



On the Cauchy Problem for the Muskat Equation. II: Critical Initial Data

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Abstract

We prove that the Cauchy problem for the Muskat equation is well-posed locally in time for any initial data in the critical space of Lipschitz functions with three-half derivative in L^2 . Moreover, we prove that the solution exists globally in time under a smallness assumption.

1 Introduction

The Muskat equation describes the dynamics of the interface separating two fluids in porous media whose velocities obey Darcy's law [22,30]. This equation belongs to the family of nonlocal parabolic equations that have attracted a lot of attention in recent years. Indeed, it has long been observed that one can reduce the Muskat equation to an evolution equation for the free surface parametrization (see [9,24,34,35]). One interesting feature of the Muskat equation is that it admits a compact formulation in terms of finite differences, as observed by Córdoba and Gancedo [19]. More precisely, assume that the free surface is the graph of some function $f = f(t, x)$ with $x \in \mathbb{R}$. Then, Córdoba and Gancedo [19] showed that the Muskat equation reduces to

$$\partial_t f = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha, \quad (1)$$

where $\Delta_\alpha f$ is the slope, defined by

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$$\Delta_\alpha f(x, t) = \frac{f(x, t) - f(x - \alpha, t)}{\alpha}. \quad (2)$$

It is easily verified that the Muskat equation is invariant by the change of unknowns:

$$f(t, x) \mapsto f_\lambda(t, x) := \frac{1}{\lambda} f(\lambda t, \lambda x) \quad (\lambda \neq 0). \quad (3)$$

Now, by a direct calculation,

$$\|f_\lambda|_{t=0}\|_{\dot{W}^{1,\infty}} = \|f_0\|_{\dot{W}^{1,\infty}}; \quad \|f_\lambda|_{t=0}\|_{\dot{H}^{\frac{3}{2}}} = \|f_0\|_{\dot{H}^{\frac{3}{2}}}.$$

This means that the spaces $\dot{W}^{1,\infty}(\mathbb{R})$ and $\dot{H}^{\frac{3}{2}}(\mathbb{R})$ are critical for the study of the Cauchy problem. Let us clarify that we denoted by $\dot{W}^{1,\infty}(\mathbb{R})$ the space of Lipschitz functions, and by $H^s(\mathbb{R})$ (resp. $\dot{H}^s(\mathbb{R})$) the classical Sobolev (resp. homogeneous Sobolev) space of order s . They are equipped with the norm defined by

$$\|u\|_{\dot{W}^{1,\infty}} := \sup_{\substack{x, y \in \mathbb{R} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|},$$

and

$$\|u\|_{\dot{H}^s} := \left(\int_{\mathbb{R}} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \quad \|u\|_{H^s}^2 = \|u\|_{\dot{H}^s}^2 + \|u\|_{L^2}^2.$$

We are interested in the study of the Cauchy problem for the latter equation. Our main result states that the Cauchy problem for the Muskat equation is well-posed locally in time for any initial data in the critical space $\dot{W}^{1,\infty}(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R})$.

Our analysis is inspired by many previous works, and we begin by reviewing the literature on this problem. The first well-posedness results were established by Yi [38], Ambrose [4,5], Córdoba and Gancedo [19], Córdoba, Córdoba and Gancedo [18], Cheng, Granero-Belinchón, Shkoller [14]. In recent years, these results were extended in several directions. In particular, the well-posedness of the Cauchy problem has been established in many sub-critical spaces: see Constantin, Gancedo, Shvydkoy and Vicol [17] for initial data in the Sobolev space $W^{2,p}(\mathbb{R})$ for some $p > 1$, Deng, Lei and Lin [23] and Camerón [10] for initial data in Hölder spaces, and Matic [29], Alazard and Lazar [2], Nguyen and Pausader [31] for initial data in $H^s(\mathbb{R})$ with $s > 3/2$.

Special features of the Muskat equations were exploited to improve the analysis of the Cauchy problem in several directions. Constantin, Córdoba, Gancedo, Rodríguez-Piazza and Strain [15] (see also [17,33]) proved a global well-posedness results assuming that the Lipschitz semi-norm is smaller than 1. Deng, Lei and Lin in [23] proved the existence of solutions whose slope can be arbitrarily large. Cameron [10] exhibited the existence of a modulus of continuity for the derivative (see also [1]) and obtained a global existence result assuming only that the product of the maximal and minimal slopes is bounded by 1. Córdoba and Lazar established in [21] the first

global well-posedness result assuming only that the initial data is sufficiently smooth and that the critical $\dot{H}^{3/2}(\mathbb{R})$ -norm is small enough (see also [25,26,28] for related global well-posedness results in Wiener spaces in the critical case, for small enough initial data). This result was extended to the 3D case by Gancedo and Lazar [27] for initial data in the critical Sobolev space $\dot{H}^2(\mathbb{R}^2)$. Eventually, in our companion paper [3], we initiated the study of the Cauchy problem for non-Lipschitz initial data.

For our subject matter, another fundamental component of the background is that the Cauchy problem is not well-posed globally in time: there are blow-up results for some large enough data by Castro, Córdoba, Fefferman, Gancedo and López-Fernández [11–13]. More precisely, they proved the existence of solutions such that at initial time $t = 0$ the interface is a graph, at a later time $t_1 > 0$ the interface is no longer a graph and then at a subsequent time $t_2 > t_1$, the interface is C^3 but not C^4 .

Our main result in this paper is the following

Theorem 1.1 (i) *For any initial data f_0 in $\dot{W}^{1,\infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})$, there exists a time $T > 0$ such that the Cauchy problem for the Muskat equation has a unique solution*

$$f \in L^\infty([0, T]; \dot{W}^{1,\infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})) \cap L^2(0, T; \dot{H}^2(\mathbb{R})).$$

(ii) *Moreover, there exists a positive constant δ such that, for any initial data f_0 in $\dot{W}^{1,\infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})$ satisfying*

$$(1 + \|f_0\|_{\dot{W}^{1,\infty}}^4) \|f_0\|_{\dot{H}^{3/2}} \leq \delta,$$

the Cauchy problem for the Muskat equation has a unique global solution

$$f \in L^\infty([0, +\infty); \dot{W}^{1,\infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})) \cap L^2(0, +\infty; \dot{H}^2(\mathbb{R})).$$

Some remarks are in order.

- Let us discuss statement (ii) about the global well-posedness component of this result. This is a 2D analogous to the recent result by Gancedo and Lazar [27] for the 3D problem; it improves on a previous result by Córdoba and Lazar [21] which proves a similar global existence result for the 2D-problem with a similar smallness assumption, but under the extra assumption that the initial data belongs to $H^{5/2}(\mathbb{R})$.
- We now come to statement (i) about the local well-posedness result for arbitrary initial data. This is, in our opinion, the main new result in this paper. Since we are working in a critical space, this result is optimal in several directions.

Firstly, it follows from the results about singularity formation by Castro, Córdoba, Fefferman, Gancedo and López-Fernández [11–13] that one cannot solve the Cauchy problem for a time T which depends only on the norm of f_0 in $\dot{W}^{1,\infty}(\mathbb{R}) \cap \dot{H}^{3/2}(\mathbb{R})$. Otherwise, one would obtain a global existence result for any initial data by an immediate scaling argument using (3). Notice that this argument does not contradict our main result: it means instead that the time of existence must depend on the initial data itself, and not only on its norm.

The previous discussion shows that one cannot prove statement (i) by using classical Sobolev energy estimates. This in turn poses new challenging questions since on the other hand the Muskat equation is a quasi-linear equation. To overcome this problem, we will estimate the solution for a norm whose definition depends on the initial data.

- We will also prove a result which elaborates on the previous discussion, stating that whenever one controls a bigger norm than the critical one, the time of existence is bounded from below on a neighborhood of the initial data.

To introduce this result, let us fix some notations.

Definition 1.2 Given a real number $s \geq 0$ and a function $\phi : [0, \infty) \rightarrow [1, \infty)$ satisfying the following assumptions:

- (H1) ϕ is increasing and $\lim \phi(r) = \infty$ when r goes to $+\infty$;
- (H2) there is a positive constant c_0 such that $\phi(2r) \leq c_0\phi(r)$ for any $r \geq 0$;
- (H3) the function $r \mapsto \phi(r) / \log(4 + r)$ is decreasing on $[0, \infty)$.

