mu(v)}\int_{\mathfrak{v}_3<0}\int_0^{t-t_{\mathbf{b}}}\frac{e^{-\frac{\nu}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \\ &\quad \times \int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}}(\mathfrak{v}, v_*)|h(s, x_{\mathbf{b}} - \frac{t - t_{\mathbf{b}} - s}{\varepsilon}\mathfrak{v}, v_*)|\mathrm{d}v_*\mathrm{d}s\frac{\sqrt{\mu(\mathfrak{v})}|\mathfrak{v}_3|}{\mathfrak{w}(x_{\mathbf{b}}, \mathfrak{v})}\mathrm{d}\mathfrak{v},\end{aligned} \tag{4.154}$$

and importantly

$$\int_0^t \frac{e^{-\frac{\nu(v)}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}}(v,v_*)\int_0^s \frac{e^{-\frac{\nu(v_*)}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa} \times \int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}}(v_*,v_{**})|h(s,x-\frac{t-s}{\varepsilon}v-\frac{s-\tau}{\varepsilon}v_*,v_{**})|\mathrm{d}v_{**}\mathrm{d}\tau\mathrm{d}v_*\mathrm{d}s. \tag{4.155}$$

We consider (4.155). We decompose the integration of $\tau\in[0,s]=[0,s-o(1)\varepsilon^2\kappa]\cup[s-o(1)\varepsilon^2\kappa,s]$. The contribution of $\int_{s-o(1)\varepsilon^2\kappa}^s\cdots\mathrm{d}\tau$ is bounded as

$$\frac{2}{\nu(v)}\big(1-e^{-\frac{\nu(v)}{2\varepsilon^2\kappa}}\big)\|\mathbf{k}_{\mathfrak{w}}(v,\cdot)\|_{L^1}\frac{o(1)\varepsilon^2\kappa}{\varepsilon^2\kappa}\|\mathbf{k}_{\mathfrak{w}}(v_*,\cdot)\|_{L^1}\sup_{0\le s\le t}\|h(s)\|_\infty \le o(1)\sup_{0\le s\le t}\|h(s)\|_\infty. \tag{4.156}$$

For the rest of term we decompose $\mathbf{k}_{\mathfrak{w}}(v_*,v_{**})=\mathbf{k}_{\mathfrak{w},N}(v_*,v_{**})+\{\mathbf{k}_{\mathfrak{w}}(v_*,v_{**})-\mathbf{k}_{\mathfrak{w},N}(v_*,v_{**})\}$ where $\mathbf{k}_{\mathfrak{w},N}(v_*,v_{**}):=\mathbf{k}_{\mathfrak{w}}(v_*,v_{**})\times\mathbf{1}_{\frac{1}{N}<|v_*-v_{**}|<N\ \&\ |v_*|<N}$. From (4.139), $\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}}(v_*,v_{**})\mathbf{1}_{|v_*|\ge N}\mathrm{d}v_{**}\lesssim 1/N$. Also from the fact $\mathbf{k}_{\mathfrak{w}}(v_*,v_{**})\le\frac{e^{-C|v_*-v_{**}|^2}}{|v_*-v_{**}|}\in L^1(\{v_*-v_{**}\in\mathbb{R}^3\})$, $\sup_{v_*}\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}}(v_*,v_{**})\{\mathbf{1}_{\frac{1}{N}\ge|v_*-v_{**}|}+\mathbf{1}_{|v_*-v_{**}|\ge N}\}\mathrm{d}v_{**}\downarrow 0$ as $N\to\infty$. Hence for $N\gg 1$

$$\begin{aligned}(4.155)\le&\int_0^t\frac{e^{-\frac{\nu(v)}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w},N}(v,v_*)\int_0^{s-o(1)\varepsilon^2\kappa}\frac{e^{-\frac{\nu(v_*)}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa}\\&\times\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w},N}(v_*,v_{**})|h(s,x-\frac{t-s}{\varepsilon}v-\frac{s-\tau}{\varepsilon}v_*,v_{**})|\mathrm{d}v_{**}\mathrm{d}\tau\mathrm{d}v_*\mathrm{d}s\\ \le& C_N\int_0^t\frac{e^{-\frac{\nu(v)}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\int_{|v_*|\le 2N}\int_0^{s-o(1)\varepsilon^2\kappa}\frac{e^{-\frac{\nu(v_*)}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa}\\&\times\int_{|v_{**}|<2N}|f_R(s,x-\frac{t-s}{\varepsilon}v-\frac{s-\tau}{\varepsilon}v_*,v_{**})|\mathrm{d}v_{**}\mathrm{d}\tau\mathrm{d}v_*\mathrm{d}s\\&+o(1)\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}},\end{aligned} \tag{4.157}$$

where we have used the fact $\sup_x\mathbf{k}_{\mathfrak{w}}(v_*,v_{**})\mathfrak{w}_{\varrho,\beta}(v_{**})\le C_N<\infty$ when $\frac{1}{N}<|v_*-v_{**}|<N$ and $|v_*|<N$ (then $|v_{**}|<2N$).

Now we decompose $f_R=\mathbf{P}f_R+(\mathbf{I}-\mathbf{P})f_R$. We first take integrations (4.157) over v_* and v_{**} and use Holder's inequality with $p=6$, $p=2$ in $1/p+1/p'=1$

for $\mathbf{P} f_R$, $(\mathbf{I}-\mathbf{P}) f_R$ respectively to derive

$$
\begin{aligned}
&(4.157)\\
&\leq (4N)^3 C_N \frac{1}{\nu(v)} \sup_{\substack{0\leq s\leq t\\ 0\leq\tau\leq s-o(1)\varepsilon^2\kappa}} \bigg(\iint_{|v_*|\leq N, |v_{**}|\leq 2N}\\
&\qquad |\mathbf{P} f_R(s, x-\frac{t-s}{\varepsilon}v-\frac{s-\tau}{\varepsilon}v_*, v_{**})|^6 \mathrm{d}v_{**}\mathrm{d}v_*\bigg)^{1/6}\\
&\quad + (4N)^3 C_N \frac{1}{\nu(v)} \sup_{\substack{0\leq s\leq t\\ 0\leq\tau\leq s-o(1)\varepsilon^2\kappa}} \bigg(\iint_{|v_*|\leq N, |v_{**}|\leq 2N}\\
&\qquad |(\mathbf{I}-\mathbf{P}) f_R(s, x-\frac{t-s}{\varepsilon}v-\frac{s-\tau}{\varepsilon}v_*, v_{**})|^2 \mathrm{d}v_{**}\mathrm{d}v_*\bigg)^{1/2}.
\end{aligned}
\tag{4.158}
$$

Now we consider a map

$$
\begin{aligned}
&v_* \in \{\mathbb{R}^3 : |v_*| \leq N\} \mapsto y\\
&:= x - \frac{t-s}{\varepsilon}v - \frac{s-\tau}{\varepsilon}v_* \in \Omega, \quad \text{where } \left|\frac{\partial y}{\partial v_*}\right| = \left|\frac{s-\tau}{\varepsilon}\right|^3 \gtrsim \varepsilon^3\kappa^3.
\end{aligned}
\tag{4.159}
$$

We note that this mapping is not one-to-one and the image can cover Ω at most N times. Therefore we have

$$
\begin{aligned}
&\left(\iint_{|v_*|\leq N, |v_{**}|\leq N} |\mathbf{P} f_R(s, x-\frac{t-s}{\varepsilon}v-\frac{s-\tau}{\varepsilon}v_*, v_{**})|^6 \mathrm{d}v_{**}\mathrm{d}v_*\right)^{1/6}\\
&\leq N^{1/6}\left(\iint_{|v_*|\leq N, |v_{**}|\leq N} |\mathbf{P} f_R(s, y, v_{**})|^6 \mathrm{d}v_{**}\frac{\mathrm{d}y}{\varepsilon^3\kappa^3}\right)^{1/6}\\
&\leq \frac{N^{1/6}}{\varepsilon^{1/2}\kappa^{1/2}} \|\mathbf{P} f_R(s)\|_{L^6_{x,v}},\\
&\left(\iint_{|v_*|\leq N, |v_{**}|\leq N} |(\mathbf{I}-\mathbf{P}) f_R(s, x-\frac{t-s}{\varepsilon}v-\frac{s-\tau}{\varepsilon}v_*, v_{**})|^2 \mathrm{d}v_{**}\mathrm{d}v_*\right)^{1/2}\\
&\leq \frac{N^{1/2}}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P}) f_R(s)\|_{L^2_{x,v}}.
\end{aligned}
$$

Therefore we conclude that

$$
\begin{aligned}
&(4.155)\\
&\leq (4N)^3 C_N (4.158) + o(1) \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty_{x,v}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (4N)^4 C_N \left\{ \frac{1}{\varepsilon^{1/2}\kappa^{1/2}} \sup_{0\leq s\leq t} \|\mathbf{P} f_R(s)\|_{L^6_{x,v}} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \sup_{0\leq s\leq t} \|(\mathbf{I}-\mathbf{P}) f_R(s)\|_{L^2_{x,v}} \right\} \\
&\quad + o(1) \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty_{x,v}} \\
&\lesssim_N \frac{1}{\varepsilon^{1/2}\kappa^{1/2}} \sup_{0\leq s\leq t} \|\mathbf{P} f_R(s)\|_{L^6_{x,v}} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \Big\{ \|(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}} \\
&\quad + \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \Big\} \\
&\quad + \frac{1}{\varepsilon^{1/2}\kappa^{3/2}} \|\partial_t u\|_{L^\infty_{t,x}} \|P f_R\|_{L^2_{t,x}} + o(1) \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty_{x,v}},
\end{aligned} \tag{4.160}
$$

where we have used (A.1) the Sobolev embedding in 1D at the last line.

Now we consider (4.153) and (4.154). We decompose $s \in [0, t-t_{\mathbf{b}}] = [0, t-t_{\mathbf{b}} - o(1)\varepsilon^2\kappa] \cup [t-t_{\mathbf{b}} - o(1)\varepsilon^2\kappa, t-t_{\mathbf{b}}]$. The contribution of $\int_{t-t_{\mathbf{b}}-o(1)\varepsilon^2\kappa}^{t-t_{\mathbf{b}}} \cdots$ is bounded as

$$
\frac{o(1)\varepsilon^2\kappa}{\varepsilon^2\kappa} \|\mathbf{k}_{\mathfrak{w}}(\mathfrak{v}, \cdot)\|_{L^1} \sup_{0\leq s\leq t} \|h(s)\|_\infty \leq o(1) \sup_{0\leq s\leq t} \|h(s)\|_\infty. \tag{4.161}
$$

For $s \in [0, t-t_{\mathbf{b}} - o(1)\varepsilon^2\kappa]$ we consider a map as (4.159)

$$
\begin{aligned}
&\mathfrak{v} \in \{\mathfrak{v} \in \mathbb{R}^3 : \mathfrak{v}_3 < 0\} \mapsto y \\
&:= x_{\mathbf{b}} - \frac{t-t_{\mathbf{b}}-s}{\varepsilon}\mathfrak{v} \in \Omega, \quad \text{where } \left|\frac{\partial y}{\partial \mathfrak{v}}\right| = \left|\frac{t-t_{\mathbf{b}}-s}{\varepsilon}\right|^3 \gtrsim \varepsilon^3\kappa^3.
\end{aligned} \tag{4.162}
$$

Following the argument to have (4.158) we bound

$$
\begin{aligned}
&\text{the contribution of } \int_0^{t-t_{\mathbf{b}}-o(1)\varepsilon^2\kappa} \cdots \text{ of (4.154)} \\
&\lesssim_N \frac{1}{\varepsilon^{1/2}\kappa^{1/2}} \|\mathbf{P} f_R(s)\|_{L^6_{x,v}} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \Big\{ \|(\mathbf{I}-\mathbf{P}) f_R\|_{L^2_{t,x,v}} \\
&\quad + \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \Big\} + \frac{1}{\varepsilon^{1/2}\kappa^{3/2}} \|\partial_t u\|_{L^\infty_{t,x}} \|P f_R\|_{L^2_{t,x}}.
\end{aligned} \tag{4.163}
$$

In conclusion, we bound $|h(t,x,v)|$ by (4.151), (4.160), (4.161), (4.163) and conclude (4.140) by choosing small enough $o(1)$ in (4.160) and (4.161). □

Proof of Proposition 11 Since many parts of the proof are overlapped with the proof of Proposition 10 we only pin point the differences. An equation for $\mathfrak{w}'\partial_t f_R$ takes the similar form of (4.132) and (4.133). We can read (3.3) for

$$
h(t,x,v) = \mathfrak{w}'(x,v)\partial_t f_R(t,x,v), \quad \text{for } \varrho' < \varrho, \tag{4.164}
$$

as (4.132) and (4.133) replacing

$$\begin{aligned}\mathcal{S}_h=&\frac{2}{\kappa}\Gamma_{\mathfrak{w}'}(\mathfrak{w}'f_2,h)+\frac{2\delta}{\varepsilon\kappa}\Gamma_{\mathfrak{w}'}(\mathfrak{w}'f_R,h)+\frac{2}{\kappa}\Gamma_{\mathfrak{w}'}(\mathfrak{w}'\partial_t f_2,\mathfrak{w}'f_R)\\&-\partial_t\Big(\frac{(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\Big)\frac{\mathfrak{w}'}{\mathfrak{w}}\mathfrak{w}f_R+\mathfrak{w}'(\mathbf{I}-\mathbf{P})\mathfrak{R}_3+\mathfrak{w}'\mathfrak{R}_4\\&-\frac{1}{\varepsilon^2\kappa}\mathfrak{w}'L_t(\mathbf{I}-\mathbf{P})f_R+\frac{1}{\varepsilon^2\kappa}\mathfrak{w}'L(\mathbf{P}_tf_R)\\&+\frac{2}{\kappa}\mathfrak{w}'\Gamma_t(f_2,f_R)+\frac{\delta}{\varepsilon\kappa}\mathfrak{w}'\Gamma_t(f_R,f_R),\\ r=&-\frac{\varepsilon}{\delta}\mathfrak{w}'(1-P_{\gamma_+})\partial_tf_2+\mathfrak{w}'r_{\gamma_+}(f_R)-\mathfrak{w}'\frac{\varepsilon}{\delta}r_{\gamma_+}(f_2),\end{aligned}\tag{4.165}$$

where $r_{\gamma_+}(g)$ has been defined in (3.9).

