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Local Well-Posedness of the (4 + 1)-Dimensional Maxwell–Klein–Gordon Equation at Energy Regularity

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Abstract This paper is the first part of a trilogy [22,23] dedicated to a proof of global well-posedness and scattering of the $(4+1)$-dimensional mass-less Maxwell–Klein–Gordon equation (MKG) for any finite energy initial data. The main result of the present paper is a large energy local well-posedness theorem for MKG in the global Coulomb gauge, where the lifespan is bounded from below by the energy concentration scale of the data. Hence the proof of global well-posedness is reduced to establishing non-concentration of energy. To deal with non-local features of MKG we develop initial data excision and gluing techniques at critical regularity, which might be of independent interest.

Keywords Maxwell-Klein-Gordon · Coulomb gauge · Local well-posedness · Energy concentration scale · Initial data gluing

1 Introduction

Let $\mathbb{R}^{1+4}$ be the $(4+1)$-dimensional Minkowski space with the metric

$$\mathbf{m}_{\mu\nu} := \mathrm{diag}\,(-1, +1, +1, +1, +1)$$

in the standard rectilinear coordinates $(t = x^0, x^1, \cdots, x^4)$. Let $L = \mathbb{R}^{1+4} \times \mathbb{C}$ be the trivial $\mathrm{U}(1)$ complex line bundle over $\mathbb{R}^{1+4}$. The *Maxwell–Klein–Gordon system*

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is a relativistic gauge field theory that describes the evolution of a pair (A, ϕ) of a connection on L and a section of L. In Section 1.1, we present the necessary background material concerning the Maxwell–Klein–Gordon system on $\mathbb{R}^{1+4}$. Readers already familiar with this equation may skip ahead to Section 1.2, where the main results and ideas of the paper are presented.

1.1 The Maxwell–Klein–Gordon System on $\mathbb{R}^{1+4}$

Let $L = \mathbb{R}^{1+4} \times \mathbb{C}$ be the trivial complex line bundle with structure group $\mathrm{U}(1) = \{e^{i\chi} \in \mathbb{C}\}$. Global sections of L are precisely $\mathbb{C}$-valued functions on $\mathbb{R}^{1+4}$. Using the trivial connection on $\mathbb{R}^{1+4}$ as a reference and employing the identification $\mathrm{u}(1) \equiv i\mathbb{R}$, any connection $\mathbf{D}_\mu$ on L can be written as

$$\mathbf{D}_\mu = \partial_\mu + i A_\mu$$

where A_μ is a real-valued 1-form on $\mathbb{R}^{1+4}$.

The (mass-less) *Maxwell–Klein–Gordon system* for a pair (A, ϕ) of a connection on L and a section of L takes the form

$$\begin{cases} \partial^\mu F_{\nu\mu} = \mathrm{Im}(\phi \overline{\mathbf{D}_\nu \phi}) \\ \Box_A \phi = 0, \end{cases} \tag{MKG}$$

where $F_{\mu\nu} := (\mathrm{d}A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the *curvature 2-form* associated to $\mathbf{D}_\mu$ and $\Box_A := \mathbf{D}^\mu \mathbf{D}_\mu$ is the covariant d'Alembertian. We are using the usual convention of raising and lowering indices using the Minkowski metric, and also of summing over repeated upper and lower indices.

We consider the initial value problem for (MKG). An *initial data set* for (MKG) consists of two pairs of 1-forms (a_j, e_j) and $\mathbb{C}$-valued functions (f, g) on $\mathbb{R}^4$. We say that (a_j, e_j, f, g) is the initial data for a solution (A, ϕ) if

$$(A_j, F_{0j}, \phi, \mathbf{D}_t \phi) \restriction_{\{t=0\}} = (a_j, e_j, f, g).$$

Note that (MKG) imposes the condition that the following equation be true for any initial data for (MKG):

$$\partial^j e_j = \mathrm{Im}(f \overline{g}). \tag{1.1}$$

This equation is the *Gauss* (or the *constraint*) *equation* for (MKG).

A basic geometric feature of the Maxwell–Klein–Gordon system is *gauge invariance*. Let χ be a *gauge transformation* for (MKG), i.e., a real-valued function on $\mathbb{R}^{1+4}$, so that $e^{i\chi} \in \mathrm{U}(1)$. Then (MKG) is invariant under the associated gauge transform $(A, \phi) \mapsto (A - \mathrm{d}\chi, e^{i\chi}\phi)$. Geometrically, a gauge transform corresponds to a change of basis in the fiber $\mathbb{C}$ of the complex line bundle L over each point in $\mathbb{R}^{1+4}$. To establish any sort of well-posedness of the initial value problem and also to reveal the hyperbolicity[1] of (MKG), the ambiguity arising from this invariance must be fixed. For this purpose we rely on the *global Coulomb gauge* condition $\sum_{j=1}^{4} \partial_j A_j = 0$ in this paper.

The Maxwell–Klein–Gordon system on $\mathbb{R}^{1+4}$ obeys the law of *conservation of energy*. The conserved energy of a solution (A, ϕ) at time t is defined as

$$\mathcal{E}_{\{t\}\times\mathbb{R}^4}[A, \phi] := \frac{1}{2} \int_{\{t\}\times\mathbb{R}^4} \sum_{0\leq\mu<\nu\leq 4} |F_{\mu\nu}|^2 + \sum_{0\leq\mu\leq 4} |\mathbf{D}_\mu \phi|^2 \, \mathrm{d}x. \tag{1.2}$$

For any sufficiently regular solution to (MKG) on $I \times \mathbb{R}^4$, where $I \subseteq \mathbb{R}$ is a connected interval, $\mathcal{E}_{\{t_1\}\times\mathbb{R}^4}[A, \phi] = \mathcal{E}_{\{t_2\}\times\mathbb{R}^4}[A, \phi]$ for every $t_1, t_2 \in I$. For a (MKG) initial data set (a, e, f, g), the conserved energy takes the form

$$\mathcal{E}_{\mathbb{R}^4}[a, e, f, g] = \frac{1}{2} \int_{\mathbb{R}^4} \sum_{1\leq j<k\leq 4}^{4} |\partial_j a_k - \partial_k a_j|^2 + \sum_{j=1}^{4} |e_j|^2 + \sum_{j=1}^{4} |\mathbf{D}_j f|^2 + |g|^2 \, \mathrm{d}x, \tag{1.3}$$

where $\mathbf{D}_j := \partial_j + i a_j$. Furthermore, given any (measurable) subset $O' \subseteq \mathbb{R}^4$, we define the local energy $\mathcal{E}_{O'}[a, e, f, g]$ by replacing the domain of integral above by O'.

The Maxwell–Klein–Gordon system can in fact be formulated on any $\mathbb{R}^{1+d}$ $(d \geq 1)$. However, the (4 + 1)-dimensional case is distinguished by the fact that the system becomes *energy critical*. That is, in $\mathbb{R}^{1+4}$ both the conserved energy (1.2) and the equations (MKG) are invariant under the scaling

$$(A, \phi) \mapsto (\widetilde{A}, \widetilde{\phi})(t, x) := (\lambda^{-1} A, \lambda^{-1}\phi)(\lambda^{-1} t, \lambda^{-1} x) \quad \text{for any } \lambda > 0.$$

1.2 Main Results and Ideas

The present paper is the first of a sequence of three papers [22,23], in which we give a complete proof of global well-posedness and scattering of (MKG) on $\mathbb{R}^{1+4}$ for any finite energy data. This theorem is analogous to the *threshold theorem* for energy critical wave maps [18,29,30,33–37]. The main result of this paper is the following local well-posedness theorem for (MKG) in the global Coulomb gauge at the energy regularity.

Theorem 1.1 (Local well-posedness of (MKG) at energy regularity, simple version). *Let E be any positive number and let (a, e, f, g) be a smooth initial data set with energy $\leq E$ satisfying the global Coulomb condition $\sum_{j=1}^{4} \partial_j a_j = 0$.*

(1) *Then there exists an open time interval $I \ni 0$ and a unique smooth solution (A, ϕ) to the initial value problem on $I \times \mathbb{R}^4$ satisfying the global Coulomb gauge condition $\sum_{j=1}^{4} \partial_j A_j = 0$.*

[1] Observe that without any choice of gauge, the the principal part of $\partial^\mu F_{\nu\mu}$ is $-\Box A_\nu + \partial_\nu \partial^\mu A_\mu$, which does not have a well-defined character.

(2) *Define the* energy concentration scale *of* (a, e, f, g) *by*

$$r_c = r_c(E)[a, e, f, g] := \sup\{r > 0 : \forall x \in \mathbb{R}^4,\ \mathcal{E}_{B_r(x)}[a, e, f, g] < \delta_0(E, \epsilon_*^2)\},$$

where $B_r(x)$ denotes the open ball of radius r centered at x, ϵ_* is a universal constant (see Theorem 1.2 below) and $\delta_0(E, \epsilon_*^2)$ is some positive function (to be specified in Section 6). Then I contains the interval $[-r_c, r_c]$.

(3) *Finally, the solution map extends continuously on compact time intervals to general finite energy initial data, with the same lifespan properties as in (2) above.*

For a more precise version, see Theorem 6.1. We remark that we do not lose any generality by restricting to initial data sets in the global Coulomb gauge, as any finite energy initial data sets can be gauge transformed into this gauge; see Section 3. We formulate our local well-posedness theorem specifically in the global Coulomb gauge in view of the rest of the series [22,23], where we show global well-posedness and scattering in this gauge.

An important feature of Theorem 1.1 is that it provides a lower bound on the lifespan in terms of the *energy concentration scale* r_c of the data. Taking the contrapositive, we see that any finite time blow up of a solution to (MKG) must be accompanied by energy concentration at a point. In [22,23], following the scheme successfully developed by one of the authors (D. Tataru) and J. Sterbenz in the context of energy critical wave maps [29,30], we establish global well-posedness of (MKG) for finite energy data by showing that such a phenomenon cannot occur. We refer to the last and the main paper of the sequence [23] for an overview of the entire series.

To prove Theorem 1.1, we rely on the following small energy global well-posedness theorem for the Maxwell–Klein–Gordon equations in the global Coulomb gauge, which was established recently by one of the authors (D. Tataru) jointly with J. Krieger and J. Sterbenz.

Theorem 1.2 (Small energy global well-posedness in Coulomb gauge [19]). *There exists an $\epsilon_* > 0$ such that the following holds. Let (a, e, f, g) be a smooth initial data on $\mathbb{R}^4$ satisfying the global Coulomb gauge condition $\sum_{\ell=1}^4 \partial_\ell a_\ell = 0$ and*

$$\mathcal{E}_{\mathbb{R}^4}[a, e, f, g] \le \epsilon_*^2.$$

(1) *Then there exists a unique smooth global solution (A, ϕ) to the initial value problem for* (MKG) *on $\mathbb{R}^{1+4}$ satisfying*

$$\|A_0\|_{Y^1(\mathbb{R}^{1+4})} + \|A_x\|_{S^1(\mathbb{R}^{1+4})} + \|\phi\|_{S^1(\mathbb{R}^{1+4})} \lesssim \sqrt{\mathcal{E}_{\mathbb{R}^4}[a, e, f, g]}, \tag{1.4}$$

where $A_x = (A_1, \dots, A_4)$.

(2) *For every compact time interval $I \subseteq \mathbb{R}$, the solution map extends continuously to general finite energy initial data after restriction[2] to $I \times \mathbb{R}^4$. More precisely,*

[2] Although this continuity statement is not explicitly stated in [19, Theorem 1], its proof can be read off from [19, Section 5.5]. We remark that continuous dependence on the data in $\mathcal{H}^1$ does not seem to hold in the global space $S^1(\mathbb{R}^{1+4})$, due to the strong dependence of the linear magnetic flow for $\Box_A$ on the low frequency part of A_x.

if $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)})$ *is a sequence of finite energy initial data sets in global Coulomb gauge whose limit is* (a, e, f, g) *in* $\mathcal{H}^1$ *(defined in Section 3.1), then*

$$\|A_0^{(n)} - A_0\|_{Y^1(I\times\mathbb{R}^4)} + \|A_x^{(n)} - A_x\|_{S^1(I\times\mathbb{R}^4)} + \|\phi^{(n)} - \phi\|_{S^1(I\times\mathbb{R}^4)} \to 0 \quad \text{as } n \to \infty, \tag{1.5}$$

where $(A^{(n)}, \phi^{(n)})$ *is the global solution to* (MKG) *with data* $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)})$.

More detailed descriptions of the function spaces S^1 and Y^1 will be given in Sections 6 and 7. In particular, S^1 is a delicate function space consisting of a number of pieces, including the energy norm, a frequency localized Strichartz norm, an $\dot{X}^{s,b}$-type norm and a null frame norm as in the energy critical wave maps problem [31,38]. The precise version of the main local well-posedness theorem (Theorem 6.1) also involves these spaces. At this point we simply remark that for any interval $I \times \mathbb{R}^4$, we have

$$\|(\varphi, \partial_t\varphi)\|_{C_t(I;\dot{H}^1_x\times L^2_x)} \lesssim \|\varphi\|_{S^1(I\times\mathbb{R}^4)}, \quad \|(\varphi, \partial_t\varphi)\|_{C_t(I;\dot{H}^1_x\times L^2_x)} \lesssim \|\varphi\|_{Y^1(I\times\mathbb{R}^4)}.$$

For a simpler energy critical semilinear wave equation, such as $\Box u = \pm u^{\frac{d+2}{d-2}}$ on $\mathbb{R}^{1+d}$, a statement analogous to Theorem 1.1 is an immediate consequence of the small energy global well-posedness theorem (Theorem 1.2 in our context) and the finite speed of propagation of the system. Roughly speaking, the proof of local well-posedness (in particular, local existence) proceeds in the following three steps (see, for instance [32, Section 5.1]):

Step 1. Truncation of the initial data set locally in space so that the energy becomes small;

Step 2. Application of small energy global well-posedness to produce the corresponding set of global solutions; and

Step 3. Patching together the resulting solutions via finite speed of propagation[3].

However, implementation of this strategy in our context is not as straightforward due to *non-local* features of the Maxwell–Klein–Gordon system in the global Coulomb gauge. One source of non-locality is the Gauss equation for initial data sets, which forbids us from naively truncating initial data to reduce to the small energy case. Another source is the global Coulomb gauge condition, which imposes a Poisson (hence non-local) equation for the component A_0 of the connection 1-form. In particular, finite speed of propagation *fails* in the global Coulomb gauge.

In this paper we develop techniques for overcoming such issues concerning non-locality of the Maxwell–Klein–Gordon equations, and employ them to prove Theorem 1.1 from Theorem 1.2 by essentially carrying out Steps 1–3 above. These techniques (in addition to Theorem 1.1 itself) are also crucially used in the last paper of the sequence [23], where we carry out a blow-up analysis of (MKG) to preclude concentration of energy and non-scattering.

To deal with the non-locality of the Gauss equation, we introduce the method of *initial data excision and gluing* at critical regularity for (MKG); see Propositions 4.1

[3] More precisely, in Step 3, by finite speed of propagation, note that the global solutions in Step 2 restricted to the domain of dependence of the truncated regions in Step 1 give rise to a family of local-in-space-time solutions, which agree with each other on the intersection of the domains.

and 4.2 for the precise formulation. Instead of naively truncating an initial data set (a, e, f, g), which would violate the Gauss equation, the idea is to *excise* the unwanted part and then *glue* another solution to the Gauss equation with the appropriate behavior. Similar techniques have been developed for the initial data sets of the Einstein equations in general relativity [4–6,8]. In our context, we need to develop a sharp version that works at the critical regularity. Our key tool is an explicit solution operator to the divergence equation [2,3,11] which preserves the compact support property; see Proposition 4.4.

The initial data excision and gluing technique allows us to carry out an analogue of Step 1. Then applying suitable gauge transformations to the resulting initial data sets to impose the global Coulomb gauge condition, we are in position to use Theorem 1.2 to produce the corresponding global solutions. This procedure is analogous to Step 2. However, we face difficulty in patching these solutions in the global Coulomb gauge (which corresponds to Step 3), since finite speed of propagation does not hold in this gauge.

We use two ideas for addressing this issue. The first is the observation that even though finite speed of propagation may fail in a particular gauge (e.g., the global Coulomb gauge), it remains true up to a gauge transformation. We refer to this fact as the *local geometric uniqueness* of (MKG); see Proposition 5.2. Hence we obtain from the global solutions produced in Step 2 a family of local-in-space-time solutions $(A_{[\alpha]}, \phi_{[\alpha]})$ to (MKG), which agree with each other on the intersection of the domains up to gauge transformations. We call such solutions *compatible pairs* (see Definition 6.15). Geometrically, these are nothing but a description of a global pair of a connection 1-form and a section of L in local trivializations.

The second idea is to patch these local descriptions together to form a single solution in the global Coulomb gauge. We begin by adapting an argument of Uhlenbeck [41, Section 3] to produce a single global-in-space solution in the desired function spaces S^1, Y^1; see Proposition 6.16. For this purpose, we develop a functional space framework for performing gauge transforms between local-in-spacetime solutions in S^1 and Y^1; see Section 6.3 and Section 7. A key point in this argument is that a gauge transformation χ between two Coulomb gauges obeys the Laplace equation $\Delta\chi = 0$, and hence enjoys improved regularity. The solution resulting from this patching argument does not necessarily satisfy the exact global Coulomb condition. Nevertheless this solution is *approximately Coulomb*, since it arose by patching together Coulomb solutions. Hence there exists a nicely behaved gauge transformation into the global Coulomb gauge, which completes the analogue of Step 3 and hence the sketch of our proof of Theorem 1.1.

Remark 1.3 The main result and the techniques developed in this paper are perturbative in nature, and hence can be easily generalized to higher dimensions, i.e., $\mathbb{R}^{1+d}$ for any $d \geq 4$. In what follows we focus on the most interesting case $\mathbb{R}^{1+4}$ for concreteness.