Then $|D|^{s,\phi}$ denotes the Fourier multiplier with symbol $|\xi|^s\phi(|\xi|)$, so that

$$\mathcal{F}(|D|^{s,\phi} f)(\xi) = |\xi|^s\phi(|\xi|)\mathcal{F}(f)(\xi).$$

Moreover, we define the space

$$\mathcal{X}^{s,\phi}(\mathbb{R}) = \{f \in \dot{W}^{1,\infty}(\mathbb{R}) \cap L^2(\mathbb{R}) : |D|^s\phi(|D_x|)f \in L^2(\mathbb{R})\},$$

equipped with the norm

$$\|f\|_{\mathcal{X}^{s,\phi}} := \|f\|_{\dot{W}^{1,\infty}} + \|f\|_{L^2} + \left(\int_{\mathbb{R}} |\xi|^{2s} (\phi(|\xi|))^2 |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Remark 1.3 The Fourier multiplier $|D|^{s,\phi}$ with $\phi(r) = \log(2 + r)^a$ was introduced and studied in [6–8] for $s \in [0, 1)$ (also see [32]).

Theorem 1.4 Consider a real number $M_0 > 0$ and a function ϕ satisfying assumptions (H1)–(H3) in Definition 1.2. Then there exists a time $T_0 > 0$ such that, for any initial data f_0 in $\mathcal{X}^{\frac{3}{2},\phi}(\mathbb{R})$ satisfying

$$\|f_0\|_{\mathcal{X}^{\frac{3}{2},\phi}} \leq M_0,$$

the Cauchy problem for the Muskat equation has a unique solution

$$f \in L^\infty([0, T_0]; \dot{W}^{1,\infty}(\mathbb{R}) \cap H^{\frac{3}{2}}(\mathbb{R})) \cap L^2(0, T_0; \dot{H}^2(\mathbb{R})). \tag{4}$$

Remark 1.5 Statement (i) in Theorem 1.1 is a consequence of Theorem 1.4. Indeed, it is easily seen that (cf [3, Lemma 3.8]), for any f_0 in the critical space $\dot{W}^{1,\infty}(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R})$, one may find a function ϕ such that f_0 belongs to $\mathcal{X}^{\frac{3}{2},\phi}(\mathbb{R})$ (and satisfying assumptions (H1)–(H3) in Definition 1.2).

Theorem 1.1 and Theorem 1.4 are proved in the next section.

2 Proof

2.1 Regularization

In order to rigorously justify the computations, we want to handle smooth functions (hereafter, a ‘smooth function’ is by definition a function that belongs to $C^1([0, T]; H^\mu(\mathbb{R}))$ for any $\mu \in [0, +\infty)$ and some $T > 0$). To do so, we must regularize the initial data and also consider an approximation of the Muskat equation. For our purposes, we further need to consider a regularization of the Muskat equation which will be compatible with the Sobolev and Lipschitz estimates. It turns out that this is a delicate technical problem.

Our strategy will consist in smoothing the equation in two different ways: (i) by introducing a cut-off function in the singular integral, removing wave-length shorter than some parameter ε and (ii) by adding a parabolic term of order 2 with a small viscosity of size $|\log(\varepsilon)|^{-1}$.

More precisely, we introduce the following Cauchy problem depending on the parameter $\varepsilon \in (0, 1]$:

$$\begin{cases} \partial_t f - |\log(\varepsilon)|^{-1} \partial_x^2 f = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \left(1 - \chi\left(\frac{\alpha}{\varepsilon}\right)\right) d\alpha, \\ f|_{t=0} = f_0 \star \chi_\varepsilon, \end{cases} \tag{5}$$

where $\chi_\varepsilon(x) = \varepsilon^{-1} \chi(x/\varepsilon)$ where χ is a smooth bump function satisfying $0 \leq \chi \leq 1$ and

$$\chi(y) = \chi(-y), \quad \chi(y) = 1 \quad \text{for } |y| \leq \frac{1}{4}, \quad \chi(y) = 0 \quad \text{for } |y| \geq 2, \quad \int_{\mathbb{R}} \chi \, dy = 1.$$

The equation (5) does not belong to a general class of parabolic equations. However, we will see that it can be studied by standard tools in functional analysis together with two estimates for the nonlinearity in the Muskat equation which plays a central role in our analysis.

Proposition 2.1 *For any ε in $(0, 1]$ and any initial data f_0 in $H^{\frac{3}{2}}(\mathbb{R})$, there exists a unique global in time solution f_ε satisfying*

$$f_\varepsilon \in C^1([0, +\infty); H^\infty(\mathbb{R})).$$

We postpone the proof of this proposition to §2.9.

2.2 An Estimate of the Lipschitz Norm

Lemma 2.2 *For any real number β_0 in $(0, 1/2)$, there exists a positive constant $C_0 \geq 1$ such that, for any $\varepsilon \in (0, 1]$ and any smooth solution $f \in C^1([0, T]; H^\infty(\mathbb{R}))$ of the*

Muskat equation (5),

$$\frac{d}{dt} \|f(t)\|_{\dot{W}^{1,\infty}} \leq C_0 \|f(t)\|_{\dot{H}^2}^2 + C_0 \varepsilon^{\beta_0} \|f(t)\|_{\dot{C}^{2,\beta_0}}, \tag{6}$$

where

$$\|u\|_{\dot{C}^{2,\beta_0}} = \|\partial_{xx}u\|_{C^{0,\beta_0}} = \sup_{\substack{x,y \in \mathbb{R} \\ x \neq y}} \frac{|(\partial_{xx}u)(x) - (\partial_{xx}u)(y)|}{|x - y|^{\beta_0}}.$$

Proof The proof is partially based on arguments from [10,20,27]. Firstly, it follows from the proof of [20, Lemma 5.1] that

$$\begin{aligned} & \partial_x \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f(x)}{1 + (\Delta_\alpha f(x))^2} d\alpha \\ &= \frac{\partial_x^2 f(t, x)}{2\pi} \int \left(\frac{1}{1 + (\Delta_\alpha f(t, x))^2} - \frac{1}{1 + (\Delta_{-\alpha} f(t, x))^2} \right) \frac{d\alpha}{\alpha} \\ & \quad - \frac{2}{\pi} \int \frac{\partial_x f(t, x) - \Delta_\alpha f(t, x)}{\alpha^2} \frac{1 + \partial_x f(t, x) \Delta_\alpha f(t, x)}{1 + (\Delta_\alpha f(t, x))^2} d\alpha. \end{aligned}$$

Moreover,

$$\begin{aligned} & \left| \partial_x \left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \chi \left(\frac{|\alpha|}{\varepsilon} \right) d\alpha \right) \right| \\ & \lesssim \int_{|\alpha| \leq 2\varepsilon} \left(|\Delta_\alpha f_{xx}| + |\Delta_\alpha f_x|^2 \right) d\alpha \\ & \lesssim \int_{\mathbb{R}} |\Delta_\alpha f_x|^2 d\alpha + \varepsilon^{\beta_0} \|f_{xx}\|_{\dot{C}^{0,\beta_0}}, \end{aligned} \tag{7}$$

where we used the notations $f_x = \partial_x f$ and $f_{xx} = \partial_{xx} f$. Thus, for any t and any x , we have

$$\begin{aligned} & (\partial_x \partial_t f)(t, x) - |\log(\varepsilon)|^{-1} \partial_x^2 f_x(t, x) \\ & \leq \frac{\partial_x^2 f(t, x)}{2\pi} \int \left(\frac{1}{1 + (\Delta_\alpha f(t, x))^2} - \frac{1}{1 + (\Delta_{-\alpha} f(t, x))^2} \right) \frac{d\alpha}{\alpha} \\ & \quad - \frac{2}{\pi} \int \frac{\partial_x f(t, x) - \Delta_\alpha f(t, x)}{\alpha^2} \frac{1 + \partial_x f(t, x) \Delta_\alpha f(t, x)}{1 + (\Delta_\alpha f(t, x))^2} d\alpha \\ & \quad + C \int |\Delta_\alpha f_x(t, x)|^2 d\alpha + C \varepsilon^{\beta_0} \|f_{xx}(t)\|_{\dot{C}^{0,\beta_0}}. \end{aligned} \tag{8}$$

Consider the function $\varphi(t) = \|\partial_x f(t)\|_{L^\infty}$ and a function $t \mapsto x_t$ such that

$$\|\partial_x f(t)\|_{L^\infty} = (\partial_x f)(t, x_t).$$

Then $(\partial_x^2 f)(t, x_t) = 0$ and $-(\partial_{xx} f_x)(t, x_t) \geq 0$. So, it follows from (8) that

$$\begin{aligned} \dot{\varphi}(t) &\leq -\frac{2}{\pi} \int \frac{\partial_x f(t, x_t) - \Delta_\alpha f(t, x_t)}{\alpha^2} d\alpha \\ &\quad - \frac{2}{\pi} \int \frac{(\partial_x f(t, x_t) - \Delta_\alpha f(t, x_t))^2}{\alpha^2} \frac{\Delta_\alpha f(t, x_t)}{1 + (\Delta_\alpha f(t, x_t))^2} d\alpha \\ &\quad + C \int |\Delta_\alpha f_x(t, x_t)|^2 d\alpha + C \varepsilon^{\beta_0} \|f_{xx}(t)\|_{\dot{C}^{0,\beta_0}}. \end{aligned}$$