We have the same equality of (4.147), (4.148) with (4.149) for h of (4.164) but replacing S_h and r of (4.165). Note that $\frac{\mathfrak{w}'(x,v)}{\mathfrak{w}(x,v)}\lesssim e^{-(\varrho-\varrho')|v|^2}$ and hence $\left|\partial_t\Big(\frac{(\partial_t+\varepsilon^{-1}v\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\Big)\frac{\mathfrak{w}'}{\mathfrak{w}}\right|\lesssim$ (3.13) from (3.13). From (3.36), (3.6), (3.7), (3.57), (3.12), (3.13), (3.58), and (4.137), we bound terms of (4.165)

$$\begin{aligned}|S_h|\lesssim&\ \nu(v)\Big\{\frac{1}{\kappa}|(3.10)|+\frac{\delta}{\kappa\varepsilon}\|\mathfrak{w}f_R\|_\infty\Big\}\|h\|_\infty+(3.6)+(3.7)\\&+\Big(\frac{\nu(v)}{\kappa}(3.11)+(3.13)+|\partial_tu|\Big(\frac{1}{\varepsilon\kappa}+\frac{\varepsilon}{\kappa}(3.10)+\frac{\delta}{\kappa}\|\mathfrak{w}f_R\|_\infty\Big)\Big)\|\mathfrak{w}f_R\|_\infty,\end{aligned}\tag{4.166}$$

$$|r|\lesssim\ \frac{\varepsilon}{\delta}(3.11)+\frac{\varepsilon^2}{\delta}|\partial_tu|(3.10)+\varepsilon|\partial_tu|\|\mathfrak{w}f_R\|_\infty.\tag{4.167}$$

Then as in (4.151)-(4.155) we derive a preliminary estimate as

$$|h(t,x,v)|$$

$$\lesssim e^{-\frac{\nu}{2\varepsilon^2\kappa}t}\|h(0)\|_\infty+\frac{\varepsilon^2\kappa}{\nu(v)}(4.166)+(4.167)\tag{4.168}$$

$$+\int_0^t\frac{e^{-\frac{\nu}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}'}(v,v_*)|h(s,x-\frac{t-s}{\varepsilon},v_*)|\mathrm{d}v_*\mathrm{d}s\tag{4.169}$$

$$\begin{aligned}&+\mathfrak{w}'(x_{\mathbf{b}}(x,v),v)c_\mu\sqrt{\mu(v)}\int_{\mathfrak{v}_3<0}\int_0^{t-t_{\mathbf{b}}(x,v)}\frac{e^{-\frac{\nu}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\\&\times\int_{\mathbb{R}^3}\mathbf{k}_{\mathfrak{w}'}(\mathfrak{v},v_*)|h(s,x_{\mathbf{b}}(x,v)\\&-\frac{t-t_{\mathbf{b}}(x,v)-s}{\varepsilon}\mathfrak{v},v_*)|\mathrm{d}v_*\mathrm{d}s\frac{\sqrt{\mu(\mathfrak{v})}|\mathfrak{v}_3|}{\mathfrak{w}'(x_{\mathbf{b}}(x,v),\mathfrak{v})}\mathrm{d}\mathfrak{v}.\end{aligned}\tag{4.170}$$

As (4.154) and (4.155), we bound (4.169) by a summation of (4.168) and

$$\int_0^t \frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\int_0^{s-o(1)\varepsilon^2\kappa}\frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa}\int_{|v_*|\le 2N} \times\int_{|v_{**}|\le 2N}|h(s,x-\frac{t-s}{\varepsilon}v-\frac{s-\tau}{\varepsilon}v_*,v_{**})|\mathrm{d}v_{**}\mathrm{d}v_*\mathrm{d}\tau\mathrm{d}s, \tag{4.171}$$

$$+\sup_{\substack{(x_{\mathbf{b}},v)\in\partial\Omega\times\mathbb{R}^3\\ t-t_{\mathbf{b}}\ge 0}}\mathfrak{w}'(x_{\mathbf{b}},v)c_\mu\sqrt{\mu(v)}\int_{\mathfrak{v}_3<0}\int_0^{t-t_{\mathbf{b}}-o(1)\varepsilon^2\kappa}\frac{e^{-\frac{\nu}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \times\int_{|v_*|\le 2N}|h(s,x_{\mathbf{b}}-\frac{t-t_{\mathbf{b}}-s}{\varepsilon}\mathfrak{v},v_*)|\mathrm{d}v_*\mathrm{d}s\frac{\sqrt{\mu(\mathfrak{v})}|\mathfrak{v}_3|}{\mathfrak{w}'(x_{\mathbf{b}},\mathfrak{v})}\mathrm{d}\mathfrak{v} \tag{4.172}$$

$$+o(1)\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}. \tag{4.173}$$

Then we follow the argument of (4.158)-(4.160) to derive that, for $p<3$,

$$|(4.171)|\lesssim\int_0^t \frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\int_0^{s-o(1)\varepsilon^2\kappa}\frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa}\frac{N^{1/3}}{\varepsilon^{3/p}\kappa^{3/p}}\|\mathbf{P}\partial_t f(\tau)\|_{L^p_{x,v}}\mathrm{d}\tau\mathrm{d}s \tag{4.174}$$

$$+\int_0^t \frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa}\int_0^{s-o(1)\varepsilon^2\kappa}\frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa}\frac{N^{1/2}}{\varepsilon^{3/2}\kappa^{3/2}}\|\mathbf{P}\partial_t f(\tau)\|_{L^2_{x,v}}\mathrm{d}\tau\mathrm{d}s. \tag{4.175}$$

Now we use the Young's inequality for temporal convolution twice to derive that, for $p<3$,

$$\begin{aligned}
&\|(4.171)\|_{L^2_t(0,T)}\\
&\lesssim\left\|\frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}|s|}}{\varepsilon^2\kappa}\right\|_{L^1_s(\mathbb{R})}\left\|\int_0^s\frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa}\left(\frac{N^{1/3}}{\varepsilon^{3/p}\kappa^{3/p}}\|\mathbf{P}\partial_t f(\tau)\|_{L^p_{x,v}}\right.\right.\\
&\qquad\left.\left.+\frac{N^{1/2}}{\varepsilon^{3/2}\kappa^{3/2}}\|(\mathbf{I}-\mathbf{P})\partial_t f(\tau)\|_{L^2_{x,v}}\right)\mathrm{d}\tau\right\|_{L^2_s(\mathbb{R})}\\
&\lesssim\left\|\frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}|s|}}{\varepsilon^2\kappa}\right\|_{L^1_s(\mathbb{R})}\left\|\frac{e^{-\frac{C_v}{2\varepsilon^2\kappa}|\tau|}}{\varepsilon^2\kappa}\right\|_{L^1_\tau(\mathbb{R})}\\
&\quad\times\left(\frac{N^{1/3}}{\varepsilon^{3/p}\kappa^{3/p}}\|P\partial_t f\|_{L^2_t((0,T);L^p_x(\Omega))}+\frac{N^{1/2}}{\varepsilon^{3/2}\kappa^{3/2}}\|(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}\right)\\
&\lesssim_N\ \frac{1}{\varepsilon^{3/p}\kappa^{3/p}}\|P\partial_t f\|_{L^2_t((0,T);L^p_x(\Omega))}+\frac{1}{\varepsilon^{3/2}\kappa^{3/2}}\|(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}.
\end{aligned} \tag{4.176}$$

As in (4.163), for (4.172) we use (4.162) to derive that, for $p < 3$,

$$\begin{aligned}
&\|(4.172)\|_{L^2_t(0,T)} \\
&\lesssim \left\| \int_0^t \frac{e^{-\frac{C_\nu}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \left(\frac{1}{\varepsilon^{3/p}\kappa^{3/p}} \|\mathbf{P}\partial_t f(s)\|_{L^p_{x,v}} \right.\right. \\
&\quad \left.\left. + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f(s)\|_{L^2_{x,v}} \right) \mathrm{d}s \right\|_{L^2_t(0,T)} \\
&\lesssim \left\| \frac{e^{-\frac{C_\nu}{2\varepsilon^2\kappa}|s|}}{\varepsilon^2\kappa} \right\|_{L^1_s(\mathbb{R})} \left\{ \frac{1}{\varepsilon^{3/p}\kappa^{3/p}} \|P\partial_t f\|_{L^2_t((0,T);L^p_x(\Omega))} \right. \\
&\quad \left. + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)} \right\} \\
&\lesssim \frac{1}{\varepsilon^{3/p}\kappa^{3/p}} \|P\partial_t f\|_{L^2_t((0,T);L^p_x(\Omega))} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)},
\end{aligned} \tag{4.177}$$

where we have used the Young's inequality for temporal convolution.

In conclusion, we bound $\|h\|_{L^2_t L^\infty_{x,v}}$ by $\|(4.168)\|_{L^2_t L^\infty_{x,v}}$, (4.176), (4.173), (4.177) and conclude (4.143) by choosing small enough $o(1)$ in (4.173). □

4.5 Proof of Theorem 2

An existence of a unique global solution F for each $\varepsilon > 0$ can be found in [12–15]. Thereby we only focus on the (a priori) estimates (2.13).

Step 1. Define $T_* > 0$ as

$$\begin{aligned}
T_* = \sup \Big\{ & t \geq 0 : \ \min\{d_2, d_{2,t}, d_6, d_3, d_{3,t}, d_\infty, d_{\infty,t}\} \geq \frac{\sigma_0}{4} \\
& \text{and } \frac{\delta\varepsilon^{1/2}}{\kappa}\sqrt{\mathcal{D}(s)} + \varepsilon\delta\|\mathfrak{w}_{\varrho,\beta} f(s)\|_{L^\infty_{x,v}} + \frac{\varepsilon^{1/2}\delta}{\kappa^{1+\mathfrak{P}}}\|Pf_R(s)\|_{L^2_x} \ll 1 \\
& \text{for all } 0 \leq s \leq t \Big\},
\end{aligned} \tag{4.178}$$

where $d_2, d_{2,t}, d_6, d_3, d_{3,t}, d_\infty, d_{\infty,t}$ are defined in (4.3), (4.5), (4.45), (4.73), (4.75), (4.141) and (4.144).

From (2.10) and (4.178) we read all the estimates of Proposition 7, Proposition 8, Proposition 10, Proposition 9, and Proposition 11 in terms of $\mathcal{E}(t)$ and $\mathcal{D}(t)$ as follows.

From (4.140), (4.178), and (2.10)

$$\begin{aligned}
&\sup_{0\leq s\leq t} \|\mathfrak{w}_{\varrho,\beta} f_R(s)\|_{L^\infty_{x,v}} \\
&\quad \lesssim \frac{1}{\varepsilon^{1/2}\kappa^{1/2}} \sup_{0\leq s\leq t} \|\mathbf{P} f_R(s)\|_{L^6_{x,v}} \\
&\qquad + \frac{1}{\varepsilon^{1/2}\kappa}\sqrt{\mathcal{D}(t)} + \frac{1}{\varepsilon^{1/2}\kappa^{1+\mathfrak{P}}}\|Pf_R\|_{L^2_{t,x}}
\end{aligned}$$

$$+\|\mathfrak{w}_{\varrho,\ss} f(0)\|_\infty + \varepsilon^{1/2}\exp\Big(\frac{3}{\kappa\mathfrak{P}'}\Big). \tag{4.179}$$

Now applying (4.179) to (4.44) we derive that

$$\begin{aligned}\sup_{0\le s\le t}\|Pf_R(s)\|_{L^6_x} &\lesssim \frac{\varepsilon}{\kappa}\exp\Big(\frac{1}{\kappa\mathfrak{P}'}\Big)\sup_{0\le s\le t}\sqrt{\mathcal{E}(s)}+\frac{1}{\kappa^{1/2}}\sqrt{\mathcal{D}(t)}+\frac{1}{\kappa^{1/2+\mathfrak{P}}}\|Pf_R\|_{L^2_{t,x}}\\ &\quad+(\varepsilon\kappa)^{\frac12}\|\mathfrak{w}_{\varrho,\ss} f(0)\|_{L^\infty_{x,v}}+\varepsilon^{1/2}\exp\Big(\frac{3}{\kappa\mathfrak{P}'}\Big).\end{aligned} \tag{4.180}$$

From (4.179), (4.180), and (4.178) and (2.10) we conclude that

$$\begin{aligned}&\sup_{0\le s\le t}\Big\{\kappa^{\frac12}\|Pf_R(s)\|_{L^6_x}+\varepsilon^{\frac12}\kappa\|\mathfrak{w}_{\varrho,\ss} f_R(s)\|_{L^\infty_{x,v}}\Big\}\\ &\lesssim \underbrace{\exp\Big(\frac{3}{\kappa\mathfrak{P}'}\Big)+\sqrt{\mathcal{F}_p(0)}+\sup_{0\le s\le t}\Big\{\sqrt{\mathcal{E}(s)}+\sqrt{\mathcal{D}(s)}\Big\}+\frac{1}{\kappa\mathfrak{P}}\|Pf_R\|_{L^2_{t,x}}}_{(4.181)_*}.\end{aligned} \tag{4.181}$$

From (4.72), (4.181), (4.178) and (2.10)

$$\begin{aligned}&\kappa^{\frac12}\|Pf_R\|_{L^2_t((0,t);L^p_x)}\\ &\lesssim \underbrace{(4.181)_*\Big\{1+\frac{\varepsilon^{1/2}\delta}{\kappa}(4.181)_*+\Big(\varepsilon^{\frac{p+2}{2(p-2)}}\kappa^{\frac{2}{p-2}}(4.181)_*\Big)^{\frac{p-2}{p}}\Big\}}_{(4.182)_*}.\end{aligned} \tag{4.182}$$

Using (4.181) and (2.9), from (4.74) and (4.143), we deduce that, for $p<3$ and $\varrho'<\varrho$,

$$\begin{aligned}&\kappa^{\frac12+\ss}\big\|P\partial_t f_R\big\|_{L^2_t((0,t);L^p_x)}+(\varepsilon\kappa)^{3/p}\kappa^{\frac12+\ss}\|\mathfrak{w}_{\varrho',\ss}\partial_t f_R\|_{L^2_t((0,t);L^\infty_{x,v})}\\ &\lesssim \underbrace{(4.182)_*\Big\{1+\varepsilon^{1-\frac{3-p}{p}}\delta\kappa^{-\frac3p}\{(4.181)_*+(4.182)_*\}\Big\}}_{(4.183)_*}.\end{aligned} \tag{4.183}$$

Step 2. Using the estimates of the previous step we will close the estimate ultimately in the basic energy estimates (4.2) and (4.4) via the Gronwall's inequality. We note that from (2.9) the multipliers of $\int_0^t\|Pf_R(s)\|^2_{L^2_x}\mathrm{d}s$ in (4.2) and $\int_0^t\|P\partial_t f_R(s)\|^2_{L^2_x}\mathrm{d}s$ in (4.4) are bounded above by

$$O(1)\kappa^{-2\mathfrak{P}}\big(1+\varepsilon\kappa^{\frac12-\mathfrak{P}}+(\varepsilon\kappa^{\frac12-\mathfrak{P}})^2\big)\lesssim\kappa^{-2\mathfrak{P}}, \tag{4.184}$$

where we have used (2.11).

In (4.2) and (4.4) we bound

$$\begin{aligned} \|\kappa^{1/2} P f_R\|_{L^2_t L^3_x} &\lesssim \kappa^{\frac{1}{2}(1-\frac{p}{3})} \|P f_R\|^{1-\frac{p}{3}}_{L^2_t L^\infty_x} \|\kappa^{1/2} P f_R\|^{\frac{p}{3}}_{L^2_t L^p_x} \\ &\lesssim_T (\varepsilon\kappa)^{-\frac{1}{2}(1-\frac{p}{3})} |(4.181)_*|^{1-\frac{p}{3}} |(4.183)_*|^{\frac{p}{3}}, \\ \|P \partial_t f_R\|_{L^2_t L^3_x} &\lesssim \|P\partial_t f_R\|^{1-\frac{p}{3}}_{L^2_t L^\infty_x} \|P\partial_t f_R\|^{\frac{p}{3}}_{L^2_t L^p_x} \\ &\lesssim \varepsilon^{-\frac{3}{p}(1-\frac{p}{3})} \kappa^{-\frac{1}{2}-\mathfrak{P}-\frac{3}{p}(1-\frac{p}{3})} |(4.183)_*|. \end{aligned} \tag{4.185}$$

We can check that the multiplier of $\|\varepsilon^{-1}\kappa^{-1/2}\sqrt{\nu}(\mathbf{I}-\mathbf{P}) f_R\|^2_{L^2_{t,x,v}}$ in (4.4) is bounded as, from (2.10) and (4.181),

$$\begin{aligned} &\Big\{ \varepsilon(1+\varepsilon\|(3.10)\|_{L^\infty_{t,x}})\|\partial_t u\|_{L^\infty_{t,x}} + \varepsilon\kappa\|\nabla_x \partial_t u\|_{L^\infty_{t,x}} + (\varepsilon\kappa^{1/2}\|(3.13)_*\|_{L^\infty_{t,x}})^2 \\ &\quad + (\varepsilon\delta\|\mathfrak{w} f_R\|_{L^\infty_{t,x,v}})^2 \Big\} \\ &\lesssim \varepsilon\kappa^{1/2-\mathfrak{P}} + \varepsilon\delta^2\kappa^{-2}|(4.181)_*|^2. \end{aligned}$$

Applying (4.181), (4.182), (4.183), (4.185) to (4.2)+o(1)(4.4), using the above bound and (2.11), and collecting the terms, we derive that

$$\begin{aligned} &\sup_{0\le s\le t} \mathcal{E}(s) + (1-\varepsilon\delta^2\kappa^{-2}|(4.181)_*|^2)\mathcal{D}(t) \\ &\lesssim \mathcal{E}(0) + \mathcal{F}(0) + \exp\left(\frac{6}{\kappa^{\mathfrak{P}'}}\right) + (4.184) \int_0^T \mathcal{E}(s)\mathrm{d}s \\ &\quad + \delta^2 \varepsilon^{-(1-\frac{p}{3})} \kappa^{-4+\frac{p}{3}} |(4.181)_*|^{4-\frac{2p}{3}} |(4.183)_*|^{\frac{2p}{3}} \\ &\quad + \delta^2 \varepsilon^{-\frac{6}{p}(1-\frac{p}{3})} \kappa^{-3-2\mathfrak{P}-\frac{6}{p}(1-\frac{p}{3})} |(4.181)_*|^2 |(4.183)_*|^2. \end{aligned} \tag{4.186}$$