1.3 Other Works on the Maxwell–Klein–Gordon Equations

Here we give a brief review of the literature on the Maxwell–Klein–Gordon problem. In dimensions 2+1 and 3+1 the Maxwell–Klein–Gordon system is *energy subcritical*, so global regularity follows from local well-posedness at the energy regularity; see Klainerman–Machedon [15] and Selberg-Tesfahun [27]. We also mention the works of Moncrief [21] and Eardley-Moncrief [9,10], where global regularity of sufficiently smooth solutions in $\mathbb{R}^{1+2}$ and $\mathbb{R}^{1+3}$ was established by a different argument; the latter two also handled the more general Yang–Mills–Higgs system on $\mathbb{R}^{1+3}$. The problem of low regularity well-posedness in $\mathbb{R}^{1+3}$ was further studied by Cuccagna [7] and then more recently by Machedon-Sterbenz [20], who reached the essentially optimal regularity $A(0), \phi(0) \in H^{\frac{1}{2}+}$. In [12], global well-posedness was established below the energy norm, more precisely for $A(0), \phi(0) \in H^{\frac{\sqrt{3}}{2}+}$

In dimension 4+1, Klainerman–Tataru [16] established an essentially optimal local well-posedness result for a model equation closely related to Maxwell–Klein–Gordon and Yang–Mills. This result was further refined by Selberg [26], who considered the full Maxwell–Klein–Gordon system on $\mathbb{R}^{1+4}$, and Sterbenz [28].

For the critical regularity problem, Rodnianski–Tao [24] made an initial breakthrough and proved global regularity for small scaling critical Sobolev data in dimensions 6 + 1 and higher. This result was greatly improved in the aforementioned work of Krieger–Sterbenz–Tataru [19] to include the energy critical dimension (4+1), which provides the starting point of the present paper.

Finally, we note that an independent proof of global well-posedness and scattering of (MKG) has recently been announced by Krieger–Lührmann [17], following a version of the Bahouri–Gérard nonlinear profile decomposition [1] and the Kenig–Merle concentration compactness/rigidity scheme [13,14], developed by Krieger–Schlag [18] for the energy critical wave maps problem.

1.4 The Structure of the Paper

After some preliminaries in Section 2, we begin with a systematic study of finite energy initial data sets for (MKG) in Section 3. We show, in particular, that every such initial data set can be gauge transformed to the global Coulomb gauge (Lemma 3.3), and also that it can be approximated by smooth data (Lemma 3.2). In Section 4, we develop the theory of excision and gluing of Maxwell–Klein–Gordon initial data sets at the energy regularity (Propositions 4.1, 4.2). In Section 5, we formulate a notion of solutions to (MKG) arising from general finite energy initial data (admissible $C_t\mathcal{H}^1$ solutions) and prove local geometric uniqueness of (MKG) in this class (Proposition 5.2). In Section 6, we give a precise statement of the main local well-posedness theorem (Theorem 6.1) and prove it up to some estimates concerning the functions spaces S^1, Y^1. Finally, in Section 7 we delve further into the structure of the spaces S^1, Y^1 and establish the function space estimates used in Section 6, thereby completing the proof of Theorem 6.1.

2 Preliminaries

2.1 Notation and Conventions

We write $A \lesssim B$ when there exists a constant $C > 0$ such that $A \le CB$. The dependence of the constant is specified by a subscript, e.g., $A \lesssim_r B$ means that there exists $C = C(r) > 0$ such that $A \leq CB$. We write $A \approx B$ when both $A \lesssim B$ and $B \lesssim A$ hold.

We employ the index notation in this paper. Unless otherwise specified, we always use the rectilinear coordinates $(t = x^0, x^1, \ldots, x^4)$. The greek indices (e.g., $\mu, \nu, \ldots$) run over $0, 1, \ldots, 4$, whereas the roman indices only run over $1, \ldots, 4$. As already mentioned in the introduction, we raise and lower indices using the Minkowski metric $\mathbf{m}_{\mu\nu}$, and use the convention of summing up repeated upper and lower indices.

We denote the open ball in $\mathbb{R}^4$ of radius r and center x by $B_r(x)$. Given a cube $R \subseteq \mathbb{R}^4$, we refer to its side length by $\ell(R)$. For a convex subset K of $\mathbb{R}^4$ (or $\mathbb{R}^{1+4}$) and $c \in (0, \infty)$, we define cK to be the dilation of K by c about the center of mass of K. For example, if $B_r(x)$ is an open ball in $\mathbb{R}^4$, then $cB_r(x)$ is the open ball with the same center and the radius c times that of B, i.e., $cB_r(x) = B_{cr}(x)$.

2.2 Dyadic Frequency Projections

Let $m_{\leq 0}(r)$ be a smooth cutoff which equals 1 on $\{r \leq 1\}$ and vanishes outside $\{r \geq 2\}$. For every $k \in \mathbb{Z}$, define $m_{\leq k}(r) := m_{\leq 0}(r/2^k)$ and $m_k(r) := m_{\leq k}(r) - m_{\leq k-1}(r)$. Then m_k is supported in the set $\{2^{k-1} \leq r \leq 2^{k+1}\}$ and forms a partition of unity, i.e.,

$$\sum_k m_k(r) = 1.$$

The following dyadic frequency (or *Littlewood–Paley*) projections are used in this paper:

$$\begin{aligned} P_k \varphi &= \mathcal{F}^{-1}[m_k(|\xi|)\mathcal{F}[\varphi]], & Q_j \varphi &= \mathcal{F}^{-1}[m_j(||\tau| - |\xi||)\mathcal{F}[\varphi]], \\ S_\ell \varphi &= \mathcal{F}^{-1}[m_\ell(|(\tau, \xi)|)\mathcal{F}[\varphi]], & T_j \varphi &= \mathcal{F}^{-1}[m_j(|\tau|)\mathcal{F}[\varphi]]. \end{aligned}$$

We also use the notation $P_{\leq k} := \sum_{k' \leq k} P_k$, $P_{(k_1, k_2]} := \sum_{k' \in (k_1, k_2]} P_{k'}$ etc.

2.3 Standard Functions Spaces on $\mathbb{R}^d$ and Domains

Unless otherwise specified, we define function spaces on a subset $\mathcal{O} \subseteq \mathbb{R}^d$ by restricting the $\mathbb{R}^d$ version, i.e.,

$$\|\varphi\|_{X(\mathcal{O})} := \inf_{\psi = \varphi \text{ on } \mathcal{O}} \|\psi\|_{X(\mathbb{R}^d)}.$$

The homogeneous Sobolev and Besov semi-norms $\|\cdot\|_{\dot{W}^{s,p}(\mathbb{R}^d)}$, $\|\cdot\|_{\dot{B}^{s,p}_r(\mathbb{R}^d)}$ on $\mathbb{R}^d$ are characterized using the Littlewood–Paley projections as follows:

$$\|\varphi\|_{\dot{W}^{s,p}(\mathbb{R}^d)} \approx \|\Big(\sum_k 2^{2sk}|P_k\varphi|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)}, \quad \|\varphi\|_{\dot{B}^{s,p}_r(\mathbb{R}^d)} \approx \Big(\sum_k 2^{rsk}\|P_k\varphi\|^r_{L^p(\mathbb{R}^d)}\Big)^{\frac{1}{r}}.$$

We define the corresponding spaces $\dot{W}^{s,p}(\mathbb{R}^d)$, $\dot{B}^{s,p}_r(\mathbb{R}^d)$ to consist of tempered distributions that are regular at zero frequency (i.e., $\|P_{\leq k}\varphi\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $k \to -\infty$)

and have finite corresponding semi-norms. We use the standard notation $\dot{W}^{s,2} = \dot{H}^s$. When $s < \frac{d}{p}$ or $s = \frac{d}{p}$ with $r = 1$ (in the Besov case) the above semi-norms are in fact norms when restricted to the space $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions on $\mathbb{R}^d$, and the corresponding spaces are obtained as the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to these norms.

3 Finite Energy Initial Data for Maxwell–Klein–Gordon

In this section we systematically develop the basic theory of finite energy initial data sets for (MKG). In Section 3.1 we define the spaces of finite energy and classical initial data sets for (MKG), and also the corresponding spaces of gauge transformations. In Section 3.2, we prove a few elementary facts about finite energy initial data sets, such as approximation by classical initial data and gauge transformation to a globally Coulomb initial data. We also show that any globally Coulomb finite energy initial data set can be approximated by classical initial data sets in the global Coulomb gauge.

3.1 Finite Energy Initial Data Sets and Gauge Transformations

Let $O \subseteq \mathbb{R}^4$ be a non-empty open set. Given 1-forms a, e and $\mathbb{C}$-valued functions f, g on O, we say that the quadruple (a, e, f, g) is a (MKG) *initial data set* if the following *Gauss* (or the *constraint*) *equation* holds:

$$\partial^\ell e_\ell = \mathrm{Im}[f\overline{g}]. \tag{3.1}$$

We define the space $\mathcal{H}^1(O)$, which consists of (MKG) initial data sets (a, e, f, g) for which the following norm is finite:

$$\|(a, e, f, g)\|_{\mathcal{H}^1(O)} := \sup_{j=1,\dots,4} \|(a_j, e_j)\|_{(\dot{H}^1_x \cap L^4_x) \times L^2_x(O)} + \|(f, g)\|_{(\dot{H}^1_x \cap L^4_x) \times L^2_x(O)}.$$

A Coulomb (gauge) initial data set is a data set (a, e, f, g) which in addition satisfies the divergence condition

$$\nabla \cdot a = \partial^\ell a_\ell = 0.$$

Given an $\mathcal{H}^1(O)$ initial data set (a, e, f, g), we define its *energy* on $O' \subseteq O$ by

$$\mathcal{E}_{O'}[a, e, f, g] := \frac{1}{2} \int_{O'} \sum_{1 \le j < k \le 4} |(\mathrm{d}a)_{jk}|^2 + \sum_{1 \le j \le 4} |e_j|^2 + \sum_{1 \le j \le 4} |\mathbf{D}_j f|^2 + |g|^2 \,\mathrm{d}x, \tag{3.2}$$

where $(\mathrm{d}a)_{jk} = \partial_j a_k - \partial_k a_j$ and $\mathbf{D}_j f = \partial_j f + i a_j f$. The space $\mathcal{H}^1(O)$ is a natural domain on which the energy functional is always finite, and for this reason $\mathcal{H}^1(O)$ will also be referred to as the space of *finite energy* initial data. In general the energy does not control the $\mathcal{H}^1$ norm, and for this reason we view the $\mathcal{H}^1$ bounds as qualitative, whereas the energy related bounds are quantitative. However, in the case of global

Coulomb data sets the situation improves and we can estimate the $\mathcal{H}^1$ norm in terms of the energy, see Lemma 3.3.

We also remark that the energy $\mathcal{E}_{O'}[a, e, f, g]$ is invariant under gauge transformations, which will be rigorously defined below.

For $N \geq 1$, we define the higher regularity space $\mathcal{H}^N(O)$ in a similar fashion with the norm

$$\|(a, e, f, g)\|_{\mathcal{H}^N(O)} := \sum_{n=1}^{N} \|(\partial_x^{(n-1)} a, \partial_x^{(n-1)} e, \partial_x^{(n-1)} f, \partial_x^{(n-1)} g)\|_{\mathcal{H}^1(O)}.$$

To define the space $\mathcal{H}^\infty(O)$ of *classical initial data sets*, we first define the space $\mathcal{H}^0(O)$ to consist of (MKG) initial data sets with finite $\mathcal{H}^0(O)$ semi-norm, which is given by

$$\|(a, e, f, g)\|_{\mathcal{H}^0(O)} := \|a\|_{L^2_x} + \|f\|_{L^2_x}$$

Then we take $\mathcal{H}^\infty(O) := \cap_{N=0}^\infty \mathcal{H}^N(O)$ and topologize it using $\{\|\cdot\|_{\mathcal{H}^N(O)}\}_{N \geq 0}$. We remark that $\mathcal{H}^\infty(O)$ initial data have not only better regularity (it is in fact smooth), but also better integrability than $\mathcal{H}^N(O)$.

Next, we define spaces of gauge transformations between initial data sets. A gauge transformation χ, which is simply an $\mathbb{R}$-valued function on O, acts on an initial data set (a, e, f, g) as follows:

$$\Gamma_\chi[a, e, f, g] := (a - \mathrm{d}\chi, e, e^{i\chi} f, e^{i\chi} g).$$

We define the space $\mathcal{G}^2(O)$ to consist of locally integrable gauge transformations such that the following semi-norm is finite:

$$\|\chi\|_{\mathcal{G}^2(O)} := \|\partial_x \chi\|_{L^4_x(O)} + \|\partial_x^2 \chi\|_{L^2_x(O)}.$$

Given an integer $N \geq 1$, we define the $\mathcal{G}^{N+1}(O)$ semi-norm as

$$\|\chi\|_{\mathcal{G}^{N+1}(O)} := \sum_{n=1}^{N} \Big(\|\partial_x^{(n)} \chi\|_{L^4_x(O)} + \|\partial_x^{(n+1)} \chi\|_{L^2_x(O)}\Big),$$

and the space $\mathcal{G}^{N+1}(O)$ to consist of locally integrable gauge transformations with finite $\mathcal{G}^{N+1}(O)$ semi-norm. Observe that $\|\chi\|_{\mathcal{G}^{N+1}(O)} = 0$ if and only if χ is a constant. Accordingly, $\mathcal{G}^{N+1}(O)$ becomes a Banach space once we mod out by constants, but we shall *not* do so in this paper. Finally, we also define

$$\mathcal{G}^\infty(O) := \bigcap_{n=1}^{\infty} \dot{H}_x^n \cap \dot{W}_x^{n-1,4}(O).$$

The space of gauge transformations between initial data sets in the class $\mathcal{H}^N(O)$ is precisely $\mathcal{G}^{N+1}(O)$. Indeed, given $\chi \in \mathcal{G}^{N+1}(O)$, it follows from the chain rule and the fact that $\sigma \mapsto e^{i\sigma}$ is a bounded smooth function that $e^{i\chi} \in \mathcal{G}^{N+1}(O)$ and

$$\|e^{i\chi}\|_{\mathcal{G}^{N+1}(O)} \lesssim \|\chi\|_{\mathcal{G}^{N+1}(O)}(1 + \|\chi\|^N_{\mathcal{G}^{N+1}(O)}).$$

From this fact, we see that if $(a, e, f, g) \in \mathcal{H}^N(O)$ and $\chi \in \mathcal{G}^{N+1}(O)$, then $\Gamma_\chi(a, e, f, g) \in \mathcal{H}^N(O)$. Conversely, if χ is a locally integrable gauge transformation on O such that we have $(a', e', f', g') = \Gamma_\chi(a, e, f, g)$ for some $(a, e, f, g), (a', e', f', g') \in \mathcal{H}^N(O)$, then it easily follows that $\chi \in \mathcal{G}^{N+1}(O)$ from the relation $\mathrm{d}\chi = a - a'$.

The map $\Gamma_\chi[a, e, f, g]$ furthermore enjoys a nice continuity property. We state a version of this property for the case $N = 1$, i.e., $(a, e, f, g) \in \mathcal{H}^1(O)$ and $\chi \in \mathcal{G}^2(O)$.

Lemma 3.1 *Let O be an open connected subset of $\mathbb{R}^4$. Let $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)})$ [resp. $\chi^{(n)}$] be a sequence of $\mathcal{H}^1(O)$ initial data sets [resp. $\mathcal{G}^2(O)$ gauge transformations] such that*

$$\|(a - a^{(n)}, e - e^{(n)}, f - f^{(n)}, g - g^{(n)})\|_{\mathcal{H}^1(O)} \to 0, \quad \|\chi - \chi^{(n)}\|_{\mathcal{G}^2(O)} \to 0,$$

for some $(a, e, f, g) \in \mathcal{H}^1(O)$ and $\chi \in \mathcal{G}^2(O)$ as $n \to \infty$. Then there exists a sequence $\chi_0^{(n)} \in \mathbb{R}$ of constant gauge transformations such that

$$\|\Gamma_\chi[a, e, f, g] - \Gamma_{\chi^{(n)}+\chi_0^{(n)}}[a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)}]\|_{\mathcal{H}^1(O)} \to 0 \text{ as } n \to \infty. \quad (3.3)$$

Proof We shall write

$$(\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g}) = \Gamma_\chi[a, e, f, g], \quad (\widetilde{a}^{(n)}, \widetilde{e}^{(n)}, \widetilde{f}^{(n)}, \widetilde{g}^{(n)}) = \Gamma_{\chi^{(n)}}[a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)}].$$

Before we begin the proof, we first make a few reductions. We first remark that the constants $\chi_0^{(n)}$ above are needed because they are not seen by the $\mathcal{G}^2(O)$ norm. We can eliminate them if we normalize $\chi^{(n)}$, e.g. by requiring that they have zero averages on some ball $B \subset O$:

$$\int_B \chi^{(n)}\,\mathrm{d}x = 0 \quad (3.4)$$

We will make this assumption from here on.

Observe further that (3.3) is easy when all $\chi^{(n)}$'s are the same. Then applying $-\chi$ to every term in the sequence, it suffices to consider the case $\chi = 0$. Finally, the convergence of $(\widetilde{a}^{(n)}, \widetilde{e}^{(n)})$ in $\dot{H}^1_x \cap L^4_x(O) \times L^2_x(O)$ is obvious, so we will focus on $(\widetilde{f}^{(n)}, \widetilde{g}^{(n)})$.

We claim that

$$\|\widetilde{\mathbf{D}}_j \widetilde{f} - \widetilde{\mathbf{D}}_j^{(n)} \widetilde{f}^{(n)}\|_{L^2_x(O)} + \|\widetilde{f} - \widetilde{f}^{(n)}\|_{L^4_x(O)} + \|\widetilde{g} - \widetilde{g}^{(n)}\|_{L^2_x(O)} \to 0 \quad \text{as } n \to \infty,$$

where $\widetilde{\mathbf{D}}_j = \partial_j + i\widetilde{a}_j$, $\widetilde{\mathbf{D}}_j^{(n)} = \partial_j + i\widetilde{a}_j^{(n)}$. Then the desired conclusion (3.3) would follow, using the claim and the $L^4_x(O)$ convergence of $\widetilde{a}^{(n)} \to \widetilde{a}$ to deduce that $\partial_x \widetilde{f}^{(n)} \to \partial_x \widetilde{f}$ in $L^2_x(O)$.