As already observed in [20] (see also [10,27]), the first term in the right-hand side has a sign since $\partial_x f(t, x_t) \geq \Delta_\alpha f(t, x_t)$ for any α . It follows that

$$\begin{aligned} \dot{\varphi}(t) &\leq \frac{1}{\pi} \int \frac{(\partial_x f(t, x_t) - \Delta_\alpha f(t, x_t))^2}{\alpha^2} d\alpha + C \int |\Delta_\alpha f_x(t, x_t)|^2 d\alpha \\ &\quad + C \varepsilon^{\beta_0} \|f_{xx}(t)\|_{\dot{C}^{0,\beta_0}}. \end{aligned}$$

We now apply Hardy’s inequality to infer that

$$\int \frac{(\partial_x f(t, x_t) - \Delta_\alpha f(t, x_t))^2}{\alpha^2} d\alpha \lesssim \int |\Delta_\alpha f_x(t, x_t)|^2 d\alpha.$$

Consequently, we end up with

$$\dot{\varphi}(t) \lesssim \int \|\Delta_\alpha f_x(t)\|_{L^\infty}^2 d\alpha + \varepsilon^{\beta_0} \|f_{xx}(t)\|_{\dot{C}^{0,\beta_0}}.$$

Introducing the difference operator $\delta_\alpha g(x) = g(x) - g(x - \alpha)$, the previous inequality is better formulated as follows:

$$\dot{\varphi}(t) \lesssim \int \|\delta_\alpha(\partial_x f)(t)\|_{L^\infty}^2 \frac{d\alpha}{|\alpha|^{1+\frac{1}{2}}} + \varepsilon^{\beta_0} \|f_{xx}(t)\|_{\dot{C}^{0,\beta_0}}.$$

Now the right-hand side is equivalent to the following homogeneous Besov norm: $\|\partial_x f(t)\|_{\dot{B}_{\infty,2}^{\frac{1}{2}}}$ (see [36,37] or Section 2 in [3]). Then it follows from Sobolev embeddings that

$$\dot{\varphi}(t) \lesssim \|f(t)\|_{\dot{H}^2}^2 + \varepsilon^{\beta_0} \|f_{xx}(t)\|_{\dot{C}^{0,\beta_0}}$$

which is the wanted result. □

2.3 Sobolev Estimates

In this paragraph we recall a generalized Sobolev energy estimate proved in our companion paper [3]. By generalized Sobolev energy estimate, we mean that, instead of

estimating the $L_t^\infty(L_x^2)$ -norm of $(-\Delta)^s f$, we shall estimate the $L_t^\infty(L_x^2)$ -norm of $|D|^{s,\phi} f$ for some function ϕ satisfying the assumptions in Definition 1.2.

There two technical results that we will borrow from [3]. The first result, which is Lemma 3.4 in [3], gives an energy estimate.

Lemma 2.3 *There exists a positive constant C such that, for any $T > 0$ and any smooth solution $f \in C^1([0, T]; H^\infty(\mathbb{R}))$ to (1), there holds*

$$\begin{aligned} \frac{d}{dt} \| |D|^{\frac{3}{2},\phi} f \|_{L^2}^2 + \int_{\mathbb{R}} \frac{|D|^{2,\phi} f|^2}{1 + (\partial_x f)^2} dx + |\log(\varepsilon)|^{-1} \int_{\mathbb{R}} | |D|^{\frac{5}{2},\phi} f|^2 dx \\ \leq C Q(f) \| |D|^{2,\phi} f \|_{L^2}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} Q(f) = & \left(\|f\|_{\dot{H}^2} + \|f\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \| |D|^{\frac{3}{2},\phi} f \|_{L^2} + \| |D|^{\frac{7}{4},\phi} f \|_{L^2} \|f\|_{\dot{H}^{\frac{7}{4}}} \\ & + \left(\|f\|_{\dot{H}^{\frac{19}{12}}}^{3/2} + \|f\|_{\dot{H}^{\frac{7}{4}}}^{1/2} \right) \| |D|^{\frac{7}{4},\phi^2} f \|_{L^2}^{1/2} \|f\|_{\dot{H}^{\frac{7}{4}}}. \end{aligned}$$

Remark 2.4 Some explanations are in order since the reader may notice several modifications compared to our paper [3]. Firstly, in [3] we considered a function ϕ whose definition depends on an extra function κ . Here we ignore this point since it is irrelevant for the present analysis. Indeed, the functions ϕ and κ are shown in [3] to be equivalent (such that $c\kappa(\lambda) \leq \phi(\lambda) \leq C\kappa(\lambda)$), and the distinction between them served only to organize the proof. Secondly, in [3] we also assume that $\phi(r)$ is bounded from below by $(\log(4+r))^a$ for some $a \geq 0$. Here we will use that this property holds with $a = 0$. Once the previous clarifications have been done, it remains to explain that in [3] we consider the equation (1) while here we work with (5). The elliptic term $(-\partial_x^2)$ is trivial to handle since in [3] we only applied an L^2 -energy estimate and since the latter operator is positive. Eventually, the cut-off function $(1 - \chi(\alpha/\varepsilon))$ is also harmless in the various computations used to prove Lemma 3.4 in [3].

Secondly, we recall two interpolation inequalities from [3, Lemma 3.5]. Hereafter, we use the notations

$$\begin{aligned} A_\phi(t) &= \| |D|^{\frac{3}{2},\phi} f(t) \|_{L^2}^2, \\ B_\phi(t) &= \| |D|^{2,\phi} f(t) \|_{L^2}^2, \\ P_\phi(t) &= \| |D|^{\frac{5}{2},\phi} f(t) \|_{L^2}^2, \end{aligned} \tag{10}$$

and

$$\mu_\phi(t) = \left(\phi \left(\frac{B(t)}{A(t)} \right) \right)^{-1}.$$

Lemma 2.5 *Consider a real number $7/4 \leq s \leq 2$. Then, there exists a positive constant C such that, for any $T > 0$, any smooth solution $f \in C^1([0, T]; H^\infty(\mathbb{R}))$ to (5) and any $t \in [0, T]$,*

$$\|f(t)\|_{\dot{H}^s} \leq C\mu_\phi(t)A_\phi(t)^{2-s}B_\phi(t)^{s-\frac{3}{2}}, \tag{11}$$

$$\||D|^{\frac{7}{4}, \phi^2} f(t)\|_{L^2} \leq C\mu_\phi(t)A_\phi(t)^{\frac{1}{4}}B_\phi(t)^{\frac{1}{4}}. \tag{12}$$

From these two lemmas, we get at once the following

Proposition 2.6 *There exist two positive constants C_1 and C_2 such that, for any $T > 0$ and any smooth solution $f \in C^1([0, T]; H^\infty(\mathbb{R}))$ of the Muskat equation (5),*

$$\begin{aligned} & \frac{d}{dt} A_\phi(t) + C_1 \frac{B_\phi(t)}{1 + \|f_x(t)\|_{L^\infty}^2} + |\log(\varepsilon)|^{-1} P_\phi(t) \\ & \leq C_2 \left(\sqrt{A_\phi(t)} + A_\phi(t) \right) \mu_\phi(t) B_\phi(t). \end{aligned} \tag{13}$$

We will also need an estimate for the L^2 -norm.