Under the assumption of

$$\begin{aligned} &\varepsilon^{1/2}\delta\kappa^{-1}(4.181)_* \ll 1, \\ \varepsilon^{\frac{p+2}{2(p-2)}} \kappa^{\frac{2}{p-2}} (4.181)_* \ll 1, \quad &[\varepsilon^{1-\frac{3-p}{p}} \delta\kappa^{-\frac{3}{p}}]^{1/2} (4.181)_* \ll 1, \\ &[\delta^2 \varepsilon^{-(1-\frac{p}{3})} \kappa^{-4+\frac{p}{3}}]^{1/4} (4.181)_* \ll 1, \\ &[\delta^2 \varepsilon^{-\frac{6}{p}(1-\frac{p}{3})} \kappa^{-3-2\mathfrak{P}-\frac{6}{p}(1-\frac{p}{3})}]^{1/4} (4.181)_* \ll 1, \end{aligned} \tag{4.187}$$

we derive that, for some constants $\mathfrak{C}_1 > 0$ and $\mathfrak{C}_2 > 0$,

$$\sup_{0\le s\le t} \mathcal{E}(s) + \mathcal{D}(t) \le \mathfrak{C}_1\Big(\mathcal{E}(0) + \mathcal{F}_p(0) + \exp\Big(\frac{6}{\kappa^{\mathfrak{P}'}}\Big)\Big) + \mathfrak{C}_2 \kappa^{-\mathfrak{P}} \int_0^t \mathcal{E}(s)\mathrm{d}s. \tag{4.188}$$

Note that among others the last condition condition is the strongest in (4.187), which can be read as, from $\delta = \sqrt{\varepsilon}$ of (2.11),

$$\delta^{\frac{1}{2}-\frac{3}{p}\left(1-\frac{p}{3}\right)}\kappa^{-\frac{3}{4}-\frac{\mathfrak{P}}{2}-\frac{3}{2p}(1-\frac{p}{3})}(4.181)_* \ll 1. \tag{4.189}$$

Applying the Gronwall's inequality to (4.188) (we may redefine $\mathcal{E}(t)$ as $\sup_{0\leq s\leq t}\mathcal{E}(s)$ if necessary), we derive that

$$\sup_{0\leq s\leq t}\mathcal{E}(s) \leq \mathfrak{C}_1\Big(\mathcal{E}(0)+\mathcal{F}_p(0)+\exp\Big(\frac{6}{\kappa\mathfrak{P}'}\Big)\Big)\Big\{1+\frac{\mathfrak{C}_2 t}{\kappa\mathfrak{P}}\exp\Big(\frac{\mathfrak{C}_2 t}{\kappa\mathfrak{P}}\Big)\Big\}.$$

Applying this estimate to the last term of (4.188) and using the fact $\mathfrak{P}' < \mathfrak{P}$ we derive that, after redefining $\mathfrak{C}_1$ if necessary,

$$\sup_{0\leq s\leq t}\mathcal{E}(s)+\mathcal{D}(t)+\mathcal{F}_p(t) \leq \mathfrak{C}_1\big(\mathcal{E}(0)+\mathcal{F}_p(0)+1\big)\exp\Big(\frac{2\mathfrak{C}_2 t}{\kappa\mathfrak{P}}\Big) \text{ for all } t\leq T_*, \tag{4.190}$$

under the assumptions of (2.10), (4.178), and (4.187).

Step 3. Now we find out the ranges of $\delta, \kappa, \varepsilon$ satisfying the assumptions of (4.178) and (4.187). From (2.12) and (4.190), if we choose δ as

$$\delta \leq \left[\frac{\kappa^{\frac{3}{2}+\mathfrak{P}+\frac{3}{p}(1-\frac{p}{3})}}{\mathfrak{C}_1\big(\mathcal{E}(0)+\mathcal{F}_p(0)+1\big)}\exp\Big(\frac{-2\mathfrak{C}_2 T}{\kappa\mathfrak{P}}\Big)\right]^{\frac{1}{1-\frac{6}{p}(1-\frac{p}{3})}}. \tag{4.191}$$

then we can achieve (4.189) and hence all conditions of (4.187). Clearly (2.11) and (2.12) ensure (4.191).

Now from (4.190) and (2.11) we derive (2.13), which implies

$$\begin{aligned}\sup_{0\leq s\leq t}&\Big\{\|\kappa^{1/2}Pf_R(s)\|_{L^6_x}+\|\varepsilon^{1/2}\kappa\mathfrak{w}_{\varrho,\text{ß}}f_R(s)\|_{L^\infty_{x,v}}\\&+\|(\varepsilon\kappa)^{3/p}\kappa^{\frac{1}{2}+\mathfrak{P}}\mathfrak{w}_{\varrho',\text{ß}}f_R(s)\|_{L^2((0,s);L^\infty_{x,v})}\Big\}\\&\lesssim\delta^{-\frac{1}{2}+\frac{3}{p}(1-\frac{p}{3})}.\end{aligned}$$

These imply $\min\{d_2, d_{2,t}, d_6, d_3, d_{3,t}, d_\infty, d_{\infty,t}\} \geq \frac{1}{4}$ and $\frac{\delta\varepsilon^{1/2}}{\kappa}\sqrt{\mathcal{D}(t)} \ll 1$ from (4.3), (4.5), (4.45), (4.73), (4.75), (4.141) and (4.144).

Then by the standard continuation argument we can verify all assumptions (4.178) up to $t \leq T$ and $T = T_*$. The estimate (2.13) follows easily.

5 Navier-Stokes Approximations of the Euler Equations

In this section we prove Theorem 3. The proof of the theorem relies on the integral representation of the solution to the Navier-Stokes equations using the Green's function for the Stokes problem in the same spirit of [47].

5.1 Elliptic Estimates and Nonlinear Estimates

In this section, we prove the estimates of the solutions of incompressible Navier-Stokes equations in large Reynolds numbers with the no slip boundary condition satisfying (1.13)-(1.15) based on recent Green's function approach using the vorticity formulation of (2.16)-(2.18) applied to the inviscid limit problem [38,44,47,54]. An advantage of working with analytic function spaces is the Cauchy estimates useful for recovery of the loss of derivatives. We recall the spaces, norms, and terminology we have defined in Section 2.

Lemma 7 ([47,54], Embeddings and Cauchy estimates) *The following holds*

(1) $\mathfrak{B}^{\lambda,\kappa t} \subset \mathfrak{L}^{1,\lambda}$ *and* $\mathfrak{B}^{\lambda,\kappa} \subset \mathfrak{L}^{1,\lambda}$.

(2) $\|g_1 g_2\|_{*,\lambda} \lesssim \|g_1\|_{\infty,\lambda}\|g_2\|_{*,\lambda}$.

(3) $\sum_{|\beta|=1} \|D^\beta g\|_{*,\lambda} \lesssim \frac{\|g\|_{*,\tilde{\lambda}}}{\tilde{\lambda}-\lambda}$, *for any* $0 < \lambda < \tilde{\lambda}$.

For (2) and (3), $\|\cdot\|_{*,\lambda}$ *can be either* $\|\cdot\|_{\infty,\lambda,\kappa}$ *or* $\|\cdot\|_{\infty,\lambda,\kappa t}$ *or* $\|\cdot\|_{\infty,\lambda,0}$ *or* $\|\cdot\|_{1,\lambda}$.

Lemma 8 ([47,54], Elliptic estimates) *Let* ϕ *be the solution of* $-\Delta\phi = \omega$ *with the zero Dirichlet boundary condition, and let* $u = \nabla \times \phi$. *Then*

$$\begin{aligned}
\|u\|_{\infty,\lambda} + \|\nabla u\|_{1,\lambda} &\lesssim \|\omega\|_{1,\lambda},\\
\|\nabla_h u\|_{\infty,\lambda} + \|\nabla u_3\|_{\infty,\lambda} &\lesssim \sum_{0\le|\beta|\le 1} \|\nabla_h^\beta \omega\|_{1,\lambda},\\
\|\partial_3 u_h\|_{\infty,\lambda} &\lesssim \sum_{0\le|\beta|\le 1} \|\nabla_h^\beta \omega\|_{1,\lambda} + \|\omega_h\|_{\infty,\lambda},\\
\|\zeta^{-1}\nabla_h^{\beta'} u_3\|_{\infty,\lambda} &\lesssim \sum_{0\le|\beta|\le 1} \|\nabla_h^{\beta+\beta'} \omega_h\|_{1,\lambda}.
\end{aligned} \tag{5.1}$$

Proof Here we only sketch the proofs. For full justification we refer to Proposition 2.3 in [47] for 2D and Section 4 of [54] for 3D and the proofs therein. From $(|\xi|^2 - \partial_z^2)\phi_\xi = \omega_\xi$ and $\phi_\xi(0) = 0$ we write

$$\begin{aligned}
\phi_\xi(z) = \int_0^z G_-(y,z)\omega_\xi(y)\mathrm{d}y + \int_z^\infty G_+(y,z)\omega_\xi(y)\mathrm{d}y,\\
\text{with } G_\pm(y,z) := \frac{-1}{2|\xi|}\Big(e^{\pm|\xi|(z-y)} - e^{-|\xi|(y+z)}\Big).
\end{aligned} \tag{5.2}$$

The first two estimates of (5.1) can be easily derived from this explicit form. For the third estimate of (5.1), we write $u_1 = \partial_2(-\Delta)^{-1}\omega_3 - \partial_3(-\Delta)^{-1}\omega_2$ and $\partial_3 u_1 =$

$\partial_3\partial_2(-\Delta)^{-1}\omega_3 - \partial_3\partial_3(-\Delta)^{-1}\omega_2$. Then the third estimate of (5.1) follows from the identity

$$\begin{aligned}\partial_z(\partial_3(-\Delta)^{-1}\omega_2)_\xi =& \frac{1}{2}\Big(\int_0^z |\xi| e^{-|\xi|(z-y)}(1-e^{-2|\xi|y})\omega_{\xi,2}(s,y)\mathrm{d}y \\ &+ \int_z^\infty |\xi| e^{-|\xi|(y-z)}(1+e^{-2|\xi|z})\omega_{\xi,2}(x,y)\mathrm{d}y \\ &+ \int_z^\infty (-2|\xi|) e^{-|\xi|(y-x)} e^{-2|\xi|z}\omega_{\xi,2}(s,y)\mathrm{d}y\Big) - \omega_{\xi,2}(z).\end{aligned}$$

Next we prove the last estimate. Note that

$$\begin{aligned}&\frac{1+z}{z}\nabla_h u_3(z) \\ &= \frac{1}{z}\int_0^z \partial_y \nabla_h u_3(x_h,y)\mathrm{d}y + \nabla_h u_3(z) \\ &= \frac{1}{z}\int_0^z \nabla_h\big(\partial_1\partial_3(-\Delta)^{-1}\omega_2 - \partial_2\partial_3(-\Delta)^{-1}\omega_1\big)(x_h,y)\mathrm{d}y + \nabla_h u_3(z).\end{aligned}$$

From (5.2) we read that for $i = 1, 2$

$$\begin{aligned}&\Big| |\xi|^{|\beta|}(\partial_3(-\Delta)^{-1}\omega_i)_\xi(s,z)\Big| \\ &\le \frac{1}{2}\Big(\int_0^z e^{-|\xi|(z-y)}(1-e^{-2|\xi|y})|\xi|^{|\beta|}|\omega_{\xi,i}(s,y)|\mathrm{d}y \\ &\quad + \int_z^\infty e^{-|\xi|(y-z)}(1+e^{-2|\xi|z})|\xi|^{|\beta|}|\omega_{\xi,i}(s,y)|\mathrm{d}y\Big) \\ &\lesssim \sup_{0\le\sigma<\lambda} \big\| |\xi|^{|\beta|}\omega_{\xi,h}\big\|_{L^1(\partial\mathcal{H}_\sigma)}.\end{aligned}$$

From the identity and estimate above we conclude the last bound of (5.1). □

As a consequence of Lemma 8, we have the following nonlinear estimates.

Lemma 9 ([47,54]) *Let u and $\tilde{u}$ be the velocity field associated with $\omega = \nabla_x \times u$ and $\tilde{\omega} = \nabla_x \times \tilde{u}$ respectively. Then*

$$\begin{aligned}\|u\cdot\nabla\tilde{\omega}\|_{1,\lambda} &\lesssim \|\omega\|_{1,\lambda}\|\nabla_h\tilde{\omega}\|_{1,\lambda} + \|(1+|\nabla_h|)\omega\|_{1,\lambda}\|\zeta\partial_z\tilde{\omega}\|_{1,\lambda}, \\ \|\omega\cdot\nabla\tilde{u}_3\|_{1,\lambda} &\lesssim \|\omega_h\|_{1,\lambda}\|\nabla_h\tilde{u}_3\|_{\infty,\lambda} + \|\omega_3\|_{1,\lambda}\|\partial_3\tilde{u}_3\|_{\infty,\lambda} \\ &\lesssim \|\omega\|_{1,\lambda}\|(1+|\nabla_h|)\tilde{\omega}\|_{1,\lambda}, \\ \|\omega\cdot\nabla\tilde{u}_h\|_{1,\lambda} &\lesssim \|\omega_h\|_{1,\lambda}\|\nabla_h\tilde{u}_h\|_{\infty,\lambda} + \|\omega_3\|_{\infty,\lambda}\|\partial_3\tilde{u}_h\|_{1,\lambda} \\ &\lesssim \|\omega\|_{1,\lambda}\big(\|\tilde{\omega}_3\|_{\infty,\lambda} + \|(1+|\nabla_h|)\omega\|_{1,\lambda}\big).\end{aligned} \tag{5.3}$$

Moreover

$$\|u\cdot\nabla\tilde{\omega}_h\|_{*,\lambda}\lesssim\|\omega\|_{1,\lambda}\|\nabla_h\tilde{\omega}_h\|_{*,\lambda}+\big(\|(1+|\nabla_h|)\omega\|_{1,\lambda}+\|\zeta\partial_z\omega_3\|_{\infty,\lambda}\big)\|\zeta\partial_z\tilde{\omega}_h\|_{*,\lambda},$$
$$\|\omega\cdot\nabla\tilde{u}_h\|_{*,\lambda}\lesssim\|\omega_3\|_{\infty,\lambda,0}\big(\|(1+|\nabla_h|)\tilde{\omega}\|_{1,\lambda}+\|\tilde{\omega}_h\|_{*,\lambda}\big)+\|\omega_h\|_{*,\lambda}\sum_{0\le|\beta|\le 1}\|\nabla_h^\beta\tilde{\omega}\|_{1,\lambda},\tag{5.4}$$

where $\|\cdot\|_{*,\lambda}$ *can be either* $\|\cdot\|_{\infty,\lambda,\kappa}$ *or* $\|\cdot\|_{\infty,\lambda,\kappa t}$.

Furthermore

$$\|u\cdot\nabla\tilde{\omega}_3\|_{\infty,\lambda,0}\lesssim\|\omega\|_{1,\lambda}\|\nabla_h\tilde{\omega}_3\|_{\infty,\lambda,0}+\|(1+|\nabla_h|)\omega\|_{1,\lambda}\|\zeta\partial_3\tilde{\omega}_3\|_{\infty,\lambda,0},$$
$$\|\omega\cdot\nabla\tilde{u}_3\|_{\infty,\lambda,0}\lesssim\|\omega_h\|_{*,\lambda}\|(1+|\nabla_h|^2)\tilde{\omega}_h\|_{1,\lambda}+\|\omega_3\|_{\infty,\lambda,0}\|(1+|\nabla_h|)\tilde{\omega}_h\|_{1,\lambda},\tag{5.5}$$

where $\|(1+|\nabla_h|^k)g\|_*=\sum_{\ell=0}^k\|\nabla_h^\ell g\|_*$.