We now prove $\widetilde{g}^{(n)} \to \widetilde{g}$ in $L^2_x(O)$; a similar argument works for $\widetilde{f}^{(n)}$ and $\widetilde{\mathbf{D}}^{(n)}_j \widetilde{f}^{(n)}$ as well. We write

$$\|\widetilde{g}-\widetilde{g}^{(n)}\|_{L^2_x(O)} = \|g-e^{i\chi^{(n)}} g^{(n)}\|_{L^2_x(O)} \leq \|(1-e^{i\chi^{(n)}})g\|_{L^2_x(O)}+\|e^{i\chi^{(n)}}(g-g^{(n)})\|_{L^2_x(O)}.$$

Since $\|e^{i\chi^{(n)}}\|_{L^\infty_x} \leq 1$, it follows that the last term vanishes as $n \to \infty$. It remains to prove

$$\|(1-e^{i\chi^{(n)}})g\|_{L^2_x(O)} \to 0. \tag{3.5}$$

By Lebesgue's dominated convergence theorem, it suffices to show that each subsequence n_k has a further subsequence n_{k_j} so that $\chi^{(n_{k_j})} \to 0$ almost everywhere in O. To see this we use Poincare's inequality. In view of the normalization (3.4), this shows that from the convergence $\|\chi^{(n)}\|_{\mathcal{G}^2(O)} \to 0$ we obtain

$$\chi^{(n)} \to 0 \qquad \text{in } L^4_{loc}(O).$$

Then the a.e. convergence on a subsequence immediately follows. □

3.2 Approximation and Gauge Transformation Lemmas

In this subsection, we record a few useful facts concerning $\mathcal{H}^1$ initial data sets on $\mathbb{R}^4$. The first result says that any $\mathcal{H}^1(\mathbb{R}^4)$ initial data set can be approximated by classical initial data sets.

Lemma 3.2 *Let* (a, e, f, g) *be an initial data set for* (MKG) *in the class* $\mathcal{H}^1(\mathbb{R}^4)$*. Then there exists a sequence* $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)})$ *of initial data sets in* $\mathcal{H}^\infty(\mathbb{R}^4)$ *which approximates* (a, e, f, g) *in* $\mathcal{H}^1(\mathbb{R}^4)$.

Proof Take any $C^\infty_0(\mathbb{R}^4)$ sequence $(\widetilde{a}^{(n)}, \widetilde{e}^{(n)}, \widetilde{f}^{(n)}, \widetilde{g}^{(n)})$ which converges to (a, e, f, g) in the $\mathcal{H}^1(\mathbb{R}^4)$ norm, and take

$$a^{(n)} = \widetilde{a}^{(n)}, \quad f^{(n)} = \widetilde{f}^{(n)}, \quad g^{(n)} = \widetilde{g}^{(n)}.$$

To satisfy the Gauss equation, we take

$$e^{(n)}_j = \widetilde{e}^{(n)}_j + (-\Delta)^{-1}\partial_j(\partial^\ell \widetilde{e}^{(n)}_\ell - \mathrm{Im}[f^{(n)}\overline{g^{(n)}}]).$$

It can be readily verified that $e^{(n)} \in H^\infty_x(\mathbb{R}^4)$. Moreover, since $\partial^\ell \widetilde{e}^{(n)}_\ell - \mathrm{Im}[f^{(n)}\overline{g^{(n)}}] \to 0$ in $\dot{H}^{-1}_x(\mathbb{R}^4)$, it follows that $e^{(n)}_j \to e_j$ in $L^2_x(\mathbb{R}^4)$, as desired. □

The second result shows that any $\mathcal{H}^1(\mathbb{R}^4)$ initial data set can be gauge transformed to a globally Coulomb initial data set.

Lemma 3.3 *Let* $(\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g})$ *be an initial data set for* (MKG) *in the class* $\mathcal{H}^1(\mathbb{R}^4)$. *Then there exists a gauge transform* $\chi \in \mathcal{G}^2(\mathbb{R}^4)$, *unique up to a constant, such that*

$$(a, e, f, g) = (\widetilde{a} - \mathrm{d}\chi, \widetilde{e}, e^{i\chi}\widetilde{f}, e^{i\chi}\widetilde{g})$$

satisfies the global Coulomb gauge condition $\partial^\ell a_\ell = 0$ *[resp.* $\partial^\ell a'_\ell = 0$*] on* $\mathbb{R}^4$. *Moreover, we have the estimate*

$$\|\chi\|_{\mathcal{G}^2(\mathbb{R}^4)} \lesssim \|\widetilde{a}\|_{\dot{H}^1_x(\mathbb{R}^4)}. \tag{3.6}$$

Proof Let

$$\omega_j = (-\Delta)^{-1}\partial_j\partial^\ell\widetilde{a}_\ell. \tag{3.7}$$

Since $\widetilde{a} \in \dot{H}^1_x(\mathbb{R}^4)$, it follows that $\omega_j \in \dot{H}^1_x(\mathbb{R}^4)$. Note moreover that

$$\partial_i\omega_j - \partial_j\omega_i = 0$$

for every $i, j = 1, 2, 3, 4$. Thus there exists[4] a real-valued function χ such that

$$\mathrm{d}\chi = \omega,$$

which furthermore satisfies $\chi \in \mathcal{G}^2(\mathbb{R}^4)$ and (3.6). Note that (a, e, f, g) defined as above satisfies the global Coulomb condition, since $\partial^\ell a_\ell = \partial^\ell\widetilde{a}_\ell + \Delta\chi = 0$. The uniqueness statement follows from the fact that the solution to $\Delta\partial_j\chi = \partial_j\partial^\ell\widetilde{a}_\ell$ in $L^4_x(O)$ is uniquely given by (3.7). □

An immediate consequence of Lemma 3.1 and the preceding two lemmas is that any Coulomb initial data set in $\mathcal{H}^1(\mathbb{R}^4)$ can be approximated in $\mathcal{H}^1(\mathbb{R}^4)$ by classical Coulomb initial data sets. We record this statement as a corollary.

Corollary 3.4 *Let* (a, e, f, g) *be a globally Coulomb initial data set for* (MKG) *in the class* $\mathcal{H}^1(\mathbb{R}^4)$. *Then there exists a sequence* $(\check{a}^{(n)}, \check{e}^{(n)}, \check{f}^{(n)}, \check{g}^{(n)})$ *of globally Coulomb initial data sets in* $\mathcal{H}^\infty(\mathbb{R}^4)$ *which approximates* (a, e, f, g) *in* $\mathcal{H}^1(\mathbb{R}^4)$.

4 Excision and Gluing of Initial Data Sets

A recurrent nuisance in gauge theory is the presence of a non-trivial constraint equation for the initial data sets. More concretely, consider the problem of localizing a (MKG) initial data set. The most naive way to proceed would be to apply a smooth cutoff; however, integrating the constraint equation (also called the *Gauss equation*)

$$\partial^\ell e_\ell = \mathrm{Im}[f\overline{g}]$$

[4] This is obvious when $\widetilde{a} \in \mathcal{S}(\mathbb{R}^4)$; the full statement follows by approximation of a by Schwartz 1-forms, using the fact that BMO is a Banach space modulo constant functions.

by parts over balls of large radius, we see that e_ℓ must in general be non-trivial on the boundary spheres even if f, g are compactly supported. This simple argument precludes the naive approach of simply cutting off (a, e, f, g).

The purpose of this section is to introduce a set of techniques for addressing this difficulty, namely *excision and gluing* of (MKG) initial data sets. In the context of localization of initial data sets, the basic idea is as follows: Instead of simply *excising* the unwanted part of the initial data set, we *glue* it to another initial data set, which has an explicit description in the excised region. For instance, in the exterior of a ball (see Proposition 4.1 below) we glue with a data set of the form $(e_{(q)j} := \frac{q}{2\pi^2}\frac{x_j}{|x|^4}, 0, 0, 0)$, which is precisely the electro-magnetic field of an electric monopole of charge q situated at the origin.

Key to our approach is a simple solution operator $\mathcal{V}$ for the divergence equation that preserves the support and obeys a sharp regularity bound. This solution operator was first used by Bogovskiĭ [2,3]. We remark that a similar solution operator was used in [11] in the context of the incompressible Euler equations.

The main results are stated in the next two propositions. The first one concerns excision and gluing of initial data sets to the exterior of a ball.

Proposition 4.1 (Excision and gluing of initial data sets to the exterior). *Let B be a ball of radius r_0 in $\mathbb{R}^4$, and $1 < \sigma_1 < \sigma_0 \leq 2$. Then there exists an operator E^{ext} from the class $\mathcal{H}^1(\sigma_0 B \setminus \overline{B})$ to the class $\mathcal{H}^1(\mathbb{R}^4 \setminus \overline{B})$ satisfying the following properties:*

(1) $(\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g}) := E^{\mathrm{ext}}[a, e, f, g]$ *is an extension of* (a, e, f, g),

$$(\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g}) = (a, e, f, g) \quad \textit{on the annulus } \sigma_1 B \setminus \overline{B}.$$

(2) *We have* $(\widetilde{a}, \widetilde{f}, \widetilde{g}) = (0, 0, 0)$ *on* $\mathbb{R}^4 \setminus \sigma_0\overline{B}$. *On the other hand, there exists a real number* $q = q(e)$, *depending continuously on* $e \in L^2(\sigma_0 B \setminus \overline{B})$, *such that*

$$\widetilde{e}_j(x) = q\frac{x^j}{r^4} \quad \textit{on } \mathbb{R}^4 \setminus \sigma_0\overline{B}.$$

(3) *The following bounds hold, with implicit constants depending on* σ_1, σ_0:

$$\|E^{\mathrm{ext}}[a, e, f, g]\|_{\mathcal{H}^1(\mathbb{R}^4\setminus\overline{B})} \lesssim \|(a, e, f, g)\|_{\mathcal{H}^1(\sigma_0 B\setminus\overline{B})}, \tag{4.1}$$

$$\mathcal{E}_{\mathbb{R}^4\setminus\overline{B}}[E^{\mathrm{ext}}[a, e, f, g]] \lesssim r_0^{-2}\|f\|^2_{L^2_x(\sigma_0 B\setminus\overline{B})} + \mathcal{E}_{\sigma_0 B\setminus\overline{B}}[a, e, f, g]. \tag{4.2}$$

(4) *The operator* E^{ext} *is continuous from* $\mathcal{H}^1(\sigma_0 B \setminus \overline{B})$ *to* $\mathcal{H}^1(\mathbb{R}^4 \setminus \overline{B})$. *Moreover,* E^{ext} *enjoys* persistence of higher regularity, *i.e., for every* $N \geq 1$, *we have* $E^{\mathrm{ext}}[\mathcal{H}^N(\sigma_0 B \setminus \overline{B})] \subseteq \mathcal{H}^N(\mathbb{R}^4 \setminus \overline{B})$.

The second proposition concerns excision and gluing of initial data in the interior of a ball.

Proposition 4.2 (Excision and gluing of initial data sets to the interior). *Let B be a ball of radius r_0 in $\mathbb{R}^4$, and $1 < \sigma_2 < \sigma_0 \leq 2$. Then there exists an operator E^{int} from the class $\mathcal{H}^1(\sigma_0 B \setminus \overline{B})$ to the class $\mathcal{H}^1(\sigma_0 B)$ satisfying the following properties:*

(1) $E^{\mathrm{int}}[a, e, f, g]$ *is an interior extension of* (a, e, f, g),

$$E^{\mathrm{int}}[a, e, f, g] = (a, e, f, g) \quad \text{on the annulus } \sigma_0 B \setminus \sigma_2 \overline{B}.$$

(2) *The following bounds hold, with implicit constants depending on* σ_2, σ_0*:*

$$\|E^{\mathrm{int}}[a, e, f, g]\|_{\mathcal{H}^1(\sigma_0 B)} \lesssim \|(a, e, f, g)\|_{\mathcal{H}^1(\sigma_0 B \setminus \overline{B})} + \|e\|^{\frac{1}{2}}_{L^2_x(\sigma_0 B \setminus \overline{B})}, \tag{4.3}$$

$$\mathcal{E}_{\sigma_0 B}[E^{\mathrm{int}}[a, e, f, g]] \lesssim \mathcal{E}_{\sigma_0 B \setminus \overline{B}}[a, e, f, g] + r_0^{-2} \|f\|^2_{L^2_x(\sigma_0 B \setminus \overline{B})}. \tag{4.4}$$

(3) *The operator* E^{int} *is continuous from* $\mathcal{H}^1(\sigma_0 B \setminus \overline{B})$ *to* $\mathcal{H}^1(\sigma_0 B)$*. Moreover,* E^{int} *enjoys* persistence of higher regularity*, i.e., for every* $N \geq 1$*, we have* $E^{\mathrm{int}}[\mathcal{H}^N(\sigma_0 B \setminus \overline{B})] \subseteq \mathcal{H}^N(\sigma_0 B)$.

Remark 4.3 There are two main difficulties in the proving these propositions. The first one is the presence of the Gauss equation, which has been discussed at the beginning of this section. The second difficulty stems from the *local energy inequalities* (4.2) and (4.4), which require, in particular, choosing a 'good gauge' before excising the initial data. To resolve this difficulty, we rely on the solvability in L^2-Sobolev spaces of the one-form Hodge system under suitable boundary conditions (see Section 4.2). This statement can be thought of as an easier abelian variant of Uhlenbeck's lemma [41] concerning existence of a gauge transformation to the Coulomb gauge.

The rest of this section is structured as follows: In Section 4.1, we introduce a solution operator $\mathcal{V} = \mathcal{V}_j[h]$ to the divergence equation $\partial^j \mathcal{V}_j[h] = h$ that, in particular, is compactly supported if h is. In Section 4.2, we briefly recall a standard result for the 1-form Hodge system on domains with smooth boundary, which will be needed later. Then in Section 4.3, we present proofs of Propositions 4.1 and 4.2.

4.1 Support-Preserving Solution Operator for the Divergence Equation

In this subsection, we define a solution operator to the divergence equation which preserves the support property of the source. This solution operator was first introduced by Bogovskiĭ [2,3]. Our construction below follows the approach of [11], in which a similar solution operator was constructed for the symmetric divergence equation $\partial_j R^{j\ell} = U^\ell$. We sharpen the estimates for $\mathcal{V}$ compared to [11] (where non-sharp estimates sufficed), which turns out to be necessary due to the criticality of our problem. The class of domains we work with is that of star-shaped domains, and unions thereof. We call a domain *strongly star-shaped* with respect to a set B if it is star-shaped with respect to any point in B.

Proposition 4.4 *Let* B *be a ball in* $\mathbb{R}^d$*,* $d \geq 2$*. Then there exists a pseudodifferential operator* $\mathcal{V} \in OPS^{-1}_{loc}(\mathbb{R}^d)$*, taking functions to* 1*-forms, which has the following properties:*

(1) *For any compact domain* D *which is star-shaped with respect to* B*, if* $h \in \mathcal{D}'$ *is supported in* D *then* $\mathcal{V}[h]$ *is also supported in* D.

(2) *Suppose that h has compact support and*

$$\int_{\mathbb{R}^d} h \, \mathrm{d}x = 0.$$

Then $\mathcal{V}[h]$ satisfies the divergence equation

$$\partial^\ell \mathcal{V}_\ell[h] = h. \tag{4.5}$$

Remark 4.5 The fact that D is star-shaped with respect to B requires that $B \subseteq D$. Thus by scaling all bounds for the operator $\mathcal{V}$ in D depend only on the ratio $\mathrm{diam}\,(D)/\mathrm{diam}\,(B)$. In particular, since $\mathcal{V} \in OPS^{-1}_{loc}(\mathbb{R}^d)$, we obtain

$$\|\mathcal{V}[h]\|_{W^{1,p}(\mathbb{R}^d)} \lesssim \|h\|_{L^p(\mathbb{R}^d)}, \qquad 1 < p < \infty. \tag{4.6}$$

Thus, by the Gagliardo-Nirenberg-Sobolev inequality we obtain the inequality

$$\|\mathcal{V}[h]\|_{L^q_x(\mathbb{R}^d)} \lesssim_{p,q} (\mathrm{diam}\, B)^{\frac{d}{q} - \frac{d}{p} + 1} \|h\|_{L^p_x(\mathbb{R}^d)} \tag{4.7}$$

whenever $1 < p \leq q < \infty$ and

$$\frac{d}{p} - 1 \leq \frac{d}{q} \leq \frac{d}{p}.$$

Before we begin the proof of Proposition 4.4 in earnest, we give a short argument that provides a solution operator $\mathcal{V}$ with the required support properties but with less regularity. We will use this to motivate the actual construction. Let h be a smooth function supported in a ball B satisfying $\int_{\mathbb{R}^d} h = 0$. Our goal is to find a solution v^ℓ to (4.5) which satisfies the support property $\mathrm{supp}\, v^\ell \subseteq B$. Note that $\int_{\mathbb{R}^d} h = 0$ is a necessary condition for such a solution to exist by the divergence theorem.

Taking the Fourier transform of h and Taylor expanding at $\xi = 0$, we get

$$\widehat{h}(\xi) = \widehat{h}(0) + \xi^\ell \int_0^1 \partial_\ell \widehat{h}(\sigma \xi) \, \mathrm{d}\sigma.$$

Since $\widehat{h}(0) = \int h \, \mathrm{d}x = 0$, we see that $\widehat{h}$ has the form of a divergence. Indeed, defining

$$v_j[h] := \mathcal{F}_x^{-1}[\int_0^1 \partial_j \widehat{h}(\sigma \xi) \, \sigma],$$

we see that $\partial^\ell v_\ell[h] = h$, as desired. More generally, we remark that if $\int_{\mathbb{R}^d} h \neq 0$, then

$$\nabla \cdot v = h - c\delta_0, \qquad c = \int_{\mathbb{R}^d} h \, \mathrm{d}x.$$

Carrying out the inverse Fourier transform, we obtain the following physical space formula for $v_j[h]$:

$$v_j[h](x) = \int_0^1 \frac{x^j}{\sigma} h\Big(\frac{x}{\sigma}\Big) \frac{\mathrm{d}\sigma}{\sigma^d}.$$

Note that the value of $v_j[h]$ at x is determined by a weighted integral of h on the radial ray $\{sx : s \geq 1\}$. In particular, the desired support property $\operatorname{supp} v \subseteq B$ immediately follows. In terms of regularity, however, integration along rays only yields radial regularity. No angular regularity at all is gained by doing this.

One can also view the above construction as arising from a mass transportation problem. The above v corresponds to transporting all the mass of h along rays to zero.

In order to produce a better solution operator, all we need to do is to expand the above Dirac mass at zero into a smooth bump function, i.e. some smooth averaging of the above construction. This idea is carried out in the following proof.