Lemma 2.7 *There holds*

$$\frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 \leq C\varepsilon^{\frac{1}{2}} \|f\|_{\dot{H}^{\frac{3}{2}}} \|f\|_{L^2}.$$

In particular,

$$\|f(t)\|_{L^2} \leq \|f_0\|_{L^2} + C\varepsilon^{\frac{1}{2}} \int_0^t \|f(\tau)\|_{\dot{H}^{\frac{3}{2}}} d\tau. \tag{14}$$

Proof Set

$$R_\varepsilon(f) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \chi\left(\frac{\alpha}{\varepsilon}\right) d\alpha. \tag{15}$$

We multiply the equation by f to obtain

$$\frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 \leq \frac{1}{\pi} \left\langle \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha, f \right\rangle + \langle R_\varepsilon(f), f \rangle.$$

Now, by [16, Section 2], the first term in the right-hand side has a sign. Indeed:

$$\begin{aligned} & \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha \right] f(x) dx \\ & = - \iint_{\mathbb{R}^2} \log \left[\sqrt{1 + \frac{(f(t, x) - f(t, x - \alpha))^2}{\alpha^2}} \right] dx d\alpha. \end{aligned}$$

It remains to estimate $R_\varepsilon(f)$. To do so, we use the estimate (19) to get

$$\begin{aligned} \|R_\varepsilon(f)\|_{L^2} & \lesssim \int_{|\alpha| \leq 2\varepsilon} \|\Delta_\alpha f_x\|_{L^2} d\alpha \\ & \lesssim \varepsilon^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|\Delta_\alpha f_x\|_{L^2}^2 d\alpha \right)^{\frac{1}{2}} \lesssim \varepsilon^{\frac{1}{2}} \|f\|_{\dot{H}^{\frac{3}{2}}}, \end{aligned} \tag{16}$$

which completes the proof. □

2.4 Estimate of the Hölder Norm

To exploit the Sobolev energy estimate (13), the main difficulty is to estimate from above the factor $1 + \|f_x(t)\|_{L^\infty}^2$. This is where we will apply Lemma 2.2. This in turn requires to estimate the Hölder norm $\|\cdot\|_{\dot{C}^{2,\beta_0}}$ of f . This is the purpose of the following result.

We will prove an estimate valid on arbitrary large time scale, which will be used later to prove a global existence result.

Proposition 2.8 *For any $0 < \beta < 1/2$, there exist two positive constant ε_0 and c_0 such that, for any $\varepsilon \in (0, \varepsilon_0]$, any smooth solution $f \in C^1([0, T]; H^\infty(\mathbb{R}))$ of the Muskat equation (5), and any time $t \leq \min\{\varepsilon^{-c_0}, T\}$, there holds*

$$\begin{aligned} \varepsilon^\beta \int_0^t \|f(\tau)\|_{C^{2,\beta}} d\tau &\leq \varepsilon^{\frac{\beta}{2}} \|f_0\|_{\dot{H}^{\frac{3}{2}}} \\ &+ \varepsilon^{\frac{\beta}{2}} \left(1 + \sup_{s \in [0,t]} \|f(s)\|_{H^{\frac{3}{2}}}\right)^2 \log \left(2 + \int_0^t \|f(s)\|_{\dot{H}^2}^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|f(s)\|_{\dot{H}^2}^2 ds\right)^{\frac{1}{2}}. \end{aligned}$$

Proof The classical Sobolev embeddings implies that

$$\|f(t)\|_{\dot{C}^{2,\beta}} \lesssim \|f(t)\|_{\dot{H}^{\frac{5}{2}+\beta}}.$$

To estimate the latter Sobolev norm, the key point will be to apply the following interpolation inequality. □

Lemma 2.9 *Consider three real numbers*

$$\gamma > 0, \quad \beta_1 > 0 \quad \text{and} \quad 0 < \beta_2 < 2.$$

Then, there exists a constant C such that, for any function $g = g(t, x)$,

$$\begin{aligned} \|g(t)\|_{\dot{H}^\gamma} &\lesssim \frac{1}{(vt)^{\frac{\beta_1}{2}}} \|g(0)\|_{\dot{H}^{\gamma-\beta_1}} \\ &+ \int_0^t \frac{1}{(v(t-s))^{\frac{\beta_2}{2}}} \|\partial_t g(s) - v\partial_{xx}g(s)\|_{\dot{H}^{\gamma-\beta_2}} ds. \end{aligned} \tag{17}$$

Proof Set $G := \partial_t g - v\partial_{xx}g$. Then, one has,

$$\hat{g}(t, \xi) = e^{-v t |\xi|^2} \hat{g}(0, \xi) + \int_0^t e^{-v(t-s)|\xi|^2} \hat{G}(s, \xi) ds.$$

The desired results then follows from Minkowski's inequality. □

Now, apply (17) with

$$\gamma = \frac{5}{2} + \beta, \quad \beta_1 = 1 + \beta, \quad \beta_2 = \frac{3}{2} + \beta, \quad \nu = |\log(\varepsilon)|^{-1},$$

to get

$$\begin{aligned} \|f(t)\|_{\dot{C}^{2,\beta}} &\lesssim \|f(t)\|_{\dot{H}^{\frac{5}{2}+\beta}} \\ &\lesssim |\log(\varepsilon)|^{\frac{1+\beta}{2}} t^{-\frac{1+\beta}{2}} \|f_0\|_{\dot{H}^{\frac{3}{2}}} \\ &\quad + \int_0^t |\log(\varepsilon)|^{\frac{3+\beta}{2}} (t-s)^{-\frac{3+\beta}{2}} \|\partial_t f - |\log(\varepsilon)|^{-1} \partial_x^2 f\|_{\dot{H}^1} ds. \end{aligned} \tag{18}$$

It remains to estimate the \dot{H}^1 -norm of $\partial_t f - |\log(\varepsilon)|^{-1} \partial_x^2 f$. In view of the equation (5), this is equivalent to bound the \dot{H}^1 -norm of

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \left(1 - \chi\left(\frac{\alpha}{\varepsilon}\right)\right) d\alpha.$$

We will split the latter term into two pieces and estimate them separately.

Firstly, directly from (7) and Minkowski’s inequality, we obtain that

$$\begin{aligned} \left\| \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \chi\left(\frac{\alpha}{\varepsilon}\right) d\alpha \right\|_{\dot{H}^1} &\lesssim \int_{|\alpha| \leq 2\varepsilon} \left(\|\Delta_\alpha f_{xx}\|_{L^2} + \|\Delta_\alpha f_x\|_{L^4}^2 \right) d\alpha \\ &\lesssim \varepsilon^{\frac{1}{2}+\beta} \left(\int_{\mathbb{R}} \|\Delta_\alpha f_{xx}\|_{L^2}^2 |\alpha|^{-2\beta} d\alpha \right)^{\frac{1}{2}} \\ &\quad + \int_{\mathbb{R}} \|\Delta_\alpha f_x\|_{L^4}^2 d\alpha. \end{aligned}$$

Now we use the following inequality:

$$\iint_{\mathbb{R}^2} |\Delta_\alpha \tilde{f}|^2 |\alpha|^{-2\beta} d\alpha dx \sim \|\tilde{f}\|_{\dot{H}^{\frac{1}{2}+\beta}}^2. \tag{19}$$

Indeed,

$$\iint_{\mathbb{R}^2} |\Delta_\alpha \tilde{f}|^2 |\alpha|^{-2\beta} d\alpha dx = \iint_{\mathbb{R}^2} \left[\frac{|\tilde{f}(x) - \tilde{f}(x-\alpha)|}{|\alpha|^{1/2+\beta}} \right]^2 \frac{d\alpha}{|\alpha|} dx \sim \|\tilde{f}\|_{\dot{H}^{\frac{1}{2}+\beta}}^2.$$

Similarly, using Sobolev embedding in Besov’s spaces, we get

$$\int_{\mathbb{R}} \|\Delta_\alpha f_x\|_{L^4}^2 d\alpha \lesssim \|f\|_{\dot{H}^{\frac{7}{4}}}^2.$$

It follows that

$$\left\| \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \chi\left(\frac{\alpha}{\varepsilon}\right) d\alpha \right\|_{\dot{H}^1} \lesssim \varepsilon^{\frac{1}{2} + \beta} \|f\|_{\dot{H}^{\frac{5}{2} + \beta}} + \|f\|_{\dot{H}^2} \|f\|_{\dot{H}^{\frac{3}{2}}}, \quad (20)$$

where we used an interpolation inequality in Sobolev spaces. On the other hand, it follows from the estimate (41) below that,

$$\begin{aligned} \left\| \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} d\alpha \right\|_{\dot{H}^1} &\lesssim \|\mathcal{T}(f)f\|_{\dot{H}^1} \\ &\lesssim \left(1 + \|f\|_{\dot{H}^{\frac{3}{2}}}\right)^2 \log\left(2 + \|f\|_{\dot{H}^2}^2\right)^{\frac{1}{2}} \|f\|_{\dot{H}^2}. \end{aligned} \quad (21)$$

By gathering the two previous estimates, we conclude that

$$\begin{aligned} &\left\| \partial_t f - |\log(\varepsilon)|^{-1} \partial_x^2 f \right\|_{\dot{H}^1} \\ &\lesssim \varepsilon^{\frac{1}{2} + \beta} \|f\|_{\dot{H}^{\frac{5}{2} + \beta}} + \left(1 + \|f\|_{\dot{H}^{\frac{3}{2}}}\right)^2 \log\left(2 + \|f\|_{\dot{H}^2}^2\right)^{\frac{1}{2}} \|f\|_{\dot{H}^2}. \end{aligned}$$