Proof Again we refer to Proposition 2.3 in [47] for 2D and Section 4 of [54] for the full justification. The bounds (5.3) and (5.4) directly follow from Lemma 8. The proof of the first estimate of (5.5) is an outcome of applying (5.1) to an easy bound

$$\|u\cdot\nabla\tilde{\omega}_3\|_{\infty,\lambda,0}\lesssim\|u_h\|_{\infty,\lambda}\|\nabla_h\tilde{\omega}_3\|_{\infty,\lambda,0}+\|\zeta(z)^{-1}u_3\|_{\infty,\lambda}\|\zeta(z)\partial_3\tilde{\omega}_3\|_{\infty,\lambda,0}.$$

For the second estimate of (5.5) it suffices to prove the bound for $\omega_h\cdot\nabla_h\tilde{u}_3$. From $|\zeta(z)(1+\phi_\kappa(z))|\lesssim 1$ or $|\zeta(z)(1+\phi_\kappa(z)+\phi_{\kappa t}(z))|\lesssim 1$,

$$\begin{aligned}\|\omega_h\cdot\nabla_h\tilde{u}_3\|_{\infty,\lambda,0}&\lesssim\|\omega_h\|_{*,\lambda}\Big\|\zeta(z)(1+\phi_\kappa(z)+\phi_{\kappa t}(z))\frac{\nabla_h\tilde{u}_3}{\zeta(z)}\Big\|_{\infty,\lambda}\\&\lesssim\|\omega_h\|_{*,\lambda}\|\zeta^{-1}\nabla_h\tilde{u}_3\|_{\infty,\lambda}.\end{aligned}$$

Then we use the last bound of (5.1) to finish the proof. □

We finally record the crucial estimate of nonlinear forcing terms $N=-u\cdot\nabla\omega+\omega\cdot\nabla u$, as an outcome of Lemma 9, that will be also crucially used to control $B=[\partial_{x_3}(-\Delta)^{-1}(-u\cdot\nabla\omega+\omega\cdot\nabla u)]\,|_{x_3=0}$ in the vorticity formulation (2.16) and (2.18).

Lemma 10 ([47,54], Nonlinear estimate) *Let* $\lambda\in(0,\lambda_0-\gamma s)$ *be given. We have the following:*

$$\|(1+|\nabla_h|)N\|_{1,\lambda}\lesssim\big(\|(1+|\nabla_h|)\omega\|_{1,\lambda}+\|(1+|\nabla_h|)\omega_3\|_{\infty,\lambda,0}\big)\|(1+|\nabla_h|^2)\omega\|_{1,\lambda}+\sum_{|\beta|=1}\|(1+|\nabla_h|)D^\beta\omega\|_{1,\lambda}\|(1+|\nabla_h|^2)\omega\|_{1,\lambda},\tag{5.6}$$

$$\sum_{|\beta|=1}\|D^\beta(1+|\nabla_h|)N\|_{1,\lambda}$$
$$\lesssim\sum_{|\beta|\le 1}\|D^\beta(1+|\nabla_h|)\omega\|_{1,\lambda}\Big(\sum_{|\beta|\le 2}\|D^\beta(1+|\nabla_h|)\omega\|_{1,\lambda})+\|(1+|\nabla_h|)\omega\|_{\infty,\lambda,0}\Big)$$

$$+\sum_{|\beta|\leq 1}\|D^{\beta}(1+|\nabla_h|)\omega_3\|_{\infty,\lambda,0}\|(1+|\nabla_h|)^2\omega\|_{1,\lambda}. \tag{5.7}$$

For $[[\,\cdot\,]]_{*,\lambda}$ *to be either* $[[\,\cdot\,]]_{\infty,\lambda,\kappa}$ *or* $[[\,\cdot\,]]_{\infty,\lambda,\kappa t}$*,*

$$[[N]]_{*,\lambda}\lesssim\|(1+|\nabla_h|^2)\omega\|_{1,\lambda}[[\omega]]_{*,\lambda}+\|(1+|\nabla_h|)\omega\|_{1,\lambda}[[D\omega]]_{*,\lambda}, \tag{5.8}$$

$$\begin{aligned}\sum_{|\beta|=1}[[D^{\beta}N]]_{*,\lambda}\lesssim&\sum_{|\beta|=1}\|(1+|\nabla_h|^{|\beta_h|+2})\omega\|_{1,\lambda}[[\omega]]_{*,\lambda}\\&+\sum_{|\beta|=1}[[D^{\beta}\omega]]_{*,\lambda}(\|(1+|\nabla_h|^2)\omega\|_{1,\lambda}+\beta_3[[D_3^{\beta_3}\omega]]_{*,\lambda})\\&+\sum_{|\beta|=2}[[D^{\beta}\omega]]_{*,\lambda}\|(1+|\nabla_h|)\omega\|_{1,\lambda}.\end{aligned} \tag{5.9}$$

The proof relies on Lemma 9. We refer to Lemma 4.2 and Lemma 4.5 in [54] for the detailed proof.

5.2 Green's Function and Integral Representation for the Vorticity Formulation

By taking the Fourier transform of (2.16)-(2.18) in $x_h\in\mathbb{T}^2$, we obtain

$$\partial_t\omega_\xi-\kappa\eta_0\Delta_\xi\omega_\xi=N_\xi\quad\text{in }\mathbb{R}_+, \tag{5.10}$$

$$\kappa\eta_0(\partial_{x_3}+|\xi|)\omega_{\xi,h}=B_\xi,\quad\omega_{\xi,3}=0\quad\text{on }x_3=0, \tag{5.11}$$

with $\omega_\xi|_{t=0}=\omega_{0\xi}$ for $\xi\in\mathbb{Z}^2$. Here

$$\Delta_\xi=-|\xi|^2+\partial_{x_3}^2, \tag{5.12}$$

and

$$\begin{aligned}N_\xi&=N_\xi(t,x_3):=(-u\cdot\nabla\omega+\omega\cdot\nabla u)_\xi\,(t,x_3),\\B_\xi&=B_\xi(t):=(\partial_{x_3}(-\Delta_\xi)^{-1}N_{\xi,h}(t))|_{x_3=0}.\end{aligned} \tag{5.13}$$

Here $(-\Delta_\xi)^{-1}$ denotes the inverse of $-\Delta_\xi$ with the zero Dirichlet boundary condition at $x_3=0$.

We give the integral representation and present key estimates on Green's function for the Stokes problem. As shown in [47,54], letting $G_\xi(t,x_3,y)$ be the Green's function for (5.10)-(5.11), the solution can be represented by the integral formula via Duhamel's principle:

$$\begin{aligned}\omega_\xi(t,x_3)=&\int_0^\infty G_\xi(t,x_3,y)\omega_{0\xi}(y)\mathrm{d}y+\int_0^t\int_0^\infty G_\xi(t-s,x_3,y)N_\xi(s,y)\mathrm{d}y\mathrm{d}s\\&-\int_0^t G_\xi(t-s,x_3,0)(B_\xi(s),0)\mathrm{d}s,\end{aligned} \tag{5.14}$$

where

$$G_\xi = \begin{bmatrix} G_{\xi h} & 0 & 0 \\ 0 & G_{\xi h} & 0 \\ 0 & 0 & G_{\xi 3} \end{bmatrix}, \tag{5.15}$$

with $G_{\xi h}$ of (5.19) and $G_{\xi 3}$ of (5.22): for $G_{\xi *}$ can be either $G_{\xi h}$ or $G_{\xi 3}$

$$\partial_t G_{\xi *}(t, x_3, y) - \kappa\eta_0 \Delta_\xi G_{\xi *}(t, x_3, y) = 0, \quad x_3 > 0, \tag{5.16}$$

$$\kappa\eta_0(\partial_{x_3} + |\xi|)G_{\xi h}(t, x_3, y) = 0, \quad x_3 = 0, \tag{5.17}$$

$$G_{\xi 3}(t, x_3, y) = 0, \quad x_3 = 0. \tag{5.18}$$

The following estimates and properties for G_ξ will be useful to show the propagation of analytic norms of ω, $\partial_t\omega$ and $\partial_t^2\omega$.

Lemma 11 ([47,54])

(1) (Bounds on $G_{\xi h}$) The Green's function $G_{\xi h}$ for the Stokes problem (5.16) and (5.17) is given by

$$G_{\xi h} = \tilde{H}_\xi + R_\xi, \tag{5.19}$$

where $\tilde{H}_\xi$ is the one dimensional Heat kernel in the half-space with the homogeneous Neumann boundary condition which takes the form of

$$\begin{aligned} \tilde{H}_\xi(t, x_3, y) &= H_\xi(t, x_3 - y) + H_\xi(t, x_3 + y) \\ &= \frac{1}{\sqrt{\kappa\eta_0 t}}\left(e^{-\frac{|x_3-y|^2}{4\kappa\eta_0 t}} + e^{-\frac{|x_3+y|^2}{4\kappa\eta_0 t}}\right)e^{-\kappa\eta_0|\xi|^2 t}, \end{aligned} \tag{5.20}$$

and the residual kernel R_ξ due to the boundary condition satisfies

$$|\partial_{x_3}^k R_\xi(t, x_3, y)| \lesssim b^{k+1}e^{-\theta_0 b(x_3+y)} + \frac{1}{(\kappa\eta_0 t)^{(k+1)/2}}e^{-\theta_0\frac{|x_3+y|^2}{\kappa\eta_0 t}}e^{-\frac{\kappa\eta_0|\xi|^2 t}{8}}, \tag{5.21}$$

with $b = |\xi| + \frac{1}{\sqrt{\kappa\eta_0}}$ and $R_\xi(t, x_3, y) = R_\xi(t, x_3 + y)$.

(2) (Formula of $G_{\xi 3}$) The Green's function $G_{\xi 3}$ for the Stokes problem (5.16) and (5.18) is given by one dimensional Heat kernel in the half-space with the homogeneous Dirichlet boundary condition as

$$\begin{aligned} G_{\xi 3}(t, x_3, y) &= H_\xi(t, x_3 - y) - H_\xi(t, x_3 + y) \\ &= \frac{1}{\sqrt{\kappa\eta_0 t}}\left(e^{-\frac{|x_3-y|^2}{4\kappa\eta_0 t}} - e^{-\frac{|x_3+y|^2}{4\kappa\eta_0 t}}\right)e^{-\kappa\eta_0|\xi|^2 t}. \end{aligned} \tag{5.22}$$

(3) *(Complex extension) The Green's function G_ξ has a natural extension to the complex domain $\mathcal{H}_\lambda$ for small $\lambda > 0$ with similar bounds in terms of Re y and Re z (cf. (3.16) in [47]). The solution ω_ξ to (5.10)-(5.11) in $\mathcal{H}_\lambda$ has a similar representation: for any $z \in \mathcal{H}_\lambda$, let σ be the positive constant so that $z \in \partial\mathcal{H}_\lambda$, then ω_ξ satisfies*

$$\omega_\xi(t,z) = \int_{\partial\mathcal{H}_\lambda} G_\xi(t,z,y)\omega_{0\xi}(y)\mathrm{d}y + \int_0^t \int_{\partial\mathcal{H}_\lambda} G_\xi(t-s,z,y)N_\xi(s,y)\mathrm{d}y\mathrm{d}s \\ - \int_0^t G_\xi(t-s,z,0)(B_\xi(s),0)\mathrm{d}s.$$

The proof of Lemma 11 can be found in Proposition 3.3 and Section 3.3 of [47]. The next lemma concerns the convolution estimates.

Lemma 12 *Let $T > 0$ be given. Recall the norms defined in Section 2. For any $0 \leq s < t \leq T$ and $k \geq 0$, there exists a constant $C_T > 0$ so that the following estimates hold: for $G_{\xi *}$ can be either $G_{\xi h}$ or $G_{\xi 3}$*

(1) *($\mathcal{L}^1_\lambda$ estimates)*

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi *}(t,z,y)g_\xi(y)\mathrm{d}y \right\|_{\mathcal{L}^1_\lambda} \leq C_T \sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}^1_\lambda}, \tag{5.23}$$

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi *}(t-s,z,y)g_\xi(y)\mathrm{d}y \right\|_{\mathcal{L}^1_\lambda} \leq C_T \sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}^1_\lambda}. \tag{5.24}$$

(2) *($\mathcal{L}^\infty_{\lambda,\kappa t}$ estimates)*

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi *}(t,z,y)g_\xi(y)\mathrm{d}y \right\|_{\mathcal{L}^\infty_{\lambda,\kappa t}} \\ \leq C_T \sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}}, \tag{5.25}$$

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi *}(t-s,z,y)g_\xi(y)\mathrm{d}y \right\|_{\mathcal{L}^\infty_{\lambda,\kappa t}} \\ \leq C_T \sum_{j=0}^k \sqrt{\frac{t}{s}} \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}^\infty_{\lambda,\kappa s}}. \tag{5.26}$$

(3) ($\mathcal{L}^\infty_{\lambda,\kappa}$ estimates) For either $\kappa = 0$ or $\kappa > 0$

$$\sum_{j=0}^{k} \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi *}(t, z, y) g_\xi(y) \mathrm{d}y \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \le C_T \sum_{j=0}^{k} \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}}, \tag{5.27}$$

$$\sum_{j=0}^{k} \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi *}(t-s, z, y) g_\xi(y) \mathrm{d}y \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \le C_T \sum_{j=0}^{k} \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}}. \tag{5.28}$$

Proof We only give a proof for $G_{\xi h}$ since $G_{\xi 3}$ can be handled easier than the other. The proof of (1) and (2) can be found in Propositions 3.7 and 3.8 of [47]. Here we present the detail for (3), the second inequality. We consider real values $y, z \in \mathbb{R}_+$ only as the complex extension follows similarly (cf. (3) in Lemma 11). Note that in view of (5.19), (5.20) and (5.21), it suffices to show

$$\sum_{j=0}^{k} \left\| (\zeta(z)\partial_z)^j \int_0^\infty R(t-s, z, y) g_\xi(y) \mathrm{d}y \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \le C_T \sum_{j=0}^{k} \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}}, \tag{5.29}$$

$$\sum_{j=0}^{k} \left\| (\zeta(z)\partial_z)^j \int_0^\infty H(t-s, z, y) g_\xi(y) \mathrm{d}y \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \le C_T \sum_{j=0}^{k} \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}}, \tag{5.30}$$

where $R(t, z, y) = be^{-b(y+z)}$ and $H(t, z, y) = \frac{1}{\sqrt{\kappa t}} e^{-\frac{|y-z|^2}{M\kappa t}}$ for some $M > 0$. We start with (5.29). Let $k = 0$ first. First note that

$$\begin{aligned} \left| \int_0^\infty R(t-s, z, y) g_\xi(y) \mathrm{d}y \right| &= \left| e^{-bz} \int_0^\infty be^{-(\bar\alpha+b)y} (1+\phi_\kappa(y)) \frac{e^{\bar\alpha y}}{1+\phi_\kappa(y)} g_\xi(y) \mathrm{d}y \right| \\ &\le e^{-bz} (1+\phi_\kappa(0)) \|g_\xi\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \int_0^\infty be^{-(\bar\alpha+b)y} \mathrm{d}y, \end{aligned}$$

since ϕ_κ is a decreasing function. The last integral is uniformly finite for all $|\xi|$ and κ. Hence,

$$\left\| \int_0^\infty R(t-s, z, y) g_\xi(y) \mathrm{d}y \right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \lesssim \sup_z \left(\frac{1+\phi_\kappa(0)}{1+\phi_\kappa(z)} e^{(\bar\alpha-b)z} \right) \|g_\xi\|_{\mathcal{L}^\infty_{\lambda,\kappa}}.$$

For $\bar\alpha > 0$, if $\kappa < \frac{1}{4\eta_0\bar\alpha^2}$ then

$$\sup_z \left(\frac{1+\phi_\kappa(0)}{1+\phi_\kappa(z)} e^{(\bar\alpha-b)z} \right)$$

$$\lesssim \frac{\sqrt{\kappa}+1}{\inf_z\left[(\sqrt{\kappa}+\frac{1}{1+|\frac{z}{\sqrt{\kappa}}|^p})e^{\frac{z}{2\sqrt{\eta_0\kappa}}}\right]} < \min\left\{1, \frac{\sqrt{\kappa}+1}{\sqrt{\kappa}+\frac{1}{(2\sqrt{\eta_0})^p p!}}\right\}, \tag{5.31}$$

where the last bound follows from the fact that

$$\frac{1}{1+|\frac{z}{\sqrt{\kappa}}|^p}e^{\frac{z}{2\sqrt{\eta_0\kappa}}} \geq \frac{1}{1+|\frac{z}{\sqrt{\kappa}}|^p}\{1+\frac{1}{p!}|\frac{z}{2\sqrt{\eta_0\kappa}}|^p\} \geq \min\left\{1, \frac{1}{(2\sqrt{\eta_0})^p p!}\right\}.$$

For $k \geq 1$, since $|\zeta(z)\partial_z R| \lesssim be^{-\frac{bz}{2}}$, the derivative estimates follow analogously. Therefore, (5.29) holds true.