Proof of Proposition 4.4 By translation and scaling we assume that B is the unit ball. Given $y \in \mathbb{R}^d$, define

$$v_{(y)j}[h](x) = \int_0^1 \frac{(x-y)^j}{\sigma} h\Big(\frac{x-y}{\sigma} + y\Big) \frac{\mathrm{d}\sigma}{\sigma^d}.$$

Let ζ be a smooth normalized bump function in B, i.e.

$$\operatorname{supp} \zeta \subseteq B, \quad \int_{\mathbb{R}^d} \zeta = 1. \tag{4.8}$$

We now define $\mathcal{V}_j[h] := \int \zeta(y) v_{(y)j}[h] \,\mathrm{d}y$, i.e.,

$$\mathcal{V}_j[h](x) = \int \int_0^1 \zeta(y) \frac{(x-y)^j}{\sigma} h\Big(\frac{x-y}{\sigma} + y\Big) \frac{\mathrm{d}\sigma}{\sigma^d} \,\mathrm{d}y. \tag{4.9}$$

As before, $\mathcal{V}[h]$ is a solution to the divergence equation

$$\partial^\ell \mathcal{V}_\ell[h] = h - c\zeta, \qquad c = \int_{\mathbb{R}^d} h \,\mathrm{d}x$$

where the second term in the right-hand side vanishes provided that h has integral zero. Moreover, from the construction it follows that we have the support property

$$\operatorname{supp}\mathcal{V}[h] \subseteq \bigcup_{x \in \operatorname{supp} h} \operatorname{Conv}(\{x\} \cup B),$$

where $\operatorname{Conv}(X)$ refers to the convex hull of X. This is exactly what we need.

It remains to prove that $\mathcal{V}$ is a regular pseudodifferential operator of order -1. For that we look at the kernel $K(x-y, y)$ of $\mathcal{V}$, which after a change of variable is written as

$$K(z, y) = \int \frac{z}{1-\sigma} \zeta\Big(\frac{1}{1-\sigma} z + y\Big) \frac{d\sigma}{(1-\sigma)^d}$$

The support condition on ζ restricts the integral to the range $|1-\sigma| \gtrsim |z|$. Then a direct integration yields the bound

$$|K(z, y)| \lesssim |z|^{1-d}$$

Similarly, we have the differentiated bounds

$$|\partial_z^{(k)} \partial_y^{(j)} K(z, y)| \leq c_{kj} |z|^{1-d-k}.$$

The symbol $a(\xi, y)$ of $\mathcal{V}$ in the right calculus[5] is obtained by taking the Fourier transform of K with respect to z. Then the preceding bound implies the homogeneous symbol bound

$$|\partial_\xi^{(k)} \partial_y^{(j)} a(\xi, y)| \lesssim_{k,j} |\xi|^{-1-k}.$$

On the other hand, taking into account the support properties of K, it follows that $|\partial_\xi^{(k)} \partial_y^{(j)} a|$ is bounded for every k, j as well. Hence the assertion $\mathcal{V} \in OPS^{-1}$ follows.

In the sequel we apply the above proposition in two situations. The first is for a ball:

Corollary 4.6 *Let B be a ball in $\mathbb{R}^d$. Then there exists a pseudodifferential operator $\mathcal{V}^B \in OPS^{-1}$, mapping distributions h supported in B to distributions $\mathcal{V}^B[h]$ supported in B, and which satisfies property (2) in Proposition 4.4.*

For this we only need to observe that B is star-shaped with respect to B.

Our second application is for an annulus:

Corollary 4.7 *Let $A = \sigma B \setminus \overline{B}$, with $\sigma > 1$, be an annulus. Then there exists a pseudodifferential operator $\mathcal{V}^A \in OPS^{-1}$, mapping distributions h supported in A to distributions $\mathcal{V}^A[h]$ supported in A, and which satisfies property (2) in Proposition 4.4. Further, all bounds are uniform for σ away from* 1.

In particular we note the following bound

$$\|\mathcal{V}^A[h]\|_{L^2_x(\mathbb{R}^4)} \lesssim \|h\|_{L_x^{\frac{4}{3}}(\mathbb{R}^4)} \tag{4.10}$$

with an implicit constant that is uniform for σ away from 1.

To show that this follows from Proposition 4.4, we cover A with three or more overlapping round sectors of identical angle θ, $A = \bigcup_{k=1}^K A_k$, so that the double-angle sectors $2A_k \subseteq A$ are star-shaped. The number of such sectors depends only on the dimension d if σ is large, but increases as $\sigma \to 1$. The closer σ gets to 1, the worse our bounds will get.

In each such sector we can apply Proposition 4.4. However, to conclude the proof of the corollary we need to also be able to distribute the zero integral condition to the sectors. This is achieved in the next lemma:

[5] We prefer the right calculus, because there the symbol is only needed for $y \in D$.

Lemma 4.8 *Consider a covering of the annulus* $A = \sigma B \setminus \overline{B}$ *with round sectors* $A = \bigcup A_k$ *of angle* θ. *Let* η_k *be an associated partition of unity in* A *whose angular support is contained in the double-angle sector* $2A_k$. *Then for each distribution* h *which satisfies*

$$\operatorname{supp} h \subseteq A \quad \text{and} \quad \int h \, \mathrm{d}x = 0.$$

there exists a linear decomposition $h = \sum_{k=1}^{K} h_k$ *so that*

$$\operatorname{supp} h_k \subseteq 2A_k, \quad \int h_k \, \mathrm{d}x = 0.$$

and the maps $h \to h_k - \eta_k h$ *are finite rank* ≤ 2 *from* $\mathcal{D}'$ *to* $\mathcal{D}$.

The previous corollary is then proved by applying Proposition 4.4 to each h_k in the sectors $2A_k$.

Proof We label the sectors A_k so that $A_k \cap A_{k+1} \neq \emptyset$. For each k, let ζ_k be a smooth function with unit mass supported in $2A_k \cap 2A_{k+1}$. For convenience, we define $\zeta_0 = 0$. The idea is to write

$$\begin{aligned} h_k &:= \eta_k h - \zeta_k \int \sum_{j \leq k} \eta_j h + \zeta_{k-1} \int \sum_{j<k} \eta_j h \quad \text{for } 1 \leq k \leq K-1, \\ h_K &:= \eta_K h + \zeta_{K-1} \int \eta_{K-1} h. \end{aligned}$$

By construction, we have $\int h_k = 0$ for $1 \leq k \leq K-1$; then it follows that $\int h_K = 0$ since $\int h = 0$. □

4.2 L^2 Hodge Theory for 1-Forms

Another ingredient in our proofs of Propositions 4.1 and 4.2 is the solvability of a boundary value problem for the 1-form Hodge system. The result that we need is as follows:

Proposition 4.9 *Let* O *be a pre-compact connected open subset of* $\mathbb{R}^4$ *with a smooth boundary* ∂O. *Assume furthermore that the first de Rham cohomology group of* O *vanishes, i.e.,* $\mathrm{H}^1_{\mathrm{deRham}}(O) = 0$. *Then for any 2-form* F *on* O *such that* $F \in H^N(O)$ *(*$N \geq 0$*), there exists a unique 1-form* $\omega \in H^{N+1}(O)$ *which solves the following boundary value problem for the 1-form Hodge system:*

$$\mathrm{d}\omega = F, \quad \partial^\ell \omega_\ell = 0, \quad \omega \restriction_{\partial O} (\mathbf{n}) = 0, \tag{4.11}$$

where $\mathbf{n}$ *is the outer-pointing normal vector field on* ∂O. *Moreover,* ω *obeys the estimate*

$$\|\omega\|_{H_x^{N+1}(O)} \lesssim \|F\|_{H_x^N(O)}. \tag{4.12}$$

This is a standard result; we refer the reader to [40, Section 5.9]. The cohomology condition ensures, by the Hodge theorem, that the kernel of the Hodge system is trivial. Then the latter fact allows us to conclude unique solvability of (4.11) by the Fredholm alternative theorem.

4.3 Proof of Propositions 4.1 and 4.2

We are ready to prove Propositions 4.1–4.2.

Proof of Proposition 4.1 Without any loss of generality, we may assume that B is centered at the origin of $\mathbb{R}^4$. For $1 < \sigma_1 < \sigma_0 \leq 2$, we define $\sigma_1 = \sigma_0^{(0)} < \sigma_0^{(1)} < \sigma_0^{(2)} < \sigma_0^{(3)} = \sigma_0$ as

$$\sigma_0^{(0)} = \sigma_1, \quad \sigma_0^{(1)} = \frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_0, \quad \sigma_0^{(2)} = \frac{1}{3}\sigma_1 + \frac{2}{3}\sigma_0, \quad \sigma_0^{(3)} = \sigma_0.$$

Below, we will write

$$E^{\mathrm{ext}}[a, e, f, g] = (\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g}).$$

Step 1. Excision of a_j. The purpose of this step is to cutoff a_j to obtain $\widetilde{a}_j$ on $\mathbb{R}^4 \setminus \overline{B}$ such that

$$\widetilde{a} = a \text{ on } \sigma_0^{(1)} B \setminus \overline{B}, \quad \widetilde{a} = 0 \text{ on } \mathbb{R}^4 \setminus \sigma_0^{(3)} \overline{B}, \tag{4.13}$$

$$\|\widetilde{a}\|_{\dot{H}^1_x \cap L^4_x(\mathbb{R}^4 \setminus \overline{B})} \lesssim_{\sigma_0} \|a\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B \setminus \overline{B})}, \tag{4.14}$$

$$\|\mathrm{d}\widetilde{a}\|^2_{L^2_x(\mathbb{R}^4 \setminus \overline{B})} \lesssim_{\sigma_0} \mathcal{E}_{\sigma_0 B \setminus \overline{B}}[a, e, f, g], \tag{4.15}$$

where $(\mathrm{d}\widetilde{a})_{jk} = \partial_j \widetilde{a}_k - \partial_k \widetilde{a}_j$. If one drops the last condition, then the simple choice $\widetilde{a} = \eta a$ for a suitable cutoff η will do the job; however, having the estimate (4.15) with only the energy of (a, e, f, g) on the annular region $\sigma_0 B \setminus \overline{B}$ on the right-hand side will be crucial for our later purposes, in particular for performing the blow-up analysis in [23]. Our idea for achieving this goal is as follows: First, we will find a gauge equivalent connection 1-form $\check{a}$ on the annular region $\sigma_0 B \setminus \overline{B}$ such that

$$\|\check{a}\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B \setminus \overline{B})} + (\operatorname{diam} B)^{-1} \|\check{a}\|_{L^2_x(\sigma_0 B \setminus \overline{B})} \lesssim_{\sigma_0} \|\mathrm{d}a\|_{L^2_x(\sigma_0 B \setminus \overline{B})} \tag{4.16}$$

We remind the reader that $\|\mathrm{d}a\|^2_{L^2_x(\sigma_0 B \setminus \overline{B})} \leq \mathcal{E}_{\sigma_0 B \setminus \overline{B}}[a, e, f, g]$. The connection 1-form $\check{a}$ can be safely excised outside $\sigma_0^{(2)} B \supset \sigma_0^{(1)} B$. Finally, we patch together a_j and $\widetilde{a}_j$ inside $\sigma_0^{(2)} B$ using a suitable gauge transformation to produce $\widetilde{a}$ satisfying (4.13)–(4.15).

We now proceed to the details. Let $O = O(\sigma_0, B)$ denote the annulus $\sigma_0 B \setminus \overline{B}$. Applying Proposition 4.9 with $F = \mathrm{d}a$ on the region O (which is possible since $H^1_{\mathbb{R}}(O) = 0$), we infer the existence of a unique 1-form $\check{a}$ which solves

$$\mathrm{d}\check{a} = \mathrm{d}a, \quad \partial^\ell \check{a}_\ell = 0, \quad \check{a} \restriction_{\partial B} (\partial_r) = \check{a} \restriction_{\partial(\sigma_0 B)} (\partial_r) = 0. \tag{4.17}$$

Moreover, $\check{a}$ obeys the estimate (4.16). To see this, first observe that this estimate follows from (4.12) and Sobolev when B is a ball of unit radius. The general case follows once we note that, for a fixed $\sigma_0 > 1$, both sides of (4.16) are invariant under scaling.

Next, we prove that $\check{a}$ is gauge equivalent to a. This amounts to finding a function χ such that

$$\check{a} = a - \mathrm{d}\chi.$$

Since $\mathrm{d}(\check{a} - a) = 0$, the existence of such a function χ on O is guaranteed by the topological fact that $\mathrm{H}^1_{\mathrm{deRham}}(O) = 0$; it is moreover unique if we furthermore require that $\int_O \chi = 0$. By Poincaré's inequality and (4.16), it follows that χ satisfies the bound

$$\|\partial_x^{(2)} \chi\|_{L^2_x(O)} + \|\partial_x \chi\|_{L^4_x(O)} + (\mathrm{diam}\, B)^{-2} \|\chi\|_{L^2_x(O)} \lesssim \|a\|_{\dot{H}^1_x \cap L^4_x(O)}. \tag{4.18}$$

We now show that, thanks to (4.16), it is safe to cut off $\check{a}$. Let $\eta_{(2)}$ be a smooth function on $\mathbb{R}^4$ such that

$$\eta_{(2)} = 1 \text{ on } \sigma_0^{(2)} B, \quad \eta_{(2)} = 0 \text{ outside } \sigma_0^{(3)} B, \quad |\partial_x^{(N)} \eta_{(2)}| \lesssim_{N,\sigma_0} (\mathrm{diam}\, B)^{-N} \quad \text{for } N \geq 0.$$

Then by (4.16), it is immediate that for any open subset $O' \subseteq O$,

$$\|\mathrm{d}(\eta_{(2)}\check{a})\|_{L^2_x(O')} \leq \|\eta_{(2)} \mathrm{d}\check{a}\|_{L^2_x(O')} + \|\partial_x \eta_{(2)}\|_{L^\infty_x(O')} \|\check{a}\|_{L^2_x(O')} \lesssim_{\sigma_0} \|\mathrm{d}a\|_{L^2_x(O')}. \tag{4.19}$$

To conclude the proof, we finally patch a and $\check{a}$ by a suitable gauge transformation to obtain $\widetilde{a}$ with the desired properties. Let $\eta_{(1)}$ be a smooth function on $\mathbb{R}^4$ such that

$$\eta_{(1)} = 1 \text{ on } \sigma_0^{(1)} B, \quad \eta_{(1)} = 0 \text{ outside } \sigma_0^{(2)} B, \quad |\partial_x^{(N)} \eta_{(1)}| \lesssim_{N,\sigma_0} (\mathrm{diam}\, B)^{-N} \quad \text{for } N \geq 0.$$

We now define $\widetilde{a}$ by the following formula:

$$\widetilde{a} := \eta_{(2)}(a - \mathrm{d}\widetilde{\chi}), \quad \text{where } \widetilde{\chi} := (1 - \eta_{(1)})\chi. \tag{4.20}$$

From (4.18), it follows that $\widetilde{\chi}$ obeys

$$\|\partial_x^{(2)} \widetilde{\chi}\|_{L^2_x(\sigma_0 B \setminus \overline{B})} + \|\partial_x \widetilde{\chi}\|_{L^4_x(\sigma_0 B \setminus \overline{B})} + (\mathrm{diam}\, B)^{-2} \|\widetilde{\chi}\|_{L^2_x(\sigma_0 B \setminus \overline{B})} \lesssim \|a\|_{\dot{H}^1_x \cap L^4_x(O)} \tag{4.21}$$

It remains to verify the properties (4.13)–(4.15). The first property (4.13) follows easily from the construction. The second property (4.14) follows from

$$\|\eta_{(2)} \mathrm{d}\widetilde{\chi}\|_{\dot{H}^1_x \cap L^4_x(\mathbb{R}^4 \setminus \overline{B})} \lesssim_{\sigma_0} \|\mathrm{d}\widetilde{\chi}\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B \setminus \overline{B})} \lesssim_{\sigma_0} \|a\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B \setminus \overline{B})}, \tag{4.22}$$

which in turn follows from (4.21). Finally, the third property (4.15) is a consequence of (4.13) and (4.19) with $O' = \sigma_0 B \setminus \sigma_0^{(1)} \overline{B}$.

Step 2. Excision of f, g. In this step, we excise (f, g) to construct $(\widetilde{f}, \widetilde{g})$ on $\mathbb{R}^4 \setminus \overline{B}$ that satisfies the following properties:

$$(\widetilde{f}, \widetilde{g}) = (f, g) \text{ on } \sigma_1 B \setminus \overline{B}, \quad (\widetilde{f}, \widetilde{g}) = 0 \text{ on } \mathbb{R}^4 \setminus \sigma_0^{(1)} B, \tag{4.23}$$

$$\|\widetilde{f}\|_{\dot{H}^1_x \cap L^4_x(\mathbb{R}^4 \setminus \overline{B})} \lesssim \|f\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B \setminus \overline{B})}, \tag{4.24}$$

$$\|\widetilde{g}\|_{L^2_x(\mathbb{R}^4 \setminus \overline{B})} \lesssim \|g\|_{L^2_x(\sigma_0 B \setminus \overline{B})}, \tag{4.25}$$

$$\sum_{j=1,\ldots 4} \|\widetilde{\mathbf{D}}_j \widetilde{f}\|^2_{L^2_x(\mathbb{R}^4 \setminus \overline{B})} \lesssim (\operatorname{diam} B)^{-2} \|f\|^2_{L^2_x(\sigma_0 B \setminus \overline{B})} + \sum_{j=1,\ldots,4} \|\mathbf{D}_j f\|^2_{L^2_x(\sigma_0 B \setminus \overline{B})}, \tag{4.26}$$

where $\widetilde{\mathbf{D}}_j = \partial_j + i\widetilde{a}_j$. These conditions are easily achieved by naively choosing $\sigma_1 = \sigma_0^{(0)}$ and cutting off f, g by a smooth function $\eta_{(0)}$ that is supported in $\sigma_0^{(1)} B$ and equals 1 on $\sigma_0^{(0)} B$.