Set

$$b = \frac{\frac{3}{2} + \beta}{2}.$$

By reporting this bound in (18), we find that

$$\begin{aligned} \|f(t)\|_{\dot{H}^{\frac{5}{2} + \beta}} &\lesssim |\log(\varepsilon)|^{\frac{1+\beta}{2}} t^{-\frac{1+\beta}{2}} \|f_0\|_{\dot{H}^{\frac{3}{2}}} \\ &+ \varepsilon^{\frac{1}{2} + \beta} |\log(\varepsilon)|^b \int_0^t (t-s)^{-b} \|f(s)\|_{\dot{H}^{\frac{5}{2} + \beta}} ds \\ &+ |\log(\varepsilon)|^b \int_0^t (t-s)^{-b} \left(1 + \|f(s)\|_{\dot{H}^{\frac{3}{2}}}\right)^2 \log\left(2 + \|f(s)\|_{\dot{H}^2}^2\right)^{\frac{1}{2}} \|f(s)\|_{\dot{H}^2} ds. \end{aligned}$$

So,

$$\begin{aligned} \int_0^t \|f(\tau)\|_{\dot{H}^{\frac{5}{2} + \beta}} d\tau &\lesssim |\log(\varepsilon)|^{\frac{1+\beta}{2}} t^{1-\frac{1+\beta}{2}} \|f_0\|_{\dot{H}^{\frac{3}{2}}} \\ &+ \varepsilon^{\frac{1}{2} + \beta} |\log(\varepsilon)|^b t^{1-b} \int_0^t \|f(s)\|_{\dot{H}^{\frac{5}{2} + \beta}} ds \\ &+ |\log(\varepsilon)|^b t^{1-b} \int_0^t \left(1 + \|f(s)\|_{\dot{H}^{\frac{3}{2}}}\right)^2 \log\left(2 + \|f(s)\|_{\dot{H}^2}^2\right)^{\frac{1}{2}} \|f(s)\|_{\dot{H}^2} ds. \end{aligned}$$

As a result, there exists $c_0 > 0$ and $\varepsilon_0 \leq 1$ such that, if $t \leq \varepsilon^{-c_0}$ and $\varepsilon \leq \varepsilon_0$,

$$\int_0^t \|f(\tau)\|_{\dot{H}^{\frac{5}{2}+\beta}} \, d\tau \leq \varepsilon^{-\frac{\beta}{2}} \|f_0\|_{\dot{H}^{\frac{3}{2}}} + |\log(\varepsilon)|^b t^{1-b} \mathcal{K}(t) \int_0^t \log\left(2 + \|f(s)\|_{\dot{H}^2}^2\right)^{\frac{1}{2}} \|f(s)\|_{\dot{H}^2} \, ds,$$

where

$$\mathcal{K}(t) = \sup_{s \in [0,t]} \left(1 + \|f(s)\|_{\dot{H}^{\frac{3}{2}}}\right)^2.$$

Now observe that

$$\begin{aligned} & \int_0^t \log\left(2 + \|f(s)\|_{\dot{H}^2}^2\right)^{\frac{1}{2}} \|f(s)\|_{\dot{H}^2} \, ds \\ & \leq (t + 1)^{\frac{1}{2}} \log\left(2 + \int_0^t \|f(s)\|_{\dot{H}^2}^2 \, ds\right)^{\frac{1}{2}} \left(\int_0^t \|f(s)\|_{\dot{H}^2}^2 \, ds\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, up to modifying the values of $c_0 > 0$ and ε_0 , we see that, for $t \leq \varepsilon^{-c_0}$ and $\varepsilon \leq \varepsilon_0$, we have

$$\begin{aligned} \varepsilon^\beta \int_0^t \|f(\tau)\|_{\dot{C}^{2,\beta}} \, d\tau & \lesssim \varepsilon^{\frac{\beta}{2}} \|f_0\|_{\dot{H}^{\frac{3}{2}}} \\ & + \varepsilon^{\frac{\beta}{2}} \mathcal{K}(t) \log\left(2 + \int_0^t \|f(s)\|_{\dot{H}^2}^2 \, ds\right)^{\frac{1}{2}} \left(\int_0^t \|f(s)\|_{\dot{H}^2}^2 \, ds\right)^{\frac{1}{2}}. \end{aligned}$$

This completes the proof. □

2.5 Global in Time Estimates, Under a Smallness Assumption

Proposition 2.10 *Let $T > 0$ and consider a smooth solution $f \in C^1([0, T], H^\infty(\mathbb{R}))$ of the Muskat equation (5). Set*

$$K = 1 + 16 \left(\frac{C_2}{C_1}\right)^2$$

and assume that

$$2 \left(K + \frac{C_0}{C_1}\right)^{\frac{1}{2}} (2 + \|\partial_x f_0\|_{L^\infty})^2 \|f_0\|_{\dot{H}^{\frac{3}{2}}} \leq 1, \tag{22}$$

where the constants C_0, C_1, C_2 are as defined in the statements of Lemma 2.2 and Proposition 2.6. Then there exists ε_0 depending only on C_0, C_1, C_2 and $\|f_0\|_{L^2}$ such that, if $\varepsilon \leq \varepsilon_0$, then

$$\sup_{0 \leq \tau \leq T} \|f(\tau)\|_{H^{\frac{3}{2}}} \leq \frac{1}{\sqrt{K}(2 + \|\partial_x f_0\|_{L^\infty})^2} \quad \text{and} \quad \int_0^T \|f(\tau)\|_{\dot{H}^2}^2 \, d\tau \leq \frac{1}{C_0}. \quad (23)$$

Proof We apply the previous *a priori* estimate (13) in the simplest case where $\phi = 1$. With this choice, the quantities A_ϕ and B_ϕ defined by (10) simplify to

$$\begin{aligned} A(t) &= \||D|^{\frac{3}{2}} f(t)\|_{L^2}^2, \\ B(t) &= \||D|^2 f(t)\|_{L^2}^2 = \|f(t)\|_{\dot{H}^2}^2. \end{aligned} \quad (24)$$

Introduce the set

$$I = \left\{ t \in [0, T]; \int_0^t B(\tau) \, d\tau \leq \frac{2}{3C_0} \text{ and } \sup_{0 \leq \tau \leq t} A(\tau) \leq \frac{1}{K(2 + \|\partial_x f_0\|_{L^\infty})^4} \right\}.$$

We want to prove that $I = [0, T]$. Since 0 belongs to I by assumption on the initial data, and since I is closed, it suffices to prove that I is open. To do so, we consider a time $t^* \in [0, T]$ which belongs to I . Our goal is to prove that

$$\int_0^{t^*} B(\tau) \, d\tau \leq \frac{1}{2C_0} \text{ and } \sup_{0 \leq \tau \leq t^*} A(\tau) \leq \frac{1}{4K(2 + \|\partial_x f_0\|_{L^\infty})^4}.$$

This will imply at once that t^* belongs to the interior of I .

Since $\mu(t) = 1$ for $\phi \equiv 1$, the estimate (13) implies that there are two positives constants C_1, C_2 such that

$$\frac{d}{dt} A(t) + C_1 \frac{B(t)}{1 + \|\partial_x f(t)\|_{L^\infty}^2} \leq C_2 \left(A(t) + \sqrt{A(t)} \right) B(t). \quad (25)$$

By combining Proposition 2.8 with Lemma 2.2, we get, for any t ,

$$\begin{aligned} \|\partial_x f(t)\|_{L^\infty} - \|\partial_x f_0\|_{L^\infty} &\leq C_0 \int_0^t B(\tau) \, d\tau + C_0 \varepsilon^{\frac{\beta}{2}} \|f_0\|_{\dot{H}^{\frac{3}{2}}} \\ &+ C_0 \varepsilon^{\frac{\beta}{2}} \left[\sup_{s \in [0, t]} \left(1 + \|f(s)\|_{H^{\frac{3}{2}}} \right)^2 \right] \log \left(2 + \int_0^t B(\tau) \, d\tau \right)^{\frac{1}{2}} \left(\int_0^t B(\tau) \, d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

By (14),

$$\sup_{s \in [0, t]} \|f(s)\|_{L^2} \leq \|f_0\|_{L^2} + C\varepsilon^{\frac{1}{2}} t \sup_{s \in [0, t]} \|f(s)\|_{\dot{H}^{\frac{3}{2}}}.$$

This implies

$$\sup_{s \in [0, t]} \|f(s)\|_{H^{\frac{3}{2}}} \leq \|f_0\|_{L^2} + (1 + C\varepsilon^{\frac{1}{2}} t) \sup_{s \in [0, t]} \|f(s)\|_{\dot{H}^{\frac{3}{2}}}. \quad (26)$$