We move onto (5.30). Let $k=0$ first. Note that, for $0 \leq s < t \leq T$ and $\kappa \lesssim 1$

$$\begin{aligned} e^{-\frac{|y-z|^2}{2M\kappa(t-s)}}e^{-\bar{\alpha}y} &= e^{-\frac{1}{2}|\frac{y-z}{\sqrt{M\kappa(t-s)}}+\bar{\alpha}\sqrt{M\kappa(t-s)}|^2} \\ e^{\frac{M}{2}\bar{\alpha}^2\kappa(t-s)}e^{-\bar{\alpha}z} &\leq e^{\frac{M}{2}\bar{\alpha}^2\kappa(t-s)}e^{-\bar{\alpha}z} \lesssim e^{-\bar{\alpha}z}, \end{aligned} \tag{5.32}$$

and thus

$$\begin{aligned} &\left|\int_0^\infty H(t-s,z,y)g_\xi(y)\mathrm{d}y\right| \\ &= \left|\int_0^\infty \frac{1}{\sqrt{\kappa(t-s)}}e^{-\frac{|y-z|^2}{M\kappa(t-s)}}e^{-\bar{\alpha}y}(1+\phi_\kappa(y))\frac{e^{\bar{\alpha}y}}{1+\phi_\kappa(y)}g_\xi(y)\mathrm{d}y\right| \\ &\lesssim e^{-\bar{\alpha}z}\|g_\xi\|_{\mathcal{L}^\infty_{\lambda,\kappa}}\int_0^\infty \frac{1}{\sqrt{\kappa(t-s)}}e^{-\frac{|y-z|^2}{2M\kappa(t-s)}}(1+\phi_\kappa(y))\mathrm{d}y. \end{aligned}$$

For the last integral, we divide the integral into two: $\int_0^\infty = \int_0^{\frac{z}{2}} + \int_{\frac{z}{2}}^\infty$. For the latter, since ϕ_κ is decreasing and the kernel is in L^1_y, we deduce

$$\int_{\frac{z}{2}}^\infty \frac{1}{\sqrt{\kappa(t-s)}}e^{-\frac{|y-z|^2}{2M\kappa(t-s)}}(1+\phi_\kappa(y))\mathrm{d}y \lesssim 1+\phi_\kappa(z).$$

For $\int_0^{\frac{z}{2}}\mathrm{d}y$, note $|y-z| \geq \frac{z}{2}$ and $1+\phi_\kappa(y) \leq 1+\phi_\kappa(0)$ for $y \in (0, \frac{z}{2})$. Hence

$$\int_0^{\frac{z}{2}} \frac{1}{\sqrt{\kappa(t-s)}}e^{-\frac{|y-z|^2}{2M\kappa(t-s)}}(1+\phi_\kappa(y))\mathrm{d}y \lesssim e^{-\frac{|z|^2}{16M\kappa(t-s)}}(1+\phi_\kappa(0)).$$

Then

$$\left\|\int_0^\infty H(t-s,z,y)g_\xi(y)\mathrm{d}y\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \lesssim \sup_z\left(\frac{1+\phi_\kappa(0)}{1+\phi_\kappa(z)}e^{-\frac{|z|^2}{16M\kappa(t-s)}}+1\right)\|g_\xi\|_{\mathcal{L}^\infty_{\lambda,\kappa}}.$$

A similar argument as in (5.31) shows that $\frac{1+\phi_\kappa(0)}{1+\phi_\kappa(z)}e^{-\frac{|z|^2}{16M\kappa(t-s)}}$ is uniformly finite in κ. This shows (5.30) for $k=0$.

For the derivative estimate, by splitting the integral into two parts and using $\partial_z H(t,z,y) = -\partial_y H(t,z,y)$, we rewrite

$$\begin{aligned}
&\int_0^\infty \zeta(z)\partial_z H(t-s,z,y)g_\xi(y)\mathrm{d}y \\
&\quad = \int_0^{\frac{z}{2}} \zeta(z)\partial_z H(t-s,z,y)g_\xi(y)\mathrm{d}y - \zeta(z)H(t-s,z,\frac{z}{2})g_\xi(\frac{z}{2}) \\
&\qquad + \int_{\frac{z}{2}}^\infty \zeta(z)H(t-s,z,y)\partial_y g_\xi(y)\mathrm{d}y.
\end{aligned}$$

For the first integral, since $|y-z| \geq \frac{z}{2}$ for $y \in (0,\frac{z}{2})$,

$$\begin{aligned}
|\zeta(z)\partial_z H(t-s,z,y)| &\lesssim \frac{z}{1+z}\frac{1}{\kappa(t-s)}e^{-\frac{|y-z|^2}{2M\kappa(t-s)}} \lesssim |y-z|\frac{1}{\kappa(t-s)}e^{-\frac{|y-z|^2}{2M\kappa(t-s)}} \\
&\lesssim \frac{1}{\sqrt{\kappa(t-s)}}e^{-\frac{|y-z|^2}{4M\kappa(t-s)}}.
\end{aligned}$$

Hence, by the same argument as in $k=0$ leads to the desired bound. For the second term,

$$\begin{aligned}
|\zeta(z)H(t-s,z,\frac{z}{2})g_\xi(\frac{z}{2})| &\lesssim \frac{z}{\sqrt{\kappa(t-s)}}e^{-\frac{|z|^2}{4M\kappa(t-s)}}e^{-\bar{\alpha}\frac{z}{2}}(1+\phi_\kappa(\frac{z}{2}))\left\|g_\xi\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \\
&\lesssim e^{-\frac{|z|^2}{8M\kappa(t-s)}}e^{-\bar{\alpha}\frac{z}{2}}(1+\phi_\kappa(z))\left\|g_\xi\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \\
&= e^{-\frac{1}{2}\left|\frac{z}{2\sqrt{M\kappa(t-s)}}-\bar{\alpha}\sqrt{M\kappa(t-s)}\right|^2}e^{\frac{M}{2}\bar{\alpha}^2\kappa(t-s)}e^{-\bar{\alpha}z} \\
&\quad (1+\phi_\kappa(z))\left\|g_\xi\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \\
&\lesssim e^{-\bar{\alpha}z}(1+\phi_\kappa(z))\left\|g_\xi\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}},
\end{aligned}$$

which leads to the desired bound. For the last integral, note that $\zeta(z) \leq 2\zeta(y)$ for $y \geq \frac{z}{2}$. Therefore the corresponding integral can be treated in the same way as in $k=0$ with $g_\xi(y)$ replaced by $\zeta(y)\partial_y g_\xi(y)$. This shows (5.30) for $k=1$. Other $k \geq 2$ can be estimated analogously. □

The next result concerns the estimates for the trace kernel.

Lemma 13 *Let* $a_\xi(s) = [\partial_z(-\Delta_\xi)^{-1}g_\xi]|_{z=0}$. *Then for any* $0 \leq s < t \leq T$ *and* $k \geq 0$, *we have the following*

$$\sum_{j=0}^{k}\left\|(\zeta(z)\partial_z)^j G_{\xi h}(t-s,z,0)a_\xi(s)\right\|_{\mathcal{L}^1_\lambda} \lesssim \left\|g_\xi\right\|_{\mathcal{L}^1_\lambda}, \tag{5.33}$$

$$\sum_{j=0}^{k}\left\|(\zeta(z)\partial_z)^j G_{\xi h}(t-s,z,0)a_\xi(s)\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \lesssim \frac{1}{\sqrt{t-s}}\left\|g_\xi\right\|_{\mathcal{L}^1_\lambda}. \tag{5.34}$$

Proof Note that from (5.19), (5.20) and (5.21), the conormal derivatives $(\zeta(z)\partial_z)^j$ of $G_{\xi h}(t-s,z,0)$ enjoy the same bounds as $G_{\xi h}(t-s,z,0)$: for some small constant c_0,

$$|(\zeta(z)\partial_z)^j G_\xi(t-s,z,0)| \lesssim be^{-c_0 bz} + \frac{1}{\sqrt{\kappa(t-s)}}e^{-c_0\frac{|z|^2}{\kappa(t-s)}}. \tag{5.35}$$

Therefore, it suffices to show the bounds for $k=0$. We first recall the representation formula for a_ξ (cf. (4.29) of [38] or (4.2) of [54]):

$$a_\xi(s) = \int_0^\infty e^{-|\xi|y} g_\xi(y)\mathrm{d}y,$$

from which we have $\|a_\xi\|_{\mathcal{L}^\infty_\lambda} \lesssim \left\|g_\xi\right\|_{\mathcal{L}^1_\lambda}$. Since the above upper bound of $G(t-s,z,0)$ is integrable in z, (5.33) follows. To show (5.34), we compute $\|G_{\xi h}(t-s,z,0)\|_{\mathcal{L}^\infty_{\lambda,\kappa}}$:

$$\|G_{\xi h}(t-s,z,0)\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \lesssim \sup_z\left[\frac{be^{(\bar\alpha-c_0b)z}}{1+\phi_\kappa(z)}\right] + \frac{1}{\sqrt{t-s}}\sup_z\left[\frac{e^{\bar\alpha z-c_0\frac{|z|^2}{\kappa(t-s)}}}{\sqrt{\kappa}+\sqrt{\kappa}\phi_\kappa(z)}\right].$$

It is a routine to check that both supremum norms are uniformly bounded in κ and $|\xi|$. Therefore (5.34) is obtained. □

5.3 Proof of Theorem 3

Our goal is to show that $\omega(t)$ indeed belongs to $C^1([0,T];\mathfrak{B}^{\lambda,\kappa})$ without the initial layer under the compatibility condition (2.34), and that $\partial_t^2\omega$ in $\mathfrak{B}^{\lambda,\kappa t}$ with the initial layer. The existence of $\omega(t)$ in $C^1([0,T];\mathfrak{B}^{\lambda,\kappa t})$ under the assumption of Theorem 3 can be proved by following the argument of [47] and [54]. For the 2D case, Theorem 1.1 of [47] indeed ensures the existence of $\omega(t)$ in $C^1([0,T];\mathfrak{B}^{\lambda,\kappa t})$ under the assumption of Theorem 3. Such a result follows from Lemma 7, Lemma 11, Lemma 12, Lemma 8, Lemma 9. A 3D result can be obtained analogously. Hence, it suffices to show the propagation of the analytic norms in (2.35).

Step 1: Propagation of analytic norms for ω. It is convenient to define

$$|||\omega(t)|||_t := |||\omega(t)|||_{\infty,\kappa} + |||\omega(t)|||_1. \tag{5.36}$$

The estimation of ω follows from the nonlinear iteration using the representation formula (5.14).

The estimates for the L^1-based norm $|||\omega(t)|||_1$ are already available in Section 5 of [54] (for 2D see Section 4.1 of [47]): From (5.23), (5.24) and (5.33), we have that for

$k = 0, 1, 2$

$$
\begin{aligned}
&\sum_{j=0}^{k} \|(\zeta(x_3)\partial_{x_3})^j \omega_\xi\|_{\mathcal{L}^1_\lambda} \\
&\quad \le \sum_{j=0}^{k} \left\|(\zeta(x_3)\partial_{x_3})^j \int_0^\infty G_\xi(t, x_3, y)\omega_{0\xi}(y)\mathrm{d}y\right\|_{\mathcal{L}^1_\lambda} \\
&\qquad + \sum_{j=0}^{k} \int_0^t \left\|(\zeta(x_3)\partial_{x_3})^j \int_0^\infty G_\xi(t-s, x_3, y)N_\xi(s, y)\mathrm{d}y\right\|_{\mathcal{L}^1_\lambda} \mathrm{d}s \\
&\qquad + \sum_{j=0}^{k} \int_0^t \left\|(\zeta(x_3)\partial_{x_3})^j G_\xi(t-s, x_3, 0)(B_\xi(s), 0)\right\|_{\mathcal{L}^1_\lambda} \mathrm{d}s \\
&\quad \lesssim \sum_{j=0}^{k} \left\|(\zeta(x_3)\partial_{x_3})^j \omega_{0\xi}\right\|_{\mathcal{L}^1_\lambda} + \sum_{j=0}^{k} \int_0^t \left\|(\zeta(x_3)\partial_{x_3})^j N_\xi(s)\right\|_{\mathcal{L}^1_\lambda} \mathrm{d}s + \int_0^t \|N_\xi(s)\|_{\mathcal{L}^1_\lambda} \mathrm{d}s.
\end{aligned}
$$

For $k = 1$, after summing up over $\xi \in \mathbb{Z}^2$, we deduce that

$$
\begin{aligned}
\sum_{0\le|\beta|\le1} \|D^\beta(1+|\nabla_h|)\omega(s)\|_{1,\lambda} \lesssim &\sum_{0\le|\beta|\le1} \|D^\beta(1+|\nabla_h|)\omega_0\|_{1,\lambda} \\
&+ \int_0^t \sum_{0\le|\beta|\le1} \|D^\beta(1+|\nabla_h|)N(s)\|_{1,\lambda}\mathrm{d}s.
\end{aligned} \tag{5.37}
$$

Using (5.6), (5.7), and the definition of $|||\cdot|||_s$ in (5.36) we derive that

$$
\begin{aligned}
\int_0^t \sum_{0\le|\beta|\le1} \|D^\beta(1+|\nabla_h|)N(s)\|_{1,\lambda}\mathrm{d}s &\lesssim \int_0^t |||\omega(s)|||_s^2\left[1 + (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha}\right]\mathrm{d}s \\
&\lesssim \left(t + \frac{1}{\gamma_0}\right) \sup_{0\le s\le t} |||\omega(s)|||_s.
\end{aligned} \tag{5.38}
$$

The second order derivatives can be treated similarly except for the contributions of N for which we apply the analyticity recovery estimate using (3) of Lemma 7 while other terms are estimated in the same way. More precisely, we have

$$
\begin{aligned}
&\sum_{|\beta|=2} \|D^\beta(1+|\nabla_h|)N(s)\|_{1,\lambda} \\
&\quad \lesssim \frac{1}{\tilde{\lambda}-\lambda} \sum_{0\le|\beta|\le1} \|D^\beta(1+|\nabla_h|)N(s)\|_{1,\tilde{\lambda}} \text{ for any } \tilde{\lambda} > \lambda,
\end{aligned} \tag{5.39}
$$

while we choose $\tilde{\lambda} = \frac{\lambda+\lambda_0-\gamma_0 s}{2}$ in particular. We note that still $\tilde{\lambda} < \lambda_0 - \gamma_0 s$ if $\lambda < \lambda_0 - \gamma_0 s$ and hence from (5.6) and (5.7)

$$\begin{aligned}
&\sum_{0\le|\beta|\le 1} \|D^\beta(1+|\nabla_h|)N(s)\|_{1,\tilde{\lambda}(s)}\\
&\lesssim \Big(\sum_{0\le|\beta|\le 1} \|D^\beta(1+|\nabla_h|)\omega(s)\|_{1,\tilde{\lambda}(s)}\Big)\Big(\sum_{0\le|\beta|\le 2} [[D^\beta\omega(s)]]_{\infty,\tilde{\lambda}(s),\kappa}\Big)\\
&\quad+\Big(\sum_{0\le|\beta|\le 2} \|D^\beta(1+|\nabla_h|)\omega(s)\|_{1,\tilde{\lambda}(s)}\Big)\Big(\sum_{0\le|\beta|\le 1} \|D^\beta(1+|\nabla_h|)\omega(s)\|_{1,\tilde{\lambda}(s)}\Big)\\
&\lesssim \big[1+(\lambda_0-\lambda-\gamma_0 s)^{-\alpha}\big]|||\omega(s)|||_s^2.
\end{aligned}$$

Therefore we derive that for $t < \frac{\lambda_0}{2\gamma_0}$ and $\lambda < \lambda_0 - \gamma_0 t$

$$\begin{aligned}
&\sum_{|\beta|=2} \|D^\beta(1+|\nabla_h|)\omega(t)\|_{1,\lambda}\\
&\lesssim \sum_{|\beta|=2} \|D^\beta(1+|\nabla_h|)\omega_0\|_{1,\lambda_0} + \int_0^t \big[1+(\lambda_0-\lambda-\gamma_0 s)^{-(\alpha+1)}\big]|||\omega(s)|||_s^2 \mathrm{d}s\\
&\lesssim \sum_{|\beta|=2} \|D^\beta(1+|\nabla_h|)\omega_0\|_{1,\lambda_0} + \Big((\lambda_0-\lambda-\gamma_0 t)^{-\alpha}\frac{1}{\gamma_0}+t\Big)\sup_{0\le s\le t}|||\omega(s)|||_s^2.
\end{aligned} \tag{5.40}$$