Step 3. Excision and gluing of e_j. In this step, we construct $\widetilde{e}_j$ that, together with $\widetilde{a}_j$, $\widetilde{f}$ and $\widetilde{g}$ constructed in the preceding steps, would satisfy the properties in Proposition 4.1. The problem of localizing of $\widetilde{e}_j$ is subtle, as it must satisfy the Gauss equation

$$\partial^\ell \widetilde{e}_\ell = \operatorname{Im}[\widetilde{f}\, \overline{\widetilde{g}}]. \tag{4.27}$$

In particular, integrating (4.27) over a ball B_r of radius $r \gg 1$, the divergence theorem implies

$$\int_{\partial B_r} \widetilde{e}_\ell \mathbf{n}^\ell = \int_{\mathbb{R}^4} \operatorname{Im}[\widetilde{f}\, \overline{\widetilde{g}}]\, \mathrm{d}x, \quad \text{where } \mathbf{n}^\ell = \frac{x^\ell}{|x|},$$

which precludes the possibility of having a compactly supported $\widetilde{e}$ in general. Instead, we will *glue* the 1-form e to another solution $e_{(q)}$ (see (4.30)) to the Gauss equation with a well-understood behavior at infinity, while keeping e unchanged in the region $\sigma_1 B$. The key to carrying out this procedure is Proposition 4.4, which allows us to solve away certain errors in the Gauss equation in a bounded region of space.

We define $\widetilde{e}$ to be

$$\widetilde{e} = \eta^2_{(0)} e + (1 - \eta^2_{(0)}) e_{(q)} + e_{(G)}, \tag{4.28}$$

where $\{e_{(q)}\}_{q \in \mathbb{R}}$ is an explicit 1-parameter family of solutions to $\partial^\ell e_{(q)\ell} = 0$ on $\mathbb{R}^4 \setminus \{0\}$, to be introduced below, and $e_{(G)}$ will be constructed to satisfy the equation

$$\partial^\ell e_{(G)\ell} = -\partial^\ell \eta^2_{(0)} (e_\ell - e_{(q)\ell}) \quad \text{with} \quad \operatorname{supp} e_{(G)} \subseteq \sigma_0^{(1)} B \setminus \sigma_1 \overline{B}. \tag{4.29}$$

For $e_{(q)}$ and $e_{(G)}$ as above, we can readily verify that (4.27) holds as follows:

$$\begin{aligned} \partial^\ell \widetilde{e}_\ell - \operatorname{Im}[\widetilde{f}\, \overline{\widetilde{g}}] &= \partial^\ell (\eta^2_{(0)} e_\ell + (1 - \eta^2_{(0)}) e_{(q)\ell} + e_{(G)\ell}) - \eta^2_{(0)} \operatorname{Im}[f \overline{g}] \\ &= \eta^2_{(0)} (\partial^\ell e_\ell - \operatorname{Im}[f \overline{g}]) + (\partial^\ell \eta^2_{(0)})(e_\ell - e_{(q)\ell}) + \partial^\ell e_{(G)\ell} = 0. \end{aligned}$$

The 1-form $e_{(q)}$ is defined on $\mathbb{R}^4 \setminus \{0\}$ component-wisely as follows:

$$e_{(q)j} = \frac{q}{2\pi^2} \frac{x^j}{|x|^4}. \tag{4.30}$$

Note that $e_{(q)}$ is precisely the electric field of a point charge at the origin given by the 4-dimensional version of Coulomb's law. Indeed, $e_{(q)}$ satisfies the free divergence equation

$$\partial^\ell e_{(q)\ell} = 0, \tag{4.31}$$

and the charge of $e_{(q)}$ measured on any sphere ∂B_r of radius r centered at the origin (in fact, any hypersurface enclosing the origin) equals q, i.e.,

$$\int_{\partial B_r} e_{(q)\ell} \mathbf{n}^\ell = q \quad \text{where } \mathbf{n}^\ell = \frac{x^\ell}{|x|}. \tag{4.32}$$

We now turn to the construction of $e_{(G)}$. We wish to apply Corollary 4.7; thus we must ensure that

$$0 = \int \partial^\ell \eta_{(0)}^2 (e_\ell - e_{(q)\ell}) \, \mathrm{d}x. \tag{4.33}$$

By (4.32) and the divergence theorem, we compute

$$\int \partial^\ell \eta_{(0)}^2 \, e_{(q)\ell} \, \mathrm{d}x = -q.$$

Thus, (4.33) dictates the following choice of q as a function of e for a fixed σ_0:

$$q[e] := -\int \partial^\ell \eta_{(0)}^2 \, e_\ell \, \mathrm{d}x. \tag{4.34}$$

Since $\partial^\ell \eta_{(0)}$ is supported in $\sigma_0^{(1)} B \setminus \sigma_1 \overline{B} \subseteq \sigma_0 B \setminus \overline{B}$, we have

$$|q| \lesssim \int_{\sigma_0 B \setminus \overline{B}} \frac{1}{|x|} |e| \, \mathrm{d}x \lesssim_{\sigma_0} (\mathrm{diam}\, B) \|e\|_{L^2_x(\sigma_0 B \setminus \overline{B})}.$$

Therefore, the L^2 norm of $e_{(q)}$ obeys the bound

$$\|e_{(q)}\|_{L^2_x(\mathbb{R}^4 \setminus \overline{B})} \lesssim |q| \Big\| \frac{1}{|x|^3} \Big\|_{L^2_x(\mathbb{R}^4 \setminus \overline{B})} \lesssim \|e\|_{L^2_x(\sigma_0 B \setminus \overline{B})}. \tag{4.35}$$

Similarly, we also have

$$\|\partial^\ell \eta_{(1)}^2 (e_\ell - e_{(q)\ell})\|_{L_x^{\frac{4}{3}}(\mathbb{R}^4)} \lesssim \|e - e_{(q)}\|_{L^2_x(\sigma_0 B \setminus \overline{B})} \lesssim \|e\|_{L^2_x(\sigma_0 B \setminus \overline{B})}.$$

Applying Corollary 4.7 with $A = \sigma_0^{(1)} B \setminus \sigma_1 \overline{B}$ we obtain a solution $e_{(G)}$ to the problem (4.29) that satisfies

$$\|e_{(G)}\|_{L^2_x(\mathbb{R}^4)} \lesssim_{\sigma_0} \|e\|_{L^2_x(\sigma_0 B \setminus \overline{B})}. \tag{4.36}$$

Combined with (4.14), (4.15), (4.24), (4.25), (4.26), (4.28) and (4.35), estimates (4.1) and (4.2) follow. The proof of Statements (2)–(3) of Proposition 4.1 is therefore complete.

Step 4. Continuity and persistence of regularity . It remains to verify Statement (4) of Proposition 4.1. Inspection of our proof so far (using also the linearity statement in Corollary 4.7) shows that $\widetilde{a}, \widetilde{e}, \widetilde{f}$ and $\widetilde{g}$ are in fact *linear* in a, e, f and g, respectively; thus the continuity statement is a triviality. Checking the persistence of regularity property is a routine exercise using the corresponding statements in Corollary 4.7 and Proposition 4.9; we omit the details.

Next, we prove Proposition 4.2. The main idea is the same as for the preceding proof of Proposition 4.1; the key difference is the choice of an 1-parameter family of solutions $e_{(p)}$ to the Gauss equation in Step 3, which now must be regular at the origin.

Proof of Proposition 4.2 As before, we may assume that B is centered at the origin of $\mathbb{R}^4$. For any given $1 < \sigma_2 < \sigma_0 \leq 2$, we define $1 = \sigma_0^{(-3)} < \sigma_0^{(-2)} < \sigma_0^{(-1)} < \sigma_0^{(0)} = \sigma_2 < \sigma_0$ as

$$\sigma_0^{(-3)} = 1, \quad \sigma_0^{(-2)} = \frac{2}{3} + \frac{1}{3}\sigma_0, \quad \sigma_0^{(-1)} = \frac{1}{2} + \frac{1}{2}\sigma_0, \quad \sigma_0^{(0)} = \sigma_2.$$

In what follows, we will write $E^{\mathrm{int}}[a, e, f, g] = (\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g})$.

Step 1. Excision of a_j. This step is very similar to Step 1 in the proof of Proposition 4.1, except that we now excise the data in the inner part of the annulus. The goal is to construct $\widetilde{a}$ on $\sigma_0 B$ such that the following properties hold:

$$\widetilde{a} = a \text{ on } \sigma_0 B \setminus \sigma_0^{(-1)} \overline{B}, \quad \widetilde{a} = 0 \text{ on } \sigma_0^{(-3)} B, \tag{4.37}$$

$$\|\widetilde{a}\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B)} \lesssim_{\sigma_0} \|a\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B \setminus \overline{B})}, \tag{4.38}$$

$$\|\mathrm{d}\widetilde{a}\|^2_{L^2_x(\sigma_0 B)} \lesssim_{\sigma_0} \mathcal{E}_{\sigma_0 B \setminus \overline{B}}[a, e, f, g]. \tag{4.39}$$

Let $O = O(\sigma_0, B)$ denote the annulus $\sigma_0 B \setminus \overline{B}$. Applying Proposition 4.9 with $F = \mathrm{d}a$ on O, we obtain a unique 1-form $\check{a}$ that satisfies (4.16)–(4.17), and also a function χ satisfying $\check{a} = a - \mathrm{d}\chi$, $\int_O \chi = 0$ and (4.18). Let $\eta_{(-3)}, \eta_{(-2)}$ be smooth function on $\mathbb{R}^4$ such that

$$\begin{aligned}
\eta_{(-3)} = 0 \text{ on } \sigma_0^{(-3)} B, \quad & \eta_{(-3)} = 1 \text{ outside } \sigma_0^{(-2)} B, \\
|\partial_x^{(N)} \eta_{(-3)}| & \lesssim_{N,\sigma_0} (\operatorname{diam} B)^{-N} \quad \text{for } N \geq 0, \\
\eta_{(-2)} = 0 \text{ on } \sigma_0^{(-2)} B, \quad & \eta_{(-2)} = 1 \text{ outside } \sigma_0^{(-1)} B, \\
|\partial_x^{(N)} \eta_{(-2)}| & \lesssim_{N,\sigma_0} (\operatorname{diam} B)^{-N} \quad \text{for } N \geq 0.
\end{aligned}$$

We define

$$\widetilde{a} := \eta_{(-3)}(a - \mathrm{d}\widetilde{\chi}), \quad \text{where } \widetilde{\chi} := (1 - \eta_{(-2)})\chi. \tag{4.40}$$

Then proceeding as before, it can be checked that $\widetilde{a}$ satisfies (4.37)–(4.39).

Step 2. Excision of f, g. We seek to construct f', g' on $\sigma_0 B$ such that

$$(f', g') = (f, g) \text{ on } \sigma_0 B \setminus \sigma_2 \overline{B}, \quad (f', g') = 0 \text{ on } \mathbb{R}^4 \setminus \sigma_0^{(-1)} B, \tag{4.41}$$

$$\|f'\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B)} \lesssim_{\sigma_0} \|f\|_{\dot{H}^1_x \cap L^4_x(\sigma_0 B \setminus \overline{B})}, \tag{4.42}$$

$$\|g'\|_{L^2_x(\sigma_0 B)} \lesssim \|g\|_{L^2_x(\sigma_0 B \setminus \overline{B})}, \tag{4.43}$$

$$\sum_{j=1,\dots 4} \|\widetilde{\mathbf{D}}_j f'\|^2_{L^2_x(\sigma_0 B)} \lesssim_{\sigma_0} \|\frac{1}{|x|} f\|^2_{L^2_x(\sigma_0 B \setminus \overline{B})} + \sum_{j=1,\dots,4} \|\mathbf{D}_j f\|^2_{L^2_x(\sigma_0 B \setminus \overline{B})}, \tag{4.44}$$

where $\widetilde{\mathbf{D}}_j = \partial_j + i\widetilde{a}_j$ and $\sigma_2 = \sigma_0^{(0)} = \frac{1+\sigma_0}{2}$.

Let $\eta_{(-1)}$ be a smooth function on $\mathbb{R}^4$ such that

$$\eta_{(-1)} = 0 \text{ on } \sigma_0^{(-1)} B, \quad \eta_{(-1)} = 1 \text{ outside } \sigma_0^{(0)} B,$$
$$|\partial_x^{(N)} \eta_{(-1)}| \lesssim_{N,\sigma_0} (\mathrm{diam}\, B)^{-N} \quad \text{for } N \geq 0.$$

We simply define

$$f' = \eta_{(-1)} f, \quad g' = \eta_{(-1)} g. \tag{4.45}$$

Then (4.41)–(4.44) can be easily verified.

Step 3. Excision and gluing of f, g **and** e_j. In this step, we finally define $\widetilde{e}$, $\widetilde{f}$ and $\widetilde{g}$ on $\sigma_0 B$. As remarked above, the basic idea is similar to that in Step 3 of the proof of Proposition 4.1. However, the 1-forms $\{e_{(q)}\}_{q \in \mathbb{R}}$ are not suitable for gluing to the cutoff of e outside a ball centered at the origin, since each $e_{(q)}$ (with $q \neq 0$) is singular at 0. Thus we need to devise a different one parameter family of initial data sets. To have a solution to the Gauss equation with a nontrivial electric charge while being regular, we need to introduce a non-trivial charge density $\mathrm{Im}[f_{(p)} \overline{g_{(p)}}]$ as well as $e_{(p)}$, where p is the charge parameter.

Let ζ be a smooth function on $\mathbb{R}^4$ such that

$$\zeta \geq 0, \quad \zeta = 0 \text{ outside } B, \quad \int_{\mathbb{R}^4} \zeta^2 \,\mathrm{d}x = 1, \quad |\partial_x^{(N)} \zeta| \lesssim_N (\mathrm{diam}\, B)^{-N-2}.$$

Then for $p \in \mathbb{R}$, we define

$$e_{(p)j} = -p(-\Delta)^{-1} \partial_j \zeta^2, \tag{4.46}$$

$$f_{(p)} = \sqrt{p} (\mathrm{diam}\, B)^{\frac{1}{2}} \zeta, \tag{4.47}$$

$$g_{(p)} = -i(\mathrm{diam}\, B)^{-1} f_{(p)} = -i\sqrt{p} (\mathrm{diam}\, B)^{-\frac{1}{2}} \zeta. \tag{4.48}$$

Note that $(e_{(p)}, f_{(p)}, g_{(p)})$ solves the Gauss equation

$$\partial^{\ell} e_{(p)\ell} = \mathrm{Im}[f_{(p)}\overline{g_{(p)}}], \tag{4.49}$$

and obeys the following properties:

$$\int_{rB} \partial^{\ell} e_{(p)\ell} = p \quad \text{for any } r > 1, \tag{4.50}$$

$$\|e_{(p)}\|_{L^2_x} \lesssim p(\mathrm{diam}\, B)^{-1}, \quad \|f_{(p)}\|_{\dot{H}^1_x \cap L^4_x(\mathbb{R}^4)} + \|g_{(p)}\|_{L^2_x(\mathbb{R}^4)} \lesssim \sqrt{p}(\mathrm{diam}\, B)^{-\frac{1}{2}}. \tag{4.51}$$

Recall the definitions of $\eta_{(-1)}$, f', g' from the previous step. We define $(\widetilde{e}, \widetilde{f}, \widetilde{g})$ as follows:

$$\begin{aligned}
\widetilde{e} &= \eta_{(-1)}^2 e + (1 - \eta_{(-1)}^2) e_{(p)} + e_{(G)} \\
\widetilde{f} &= f' + f_{(p)} \\
\widetilde{g} &= g' + g_{(p)}
\end{aligned}$$

where $e_{(G)}$ will be constructed so that

$$\partial^{\ell} e_{(G)\ell} = -\partial^{\ell} \eta_{(-1)}^2 (e_{\ell} - e_{(p)\ell}) \quad \text{with} \quad \mathrm{supp}\, e_{(G)} \subseteq \sigma_0^{(0)} B \setminus \overline{B}. \tag{4.52}$$

Note that

$$\mathrm{supp}\, f' \cup \mathrm{supp}\, g' \subseteq \mathrm{supp}\, \eta_{(-1)} \quad \text{and} \quad \mathrm{supp}\, \eta_{(-1)} \cap (\mathrm{supp}\, f_{(p)} \cup \mathrm{supp}\, g_{(p)}) = \emptyset.$$

Using these properties, we can verify that $(\widetilde{e}, \widetilde{f}, \widetilde{g})$ solves the Gauss equation as follows:

$$\begin{aligned}
\partial^{\ell} \widetilde{e}_{\ell} - \mathrm{Im}[\widetilde{f}\,\overline{\widetilde{g}}] &= \partial^{\ell} (\eta_{(-1)}^2 e_{\ell} + (1 - \eta_{(-1)}^2) e_{(p)\ell} + e_{(G)\ell}) \\
&\quad - \eta_{(-1)}^2 \mathrm{Im}[f\overline{g}] - (1 - \eta_{(-1)}^2) \mathrm{Im}[f_{(p)}\overline{g_{(p)}}] \\
&= \partial^{\ell} \eta_{(-1)}^2 (e_{\ell} - e_{(p)\ell}) + \partial^{\ell} e_{(G)\ell} = 0.
\end{aligned}$$

In order to apply Corollary 4.7, we need

$$0 = \int \partial^{\ell} \eta_{(-1)}^2 (e_{\ell} - e_{(p)\ell})\, \mathrm{d}x,$$

which enforces the following choice of p as a function of e for a fixed σ_0:

$$p[e] := \int \partial^{\ell} \eta_{(-1)}^2 e_{\ell}\, \mathrm{d}x. \tag{4.53}$$

As before, p obeys the bound

$$|p| \lesssim_{\sigma_0} (\operatorname{diam} B)\|e\|_{L^2_x(\sigma_0 B \setminus \overline{B})}, \tag{4.54}$$

and therefore

$$\|\partial^\ell \eta^2_{(-1)}(e_\ell - e_{(p)\ell})\|_{L^{\frac{4}{3}}_x(\mathbb{R}^4)} \lesssim_{\sigma_0} \|e - e_{(p)}\|_{L^2_x(\sigma_0 B \setminus \overline{B})} \lesssim_{\sigma_0} \|e\|_{L^2_x(\sigma_0 B \setminus \overline{B})}.$$

Now applying Corollary 4.7 with $A = \sigma_0^{(0)} B \setminus \sigma_0^{(-1)} \overline{B}$ we obtain a solution $e_{(G)}$ to the problem (4.52) such that

$$\|e_{(G)}\|_{L^2_x(\mathbb{R}^4)} \lesssim_{\sigma_0} \|e\|_{L^2_x(\sigma_0 B \setminus \overline{B})}. \tag{4.55}$$

From (4.38), (4.39), (4.42), (4.43), (4.44), (4.51), (4.54) and (4.55), estimates (4.3) and (4.4) follow. Thus the proof of Statements (1)–(2) of Proposition 4.2 is complete.