If $t \leq t^*$, then the bound on the integral of B , $\sup_{0 \leq \tau \leq t^*} \|f\|_{\dot{H}^{\frac{3}{2}}} \leq 1$ and (26) imply that

$$\begin{aligned} \|\partial_x f(t)\|_{L^\infty} - \|\partial_x f_0\|_{L^\infty} &\leq \frac{1}{2} + C_0 \varepsilon^{\frac{\beta}{2}} + C_0 \varepsilon^{\frac{\beta}{2}} (1 + \|f_0\|_{L^2} \\ &\quad + (1 + C \varepsilon^{\frac{1}{2}} t^*))^2 \log(3)^{\frac{1}{2}}. \end{aligned}$$

For ε small enough, we conclude that

$$\|\partial_x f(t)\|_{L^\infty} \leq \frac{2}{3} + \|\partial_x f_0\|_{L^\infty}.$$

On the other hand, if $t^* \in I$, then for any $t \leq t^*$ we have

$$A(t) + \sqrt{A(t)} \leq 2\sqrt{A(t)} \leq \frac{2}{\sqrt{K}(2 + \|\partial_x f_0\|_{L^\infty})^2}.$$

Consequently, for any $t \leq t^*$, (25) gives

$$\frac{d}{dt} A(t) + C_1 \frac{B(t)}{(2 + \|\partial_x f_0\|_{L^\infty})^2} \leq \frac{2C_2}{\sqrt{K}(2 + \|\partial_x f_0\|_{L^\infty})^2} B(t).$$

By definition of K , we have

$$K \geq \frac{16C_2^2}{C_1^2},$$

so, for any $t \leq t^*$,

$$\frac{d}{dt} A(t) + \frac{C_1}{2} \frac{B(t)}{(2 + \|\partial_x f_0\|_{L^\infty})^2} \leq 0. \tag{27}$$

Integrate this on the time interval $[0, t^*]$, to infer that

$$\sup_{t \in [0, t^*]} A(t) + \frac{C_1}{2(2 + \|\partial_x f_0\|_{L^\infty})^2} \int_0^{t^*} B(t) dt \leq A(0).$$

Using the smallness assumption (22), the previous inequality (27) implies at once that

$$\begin{aligned} \sup_{t \in [0, t^*]} A(t) \leq A(0) &\leq \frac{1}{4K(2 + \|\partial_x f_0\|_{L^\infty})^4}, \\ \int_0^{t^*} B(t) dt &\leq \frac{2(2 + \|\partial_x f_0\|_{L^\infty})^2}{C_1} A(0) \leq \frac{1}{2C_0}. \end{aligned}$$

These are the wanted bootstrap inequalities. As explained above, by connexity, this proves that $I = [0, T]$, which implies the desired results in (23). \square

2.6 A Priori Estimates Locally in Time, for Arbitrary Initial Data

Proposition 2.11 Consider ϕ satisfying assumptions (H1)–(H3) in Definition 1.2. Let $T > 0$ and consider a smooth solution $f \in C^1([0, T], H^\infty(\mathbb{R}))$ of the Muskat equation (5). For any $M_0 > 0$ there exists $\varepsilon_0 > 0$ and $T_0 > 0$ such that the following properties holds. If $\varepsilon \in (0, \varepsilon_0]$ and

$$\| |D|^{\frac{3}{2}, \phi} f(0) \|_{L^2}^2 \leq M_0,$$

then, with $T^* = \min\{T, T_0\}$, there holds

$$\sup_{t \in [0, T^*]} A_\phi(t) \leq 5M_0, \quad \int_0^{T^*} \mu_\phi(t)^2 B_\phi(t) dt \leq \frac{1}{C_0},$$

where A_ϕ, B_ϕ, μ_ϕ are defined in (10) while C_0 is given by Lemma 2.2.

Proof For this proof we skip the index ϕ and write simply A, B, μ .

Since (see (11)),

$$\|f(t)\|_{\dot{H}^2} \leq C\mu(t)B(t)^{\frac{1}{2}}.$$

We then apply Proposition 2.8 for some fixed parameter $\beta > 0$. Then, it follows from (13) that

$$\frac{d}{dt} A(t) + C_1 \frac{B(t)}{\nu(t)^2} \leq C_2 \left(\sqrt{A(t)} + A(t) \right) \mu(t)B(t), \tag{28}$$

where

$$\begin{aligned} \nu(t) = & 1 + \|\partial_x f_0\|_{L^\infty} + C_0 \int_0^t \mu(\tau)^2 B(\tau) d\tau + C_0 \varepsilon^{\frac{\beta}{2}} \|f_0\|_{\dot{H}^{\frac{3}{2}}} \\ & + C_0 \varepsilon^{\frac{\beta}{2}} \left[\sup_{\tau \in [0, t]} \left(1 + \|f(\tau)\|_{\dot{H}^{\frac{3}{2}}} \right)^2 \right] \log \left(2 + \int_0^t \mu(\tau)^2 B(\tau) d\tau \right)^{\frac{1}{2}} \\ & \left(\int_0^t \mu(\tau)^2 B(\tau) d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Given a positive number T_0 to be determined, introduce the set

$$I(T_0) = \left\{ t \in [0, \min\{T, T_0\}]; \int_0^t \mu(\tau)^2 B(\tau) d\tau \leq \frac{2}{3C_0} \text{ and } \sup_{0 \leq \tau \leq t} A(\tau) \leq 5M_0 \right\}.$$

We want to prove that $I(T_0) = [0, \min\{T, T_0\}]$. Since 0 belongs to $I(T_0)$ by assumption on the initial data, and since $I(T_0)$ is closed, it suffices to prove that $I(T_0)$ is open.

To do so, we consider a time $t^* \in [0, \min\{T, T_0\})$ which belongs to $I(T_0)$. Our goal is to prove that

$$\int_0^{t^*} \mu(\tau)^2 B(\tau) \, d\tau \leq \frac{1}{2C_0} \text{ and } \sup_{0 \leq \tau \leq t^*} A(\tau) \leq 4M_0.$$

This will imply at once that t^* belongs to the interior of $I(T_0)$.

As in the previous proof, we use (26) to write

$$\sup_{s \in [0, t]} \|f(s)\|_{H^{\frac{3}{2}}} \leq \|f_0\|_{L^2} + (1 + C\varepsilon^{\frac{1}{2}}t) \sup_{s \in [0, t]} \|f(s)\|_{\dot{H}^{\frac{3}{2}}}. \tag{29}$$

It $t \leq t^*$ with $t^* \in I(T_0)$, then

$$\begin{aligned} v(t) &\leq 1 + \|\partial_x f_0\|_{L^\infty} + \frac{2}{3} + C_0 \varepsilon^{\frac{\beta}{2}} M_0 \\ &\quad + C_0 \varepsilon^{\frac{\beta}{2}} (1 + \|f_0\|_{L^2} + 6M_0)^2 \log \left(2 + \frac{2}{3C_0} \right)^{\frac{1}{2}} \left(\frac{2}{3C_0} \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, one can define ε_0 small enough, depending only on $M_0, \|f_0\|_{L^2}$ and the fixed parameter β , such that if $\varepsilon \leq \varepsilon_0$ and if $t^* \in I(T_0)$, then for any $t \in [0, t^*]$, we have

$$v(t) \leq 2 + \|\partial_x f_0\|_{L^\infty}.$$

Consequently

$$\frac{d}{dt} A(t) + C_1 \frac{B(t)}{(2 + \|\partial_x f_0\|_{L^\infty})^2} \leq C_2 \left(A(t) + \sqrt{A(t)} \right) \mu(t) B(t).$$

Introduce the function

$$\mathcal{E}(r, m) := \sup_{\rho \geq 0} \left\{ C_2 (\sqrt{r} + r) \left(\phi \left(\frac{\rho}{r} \right) \right)^{-1} \rho - \frac{C_1 \rho}{2m} \right\}.$$

Then, for any $t \in [0, t^*]$, we have

$$\frac{d}{dt} A(t) + \frac{C_1}{2} \frac{B(t)}{(2 + \|\partial_x f_0\|_{L^\infty})^2} \leq \mathcal{E} \left(A(t), \|\partial_x f_0\|_{L^\infty} \right).$$

Assume that the number T_0 satisfies

$$T_0 \leq \frac{A(0)}{4\mathcal{E}(4A(0), \|\partial_x f_0\|_{L^\infty})}.$$

Then, for any $t \leq t^*$, we get that

$$\sup_{\tau \leq t} A(\tau) + \frac{C_1}{2} \frac{1}{(2 + \|\partial_x f_0\|_{L^\infty})^2} \int_0^t B(\tau) \, d\tau \leq 4A(0).$$