Therefore, we conclude that, from (5.37) with (5.38), and (5.40)

$$|||\omega(t)|||_1 \lesssim \sum_{0\le|\beta|\le 2} \|D^\beta(1+|\nabla_h|)\omega_0\|_{1,\lambda_0} + (t+\frac{1}{\gamma_0})\sup_{0\le s\le t}|||\omega(s)|||_s^2 \quad \text{for } t < \frac{\lambda_0}{2\gamma_0}. \tag{5.41}$$

The propagation of the boundary layer norm $|||\omega(t)|||_{\infty,\kappa}$ can be shown analogously using $\mathcal{L}^\infty_{\lambda,\kappa}$ estimates of Lemma 12 and Lemma 13: For $k = 0, 1, 2$ and $\kappa > 0$ for $i = 1, 2$ and $\kappa = 0$ for $i = 3$ we have

$$\begin{aligned}
&\sum_{j=0}^k \|(\zeta(x_3)\partial_{x_3})^j\omega_{\xi,i}\|_{\mathcal{L}^\infty_{\lambda,\kappa}}\\
&\le \sum_{j=0}^k \Big\|(\zeta(x_3)\partial_{x_3})^j\int_0^\infty G_{\xi i}(t,x_3,y)\omega_{0\xi,i}(y)\mathrm{d}y\Big\|_{\mathcal{L}^\infty_{\lambda,\kappa}}\\
&\quad+\sum_{j=0}^k\int_0^t \Big\|(\zeta(x_3)\partial_{x_3})^j\int_0^\infty G_{\xi,i}(t-s,x_3,y)N_{\xi,i}(s,y)\mathrm{d}y\Big\|_{\mathcal{L}^\infty_{\lambda,\kappa}}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{k}\int_0^t \left\|(\zeta(x_3)\partial_{x_3})^j G_{\xi,i}(t-s,x_3,0)B_{\xi,i}(s)\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \mathrm{d}s\\
&\lesssim \sum_{j=0}^{k}\left\|(\zeta(x_3)\partial_{x_3})^j \omega_{0\xi,i}\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} + \sum_{j=0}^{k}\int_0^t \left\|(\zeta(x_3)\partial_{x_3})^j N_{\xi,i}(s)\right\|_{\mathcal{L}^\infty_{\lambda,\kappa}} \mathrm{d}s\\
&+(1-\delta_{i3})\int_0^t \frac{1}{\sqrt{t-s}}\|N_{\xi,i}\|_{\mathcal{L}^1_\lambda}.
\end{aligned}
$$

Let $k=1$. After summing up over $\xi\in\mathbb{Z}$ and $i=1,2$ (with $\kappa>0$) and $i=3$ (with $\kappa=0$) , we deduce that

$$
\begin{aligned}
\sum_{0\le|\beta|\le1}[[D^\beta\omega(t)]]_{\infty,\lambda,\kappa} \lesssim \sum_{0\le|\beta|\le1}[[D^\beta\omega_0]]_{\infty,\lambda_0,\kappa} + \int_0^t \sum_{0\le|\beta|\le1}[[D^\beta N(s)]]_{\infty,\lambda,\kappa}\mathrm{d}s\\
+\int_0^t \frac{1}{\sqrt{t-s}}|||\omega(s)|||_1^2\mathrm{d}s.
\end{aligned}
$$

Using the definition of $|||\cdot|||_s$ in (5.36), and applying Lemma 10 with (5.6), (5.8), and (5.9), we derive

$$
\begin{aligned}
\sum_{0\le|\beta|\le1}[[D^\beta N(s)]]_{\infty,\lambda,\kappa} &\lesssim \Big(\sum_{0\le|\beta|\le2}\|D^\beta(1+|\nabla_h|)\omega(s)\|_{1,\lambda}\Big)\\
&\quad \Big(\sum_{0\le|\beta|\le1}[[D^\beta\omega(s)\|_{\infty,\lambda,\kappa}\Big)\\
&\quad +\|(1+|\nabla_h|)\omega(s)\|_{1,\lambda}\sum_{|\beta|=2}[[D^\beta\omega(s)]]_{\infty,\lambda,\kappa}\\
&\lesssim \big[1+(\lambda_0-\lambda-\gamma_0 s)^{-\alpha}\big]|||\omega(s)|||_s^2.
\end{aligned}
$$

Therefore we derive that

$$
\begin{aligned}
&\sum_{0\le|\beta|\le1}[[D^\beta\omega(t)]]_{\infty,\lambda,\kappa}\\
&\quad\lesssim \sum_{0\le|\beta|\le1}[[D^\beta\omega_0]]_{\infty,\lambda_0,\kappa} + \int_0^t |||\omega(s)|||_s^2\big[1+(\lambda_0-\lambda-\gamma_0 s)^{-\alpha}\big]\mathrm{d}s\\
&\quad+\int_0^t \frac{1}{\sqrt{t-s}}|||\omega(s)|||_s^2 ds\\
&\lesssim \sum_{0\le|\beta|\le1}[[D^\beta\omega_0]]_{\infty,\lambda_0,\kappa} + \Big(\sqrt{t}+\frac{1}{\gamma_0}\Big)\sup_{0\le s\le t}|||\omega(s)|||_s^2.
\end{aligned}
\tag{5.42}
$$

Now we control the second order derivatives similarly except for the N. As in (5.39) we use the analyticity recovery estimate using Lemma 7

$$\sum_{|\beta|=2}[[D^\beta N(s)]]_{\infty,\lambda,\kappa} \lesssim \frac{1}{\tilde\lambda-\lambda}\sum_{0\le|\beta|\le1}[[D^\beta N(s)]]_{\infty,\tilde\lambda,\kappa} \text{ for any } \tilde\lambda>\lambda, \quad (5.43)$$

while again we choose $\tilde\lambda = \frac{\lambda+\lambda_0-\gamma_0 s}{2}$ in particular. We note that still $\tilde\lambda < \lambda_0-\gamma_0 s$ if $\lambda<\lambda_0-\gamma_0 s$ and hence from (5.8) and (5.9)

$$\sum_{0\le|\beta|\le1}[[D^\beta N(s)]]_{\infty,\tilde\lambda(s),\kappa} \lesssim (\lambda_0-\lambda-\gamma_0 s)^{-\alpha}|||\omega(s)|||_s^2.$$

Therefore we derive that for $t<\frac{\lambda_0}{2\gamma_0}$ and $\lambda<\lambda_0-\gamma_0 t$

$$\begin{aligned}
&\sum_{|\beta|=2}[[D^\beta\omega(t)]]_{\infty,\lambda,\kappa}\\
&\lesssim \sum_{|\beta|=2}\|D^\beta\omega_0\|_{\infty,\lambda_0,\kappa} + \int_0^t(\lambda_0-\lambda-\gamma_0 s)^{-(\alpha+\frac32)}|||\omega(s)|||_s^2 ds\\
&\quad + \int_0^t\frac{1}{\sqrt{t-s}}|||\omega(s)|||_1^2 ds\\
&\lesssim \sum_{|\beta|=2}\|D^\beta\omega_0\|_{\infty,\lambda_0,\kappa} + (\lambda_0-\lambda-\gamma_0 t)^{-\alpha}\\
&\quad\left(\int_0^t(\lambda_0-\lambda-\gamma_0 s)^{-\frac32}ds\right)\sup_{0\le s\le t}|||\omega(s)|||_s^2\\
&\quad + \sqrt{t}\sup_{0\le s\le t}|||\omega(s)|||_s^2\\
&\lesssim \sum_{|\beta|=2}\|D^\beta\omega_0\|_{\infty,\lambda_0,\kappa} + \left((\lambda_0-\lambda-\gamma_0 t)^{-\alpha}\frac{1}{\gamma_0}+\sqrt{t}\right)\sup_{0\le s\le t}|||\omega(s)|||_s^2.
\end{aligned} \quad (5.44)$$

Therefore we conclude that, from (5.42) and (5.44),

$$|||\omega(t)|||_{\infty,\kappa} \lesssim \sum_{0\le|\beta|\le2}\|D^\beta\omega_0\|_{\infty,\lambda_0,\kappa} + \left(\sqrt{t}+\frac{1}{\gamma_0}\right)\sup_{0\le s\le t}|||\omega(s)|||_s^2 \text{ for } t<\frac{\lambda_0}{2\gamma_0}. \quad (5.45)$$

In conclusion, from (5.41), (5.45), and by a standard continuity argument we obtain for sufficiently large γ_0

$$\sup_{0\le t<\frac{\lambda_0}{2\gamma_0}}|||\omega(t)|||_t \lesssim \sum_{0\le|\beta|\le2}\|D^\beta\omega_0\|_{\infty,\lambda_0,\kappa} + \sum_{0\le|\beta|\le2}\|D^\beta(1+|\nabla_h|)\omega_0\|_{1,\lambda_0}. \quad (5.46)$$

Step 2: Propagation of analytic norms for $\partial_t\omega$. The continuity of $\omega(t)$ in t follows from the mild solution form (5.14) of $\omega_\xi(t)$. We claim that $\omega(t)\in C^1([0,T];\mathfrak{B}^{\lambda,\kappa})$ and moreover $\|\!|\partial_t\omega(t)|\!\|_t$ is bounded. To this end, we first derive the mild form of $\partial_t\omega_\xi$ from (5.14):

$$\begin{aligned}\partial_t\omega_\xi(t,x_3)=&\int_0^\infty G_\xi(t,x_3,y)\partial_t\omega_{0\xi}(y)\mathrm{d}y\\&\qquad+\int_0^t\int_0^\infty G_\xi(t-s,x_3,y)\partial_s N_\xi(s,y)\mathrm{d}y\mathrm{d}s\\&-\int_0^t G_\xi(t-s,x_3,0)(\partial_s B_\xi(s),0)\mathrm{d}s,\end{aligned}\tag{5.47}$$

where we recall $\partial_t\omega_0$ in (2.32). To justify this formula, we first recall (5.16)-(5.18). We start with the horizontal part of the formula (5.47) for $\partial_t\omega_{\xi,h}$. From Lemma 11, $G_{\xi h}(t,x_3,y)=H_\xi(t,x_3-y)+H_\xi(t,x_3+y)+R_\xi(t,x_3+y)$. Then by using the fact that $H'_\xi(t,\cdot)$ is an odd function, we see $\partial_{x_3}G_{\xi h}(t,x_3,y)|_{x_3=0}=R'_\xi(t,y)$. Now we read (5.17) as

$$\kappa\eta_0 R'_\xi(t,y)+\kappa\eta_0|\xi|G_{\xi h}(t,0,y)=0,\quad \kappa\eta_0 R'_\xi(t,x_3)+\kappa\eta_0|\xi|G_{\xi h}(t,x_3,0)=0,\tag{5.48}$$

where we have used that $H_\xi(t,\cdot)$ is an even function for the second relation. On the other hand, since we also have $\partial_{y_3}G_{\xi h}(t,x_3,y)|_{y=0}=R'_\xi(t,x_3)$, we deduce that

$$\kappa\eta_0(\partial_{y_3}+|\xi|)G_{\xi h}(t,x_3,y_3)=0,\quad y_3=0.\tag{5.49}$$

It is straightforward to see $\Delta_\xi G_{\xi h}=\partial^2_{x_3}G_{\xi h}-|\xi|^2G_{\xi h}=\partial^2_y G_{\xi h}-|\xi|^2G_{\xi h}$.

We now take ∂_t of (5.14):

$$\begin{aligned}\partial_t&\int_0^\infty G_{\xi h}(t,x_3,y)\omega_{0\xi,h}(y)\mathrm{d}y=\int_0^\infty \partial_t G_{\xi,h}(t,x_3,y)\omega_{0\xi,h}(y)\mathrm{d}y\\&=\int_0^\infty \kappa\eta_0(\partial_y^2-|\xi|^2)G_{\xi h}(t,x_3,y)\omega_{0\xi,h}(y)\mathrm{d}y\\&=\big[\kappa\eta_0\partial_y G_{\xi h}(t,x_3,y)\omega_{0\xi,h}(y)\big]_{y=0}^{y=\infty}-\int_0^\infty\kappa\eta_0|\xi|^2G_{\xi h}(t,x_3,y)\omega_{0\xi,h}(y)\mathrm{d}y\\&\quad-\int_0^\infty\kappa\eta_0\partial_y G_{\xi h}(t,x_3,y)\partial_y\omega_{0\xi,h}(y)\mathrm{d}y\\&=\big[\kappa\eta_0\partial_y G_{\xi h}(t,x_3,y)\omega_{0\xi,h}(y)\big]_{y=0}^{y=\infty}-\big[\kappa\eta_0 G_{\xi h}(t,x_3,y)\partial_y\omega_{0\xi,h}(y)\big]_{y=0}^{y=\infty}\\&\quad+\int_0^\infty G_{\xi h}(t,x_3,y)\kappa\eta_0\Delta_\xi\omega_{0\xi,h}(y)\mathrm{d}y\\&=-\kappa\eta_0\partial_y G_{\xi h}(t,x_3,0)\omega_{0\xi,h}(0)+\kappa\eta_0 G_{\xi h}(t,x_3,0)\partial_y\omega_{0\xi,h}(0)\\&\quad+\int_0^\infty G_{\xi h}(t,x_3,y)\kappa\eta_0\Delta_{\xi h}\omega_{0\xi,h}(y)\mathrm{d}y,\end{aligned}$$

and

$$
\begin{aligned}
&\partial_t \int_0^t \int_0^\infty G_{\xi h}(t-s, x_3, y) N_{\xi,h}(s, y)\mathrm{d}y\mathrm{d}s \\
&\quad = \int_0^\infty G_{\xi h}(t, x_3, y) N_{\xi,h}(0, y)\mathrm{d}y \\
&\qquad + \int_0^t \int_0^\infty G_{\xi h}(s, x_3, y)\partial_t N_{\xi,h}(t-s, y)\mathrm{d}y\mathrm{d}s, \\
&\partial_t \int_0^t G_{\xi h}(t-s, x_3, 0) B_\xi(s)\mathrm{d}s = G_{\xi h}(t, x_3, 0) B_\xi(0) \\
&\qquad + \int_0^t G_{\xi h}(t-s, x_3, 0)\partial_s B_\xi(s)\mathrm{d}s
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\partial_t \omega_{\xi,h}(t, x_3) = &-\kappa\eta_0 \partial_y G_{\xi h}(t, x_3, 0)\omega_{0\xi,h}(0) + \kappa\eta_0 G_{\xi h}(t, x_3, 0)\partial_y \omega_{0\xi,h}(0) \\
&- G_{\xi h}(t, x_3, 0) B_\xi(0) \\
&+ \int_0^\infty G_{\xi h}(t, x_3, y)\{\kappa\eta_0 \Delta_\xi \omega_{0\xi,h}(y) + N_{\xi,h}(0, y)\}\mathrm{d}y \\
&+ \int_0^t \int_0^\infty G_{\xi h}(t-s, x_3, y)\partial_s N_{\xi,h}(s, y)\mathrm{d}y\mathrm{d}s \\
&- \int_0^t G_{\xi h}(t-s, x_3, 0)\partial_s B_\xi(s)\mathrm{d}s.
\end{aligned} \tag{5.50}
$$

Next we show that the first line in the right-hand side is 0. From (5.49)

$$
\begin{aligned}
&-\kappa\eta_0 \partial_y G_{\xi h}(t, x_3, 0)\omega_{0\xi,h}(0) + \kappa\eta_0 G_{\xi h}(t, x_3, 0)\partial_y \omega_{0\xi,h}(0) \\
&\quad = G_{\xi h}(t, x_3, 0)\kappa\eta_0(|\xi| + \partial_y)\omega_{0\xi,h}(0),
\end{aligned}
$$

and hence the first line of (5.50) reads

$$
G_{\xi h}(t, x_3, 0)\left[\kappa\eta_0(|\xi| + \partial_{x_3})\omega_{0\xi,h}(0) - B_\xi(0)\right], \tag{5.51}
$$

which is zero due to the first compatibility condition of (2.34). Recalling $\partial_t \omega_0$ in (2.32), the formula (5.47) for $\partial_t \omega_{\xi,h}$ has been established. We may follow the same procedure to verify the vertical part of the formula (5.47) for $\partial_t \omega_{\xi,3}$ by noting that the second compatibility condition of (2.34) removes the term $-\kappa\eta_0 \partial_y G_{\xi 3}(t, x_3, 0)\omega_{0\xi,3}(0)$ which would create the initial layer otherwise because $\partial_y G_{\xi 3}(t, x_3, 0)$ does not vanish.