Step 4. Continuity and persistence of regularity. To complete the proof, we need to establish Statement (3) of Proposition 4.2. As in Proposition 4.1, this task is a routine exercise of inspecting the proofs so far; we omit the details. □

5 Local Geometric Uniqueness of Maxwell–Klein–Gordon

In this section we formulate and prove local geometric uniqueness (i.e., uniqueness up to a gauge transformation) of Maxwell–Klein–Gordon equations at the energy regularity. In Section 5.1, we formulate the notion of an admissible $C_t\mathcal{H}^1$ solution and the associated class $C_t\mathcal{G}^2$ of gauge transformations, which provides an adequate setting for local geometric uniqueness. Then in Section 5.2, we state and prove the local geometric uniqueness of (MKG) in the class of admissible $C_t\mathcal{H}^1$ solutions (Proposition 5.2).

5.1 Admissible $C_t\mathcal{H}^1$ Solutions and Gauge Transformations

Here we introduce the notions of *classical* and *admissible* $C_t\mathcal{H}^1$ *solutions* to (MKG). Classical solutions refer to smooth solutions to (MKG) with sufficient spatial decay, and admissible $C_t\mathcal{H}^1$ solutions are defined as local-in-time limits of classical solutions in the *energy topology* $C_t\mathcal{H}^1$, to be defined below. We also define the associated classes of gauge transformations.

Given an open set $\mathcal{O} \subseteq \mathbb{R}^{1+4}$ and a pair (A_μ, ϕ), we define the $C_t\mathcal{H}^1(\mathcal{O})$ norm of (A_μ, ϕ) to be

$$\|(A_\mu, \phi)\|_{C_t\mathcal{H}^1(\mathcal{O})} := \sup_{t \in I(\mathcal{O})} (\|(A_\mu, \phi)\|_{\dot{H}^1_x \cap L^4_x(\mathcal{O}_t)} + \|(\partial_t A_\mu, \partial_t \phi)\|_{L^2_x(\mathcal{O}_t)}),$$

where $\mathcal{O}_t := \mathcal{O} \cap (\{t\} \times \mathbb{R}^4)$ and $I(\mathcal{O}) := \{t \in \mathbb{R} : \mathcal{O}_t \neq \emptyset\}$. Similarly, we define the $C_t\mathcal{G}^2(\mathcal{O})$ norm to be

$$\|\chi\|_{C_t\mathcal{G}^2(\mathcal{O})} := \sup_{t \in I(\mathcal{O})} (\|\chi\|_{\dot{H}^2_x \cap \dot{W}^{1,4}_x \cap \mathrm{BMO}(\mathcal{O}_t)} + \|\partial_t \chi\|_{\dot{H}^1_x \cap L^4_x(\mathcal{O}_t)} + \|\partial_t^2 \chi\|_{L^2_x(\mathcal{O}_t)})$$

We will say that a smooth solution (A, ϕ) is *classical*, and write $(A, \phi) \in C_t^\infty \mathcal{H}^\infty(\mathcal{O})$, if

$$(\partial_{t,x}^{(N)} A_\mu, \partial_{t,x}^{(N)} \phi) \in C_t \mathcal{H}^1(\mathcal{O}) \text{ for all } N \geq 0 \quad \text{and} \quad (A_\mu, \phi) \in C_t(I(\mathcal{O}); L_x^2(\mathcal{O}_t)).$$

We similarly define the space $C_t^\infty \mathcal{G}^\infty(\mathcal{O})$ of *classical gauge transformations* by saying that $\chi \in C_t^\infty \mathcal{G}^\infty(\mathcal{O})$ if and only if

$$\chi \in C_t(I(\mathcal{O}); L_x^4(\mathcal{O}_t)) \quad \text{and} \quad \partial_{t,x}^{(N)} \chi \in C_t(I(\mathcal{O}); L_x^2(\mathcal{O}_t)) \text{ for every } N \geq 1.$$

We define the notion of a *admissible $C_t \mathcal{H}^1$ solution* to (MKG) and gauge equivalence between two such solutions as follows.

Definition 5.1 (Admissible $C_t \mathcal{H}^1$ solutions). Let $\mathcal{O}$ be an open subset of $\mathbb{R}^{1+4}$.

(1) We say that a pair $(A_\mu, \phi) \in C_t \mathcal{H}^1(\mathcal{O})$ is an *admissible $C_t \mathcal{H}^1$ solution* to (MKG) on $\mathcal{O}$ (or *admissible $C_t \mathcal{H}^1(\mathcal{O})$ solution*) if it can be approximated by a sequence $(A_\mu^{(n)}, \phi^{(n)})$ of classical solutions to (MKG) locally in time with respect to the $C_t \mathcal{H}^1$ norm. More precisely, for every compact interval $J \subseteq I(\mathcal{O})$, we have as $n \to \infty$,

$$\|(A_\mu, \phi) - (A_\mu^{(n)}, \phi^{(n)})\|_{C_t \mathcal{H}^1(\mathcal{O} \cap (J \times \mathbb{R}^4))} \to 0.$$

(2) We say that two admissible $C_t \mathcal{H}^1(\mathcal{O})$ solutions (A_μ, ϕ) and (A'_μ, ϕ') are *gauge equivalent* if there exists a gauge transform $\chi \in C_t \mathcal{G}^2(\mathcal{O})$ such that $A_\mu = A'_\mu - \partial_\mu \chi, \phi = \phi' e^{i\chi}$.

5.2 Local Geometric Uniqueness of an Admissible $C_t \mathcal{H}^1$ Solution

In this subsection, we state and prove the geometric uniqueness of an admissible $C_t \mathcal{H}^1$ solution of (MKG). As discussed earlier, this statement can be thought of as the gauge invariant version of finite speed of propagation for (MKG).

Before stating the main result (Proposition 5.2), we need to make a few definitions. Given a point $(t_0, x_0) \in \mathbb{R}^{1+4}$, we define its *causal past* $J^-(t_0, x_0)$ to be the past-directed light cone with (t_0, x_0) as the tip, i.e.,

$$J^-(t_0, x_0) := \{(t, x) \in \mathbb{R}^{1+4} : t \leq t_0, \ |x - x_0| \leq t_0 - t\}.$$

For an open subset $B \subseteq \{t_0\} \times \mathbb{R}^4$, we define its *future domain of dependence* $\mathcal{D}^+(B)$ to be

$$\mathcal{D}^+(B) = \{(t, x) \in \mathcal{O} : J^-(t, x) \cap (\{t_0\} \times \mathbb{R}^4) \subseteq B\}.$$

For example, when B is an open ball of radius $r_0 > 0$ in $\{t_0\} \times \mathbb{R}^4$ centered at x_0, its future domain of dependence is the cone given by $\mathcal{D}^+(B) = \{(t, x) : t_0 \leq t < t_0 + r_0, 0 \leq |x - x_0| < r_0 - (t - t_0)\}$. The causal future $J^+(t_0, x_0)$ and past domain of dependence $\mathcal{D}^-(B)$ can be defined analogously.

We now state our local geometric uniqueness result.

Proposition 5.2 (Local geometric uniqueness at energy regularity). *Let $T_0 > 0$ and let B be an open ball in $\mathbb{R}^4$. Let (A, ϕ), (A', ϕ') be admissible $C_t \mathcal{H}^1$ solutions on the region*

$$\mathcal{D} := \mathcal{D}^+(\{0\} \times B) \cap \big([0, T_0) \times \mathbb{R}^4\big).$$

Suppose that the initial data (a, e, f, g) and (a', e', f', g') for (A, ϕ) and (A', ϕ'), respectively, are gauge equivalent on B, i.e., there exists a gauge transformation $\underline{\chi} \in \mathcal{G}^2(B)$ such that

$$(a, e, f, g) = (a' - \mathrm{d}\underline{\chi}, e', e^{i\underline{\chi}} f', e^{i\underline{\chi}} g').$$

Then there exists a unique gauge transformation $\chi \in C_t \mathcal{G}^2(\mathcal{D})$ such that $\chi \restriction_{\{0\}\times B} = \underline{\chi}$ and

$$(A, \phi) = (A' - \mathrm{d}\chi, e^{i\chi} \phi') \text{ on } \mathcal{D}.$$

When the energy is small, this proposition is a rather quick consequence of Lemma 3.3, the small energy well-posedness theorem (Theorem 1.2) and the following local geometric uniqueness for classical solutions.

Lemma 5.3 (Local geometric uniqueness of a classical solution). *Let $T_0 > 0$ and let B be an open ball in $\mathbb{R}^4$. Let (A, ϕ), (A', ϕ') be classical solutions on the region $\mathcal{D}$ as in Proposition 5.2. Suppose that the initial data (a, e, f, g) and (a', e', f', g') for (A, ϕ) and (A', ϕ'), respectively, are gauge equivalent on B by a gauge transformation $\underline{\chi} \in \mathcal{G}^\infty(B)$. Then there exists a unique gauge transformation $\chi \in C_t^\infty \mathcal{G}^\infty(\mathcal{D})$ such that $\chi \restriction_{\{0\}\times B} = \underline{\chi}$ and $(A, \phi) = (A' - \mathrm{d}\chi, e^{i\chi} \phi')$ on $\mathcal{D}$.*

This lemma can be proved by applying a gauge transformation to both solutions (A, ϕ), (A', ϕ') so that they have the same initial data and lie in a gauge where some higher regularity local well-posedness (hence uniqueness) and the finite speed of propagation property holds. An example of such a gauge is the *temporal gauge* $A_0 = 0$ [9,10,25]. We omit the straightforward details.

Our idea for proving Proposition 5.2, which foreshadows the strategy behind establishing the local well-posedness theorem (Theorem 6.1) in Section 6, is essentially to piece together the aforementioned small energy uniqueness by exploiting finite speed of propagation. An immediate obstacle is that Theorem 1.2 requires using a non-local gauge (i.e., the global Coulomb gauge), with respect to which finite speed of propagation breaks down. To get around this, we will rely on the excision and gluing techniques developed in Section 4.

Proof of Proposition 5.2 For simplicity of the exposition, we will assume that $T_0 \geq 1$, so that $\mathcal{D} = \mathcal{D}^+(B)$. The general case $T_0 > 0$ can be handled with a little modification of the argument below. Given a subset $O \subseteq \mathbb{R}^4$, we will abuse the notation for convenience and use O and $\{0\} \times O$ interchangeably.

Step 1. We begin the proof of Proposition 5.2 by reducing it to the following claim:

Claim 1 Let $\delta > 0$ and let B be an open ball in $\mathbb{R}^4$. Let (A, ϕ) and (A', ϕ') be admissible $C_t \mathcal{H}^1$ solutions on $\mathcal{D}$ with gauge equivalent initial data on B as in the hypothesis of

Proposition 5.2. Then there exists a unique gauge transform $\chi \in C_t\mathcal{G}^2(\mathcal{D}^+((1-\delta)B))$ such that $(A,\phi) = (A' - \mathrm{d}\chi, e^{i\chi}\phi')$ on $\mathcal{D}^+((1-\delta)B)$ and $\chi\restriction_{\{0\}\times(1-\delta)B} = \underline{\chi}\restriction_{(1-\delta)B}$.

Indeed, once Claim 1 is proved, Proposition 5.2 would immediately follow by taking $\delta \to 0$. Note that we have an apriori bound on the gauge transformation χ between (A,ϕ) and (A',ϕ') in $C_t\mathcal{G}^2(\mathcal{D})$ simply from the fact that $(A,\phi), (A',\phi') \in C_t\mathcal{H}^1(\mathcal{D})$.

The advantage of establishing Claim 1 instead of directly proving the proposition is that we have gained an extra room $\mathcal{D}^+(B) \setminus \mathcal{D}^+((1-\delta)B)$, which will serve as a 'cushion' for performing the excision and gluing procedure developed in Section 4.

Step 2. In this step, we show that Claim 1 follows from a more local statement, namely Claim 2 to be stated below. By translation and scaling symmetries, we may assume that B is the unit ball $\{|x| < 1\}$ in $\mathbb{R}^4$. Let (A,ϕ) be an admissible $C_t\mathcal{H}^1$ solution to (MKG) on $\mathcal{D}$, and let (a,e,f,g) be its initial data on $\{0\}\times B$.

We make the following claim:

Claim 2 There exists $0 < \epsilon \leq \frac{1}{1+\delta}$, which depends only on (a,e,f,g) and $\delta > 0$, such that the following holds: For every ball B_ϵ of radius ϵ such that $(1+\delta)B_\epsilon \subseteq B$, there exists an admissible $C_t\mathcal{H}^1$ solution $(\check{A}_{[B_\epsilon]}, \check{\phi}_{[B_\epsilon]})$ to (MKG) on $\mathcal{D}^+(B_\epsilon)$ such that $(A,\phi)\restriction_{\mathcal{D}^+(B_\epsilon)}$ is gauge equivalent to $(\check{A}_{[B_\epsilon]}, \check{\phi}_{[B_\epsilon]})$. Moreover, for a fixed $\delta > 0$, $(\check{A}_{[B_\epsilon]}, \check{\phi}_{[B_\epsilon]})$ is uniquely determined by (a,e,f,g).

In the rest of this step, we give a proof of Claim 1 assuming Claim 2. In what follows, we will write B_ϵ to denote a ball of radius ϵ whose center may vary.

Let (a',e',f',g') be the initial data set for (A',ϕ') on $\{0\}\times B$. By hypothesis, there exists $\underline{\chi} \in \mathcal{G}^2(B)$ such that

$$(a_j, e_j, f, g) = (a'_j - \partial_j\underline{\chi}, e'_j, e^{i\underline{\chi}}f', e^{i\underline{\chi}}g').$$

We extend $\underline{\chi}$ to $\mathcal{D}$ by imposing the condition $\partial_t\underline{\chi} = 0$; abusing the notation a bit, we will denote the extension still by $\underline{\chi}$. We then define $(A'',\phi'') := (A' - \mathrm{d}\underline{\chi}, e^{i\underline{\chi}}\phi')$. Note that $\underline{\chi} \in C_t\mathcal{G}^2(\mathcal{D})$, $(A'',\phi'') \in C_t\mathcal{H}^1(\mathcal{D})$, and that the initial data for (A,ϕ) and (A'',ϕ'') coincide on $\{0\}\times B$. Applying Claim 2 to (A,ϕ) and (A'',ϕ'') separately, observe that we obtain the same solution $(\check{A}_{[B_\epsilon]}, \check{\phi}_{[B_\epsilon]})$ for each B_ϵ such that $(1+\delta)B_\epsilon \subseteq B$, because the initial data are identical. Since gauge equivalence is a transitive relation, it follows that for every $(1+\delta)B_\epsilon \subseteq B$, there exists $\chi_{[B_\epsilon]} \in C_t\mathcal{G}^2(\mathcal{D}^+(B_\epsilon))$ such that

$$(A,\phi) = (A'' - \mathrm{d}\chi_{[B_\epsilon]}, e^{i\chi_{[B_\epsilon]}}\phi'') \quad \text{on } \mathcal{D}^+(B_\epsilon)$$

with $\chi_{[B_\epsilon]} = 0$ on $\{0\}\times B_\epsilon$. Note that

$$\mathcal{D}^+((1-\delta\epsilon)B) \cap \big([0,\epsilon)\times\mathbb{R}^4\big) = \bigcup_{(1+\delta)B_\epsilon\subseteq B} \mathcal{D}^+(B_\epsilon).$$

Since $\partial_t\chi_{[B_\epsilon]} = A''_0 - A_0$ for each B_ϵ, we deduce that there exists a gauge transform χ' on $\mathcal{D}^+((1-\delta\epsilon)B) \cap ([0,\epsilon)\times\mathbb{R}^4)$ that coincides with each $\chi_{[B_\epsilon]}$ on $\mathcal{D}^+(B_\epsilon)$ and thus

$$(A, \phi) = (A'' - \mathrm{d}\chi', e^{i\chi'}\phi'') \quad \text{on } \mathcal{D}^+((1-\delta\epsilon)B) \cap \big([0,\epsilon) \times \mathbb{R}^4\big).$$

Note also that $\chi' \in C_t\mathcal{G}^2(\mathcal{D}^+((1-\delta\epsilon)B) \cap ([0,\epsilon) \times \mathbb{R}^4))$, since (A, ϕ) and (A'', ϕ'') are in $C_t\mathcal{H}^1$. Moreover, we have $\chi' \restriction_{\{0\}\times(1-\delta\epsilon)B} = 0$, since each $\chi_{[B_\epsilon]}$ equals 0 on $\{0\} \times B_\epsilon$. Defining $\chi = \chi' + \underline{\chi}$ on $\mathcal{D}^+((1-\delta\epsilon)B) \cap ([0,\epsilon) \times \mathbb{R}^4)$, it follows that

$$(A, \phi) = (A' - \mathrm{d}\chi, e^{i\chi}\phi') \quad \text{on } \mathcal{D}^+((1-\delta\epsilon)B) \cap \big([0,\epsilon) \times \mathbb{R}^4\big). \tag{5.1}$$

and $\chi \restriction_{\{0\}\times(1-\delta\epsilon)B} = \underline{\chi}$.

We now conclude with a continuity argument. Consider the set

$$\begin{aligned}\mathcal{T} = \{T \in [0,1] :\ & \exists \chi \in C_t\mathcal{G}^2 \text{ s.t. } (A, \phi) \\ & = (A' - \mathrm{d}\chi, e^{i\chi}\phi') \quad \text{on } \mathcal{D}^+((1-\delta T)B) \cap \big([0,T] \times \mathbb{R}^4\big) \\ & \quad \text{and} \quad \chi \restriction_{\{0\}\times(1-\delta T)B} = \underline{\chi}\ \}.\end{aligned}$$

Clearly $\mathcal{T}$ is an interval containing 0. We claim that $\sup \mathcal{T} = 1$. Indeed, by continuity of (A, ϕ) and (A', ϕ'), we have $\sup \mathcal{T} \in \mathcal{T}$ so $\mathcal{T}$ is closed. On the other hand, if $T < 1$ is in $\mathcal{T}$, then by (5.1) (suitably rescaled), we see that there exists some $\epsilon > 0$ such that $T + \epsilon \in \mathcal{T}$. Thus $\mathcal{T}$ is open in $[0, 1]$. As it is both open and closed, we must have $\mathcal{T} = [0, 1]$. Claim 1 now follows.