In particular, for $t = t^*$, this gives

$$\sup_{t \leq t^*} A(t) \leq 4A(0), \quad \int_0^{t^*} B(t) \, dt \leq \frac{8A(0)}{C_1} (2 + \|\partial_x f_0\|_{L^\infty})^2. \tag{30}$$

To get the result, we must show that

$$C_0 \int_0^T \mu(t)^2 B(t) \, dt \leq \frac{1}{2}. \tag{31}$$

Recall that

$$\mu(t) = \left(\phi \left(\frac{B(t)}{A(t)} \right) \right)^{-1}.$$

Since ϕ is increasing and since $A(t) \leq 4A(0)$, we have

$$\mu(t) \leq \left(\phi \left(\frac{B(t)}{4A(0)} \right) \right)^{-1}.$$

Now, we claim that the function $F : [0, +\infty) \rightarrow [0, +\infty)$, defined by

$$F(r) = \left(\phi \left(\frac{r}{4A(0)} \right) \right)^{-1} r,$$

is increasing. To see this decompose $F(r)$ under the form $F(r) = F_1(r) (F_2(r))^2$ with

$$F_1(r) = \frac{r}{(\log(\lambda_0 + r))^2} \quad F_2(r) = \frac{\log(\lambda_0 + r)}{\phi(r/4A(0))}.$$

Then

$$\begin{aligned} \int_0^{t^*} \mu(t) B(t) \, dt &\leq \int_0^{t^*} \left(\phi \left(\frac{B(t)}{4A(0)} \right) \right)^{-2} B(t) \, dt \\ &\leq \int_0^{t^*} \left(\phi \left(\frac{r}{4A(0)} \right) \right)^{-2} r \, dt + \int_0^{t^*} \left(\phi \left(\frac{r}{4A(0)} \right) \right)^{-2} B(t) \, dt \\ &\stackrel{(30)}{\leq} t^* \left(\phi \left(\frac{r}{4A(0)} \right) \right)^{-2} r + \left(\phi \left(\frac{r}{4A(0)} \right) \right)^{-2} 8A(0) (2 + \|\partial_x f_0\|_{L^\infty})^2, \end{aligned}$$

for any $r \geq 1$. Now we successively determine two numbers $r_0 > 1$ and $T_0 > 0$ such that

$$C_0 \left(\phi \left(\frac{r_0}{4A(0)} \right) \right)^{-2} 8A(0) (2 + \|\partial_x f_0\|_{L^\infty})^2 = \frac{1}{4}, \tag{32}$$

and

$$T_0 \left(\phi \left(\frac{r_0}{4A(0)} \right) \right)^{-2} r_0 = \frac{1}{4}. \tag{33}$$

With this choice we get (31) and we obtain that $I(T_0) = [0, \min\{T, T_0\}]$, which is equivalent to the statement of the proposition. \square

2.7 Transfer of Compactness

Previously, we have proven *a priori* estimates for the spatial derivatives. In this paragraph, we gather results from which we will infer estimates for the time derivative as well as for the nonlinearity in the Muskat equation. These estimates serve to pass to the limit the equation (which is needed to regularize the solutions).

The Muskat equation (1) can be written under the form

$$\partial_t f + |D| f = \mathcal{T}(f)f, \tag{34}$$

where $\mathcal{T}(f)$ is the operator defined by

$$\mathcal{T}(f)g = -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_\alpha g) \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} d\alpha. \tag{35}$$

We recall the following result from Proposition 2.3 in [2] and from Remark 2.9 and Propositions 2.10 and 2.13 in [3].

Proposition 2.12 (i) *For all $\delta \in [0, 1/2)$, there exists a constant $C > 0$ such that, for all functions f_1, f_2 in $\dot{H}^{1-\delta}(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}+\delta}(\mathbb{R})$,*

$$\|(\mathcal{T}(f_1) - \mathcal{T}(f_2))f_2\|_{L^2} \leq C \|f_1 - f_2\|_{\dot{H}^{1-\delta}} \|f_2\|_{\dot{H}^{\frac{3}{2}+\delta}}.$$

(ii) *One can decompose the nonlinearity under the form*

$$\mathcal{T}(f)g = \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} |D|g + V(f)\partial_x g + R(f, g), \tag{36}$$

where the coefficient $V(f)$ and the remainder term $R(f, g)$ satisfy the following estimates:

$$\|V(f)\|_{L^\infty} \leq C \int_{\mathbb{R}} |\xi| |\hat{f}(\xi)| d\xi, \tag{37}$$

$$\|R(f, g)\|_{L^2} \leq C \|g\|_{\dot{H}^{\frac{3}{4}}} \|f\|_{\dot{H}^{\frac{7}{4}}}, \tag{38}$$

for some absolute constant C . Moreover,

$$\|T(f)f\|_{\dot{H}^1} \leq C \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + 1 + \|V(f)\|_{L^\infty} \right) \|f\|_{\dot{H}^2}, \tag{39}$$

and,

$$|(V(f)\partial_x g, |D|g)| \leq C \left(\|f\|_{\dot{H}^2} + \|f\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1}. \tag{40}$$

For later purpose, we need a refinement of (39).

Proposition 2.13 *There exists a positive constant $C > 0$ such that, for all function $f \in H^2(\mathbb{R})$,*

$$\|T(f)f\|_{\dot{H}^1} \leq C \left(1 + \|f\|_{\dot{H}^{\frac{3}{2}}} \right)^2 \log \left(2 + \|f\|_{\dot{H}^2}^2 \right)^{\frac{1}{2}} \|f\|_{\dot{H}^2}. \tag{41}$$

Proof In view of (39) and (37), it is sufficient to estimate the L^1 -norm of $|\xi| \hat{f}$. Write,

$$\begin{aligned} \int_{\mathbb{R}} |\xi| |\hat{f}| \, d\xi &= \int_{|\xi| > \lambda} |\xi|^{-1} |\xi|^2 |\hat{f}| \, d\xi + \int_{|\xi| \leq \lambda} (|\xi| + 1)^{-\frac{1}{2}} |\xi| (1 + |\xi|)^{\frac{1}{2}} |\hat{f}| \, d\xi \\ &\lesssim \left(\int_{|\xi| > \lambda} \frac{1}{|\xi|^2} \, d\xi \right)^{\frac{1}{2}} \|f\|_{\dot{H}^2} \\ &\quad + \left(\int_{|\xi| \leq \lambda} \frac{1}{(|\xi| + 1)} \, d\xi \right)^{\frac{1}{2}} \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{L^2} \right) \\ &\lesssim \lambda^{-\frac{1}{2}} \|f\|_{\dot{H}^2} + \log(1 + \lambda)^{\frac{1}{2}} \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{L^2} \right). \end{aligned}$$

Choosing $\lambda = \|f\|_{\dot{H}^2}^2$, we obtain

$$\int_{\mathbb{R}} |\xi| |\hat{f}| \, d\xi \lesssim 1 + \log(1 + \|f\|_{\dot{H}^2}^2)^{\frac{1}{2}} \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{L^2} \right).$$

By reporting this in (37) and then using (39), we get the desired result (41). □

By using the equation (34), we deduce at once the following bound.

Corollary 2.14 *There exists a non-decreasing function $\mathcal{F}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that, for any $T > 0$, any ε and any smooth solution f in $C^1([0, T]; H^\infty(\mathbb{R}))$ of the Muskat equation (5), if one sets*

$$\begin{aligned} M_\varepsilon(T) &= \sup_{t \in [0, T]} \left(\|f(t)\|_{\dot{H}^{\frac{3}{2}}}^2 + \|f(t)\|_{L^2}^2 \right) \\ &\quad + \int_0^T \|f(t)\|_{\dot{H}^2}^2 \, dt + |\log(\varepsilon)|^{-1} \int_0^T \|f(t)\|_{\dot{H}^{\frac{5}{2}}}^2 \, dt \end{aligned}$$

then,

$$\int_0^T \frac{\|\mathcal{T}(f)f\|_{\dot{H}^1}^2}{\log(2 + \|\mathcal{T}(f)f\|_{\dot{H}^1})} dt \leq \mathcal{F}(M_\varepsilon(T)), \tag{42}$$

and

$$\int_0^T \frac{\|\partial_t f\|_{\dot{H}^1}^2}{\log(2 + \|\partial_t f\|_{\dot{H}^1}^2)} dt \leq \mathcal{F}(M_\varepsilon(T)). \tag{43}$$

Proof Let C be the constant given by Proposition 2.13 and set $\tilde{C} = \max\{C, 1\}$. We claim that

$$\frac{\|\mathcal{T}(f)f\|_{\dot{H}^1}^2}{\log(2 + \|\mathcal{T}(f)f\|_{\dot{H}^1})} \leq \tilde{C}^2 \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + \|f\|_{\dot{H}^1} + 1 \right)^2 \|f\|_{\dot{H}^2}^2.$$

If $\|\mathcal{T}(f)f\|_{\dot{H}^1} \leq \|f\|_{\dot{H}^2}$, then this is obvious. Otherwise, this follows at once from (41). This implies (42).