We may now repeat Step 1 for $\partial_t \omega$ using the representation formula (5.47). The estimates are obtained in the same fashion. For the nonlinear terms, since $\partial_t N = -u \cdot \nabla \partial_t \omega - \partial_t u \cdot \nabla \omega + \omega \cdot \nabla \partial_t u + \partial_t \omega \cdot \nabla u$, the structure of $\partial_t N$ with respect to $\partial_t \omega$ is consistent with the one of N with respect to ω and we can use the bilinear estimates

(5.3) and (5.4). In summary, one can derive that for $t < \frac{\lambda_0}{2\gamma_0}$

$$
\begin{aligned}
|||\partial_t\omega(t)|||_1 &\lesssim \sum_{0\le|\beta|\le 2} \|D^\beta(1+|\nabla_h|)\partial_t\omega_0\|_{1,\lambda_0} \\
&\quad + \left(t+\frac{1}{\gamma_0}\right)\sup_{0\le s\le t}|||\omega(s)|||_s \sup_{0\le s\le t}|||\partial_t\omega(s)|||_s, \qquad (5.52)\\
|||\partial_t\omega(t)|||_{\infty,\kappa} &\lesssim \sum_{0\le|\beta|\le 2} \|D^\beta\partial_t\omega_0\|_{\infty,\lambda_0,\kappa} \\
&\quad + \left(\sqrt{t}+\frac{1}{\gamma_0}\right)\sup_{0\le s\le t}|||\omega(s)|||_s \sup_{0\le s\le t}|||\partial_t\omega(s)|||_s, \qquad (5.53)
\end{aligned}
$$

which lead to the desired bounds for $\partial_t\omega(t)$ by choosing sufficiently large γ_0.

Step 3: Propagation of analytic norms for $\partial_t^2\omega_\xi$. As a consequence of Step 2, $\partial_t\omega_\xi(t,x_3)$ solves the following system

$$
\partial_t^2\omega_\xi - \kappa\eta_0\Delta_\xi\partial_t\omega_\xi = \partial_t N_\xi \quad \text{in } \mathbb{R}_+, \tag{5.54}
$$

$$
\kappa\eta_0(\partial_{x_3}+|\xi|)\partial_t\omega_{\xi,h} = \partial_t B_\xi \quad \text{on } x_3=0, \tag{5.55}
$$

$$
\partial_t\omega_{\xi,3} = 0 \quad \text{on } x_3=0, \tag{5.56}
$$

with $\partial_t\omega_\xi|_{t=0} = \partial_t\omega_{0\xi}$ for $\xi\in\mathbb{Z}^2$ where $\partial_t\omega_0$ is defined in (2.32). Then as done in Step 2, by using the properties of G_ξ and integration by parts and by the last compatibility condition of (2.34), we can derive the representation formula for $\partial_t^2\omega$:

$$
\begin{aligned}
\partial_t^2\omega_\xi(t,x_3) &= (G_{\xi h}(t,x_3,0)\left[\kappa\eta_0(|\xi|+\partial_{x_3})\partial_t\omega_{0\xi,h}(0)-\partial_t B_\xi(0)\right],0) \\
&\quad + \int_0^\infty G_\xi(t,x_3,y)\partial_t^2\omega_{0\xi}(y)\mathrm{d}y + \int_0^t\int_0^\infty G_\xi(t-s,x_3,y)\partial_s^2 N_\xi(s,y)\mathrm{d}y\mathrm{d}s \\
&\quad - \int_0^t G_\xi(t-s,x_3,0)(\partial_s^2 B_\xi(s),0)\mathrm{d}s,
\end{aligned} \tag{5.57}
$$

where we recall $\partial_t^2\omega_0$ in (2.32). As we do not require higher order compatibility condition for the horizontal vorticity, a new term representing the initial-boundary layer emerges. We first examine $G_\xi(t,z,0)$. Recall (5.35).

Similar to Lemma 13, we have for $C_0<\infty$

$$
\sum_{j=0}^k \left\|(\zeta(z)\partial_z)^j G_\xi(t,z,0)\right\|_{\mathcal{L}^1_\lambda} \lesssim C_0, \quad \sum_{j=0}^k \left\|(\zeta(z)\partial_z)^j G_\xi(t,z,0)\right\|_{\mathcal{L}^\infty_{\lambda,\kappa t}} \lesssim C_0. \tag{5.58}
$$

From (5.58), (5.33) and (5.3)

$$\sum_{0\le|\beta|\le 2}\sum_{\xi\in\mathbb{Z}^2} e^{\lambda|\xi|}\Big\| D_\xi^\beta(1+|\xi|) \Big[(G_{\xi h}(t,x_3,0)\big(\kappa\eta_0(|\xi|+\partial_{x_3})\partial_t\omega_{0\xi,h}(0)-\partial_t B_\xi(0)\big),0)\Big]\Big\|_{\mathcal{L}^1_\lambda}$$
$$\lesssim \kappa\eta_0\sum_{0\le|\beta|\le 2}\|\nabla_h^\beta(1+|\nabla_h|)\nabla\partial_t\omega_{0,h}\|_{1,\lambda}+\sum_{0\le|\beta|\le 2}\|\nabla_h^\beta(1+|\nabla_h|)\partial_t N(0)\|_{1,\lambda}$$
$$\lesssim \kappa\eta_0\|(1+|\nabla_h|^3)\nabla\partial_t\omega_0\|_{1,\lambda}$$
$$+\|(1+|\nabla_h|^4)\partial_t\omega_0\|_{1,\lambda}\sum_{0\le|\beta|\le 1}\|D^\beta(1+|\nabla_h|^3)\partial_t\omega_0\|_{1,\lambda}.$$

Hence an L^1-based analytic norm is easily obtained as

$$\Big|\Big|\Big|\partial_t^2\omega(t)\Big|\Big|\Big|_1$$
$$\lesssim \kappa\eta_0\|(1+|\nabla_h|^3)\nabla\partial_t\omega_0\|_{1,\lambda}+\|(1+|\nabla_h|^4)\partial_t\omega_0\|_{1,\lambda}\sum_{0\le|\beta|\le 1}\|D^\beta(1$$
$$+|\nabla_h|^3)\partial_t\omega_0\|_{1,\lambda}$$
$$+\sum_{0\le|\beta|\le 2}\|D^\beta(1+|\nabla_h|)\partial_t^2\omega_0\|_{1,\lambda_0}+(t+\frac{1}{\gamma_0})\sup_{0\le s\le t}|||\omega(s)|||_s\sup_{0\le s\le t}\Big|\Big|\Big|\partial_t^2\omega(s)\Big|\Big|\Big|_s$$
$$+(t+\frac{1}{\gamma_0})\sup_{0\le s\le t}|||\partial_t\omega(s)|||_s^2. \tag{5.59}$$

Now we move to the L^∞-based analytic norm bound. We compute $\|G_\xi(t,z,0)\|_{\mathcal{L}^\infty_{\lambda,\kappa t}}$:

$$\|(\zeta(z)\partial_z)^j G_\xi(t,z,0)\|_{\mathcal{L}^\infty_{\lambda,\kappa t}}\lesssim \sup_z\left[\frac{be^{(\bar\alpha-c_0b)z}}{1+\phi_\kappa(z)+\phi_{\kappa t}(z)}\right]$$
$$+\sup_z\left[\frac{e^{\bar\alpha z-c_0\frac{|z|^2}{\kappa t}}}{\sqrt{\kappa t}+\sqrt{\kappa t}\phi_\kappa(z)+\sqrt{\kappa t}\phi_{\kappa t}(z)}\right].$$

It is a routine to check that both supremum norms are uniformly bounded in κ and $|\xi|$. Hence (5.58) shows that the kernel $G_\xi(t,z,0)$ is well-behaved in $\mathcal{L}^1_\lambda$ and the initial-boundary layer analytic space $\mathcal{L}^\infty_{\lambda,\kappa t}$. Then we run the same argument as in Step 2 but with $\mathcal{L}^\infty_{\lambda,\kappa t}$ in place of $\mathcal{L}^\infty_{\lambda,\kappa}$. Thanks to (5.58), the estimates of the first term in (5.57) are bounded by the initial norm (2.33):

$$\sum_{0\le|\beta|\le 2}\sum_{\zeta\in\mathbb{Z}^2} e^{\lambda|\xi|}\Big\| D_\xi^\beta\Big[(G_{\xi h}(t,x_3,0)\big(\kappa\eta_0(|\xi|+\partial_{x_3})\partial_t\omega_{0\xi,h}(0)-\partial_t B_\xi(0)\big),0)\Big]\Big\|_{\mathcal{L}^\infty_{\lambda,\kappa t}}$$

$$\lesssim \kappa\eta_0 \sum_{0\le|\beta|\le 2} \|\nabla_h^\beta \nabla \partial_t \omega_0\|_{\infty,\lambda} + \sum_{0\le|\beta|\le 2} \|\nabla_h^\beta \partial_t N(0)\|_{1,\lambda}$$
$$\lesssim \kappa\eta_0 \sum_{0\le|\beta|\le 2} \|\nabla_h^\beta \nabla \partial_t \omega_0\|_{\infty,\lambda}$$
$$+ \|(1+|\nabla_h|^3)\partial_t\omega_0\|_{1,\lambda} \sum_{0\le|\beta|\le 1} \|D^\beta(1+|\nabla_h|^2)\partial_t\omega_0\|_{1,\lambda}.$$

Other three terms are estimated in the same way as in [47] or [54] and we arrive that

$$\begin{aligned}
&\left|\left|\left|\partial_t^2\omega(t)\right|\right|\right|_{\infty,\kappa t} \\
&\quad\lesssim \kappa\eta_0 \sum_{0\le|\beta|\le 2} \|\nabla_h^\beta \nabla \partial_t \omega_0\|_{\infty,\lambda} \\
&\quad+ \|(1+|\nabla_h|^3)\partial_t\omega_0\|_{1,\lambda} \sum_{0\le|\beta|\le 1} \|D^\beta(1+|\nabla_h|^2)\partial_t\omega_0\|_{1,\lambda} \\
&\quad+ \sum_{0\le|\beta|\le 2} \|D^\beta\partial_t^2\omega_0\|_{\infty,\lambda_0,\kappa t} + (\sqrt{t}+\frac{1}{\gamma_0}) \sup_{0\le s\le t} |||\omega(s)|||_s \sup_{0\le s\le t} \left|\left|\left|\partial_t^2\omega(s)\right|\right|\right|_s \\
&\quad+ (\sqrt{t}+\frac{1}{\gamma_0}) \sup_{0\le s\le t} |||\partial_t\omega(s)|||_s^2.
\end{aligned} \tag{5.60}$$

Finally combining (5.59) and (5.60) and then choosing sufficiently large γ_0 we derive a desired estimate for $\left|\left|\left|\partial_t^2\omega(t)\right|\right|\right|_t$ for $t \in (0, \frac{\lambda_0}{2\gamma_0})$.

Altogether from (5.46), (5.52), (5.53), (5.59), and (5.60), we finish the proof of the estimate (2.35).

Step 4: Estimate (1), vorticity estimates. Both (2.36) and (2.37) are direct consequences of (2.35). To show (2.38), we first note that the boundedness of $\omega(t)$ norms implies $|\partial_{x_3}\omega_\xi(t,x_3)| \lesssim e^{-\bar{\alpha}x_3}e^{-\lambda|\xi|}$ for all $|\xi|$ and $x_3 \ge 1$ (away from the boundary). When $x_3 \le 1$, we draw on the equation (5.10) to rewrite $\partial_{x_3}^2\omega_{\xi,h} = \frac{1}{\kappa\eta_0}\{\partial_t\omega_{\xi,h} + \kappa\eta_0|\xi|^2\omega_{\xi,h} - N_{\xi,h}\}$ and the boundary condition (5.11):

$$\begin{aligned}
\partial_{x_3}\omega_{\xi,h}(t,x_3) &= \partial_{x_3}\omega_{\xi,h}(t,0) + \int_0^{x_3} \partial_{x_3}^2\omega_{\xi,h}(t,y)\mathrm{d}y \\
&= -|\xi|\omega_{\xi,h}(t,0) + \frac{1}{\kappa\eta_0}B_\xi(t) + \int_0^{x_3} \frac{1}{\kappa\eta_0}[\partial_t\omega_{\xi,h} + \kappa\eta_0|\xi|^2\omega_{\xi,h} \\
&\quad - N_{\xi,h}](t,y)\mathrm{d}y.
\end{aligned} \tag{5.61}$$

We now appeal to $|B_\xi(t)| \leq \|N_\xi(t)\|_{\mathcal{L}^1_\lambda}$ and $\sum_{0\leq\ell\leq1}(\left|\!\left|\!\left|\partial_t^\ell\omega(t)\right|\!\right|\!\right|_{\infty,\kappa}+\left|\!\left|\!\left|\partial_t^\ell\omega(t)\right|\!\right|\!\right|_1) < \infty$ to obtain that for all $x_3 \in \mathbb{R}_+$

$$|\partial_{x_3}\omega_{\xi,h}(t,x_3)| \lesssim \frac{1}{\kappa}e^{-\bar{\alpha}x_3}e^{-\lambda|\xi|} \text{ for } 0<\lambda<\lambda_0, \tag{5.62}$$

which proves (2.38) for ω_h and $\ell = 0$. The remaining case can be estimated similarly. Near $O(1)$ boundary, from (5.54) and (5.55), we derive

$$\begin{aligned}&\partial_{x_3}\partial_t\omega_{\xi,h}(t,x_3)\\&\quad= -|\xi|\partial_t\omega_{\xi,h}(t,0)+\frac{1}{\kappa\eta_0}\partial_t B_\xi(t)\\&\qquad+\int_0^{x_3}\frac{1}{\kappa\eta_0}[\partial_t^2\omega_{\xi,h}+\kappa\eta_0|\xi|^2\partial_t\omega_{\xi,h}-\partial_t N_{\xi,h}](t,y)\mathrm{d}y.\end{aligned} \tag{5.63}$$

Together with $\sum_{0\leq\ell\leq1}\left|\!\left|\!\left|\partial_t^\ell\omega(t)\right|\!\right|\!\right|_{\infty,\kappa}+\sum_{0\leq\ell\leq2}\left|\!\left|\!\left|\partial_t^\ell\omega(t)\right|\!\right|\!\right|_1 < \infty$ we deduce (2.38) for ω_h and $\ell = 1$. For ω_3 we use $\nabla\cdot\omega = 0$ to write $\partial_3\omega_3 = -\partial_1\omega_1 - \partial_2\omega_2$. Now (2.38) for ω_3 follows from (2.36).

Step 5: Estimate (2), velocity estimates, except (2.42). From (5.2)

$$|\xi|^{\beta_h}|\partial_z^{\beta_3}\partial_t^\ell\phi_\xi(t,z)| \lesssim \int_{\partial\mathcal{H}_\lambda}|\xi|^{|\beta|-1}e^{-|\xi||y-z|}|\partial_t^\ell\omega_\xi(t,y)||\mathrm{d}y| \ \text{ for } \beta_3\leq 1. \tag{5.64}$$

For $|\beta| = |\beta_h|+\beta_3 = 1$ we bound (5.64) by $e^{-\lambda|\xi|}\|\partial_t^\ell\omega(t)\|_{1,\lambda}$. Then from (2.35) we conclude (2.39).

For $|\beta|\geq 2$ and $\beta_3\leq 1$, we bound (5.64) by

$$\begin{aligned}(5.64) &\lesssim \int_{\partial\mathcal{H}_\lambda}|\xi|^{|\beta|-2}|\xi|e^{-|\xi||y-z|}e^{-\bar{\alpha}\mathrm{Re}\,y}e^{-\lambda|\xi|}\big(1+\phi_\kappa(y)+\phi_{\kappa t}(y)\big)|\mathrm{d}y|\\&\lesssim |\xi|^{|\beta|-2}e^{-\lambda|\xi|}e^{-\min(1,\frac{\bar{\alpha}}{2})x_3}\int_{\partial\mathcal{H}_\lambda}e^{-\frac{\bar{\alpha}}{2}\mathrm{Re}\,y}\big(1+\phi_\kappa(y)+\phi_{\kappa t}(y)\big)|\mathrm{d}y|\\&\lesssim |\xi|^{|\beta|-2}e^{-\lambda|\xi|}e^{-\min(1,\frac{\bar{\alpha}}{2})x_3} \ \text{ for } |\beta|\geq 2, \text{ and } \beta_3\leq 1,\\&\quad \text{and } \ell = 0,1,2, \text{ and } t\in[0,T],\end{aligned} \tag{5.65}$$

where we have used $|\xi||y-z|+\frac{\bar{\alpha}}{2}\mathrm{Re}\,y \geq \min(1,\frac{\bar{\alpha}}{2})x_3$ for $|\xi|\geq 1$ and (2.35).