Step 3. Proof of Claim 2. To finish the proof, it remains to establish Claim 2. The key ingredients are the local geometric uniqueness statement for classical solutions, Theorem 1.2 and the excision and gluing techniques in Section 4.

Fix $\sigma_0 := 1 + \delta$ and $\sigma_1 = 1 + \delta/2$. We select $\epsilon > 0$ so that for every $\sigma_0 B_\epsilon \subseteq B$ we have

$$\|(a, e, f, g)\|^2_{\mathcal{H}^1(\sigma_0 B_\epsilon)} < \frac{1}{10C_1^2}\epsilon_*^2. \tag{5.2}$$

where $C_1 = C_1(\sigma_0, \sigma_1) \geq 1$ is the implicit constant from (4.1) in Proposition 4.1. Since $(a, e, f, g) \in \mathcal{H}^1(B)$ and $\sigma_0 B_\epsilon \subseteq B$, it is not difficult to see that a non-zero choice of ϵ is always possible, and it depends only on $\delta > 0$ (through $\sigma_0 = 1 + \delta$) and (a, e, f, g) on B.

Next, by the definition of an admissible solution, there exists a sequence $(A^{(n)}, \phi^{(n)})$ of classical solutions on $\mathcal{D}$ which converges to (A, ϕ) in the $C_t\mathcal{H}^1$ norm. Denoting their initial data on $\{0\} \times B$ by $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)})$, we may assume (by throwing away finitely many terms) that

$$\|(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)})\|^2_{\mathcal{H}^1(\sigma_0 B_\epsilon)} < \frac{1}{10C_1^2}\epsilon_*^2 \quad \text{for all } n \in \mathbb{Z}_+. \tag{5.3}$$

Now we apply Proposition 4.1 to $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)})$ [resp. (a, e, f, g)] on $\sigma_0 B_\epsilon \setminus \overline{B_\epsilon}$, from which we obtain an initial data set $(\widetilde{a}^{(n)}, \widetilde{e}^{(n)}, \widetilde{f}^{(n)}, \widetilde{g}^{(n)})$ [resp. $(\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g})$] on $\mathbb{R}^4$ such that

$$\mathcal{E}[\widetilde{a}^{(n)}, \widetilde{e}^{(n)}, \widetilde{f}^{(n)}, \widetilde{g}^{(n)}] < \frac{1}{2}\epsilon_*^2, \quad (\widetilde{a}^{(n)}, \widetilde{e}^{(n)}, \widetilde{f}^{(n)}, \widetilde{g}^{(n)}) \to (\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g}) \quad \text{in } \mathcal{H}^1(\mathbb{R}^4). \tag{5.4}$$

Applying Lemma 3.3 and imposing some condition to fix the constant gauge transformation ambiguity (e.g., requiring the integral of the gauge transformation on B_ϵ to vanish), we arrive at a globally Coulomb initial data set $(\check{a}^{(n)}, \check{e}^{(n)}, \check{f}^{(n)}, \check{g}^{(n)})$ [resp. $(\check{a}, \check{e}, \check{f}, \check{g})$] which is gauge equivalent to $(\widetilde{a}^{(n)}, \widetilde{e}^{(n)}, \widetilde{f}^{(n)}, \widetilde{g}^{(n)})$ [resp. $(\widetilde{a}, \widetilde{e}, \widetilde{f}, \widetilde{g})$] and satisfies (5.4). Then by Theorem 1.2, there exists a sequence of global classical solutions $(\check{A}^{(n)}, \check{\phi}^{(n)})$ with initial data $(\check{a}^{(n)}, \check{e}^{(n)}, \check{f}^{(n)}, \check{g}^{(n)})$ in the global Coulomb gauge, which converges in $S^1 \subseteq C_t\mathcal{H}^1$ locally in time to a solution $(\check{A}, \check{\phi})$ with initial data $(\check{a}, \check{e}, \check{f}, \check{g})$.

Observe that $(\check{a}^{(n)}, \check{e}^{(n)}, \check{f}^{(n)}, \check{g}^{(n)}) \restriction_{B_\epsilon}$ is gauge equivalent to $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)}) \restriction_{B_\epsilon}$ by construction. By classical geometric well-posedness, it follows that $(\check{A}^{(n)}, \check{\phi}^{(n)}) \restriction_{\mathcal{D}^+(B_\epsilon)}$ is gauge equivalent to $(A^{(n)}, \phi^{(n)}) \restriction_{\mathcal{D}^+(B_\epsilon)}$ for each n. As $(A^{(n)}, \phi^{(n)}) \to (A, \phi)$ and $(\check{A}^{(n)}, \check{\phi}^{(n)}) \to (\check{A}, \check{\phi})$ in $C_t\mathcal{H}^1(\mathcal{D}^+(B_\epsilon))$, we can take the limit of the gauge transformations and conclude that there exists $\chi \in C_t\mathcal{G}^2(\mathcal{D}^+(B_\epsilon))$ such that

$$(A, \phi) = (\check{A} - \mathrm{d}\chi, e^{i\chi}\check{\phi}) \quad \text{on } \mathcal{D}^+(B_\epsilon).$$

Defining $(\check{A}_{[B_\epsilon]}, \check{\phi}_{[B_\epsilon]}) := (\check{A}, \check{\phi}) \restriction_{\mathcal{D}^+(B_\epsilon)}$, Claim 2 follows. □

6 Finite Energy Local Well-Posedness in Global Coulomb Gauge

The purpose of this section is to establish local well-posedness of the $(4+1)$-dimensional (MKG) for finite energy Coulomb initial data in the class of admissible solutions in the *global Coulomb gauge* (to be defined precisely below). As the energy regularity is critical respect to the scaling property of (MKG), the lifespan of the solution *cannot* depend only on the size of the initial energy. However, given an initial data (a, e, f, g) with $\mathcal{E}[a, e, f, g] \leq E$, we shall prove a lower bound on the lifespan that is proportional to the *energy concentration scale* r_c of the initial data, defined as

$$r_c[a, e, f, g] := \sup\{r \geq 0 : \forall x \in \mathbb{R}^4,\ \mathcal{E}_{B_r(x)}[a, e, f, g] < \delta_0(E, \epsilon_*^2)\}, \tag{6.1}$$

where $\delta_0(E, \epsilon_*^2) > 0$ is a fixed function to be determined below (see Proposition 6.7) and ϵ_*^2 is the threshold energy for small data global well-posedness (Theorem 1.2). Note that for any choice of δ_0 and $(a, e, f, g) \in \mathcal{H}^1(\mathbb{R}^4)$, we always have $r_c[a, e, f, g] > 0$.

We define the *energy profile* ρ of (a, e, f, g) to be

$$\rho(x) = \rho[a, e, f, g](x) := \frac{1}{2}(|\mathrm{d}a|^2 + |e|^2 + |\mathbf{D}f|^2 + |g|^2)(x), \tag{6.2}$$

so that $\int_S \rho \, \mathrm{d}x = \mathcal{E}_S[a, e, f, g]$ for any measurable set $S \subseteq \mathbb{R}^4$. We say that an admissible $C_t\mathcal{H}^1$ solution (A_μ, ϕ) on a time interval $I \times \mathbb{R}^4$ obeys the *global Coulomb gauge condition* if

$$\partial^\ell A_\ell = 0 \quad \text{on } I \times \mathbb{R}^4. \tag{6.3}$$

The precise statement of our local well-posedness theorem in global Coulomb gauge is as follows.

Theorem 6.1 (Local well-posedness of (MKG) at energy regularity, complete version). *Let (a, e, f, g) be an $\mathcal{H}^1(\mathbb{R}^4)$ initial data set satisfying the global Coulomb condition $\partial^\ell a_\ell = 0$ with energy $\mathcal{E}[a, e, f, g] \le E$. Let $r_c = r_c[a, e, f, g]$ be defined as in* (6.1)*. Then the following statements hold.*

(1) *There exists a unique $C_t\mathcal{H}^1$ admissible solution (A, ϕ) to* (MKG) *on $[-r_c, r_c]\times\mathbb{R}^4$ with (a, e, f, g) as its data at $t = 0$, which obeys the global Coulomb gauge condition* (6.3).
(2) *We have the additional regularity properties*

$$A_0 \in Y^1([-r_c, r_c] \times \mathbb{R}^4), \quad A_x, \phi \in S^1([-r_c, r_c] \times \mathbb{R}^4), \tag{6.4}$$

with bounds depending only on the energy profile ρ, where the spaces Y^1 and S^1 will be defined in Section 6.3 below.
(3) *The solution (A, ϕ) is more regular if the initial data set (a, e, f, g) is. In particular, (A, ϕ) is classical if (a, e, f, g) is a classical initial data set.*
(4) *Consider a sequence $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)})$ of $\mathcal{H}^1$ globally Coulomb initial data sets such that $(a^{(n)}, e^{(n)}, f^{(n)}, g^{(n)}) \to (a, e, f, g)$ in $\mathcal{H}^1(\mathbb{R}^4)$ as $n \to \infty$. Denote the corresponding solutions to* (MKG) *given by Statement (1) by $(A^{(n)}, \phi^{(n)})$. Then the lifespan of $(A^{(n)}, \phi^{(n)})$ eventually contains $[-r_c, r_c]$. Moreover, we have*

$$\|A_0 - A_0^{(n)}\|_{Y^1[-r_c,r_c]} + \|A_x - A_x^{(n)}\|_{S^1[-r_c,r_c]} + \|\phi - \phi^{(n)}\|_{S^1[-r_c,r_c]} \to 0 \tag{6.5}$$

as $n \to \infty$.

Remark 6.2 In fact our proof below yields an a-priori bound for the S^1 norm of (A_x, ϕ) and the Y^1 norm of A_0 that depends only on the energy E, the energy concentration scale r_c and the tail of the energy profile ρ, i.e., the smallest radius $r_0 > 0$ such that there exists $x_0 \in \mathbb{R}^4$ satisfying

$$\int_{\mathbb{R}^4\setminus B_{\frac{1}{54}r_0}(x_0)} \rho \, \mathrm{d}x < \delta_0(E, \epsilon_*^2). \tag{6.6}$$

We refer to Remark 6.19 for a further discussion.

As mentioned in the introduction, a theorem of this type is usually proved by exploiting finite speed of propagation, patching together local solutions with small initial data. However, while implementing this strategy in our context, one is faced with difficulties due to non-local features of (MKG). One source of non-locality is the presence of the Gauss (or constraint) equation; another is the elliptic nature of the global Coulomb gauge. To address the first issue, we use the technique of excision and gluing initial data sets developed in Section 4. To deal with the second issue, we introduce a procedure for patching rough local solutions together to produce a local-in-time but *global-in-space* solution, inspired by similar ideas in elliptic gauge theories.

The rest of this section is structured as follows. In Section 6.1, the uniqueness statement of Theorem 6.1 is established using the local geometric uniqueness result proved in Section 5. In Section 6.2 we consider the question of partitioning the initial surface $\mathbb{R}^4$ into regions which carry a small energy. Section 6.3, we introduce the function space framework for patching up local (MKG) solutions. Using this framework, we establish Proposition 6.16 in Section 6.4, which is an abstract statement that contains the essence of our patching argument. Finally, in Section 6.5, we put together the tools developed in the previous subsections to prove Theorem 6.1.

6.1 Uniqueness in the Global Coulomb Gauge

In this brief subsection, we prove the uniqueness statement in Theorem 6.1 (i.e., uniqueness of an admissible $C_t\mathcal{H}^1(I \times \mathbb{R}^4)$ solution in the global Coulomb gauge) using Proposition 5.2.

Patching together Proposition 5.2 on balls covering $\mathbb{R}^4$, it follows that two admissible $C_t\mathcal{H}^1$ solutions (A, ϕ) and (A', ϕ') on $[0, T_0) \times \mathbb{R}^4$ are gauge equivalent if their initial data sets are gauge equivalent. We then make the following observation:

Lemma 6.3 *Let $I \subset \mathbb{R}$ be an open interval. Let (A_μ, ϕ) and (A'_μ, ϕ') be admissible $C_t\mathcal{H}^1$ solutions on $I \times \mathbb{R}^4$, which are gauge equivalent and obey the global Coulomb gauge condition (6.3). Then there exists a constant $\chi_0 \in \mathbb{R}$ such that $(A'_\mu, \phi') \equiv (A_\mu, \phi e^{i\chi_0})$ on $I \times \mathbb{R}^4$.*

Proof Note that in the global Coulomb gauge, $A \in C_t^0 \dot{H}_x^1$ is determined uniquely from $\partial^\ell A_\ell = 0$, $\mathrm{d}A = F$ and (MKG). This observation fixes the gauge transformation χ between (A, ϕ) and (A', ϕ') up to a constant, at which point we are done. □.

Therefore, to complete the proof of Theorem 6.1, it suffices to prove the local existence, persistence of regularity and continuous dependence on the initial data.

6.2 Energy Concentrations Scales

Here we consider the energy distribution of initial data (a, e, f, g) in the global Coulomb gauge, and show that we can cover $\mathbb{R}^4$ with small energy cubes with side length bounded from below by $4r_c$. We also ensure that the covering is slowly varying, in the sense that neighboring cubes have comparable side lengths. This condition is needed for an effective control of the constants in the patching procedure in Section 6.4. The number of such cubes, which we denote by K, can be trivially bounded by $(r_0/r_c)^4$, where r_0 is defined by the condition (6.6); this number will enter in the final a-priori S^1 regularity bound in (6.4). As a part of our analysis here, we also specify the constant $\delta_0(E, \epsilon_*^2)$ in (6.1). See Proposition 6.7 below for a more precise statement.

We begin with a preliminary result, which shows that for Coulomb data the energy controls the full $\mathcal{H}^1$ norm:

Proposition 6.4 *Let* $(a, e, f, g) \in \mathcal{H}^1(\mathbb{R}^4)$ *be a Coulomb initial data set with energy* E*. Then we have the bound*

$$\|(a, e, f, g)\|^2_{\mathcal{H}^1(\mathbb{R}^4)} \lesssim E + E^2. \tag{6.7}$$

Proof We need to obtain bounds for A and f in $\dot{H}^1_x$. We begin with a, where the Coulomb condition $\nabla \cdot a = 0$ allows us to estimate in linear elliptic fashion

$$\|a\|_{\dot{H}^1_x} \lesssim \|\mathrm{d}a\|_{L^2_x} \lesssim E^{\frac{1}{2}}.$$

For f we first use the diamagnetic inequality and Sobolev embeddings to obtain

$$\|f\|_{L^4_x} \lesssim \|\nabla|f|\|_{L^2_x} \lesssim \|\mathbf{D}f\|_{L^2_x} \lesssim E^{\frac{1}{2}}$$

and then, splitting the covariant derivative,

$$\|\nabla f\|_{L^2_x} \lesssim \|\mathbf{D}f\|_{L^2_x} + \|f\|_{L^4_x}\|a\|_{L^4_x} \lesssim E^{\frac{1}{2}} + E,$$

which completes the proof. □

Next, we give an improvement of Hardy's inequality

$$\||x - x_0|^{-1} f\|_{L^2_x} \lesssim \|\nabla|f|\|_{L^2_x} \leq \|\mathbf{D}f\|_{L^2_x}, \tag{6.8}$$

which is our tool for obtaining smallness of the weighted L^2 norm in (4.2). We state a general version on $\mathbb{R}^d$.

Lemma 6.5 (Improved Hardy's inequality). *Let* $a_j, f \in \dot{H}^1(\mathbb{R}^d)$ *where* $d \geq 3$*. Then for any ball* $B = B_{r_0}(x_0)$ *and* $\sigma_0 \geq 2$*, we have the bounds*

$$\left\| \frac{1}{|x - x_0|} f \right\|_{L^2_x(2B\setminus\overline{B})} \lesssim \|\mathbf{D}f\|_{L^2_x(\sigma_0 B\setminus\overline{B})} + \sigma_0^{-\frac{d-2}{2}} \|\mathbf{D}f\|_{L^2_x(\mathbb{R}^d\setminus\overline{\sigma_0 B})}, \tag{6.9}$$

$$\left\| \frac{1}{|x - x_0|} f \right\|_{L^2_x(2B)} \lesssim \|\mathbf{D}f\|_{L^2_x(\sigma_0 B)} + \sigma_0^{-\frac{d-2}{2}} \|\mathbf{D}f\|_{L^2_x(\mathbb{R}^d\setminus\overline{\sigma_0 B})}. \tag{6.10}$$

Remark 6.6 In this paper, we only use the inequality (6.10) on balls. The version (6.9) will be useful in the third paper [23] of the series.

Proof By translation and scaling, we may assume that $B = B_1(0)$. We begin by splitting $g := |f|$ into spherical harmonics. In the case of non-spherically-symmetric modes, by Poincaré's inequality on spheres and the diamagnetic inequality, we have

$$\|\frac{1}{|x|} g\|_{L^2_x(2B\setminus\overline{B})} \lesssim \|\not\nabla|f|\|_{L^2_x(2B\setminus\overline{B})} \lesssim \|\mathbf{D}f\|_{L^2_x(2B\setminus\overline{B})},$$

where $|\not\nabla|f||$ denotes the size of the angular derivatives under the induced metric on the sphere $\{|x| = const\}$. Hence we are reduced to the case when g is radial.

By the one-dimensional Hardy inequality, we have

$$\sigma_0^{\frac{d-2}{2}} |g(\sigma_0)| \lesssim \|r^{\frac{d-1}{2}} g'\|_{L^2(\sigma_0,\infty)} \lesssim \|\mathbf{D} f\|_{L^2_x(\mathbb{R}^d\setminus\overline{\sigma_0 B})}.$$

Moreover, by the fundamental theorem of calculus and the diamagnetic inequality, we have the one-dimensional dyadic bounds

$$\sigma^{-\frac{1}{2}} \sup_{\frac{1}{2}\sigma\leq r,r'\leq\sigma} |g(r) - g(r')| \lesssim \|g'\|_{L^2(\frac{1}{2}\sigma,\sigma)} \lesssim \sigma^{-\frac{d-1}{2}} \|\mathbf{D} f\|_{L^2_x(\sigma B\setminus\frac{1}{2}\sigma\overline{B})} \quad \text{for all } \sigma \geq 2. \tag{6.11}$$

Then by summing up the dyadic bounds for $2 \leq \sigma \lesssim \sigma_0$, we then obtain the L^∞ bound

$$\|g\|_{L^\infty(1,2)} \lesssim \|\mathbf{D} f\|_{L^2_x(\sigma_0 B\setminus\overline{B})} + \sigma_0^{-\frac{d-2}{2}} \|\mathbf{D} f\|_{L^2_x(\mathbb{R}^d\setminus\overline{\sigma_0 B})}. \tag{6.12}$$

Applying Hölder's inequality, the desired estimate (6.9) follows.