The proof of (43) follows from similar argument, using the equation to estimate $\partial_t f$ in terms of $\mathcal{T}(f)f$. □

It follows from the previous results that one can extract from the solutions of the approximate Cauchy problems (5) a sub-sequence converging to a solution of the Muskat equation (1). Since it is rather classical, we do not include the details and refer for instance to [19,21].

2.8 Uniqueness

To prove the uniqueness of the solution to the Cauchy problem for rough initial data, we shall prove an estimate for the difference of two solutions.

Proposition 2.15 *Let $T > 0$ and consider two solutions f_1, f_2 of the Muskat equation, with initial data $f_{1,0}, f_{2,0}$ respectively, satisfying*

$$f_k \in C^0([0, T]; \dot{W}^{1,\infty}(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R})) \cap C^1([0, T]; \dot{H}^{\frac{1}{2}}(\mathbb{R})) \cap L^2(0, T; \dot{H}^2(\mathbb{R})), \quad k = 1, 2.$$

Assume that

$$\sup_{t \in [0, T]} \left(\|f_k(t)\|_{\dot{H}^{\frac{3}{2}}}^2 + \|f_k(t)\|_{\dot{W}^{1,\infty}}^2 \right) + \int_0^T \|f_k\|_{\dot{H}^2}^2 dt \leq M < \infty, \quad k = 1, 2. \tag{44}$$

Then the difference $g = f_1 - f_2$ is estimated by

$$\sup_{t \in [0, T]} \|g(t)\|_{\dot{H}^{\frac{1}{2}}} \leq \|g(0)\|_{\dot{H}^{\frac{1}{2}}} \exp \left(C(M + 1)^5 \int_0^T (\|f_1\|_{\dot{H}^2}^2 + \|f_2\|_{\dot{H}^2}^2) dt \right). \tag{45}$$

Proof Since $\partial_t f_k + |D| f_k = \mathcal{T}(f_k) f_k$, it follows from the decomposition (36) of $\mathcal{T}(f_k) f_k$ that the difference $g = f_1 - f_2$ satisfies

$$\partial_t g + \frac{|D|g}{1 + (\partial_x f_1)^2} = V(f_1)\partial_x g + R(f_1, g) + (\mathcal{T}(f_2 + g) - \mathcal{T}(f_2)) f_2.$$

Since g belongs to $C^1([0, T]; \dot{H}^{\frac{1}{2}}(\mathbb{R}))$, we may take the L^2 -scalar product of this equation with $|D|g$ to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + \int \frac{(|D|g)^2}{1 + (\partial_x f_1)^2} dx &\leq |(V(f_1)\partial_x g, |D|g)| + \|R(f_1, g)\|_{L^2} \|g\|_{\dot{H}^1} \\ &\quad + \|(\mathcal{T}(f_2 + g) - \mathcal{T}(f_2)) f_2\|_{L^2} \|g\|_{\dot{H}^1}. \end{aligned}$$

It follows from Proposition 2.12 that

$$\begin{aligned} \frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + M^{-1} \|g\|_{\dot{H}^1}^2 &\lesssim \left(\|f_1\|_{\dot{H}^2} + \|f_1\|_{\dot{H}^{\frac{7}{4}}}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} \\ &\quad + \|f_2\|_{\dot{H}^{\frac{7}{4}}} \|g\|_{\dot{H}^{\frac{3}{4}}} \|g\|_{\dot{H}^1}. \end{aligned}$$

By Gagliardo-Nirenberg interpolation inequality

$$\begin{aligned} \frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + M^{-1} \|g\|_{\dot{H}^1}^2 &\lesssim \|f_1\|_{\dot{H}^2} \left(1 + \|f_1\|_{\dot{H}^{\frac{3}{2}}} \right) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} \\ &\quad + \|f_2\|_{\dot{H}^2}^{\frac{1}{2}} \|f_2\|_{\dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1}^{\frac{3}{2}} \\ &\lesssim \|f_1\|_{\dot{H}^2} (1 + M) \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1} \\ &\quad + M^{\frac{1}{2}} \|f_2\|_{\dot{H}^2}^{\frac{1}{2}} \|g\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^1}^{\frac{3}{2}}. \end{aligned}$$

Thus, thanks to Holder’s inequality, one gets

$$\frac{d}{dt} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{1}{2M} \|g\|_{\dot{H}^1}^2 \leq C(M + 1)^5 \left(\|f_1\|_{\dot{H}^2}^2 + \|f_2\|_{\dot{H}^2}^2 \right) \|g\|_{\dot{H}^{\frac{1}{2}}}^2$$

which in turn implies (45). □

2.9 The Cauchy Problem for the Approximate Equations

It remains to prove Proposition 2.1.

Rewrite the equation (5) under the form

$$\partial_t f - |\log(\varepsilon)|^{-1} \partial_x^2 f = N_\varepsilon(f), \tag{46}$$

with

$$N_\varepsilon(f) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \left(1 - \chi \left(\frac{|\alpha|}{\varepsilon} \right) \right) d\alpha.$$

The next proposition shows that Equation (46) can be seen as a sub-critical parabolic equation.

Lemma 2.16 *There holds*

$$\|N_\varepsilon(f)\|_{\dot{H}^1} \lesssim \varepsilon^{\frac{1}{2}} \|f\|_{\dot{H}^{\frac{5}{2}}} + \left(1 + \|f\|_{\dot{H}^{\frac{3}{2}}} \right)^2 \log \left(2 + \|f\|_{\dot{H}^2}^2 \right)^{\frac{1}{2}} \|f\|_{\dot{H}^2}, \quad (47)$$

and

$$\|N_\varepsilon(f)\|_{L^2} \leq C \left(1 + \|f\|_{\dot{H}^{\frac{3}{2}}} \right)^2. \quad (48)$$

Proof The estimate (47) follows at once from (20) and (21). To prove (48), we decompose $N_\varepsilon(f) = -|D|f + \mathcal{T}(f)f + R_\varepsilon(f)$ where $\mathcal{T}(f)$ is the operator already introduced in §2.7 and the remainder $R_\varepsilon(f)$ is as defined by (15). Recall from Proposition 2.3 in [2] that

$$\|\mathcal{T}(f)f\|_{L^2} \lesssim \|f\|_{\dot{H}^1} \|f\|_{\dot{H}^{\frac{3}{2}}}.$$

So the wanted conclusion follows from the estimate (16) for $R_\varepsilon(f)$. □

Multiply the latter equation by $(I - \Delta)^{3/2} f$ and integrate in time, to obtain

$$\frac{1}{2} \frac{d}{dt} \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + |\log(\varepsilon)|^{-1} \||D|f\|_{\dot{H}^{\frac{3}{2}}}^2 \leq \|N_\varepsilon(f)\|_{\dot{H}^1} \|f\|_{\dot{H}^2}. \quad (49)$$

Recall that

$$\|N_\varepsilon(f)\|_{\dot{H}^1} \lesssim \varepsilon^{\frac{1}{2}} \|f\|_{\dot{H}^{\frac{5}{2}}} + \left(1 + \|f\|_{\dot{H}^{\frac{3}{2}}} \right)^2 \log \left(2 + \|f\|_{\dot{H}^2}^2 \right)^{\frac{1}{2}} \|f\|_{\dot{H}^2}, \quad (50)$$

Since $\varepsilon^{\frac{1}{2}} \ll |\log(\varepsilon)|^{-1}$ for $\varepsilon \ll 1$, we can absorb the contribution of $\varepsilon^{\frac{1}{2}} \|f\|_{\dot{H}^{\frac{5}{2}}}$ in the right-hand side of (50) by the left-hand side of (49). On the other hand, since $5/2 > 2$, one can absorb the contribution of the other terms by using the Hölder’s inequality. This proves an *a priori* estimate for (46). We also get easily a contraction estimate similar to (but much simpler) the one given by Proposition 2.15. Then by using classical tools for semi-linear equations, we conclude that the Cauchy problem for (46) can be solved by standard iterative scheme.

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