For $\beta_3 = 2,3$ we use $\partial_z^2\partial_t^\ell\phi_\xi = |\xi|^2\partial_t^\ell\phi_\xi + \partial_t^\ell\omega_\xi$. Then following the same argument of (5.65), we derive

$$\begin{aligned}&|\xi|^{\beta_h}|\partial_z^{\beta_3}\partial_t^\ell\phi_\xi(t,z)|\\&\quad\lesssim |\xi|^{|\beta_h|+2}|\partial_z^{\beta_3-2}\partial_t^\ell\phi_\xi(t,z)|+|\xi|^{\beta_h}|\partial_z^{\beta_3-2}\partial_t^\ell\omega_\xi(t,z)|\\&\quad\lesssim \begin{cases}(|\xi|^{|\beta|-2}+|\xi|^{\beta_h})e^{-\lambda|\xi|}e^{-\min(1,\frac{\bar{\alpha}}{2})\mathrm{Re}\,z}(1+\phi_\kappa(z)) \ \text{ for } \ell=0,1, \text{ and } \beta_3=2,\\(|\xi|^{|\beta|-2}+|\xi|^{\beta_h})e^{-\lambda|\xi|}e^{-\min(1,\frac{\bar{\alpha}}{2})\mathrm{Re}\,z}\kappa^{-1} \ \text{ for } \ell=0,1, \text{ and } \beta_3=3,\\(|\xi|^{|\beta|-2}+|\xi|^{\beta_h})e^{-\lambda|\xi|}e^{-\min(1,\frac{\bar{\alpha}}{2})\mathrm{Re}\,z}(1+\phi_\kappa(z)+\phi_{\kappa t}(z)) \ \text{ for } \ell=2, \text{ and } \beta_3=2.\end{cases}\end{aligned}$$

(5.66)

Finally from (5.65) and (5.66) we conclude (2.40) and (2.41).

Step 6: Estimate (3), pressure estimates and (2.42). We next turn to the pressure. Taking the divergence to (1.13) and using (1.14), we deduce

$$-\Delta p=\sum_{\ell,m=1}^{3}\partial_\ell u_m\partial_m u_\ell. \tag{5.67}$$

We obtain the boundary condition of p by reading the third component of (1.13), and then using (1.14) and (1.15),

$$\begin{aligned}\partial_3 p&=\kappa\eta_0\Delta u_3=\kappa\eta_0\partial_3\partial_3 u_3=-\kappa\eta_0\partial_1\partial_3 u_1-\kappa\eta_0\partial_2\partial_3 u_2\\&=-\kappa\eta_0\partial_1(\omega_2+\partial_1 u_3)-\kappa\eta_0\partial_2(-\omega_1+\partial_2 u_3)\\&=-\kappa\eta_0\partial_1\omega_2+\kappa\eta_0\partial_2\omega_1\ \text{ for } x_3=0,\end{aligned} \tag{5.68}$$

where $\omega_1=\partial_2 u_3-\partial_3 u_2$ and $\omega_2=-\partial_1 u_3+\partial_3 u_1$.

In the Fourier side we read the problem as

$$\begin{aligned}(|\xi|^2-\partial_3^2)p_\xi(t,x_3)=g_\xi(t,x_3):=\sum_{\ell,m=1}^{3}(\partial_\ell u_m\partial_m u_\ell)_\xi(t,x_3)\ \text{ for } x_3\in\mathbb{R}_+,\\ \partial_3 p_\xi(t,0)=-i\kappa\eta_0\xi_1\omega_{\xi,2}(t,0)+i\kappa\eta_0\xi_2\omega_{\xi,1}(t,0).\end{aligned} \tag{5.69}$$

A representation of $p_\xi(t,x_3)$ is given by

$$\begin{aligned}p_\xi(t,x_3)=&-\int_0^{x_3}\frac{1}{2|\xi|}e^{-|\xi|(x_3-y)}g_\xi(y)\mathrm{d}y-\int_{x_3}^{\infty}\frac{1}{2|\xi|}e^{-|\xi|(y-x_3)}g_\xi(y)\mathrm{d}y\\&-\int_0^{\infty}\frac{1}{2|\xi|}e^{-|\xi|(y+x_3)}g_\xi(y)\mathrm{d}y\\&-\frac{1}{|\xi|}e^{-|\xi|x_3}(-i\kappa\eta_0\xi_1\omega_{\xi,2}(t,0)+i\kappa\eta_0\xi_2\omega_{\xi,1}(t,0)),\end{aligned} \tag{5.70}$$

which is valid for all $\xi\neq 0$. When $\xi=0$, by integrating (5.69) and by using the boundary conditions $\partial_3 p_0(t,0)=0$, $u(t,x_h,0)=0$ and the divergence free condition

$\nabla \cdot u = 0$, we first obtain

$$\begin{aligned}\partial_3 p_0(t, x_3) &= -\frac{1}{(2\pi)^2} \int_0^{x_3} \iint_{\mathbb{T}^2} \sum_{\ell,m=1}^{3} \partial_\ell u_m \partial_m u_\ell \mathrm{d}x_h \mathrm{d}y_3 \\ &= -\frac{1}{(2\pi)^2} \iint_{\mathbb{T}^2} (u \cdot \nabla u_3)(t, x_h, x_3) \mathrm{d}x_h \\ &= -\frac{2}{(2\pi)^2} \iint_{\mathbb{T}^2} (u_3 \partial_3 u_3)(t, x_h, x_3) \mathrm{d}x_h,\end{aligned} \tag{5.71}$$

where we have used the integration by parts and $\nabla \cdot u = 0$ at the last step.

Observe that $\partial_3 p_0$ decays exponentially in x_3, and in particular $\int_0^\infty |\partial_3 p_0(t, x_3)| \mathrm{d}x_3 < \infty$. The integration yields

$$p_0(t, x_3) = p_0(t, 0) - \int_0^{x_3} \frac{2}{(2\pi)^2} \iint_{\mathbb{T}^2} (u_3 \partial_3 u_3)(t, x_h, y_3) \mathrm{d}x_h \mathrm{d}y_3.$$

Since $p_0(t, 0)$ is a free constant in x_3, we fix $p_0(t, x_3)$ by choosing

$$p_0(t, 0) = \frac{2}{(2\pi)^2} \int_0^\infty \iint_{\mathbb{T}^2} (u_3 \partial_3 u_3)(t, x_h, y_3) \mathrm{d}x_h \mathrm{d}y_3 < \infty,$$

such that

$$p_0(t, x_3) = \frac{2}{(2\pi)^2} \int_{x_3}^\infty \iint_{\mathbb{T}^2} (u_3 \partial_3 u_3)(t, x_h, y_3) \mathrm{d}x_h \mathrm{d}y_3. \tag{5.72}$$

The pressure p is then recovered by

$$p(t, x_h, x_3) = p_0(t, x_3) + \sum_{|\xi| \geq 1, \xi \in \mathbb{Z}^2} p_\xi(t, x_3) e^{i x_h \cdot \xi}, \tag{5.73}$$

where $p_0(t, x_3)$ and $p_\xi(t, x_3)$ are given in (5.72) and (5.70).

Now the pressure estimate follows readily from the velocity and vorticity estimates. To show (2.43), we first note from (2.39) and (2.40) $|p_0(t, x_3)| \lesssim |u_3(t, x)| \int_\Omega |\partial_3 u_3(t, x)| \mathrm{d}x \lesssim 1$ and from Lemma 8

$$\begin{aligned}|g_\xi| &\lesssim e^{-\lambda|\xi|} \sum_{i=1}^{2} \left(\|\partial_i u_h\|_{\infty,\lambda}^2 + \|\zeta^{-1} \partial_i u_3\|_{\infty,\lambda} \|\zeta \partial_3 u_i\|_{\infty,\lambda} \right) \\ &\lesssim e^{-\lambda|\xi|} \left[\sum_{0 \leq |\beta| \leq 1} \|\nabla_h^\beta \omega\|_{1,\lambda}^2 + \sum_{1 \leq |\beta| \leq 2} \|\nabla_h^\beta \omega_h\|_{1,\lambda} \right. \\ &\qquad \left. \times \left(\sum_{0 \leq |\beta| \leq 1} \|\nabla_h^\beta \omega\|_{1,\lambda} + \|\zeta \omega_h\|_{\infty,\lambda} \right) \right],\end{aligned}$$

from which we deduce $|p(t, x_h, x_3)| \lesssim 1$. The estimation of $\partial_t p$ and $\partial_t^2 p$ follows analogously.

For the decay estimates (2.44), we start with $\ell = 0$ and $\beta = 0$. Due to our choice of $p_0(t, x_3)$ in (5.72), using (2.39) and (2.40), we have the spatial decay for $p_0(t, x_3)$:

$$|p_0(t, x_3)| \lesssim \int_{x_3}^{\infty} \iint_{\mathbb{T}^2} (1 + \phi_\kappa(y_3)) e^{-\min(1, \frac{\bar{\alpha}}{2}) y_3} \mathrm{d}x_h \mathrm{d}y_3 \lesssim \kappa^{-\frac{1}{2}} e^{-\min(1, \frac{\bar{\alpha}}{2}) x_3}.$$

For $\xi \neq 0$, we use another estimate for $|g_\xi|$ and Lemma 8

$$\begin{aligned} |g_\xi(y)| &\lesssim \sum_{\ell,m=1}^{3} \sum_{\eta \in \mathbb{Z}^2} e^{-\lambda|\xi-\eta|} e^{-\min(1, \frac{\bar{\alpha}}{2}) y} (1 + \phi_\kappa(y)) |(\partial_m u_\ell)_\eta(y)| \\ &\lesssim \kappa^{-\frac{1}{2}} e^{-\lambda|\xi|} e^{-\min(1, \frac{\bar{\alpha}}{2}) y} \sum_{\ell,m=1}^{3} \sum_{\eta \in \mathbb{Z}^2} e^{\lambda|\eta|} |(\partial_m u_\ell)_\eta(y)|, \end{aligned} \tag{5.74}$$

from which we deduce that $|p_\xi(t, x_3)| \lesssim \kappa^{-\frac{1}{2}} e^{-\min(1, \frac{\bar{\alpha}}{2}) x_3}$. Hence (2.44) holds for $\ell = 0$ and $\beta = 0$. For the pressure gradient estimate when $|\beta| = 1$, from (5.71) and (2.40) we first note

$$|\partial_3 p_0(t, x_3)| \lesssim \sup_{x_h \in \mathbb{T}^2} (|u_3||\partial_3 u_3|) \lesssim (1 + \phi_\kappa(x_3)) e^{-\min(1, \frac{\bar{\alpha}}{2}) x_3}.$$

For $\xi \neq 0$, by (5.74) it is easy to see that $|\xi p_\xi(t, x_3)| \lesssim \kappa^{-\frac{1}{2}} e^{-\min(1, \frac{\bar{\alpha}}{2}) x_3}$. Note that $\partial_3 p_\xi(t, x_3)$ has a similar integral form as $|\xi| p_\xi(t, x_3)$ and the estimate follows in the same way, which results in $|\partial_3 p_\xi(t, x_3)| \lesssim \kappa^{-\frac{1}{2}} e^{-\min(1, \frac{\bar{\alpha}}{2}) x_3}$. This finishes (2.44) for $\ell = 0$ and $|\beta| = 1$. The remaining cases for $\ell = 1$ and $|\beta| = 0, 1$ can be treated in the same way.

For the decay estimate of $\partial_t^2 p$, we take into account the initial layer which occurs at $\partial_t^2 \omega$ and $\nabla \partial_t^2 u$. First using (2.39), (2.40) and (2.41) we have

$$\begin{aligned} |\partial_t^2 p_0(t, x_3)| &\lesssim \left| \int_{x_3}^{\infty} \iint_{\mathbb{T}^2} (u_3 \partial_3 \partial_t^2 u_3 + \partial_t^2 u_3 \partial_3 u_3 + 2\partial_t u_3 \partial_3 \partial_t u_3)(t, x_h, y_3) \mathrm{d}x_h \mathrm{d}y_3 \right| \\ &\lesssim \big(1 + \phi_\kappa(x_3) + \phi_{\kappa t}(x_3)\big) e^{-\min(1, \frac{\bar{\alpha}}{2}) x_3}, \end{aligned}$$

while for $|\xi| \neq 0$ we have

$$\begin{aligned} |\partial_t^2 g_\xi(y)| &\lesssim \sum_{\ell,m=1}^{3} \sum_{\eta \in \mathbb{Z}^2} e^{-\lambda|\xi-\eta|} e^{-\min(1, \frac{\bar{\alpha}}{2}) y} (1 + \phi_\kappa(y)) |(\partial_m u_\ell)_\eta(y)| \\ &\lesssim \kappa^{-\frac{1}{2}} e^{-\lambda|\xi|} e^{-\min(1, \frac{\bar{\alpha}}{2}) y} \sum_{i=1}^{2} \sum_{\ell,m=1}^{3} \sum_{\eta \in \mathbb{Z}^2} e^{\lambda|\eta|} |(\partial_m \partial_t^i u_\ell)_\eta(y)|, \end{aligned}$$

from which we deduce (2.45).

The last estimate for $\partial_t^\ell u$ for $\ell = 1, 2$ follows from the equation: $\partial_t u = \kappa\eta_0 \Delta u - u \cdot \nabla u - \nabla p$ and $\partial_t^2 u = \kappa\eta_0 \Delta \partial_t u - u \cdot \nabla \partial_t u - \partial_t u \cdot \nabla u - \nabla \partial_t p$.

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Appendix A. Sobolev Embedding in 1D

Often we have used a standard 1D embedding: For $T > 0$,

$$|g(t)|^2 \lesssim_T \int_0^T |g(s)|^2 \mathrm{d}s + \int_0^T |g'(s)|^2 \mathrm{d}s \quad \text{for } t \in [0, T]. \tag{A.1}$$

A proof is based on an equality:

$$|g(t)|^2 = \frac{1}{T/2} \int_t^{t+T/2} \Big(g(s) - \int_t^s g'(\tau) \mathrm{d}\tau \Big)^2 \mathrm{d}s.$$

For $0 < t \leq T/2$,

$$\begin{aligned}
|g(t)|^2 &\leq \frac{1}{T/2} \int_t^{t+T/2} \Big(2|g(s)|^2 + 2\Big| \int_t^s g'(\tau) \mathrm{d}\tau \Big|^2 \Big) \mathrm{d}s \\
&\leq \frac{1}{T/2} \int_t^{t+T/2} \Big(2|g(s)|^2 + 2|s-t| \int_t^s |g'(\tau)|^2 \mathrm{d}\tau \Big) \mathrm{d}s \\
&\leq \frac{2}{T/2} \int_t^{t+T/2} |g(s)|^2 \mathrm{d}s + \frac{2}{T/2} \int_t^{t+T/2} |s-t| \int_t^s |g'(\tau)|^2 \mathrm{d}\tau \mathrm{d}s \\
&\leq \frac{2}{T/2} \int_t^{t+T/2} |g(s)|^2 \mathrm{d}s + T \int_t^{t+T/2} |g'(s)|^2 \mathrm{d}s \\
&\lesssim_T \int_0^T |g(s)|^2 \mathrm{d}s + \int_0^T |g'(s)|^2 \mathrm{d}s.
\end{aligned}$$

For $T/2 < t \leq T$, using

$$|g(t)|^2 = \frac{1}{T/2} \int_{t-T/2}^t \Big(g(s) - \int_t^s g'(\tau) \mathrm{d}\tau \Big)^2 \mathrm{d}s,$$

we derive that

$$
\begin{aligned}
|g(t)|^2 &\leq \frac{1}{T/2}\int_{t-T/2}^{t}\Big(2|g(s)|^2 + 2\Big|\int_t^s g'(\tau)\mathrm{d}\tau\Big|^2\Big)\mathrm{d}s \\
&\leq \frac{2}{T/2}\int_{t-T/2}^{t}|g(s)|^2\mathrm{d}s + T\int_{t-T/2}^{t}|g'(s)|^2\mathrm{d}s \\
&\lesssim_T \int_0^T |g(s)|^2\mathrm{d}s + \int_0^T |g'(s)|^2\mathrm{d}s.
\end{aligned}
$$

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