We now turn to the bound (6.10) on the full ball $2B$. Again splitting $g = |f|$ into spherical harmonics, we are reduced to the case of a radial function g. But in this case we have the one-dimensional Hardy inequality

$$\|r^{\frac{d-3}{2}} g\|_{L^2(0,1)} \lesssim \|r^{\frac{d-1}{2}} g'\|_{L^2(0,1)} + |g(1)| \lesssim \|\mathbf{D} f\|_{L^2_x(B)} + |g(1)|. \tag{6.13}$$

Combined with (6.12), the desired inequality (6.10) follows. □

We are now ready to state and prove the main covering result of this section. We also settle the choice of $\delta_0(E, \epsilon_*^2)$.

Proposition 6.7 *Assume that $\delta_0(E, \epsilon_*^2)$ is chosen so that*

$$\delta_0(E, \epsilon_*^2) = c^2\epsilon_*^2 \min\{1, \epsilon_*^4 E^{-2}\}, \tag{6.14}$$

with a small universal constant c. Let r_0 and x_0 be as in (6.6). Then there exists a dyadic cube R_0 of side length $\approx r_0$ and a partition of it into smaller dyadic cubes

$$R_0 = \bigcup_{\alpha\in\mathcal{A}} R_\alpha$$

with the following properties:

(1) *Small energy: The following bound holds for $Q = 18R_\alpha$ and $Q = (\frac{1}{18}R_0)^c$:*

$$\mathcal{E}_Q[a, e, f, g] + \frac{1}{\ell(Q)^2}\|f\|^2_{L^2_x(Q)} \ll \epsilon_*^2, \tag{6.15}$$

where we use the convention that $\ell(Q) = \ell(\frac{1}{18}R_0)$ when $Q = (\frac{1}{18}R_0)^c$.

(2) *Size of cubes: The side length of the cubes* $\{R_\alpha\}$ *is bounded from below by* $4r_c$.
(3) *Number of cubes: The number of cubes* $\{R_\alpha\}$ *is bounded by* $K := |\mathcal{A}| \lesssim (r_0/r_c)^4$.
(4) *Slow variance: The size of all pairs of neighboring cubes may differ at most by a factor of* 2*, and all cubes adjacent to the boundary of* R_0 *have size at most* $\ell(R_0)/64$.

Proof Let r_0 and x_0 be as in (6.6). It suffices to consider the case $E > \epsilon_*^2$, since the proposition is trivial in the other case. We may also assume that $r_0 \geq 200\, r_c$, as otherwise we can simply choose $\{R_\alpha\} = \{R_0\}$ where R_0 is the cube of side length $2r_0$ centered at x_0. By translation and scaling, we may henceforth take $x_0 = 0$ and $r_c = 1$.

We choose the large cube R_0 centered at 0 so that $B_{r_0}(0) \subseteq R_0 \subseteq 3B_{r_0}(0)$ and $\ell(R_0) \in 2^{\mathbb{Z}}$. This cube will set the coordinates for our dyadic grid; more precisely, subsequent cubes will be obtained by repeatedly subdividing the sides of R_0 in half. To ensure slow variance, we use the following procedure to construct the collection $\mathcal{R} := \{R_\alpha\}_{\alpha \in \mathcal{A}}$:

- In the first step, we add to the collection $\mathcal{R}$ the cubes of side length $\frac{1}{64}\ell(R_0)$ adjacent to R_0;
- Then we recursively add to the collection $\mathcal{R}$ the cubes which are disjoint from but adjacent to the existing collection, with half the side length of the cubes added in the previous step;
- We repeat this process until we arrive at cubes of side length $\frac{1}{4}$. Then we cover the rest of R_0 with dyadic cubes of side length $\frac{1}{4}$.

We call R_0 the *initial cube*, the cubes of side length between $\frac{1}{2}$ and $\frac{1}{64}\ell(R_0)$ the *intermediate cubes*, and the cubes of side length $\frac{1}{4}$ the *final cubes*. Note that all intermediate cubes are contained in $R_0 \setminus (\frac{15}{16}R_0)^c$.

From the construction, it is obvious to see that Properties (2), (3) and (4) hold. The condition (6.15) clearly holds for the initial cube $Q = (\frac{1}{18}R_0)^c$, by (6.6) and the localized Hardy's inequality

$$\left\| \frac{1}{|x|} f \right\|^2_{L^2_x(\mathbb{R}^4 \setminus \frac{1}{54}B_{r_0})} \lesssim \|\mathbf{D} f\|^2_{L^2_x(\mathbb{R}^4 \setminus \frac{1}{54}B_{r_0})} < \delta_0(E, \epsilon_*^2) = c\epsilon_*^4 E^{-1}. \tag{6.16}$$

Moreover, we claim that the final cubes also satisfy (6.15). Indeed, the energy term $\mathcal{E}_Q$ in (6.15) follows from the definition of r_c. To control the weighted L^2_x norm, we apply Lemma 6.5 and use the fact that we scaled $r_c = 1$ to obtain

$$\frac{1}{\ell(18R_\alpha)^2}\|f\|^2_{L^2_x(18R_\alpha)} \lesssim \sigma_0^4 \delta_0(E, \epsilon_*^2) + \sigma_0^{-2} E$$

Then choosing δ_0 as in (6.14) and $\sigma_0^2 = c_0^{-2}\epsilon_*^{-2}E$ for some small universal constant $c_0 > 0$, the desired estimate (6.15) follows.

Finally, for the intermediate cubes $R_\alpha \in \mathcal{R}$ of side length between $\frac{1}{4}$ and $\frac{1}{64}\ell(R_0)$, the smallness for the energy $\mathcal{E}_Q$ in (6.15) follows immediately from (6.6). Hence it only remains to justify the weighted L^2_x bound in (6.6) for these intermediate cubes. We split our argument into two cases:

(a) When $\ell(R_\alpha) \geq \frac{1}{100}\sigma_0^{-1}\ell(R_0)$, we use (6.16) to estimate

$$\frac{1}{\ell(18R_\alpha)^2}\|f\|^2_{L^2_x(18R_\alpha)} \lesssim \frac{\sigma_0^2}{\ell(R_0)^2}\|f\|^2_{L^2_x(18R_\alpha)} \lesssim (c/c_0)^2\epsilon_*^4 E^{-1},$$

which is good.

(b) When $\ell(R_\alpha) < \frac{1}{100}\sigma_0^{-1}\ell(R_0)$, observe that we have

$$18R_\alpha \subseteq B_\alpha \subseteq \sigma_0 B_\alpha \subseteq \mathbb{R}^4 \setminus \frac{1}{54}B_{r_0}$$

where B_α is the ball of radius $\ell(18R_\alpha)$ with the same center as R_α. This chain of inclusions is a consequence of the fact that all intermediate cubes belong to $R_0 \setminus (\frac{15}{16}R_0)^c$, which are all at distance at least $\frac{1}{4}\ell(R_0)$ from $\frac{1}{54}B_{r_0}$. Therefore, by (6.6) and application of Lemma 6.5, we obtain

$$\frac{1}{\ell(18R_\alpha)^2}\|f\|^2_{L^2_x(18R_\alpha)} \lesssim \frac{1}{\ell(18R_\alpha)^2}\|f\|^2_{L^2_x(B_\alpha)} \lesssim \delta_0(E, \epsilon_*^2) + \sigma_0^{-2}E,$$

which implies the desired bound. □

6.3 Functions Spaces and Gauge Transformation Estimates

In this subsection we introduce the function spaces that will be used in the proof of existence of finite energy solutions to (MKG) in the Coulomb gauge.

The first two such spaces are the spaces Y^1 and S^1, which were used in [19] to control the elliptic component (i.e., A_0), respectively the hyperbolic components (i.e. A_x and ϕ) of small energy solutions in the global Coulomb gauge. These functions spaces are defined in [19] in the whole space-time $\mathbb{R}^{1+4}$.

We start with the space Y^s for the elliptic component (i.e., A_0) of a solution to (MKG) in the global Coulomb gauge. Let s be a non-negative integer and $q \in [1, \infty]$. Given a tempered distribution φ on $\mathbb{R}^{1+4}$, we define its $Y^{s,q}$ norm to be

$$\|\varphi\|_{Y^{s,q}(\mathbb{R}^{1+4})} := \|\partial^s_{t,x}\varphi\|_{L^q_t \dot{H}^{1/q}_x(\mathbb{R}^{1+4})}$$

where we take $L^q_t \dot{H}^{1/q}_x = L^\infty_t L^2_x$ when $q = \infty$. Then the Y^s space is defined as the space of tempered distributions for which the following norm is finite:

$$\|\varphi\|_{Y^s(\mathbb{R}^{1+4})} := \|\varphi\|_{Y^{s,2}(\mathbb{R}^{1+4})} + \|\varphi\|_{Y^{s,\infty}(\mathbb{R}^{1+4})}$$

Observe that $\|\cdot\|_{Y^{0,q}(\mathbb{R}^{1+4})}$ (and thus $\|\cdot\|_{Y^0(\mathbb{R}^{1+4})}$) scales the same way as the $L^\infty_t L^2_x$ norm. In particular, $\|\cdot\|_{Y^1(\mathbb{R}^{1+4})}$ scales like the $L^\infty_t \dot{H}^1_x$ norm.

Next, we introduce the S^1 norm on $\mathbb{R}^{1+4}$, which was used in [19] to measure the size of the hyperbolic components (i.e., A_x and ϕ) of solutions to (MKG) in the global Coulomb gauge. The precise definition of this norm involves *null frame* *spaces* [31,38], and is rather technical to state. The fine structure of this norm, though crucial for establishing the small data theory of (MKG) at the energy regularity, is not necessary for the purpose of the present section. Hence, here we will be content with simply stating the necessary properties of the S^1 norm; a more detailed description of S^1 will be recalled from [19] in Section 7, where the proof of these properties will be given.

We begin by introducing the norms $X_r^{s,b}$ and $\underline{X}$ (where $s, b \in \mathbb{R}$, $1 \leq r \leq \infty$), defined by

$$\|\varphi\|_{X^{s,b}_{k;r}(\mathbb{R}^{1+4})} := 2^{sk} \Big(\sum_j (2^{bj} \|Q_j \varphi\|_{L^2_{t,x}(\mathbb{R}^{1+4})})^r \Big)^{\frac{1}{r}}, \tag{6.17}$$

$$\|\varphi\|_{X^{s,b}_{r}(\mathbb{R}^{1+4})} := \|\varphi\|_{\ell^2 X^{s,b}_{r}(\mathbb{R}^{1+4})} = \Big(\sum_k \|P_k \varphi\|^2_{X^{s,b}_{k;r}(\mathbb{R}^{1+4})} \Big)^{\frac{1}{2}}, \tag{6.18}$$

$$\|\varphi\|_{\underline{X}(\mathbb{R}^{1+4})} := \|\Box \varphi\|_{L^2_t \dot{H}^{-\frac{1}{2}}_x(\mathbb{R}^{1+4})}, \tag{6.19}$$

with the obvious modification in the case $r = \infty$.

The S^1 norm, to first approximation, is an intermediate norm between $C^0_t \dot{H}^1_x \cap X^{1,\frac{1}{2}}_{\infty}$ and $X^{1,\frac{1}{2}}_{1} \cap \underline{X}$. More precisely, we have

$$\|\partial_{t,x}\varphi\|_{L^\infty_t L^2_x} + \|\partial_{t,x}\varphi\|_{X^{0,\frac{1}{2}}_{\infty}} \lesssim \|\varphi\|_{S^1} \lesssim \|\partial_{t,x}\varphi\|_{X^{0,\frac{1}{2}}_{1}} + \|\varphi\|_{\underline{X}}, \tag{6.20}$$

where all norms are defined on $\mathbb{R}^{1+4}$. Further properties of S^1 will be stated in the course of this subsection.

The spaces Y^1 and S^1 have an ℓ^2 dyadic structure in frequency. However, it is also useful to work with different dyadic summations. Precisely, we introduce the notation $\ell^r X$ for any function space X on $\mathbb{R}^{1+4}$, where

$$\|\varphi\|_{\ell^r X} := \Big(\sum_k \|P_k \varphi\|^r_X \Big)^{\frac{1}{r}}.$$

Remark 6.8 One motivation for this is the observation, heavily used in in [19], that certain portions of small data MKG waves exhibit better dyadic summability properties, as follows:

- The elliptic portion A_0 of the solution is in the smaller space $\ell^1 Y^1$.
- The hyperbolic component A_x, admits a decomposition $A_x = A_x^{free} + A_x^{nl}$, where A_x^{free} represents the free wave matching the initial data, while the nonlinear portion A^{nl} has the better regularity $A^{nl} \in \ell^1 S^1$.
- The high modulation part of both A_x and ϕ has better dyadic summability, $(A - x, \phi) \in \ell^1 \underline{X}$.

We further remark that the $\ell^1 \underline{X}$ norm was included in S^1 in [19]. For the sake of uniformity in notation we do not do this in our series of papers.

In addition to Y^1 and S^1, in this paper we also need function spaces to describe the class of gauge transformations we use in order to assemble local solutions to (MKG). The main space we use for this is $\mathcal{Y} := \ell^1 Y^2(\mathbb{R}^{1+4})$, with norm

$$\|\varphi\|_{\mathcal{Y}(\mathbb{R}^{1+4})} = \sum_k \sum_{N=0}^{2} \Big(2^{(\frac{5}{2}-N)k} \|\partial_t^N P_k \varphi\|_{L^2_{t,x}(\mathbb{R}^{1+4})} + 2^{(2-N)k} \|\partial_t^N P_k \varphi\|_{L^\infty_t L^2_x(\mathbb{R}^{1+4})} \Big). \tag{6.21}$$

For technical reasons we will also consider a variant of $\mathcal{Y}$, namely the $\widehat{\mathcal{Y}}$ space. Its norm is defined as

$$\|\eta\|_{\widehat{\mathcal{Y}}(\mathbb{R}^{1+4})} := \|\eta\|_{Y^{2,2}(\mathbb{R}^{1+4})} + \sum_k 2^{2k} \|P_k \eta\|_{L^\infty_t L^2_x(\mathbb{R}^{1+4})}.$$

It is easy to see that $\widehat{\mathcal{Y}}(\mathbb{R}^{1+4})$ is weaker than $\mathcal{Y}(\mathbb{R}^{1+4})$, i.e.,

$$\|\chi\|_{\widehat{\mathcal{Y}}(\mathbb{R}^{1+4})} \lesssim \|\chi\|_{\mathcal{Y}(\mathbb{R}^{1+4})}. \tag{6.22}$$

Insofar, we have defined our function spaces on the whole space $\mathbb{R}^{1+4}$. Here we also need to use them on on compact time intervals $I \times \mathbb{R}^4$ or more generally on open sets. For this it suffices to take the easy way out and use the method of restrictions. Precisely, Let X be any one of Y^1, S^1, $\mathcal{Y}$, $\widehat{\mathcal{Y}}$ or $\dot{B}^{\frac{5}{2},2}_1$, etc. For an open subset $\emptyset \neq \mathcal{O} \subseteq \mathbb{R}^{1+4}$, we define the space $X(\mathcal{O})$ to consist of restrictions of elements in $X(\mathbb{R}^{1+4})$ to $\mathcal{O}$, with the norm given by

$$\|\phi\|_{X(\mathcal{O})} := \inf\{\|\widetilde{\phi}\|_{X(\mathbb{R}^{1+4})} : \widetilde{\phi} \in X(\mathbb{R}^{1+4}),\ \widetilde{\phi} = \chi \text{ on } \mathcal{O}\}.$$

Given two non-empty open sets $\mathcal{O}_1 \supseteq \mathcal{O}_2$, the restriction map $\mathcal{Y}(\mathcal{O}_1) \to \mathcal{Y}(\mathcal{O}_2)$ is a bounded surjection.

In particular, for X as above and a time interval I we will denote by $X[I]$ the restrictions to $I \times \mathbb{R}^4$ of X functions. We refer the reader to the second paper in our series [22] for further discussion of the $S^1[I]$ and $Y[I]$ spaces.

We remark that, in view of the above definition, all algebraic estimates involving our spaces in $\mathbb{R}^{1+4}$ easily carry over to any nonempty open subsets. In particular this applies to all of the estimates below in this subsection.

The space $\mathcal{Y}$ (more precisely, its local version defined below) will be the main function space that contains the local gauge transformations in the proof of Theorem 6.1. It has the desirable property that if $\chi \in \mathcal{Y}$ and (A, ϕ) is a solution to (MKG) such that $A_0 \in Y^1$, $A_x, \phi \in S^1$, then the gauge transformed solution $(A', \phi') = (A - \mathrm{d}\chi, e^{i\chi}\phi)$ also belong to the same functions spaces. The following lemma justifies a half of this statement, precisely the part dealing with A. The other half is in Lemma 6.10.

Lemma 6.9 *For $\chi \in \mathcal{Y}(\mathbb{R}^{1+4})$ we have*

$$\|\partial_t \chi\|_{\ell^1 Y^1(\mathbb{R}^{1+4})} + \|\partial_x \chi\|_{\ell^1 S^1(\mathbb{R}^{1+4})} + \|\chi\|_{L^\infty_{t,x}(\mathbb{R}^{1+4})} \lesssim \|\chi\|_{\mathcal{Y}(\mathbb{R}^{1+4})}. \tag{6.23}$$

Proof Due to the ℓ^1 dyadic summation in the $\mathcal{Y}$ norm, we can assume without loss of generality that χ has dyadic frequency localization at frequency 2^k. Then the estimate for $\|\partial_t \chi\|_{Y^1} \lesssim 1$ is straightforward, while the L^∞ bound is a consequence of Bernstein's inequality.

To prove the bound for $\|\partial_x \chi\|_{\ell^1 S^1}$, it suffices to verify the following two bounds for functions χ at frequency 2^k:

$$2^k \|Q_{\leq k+10}\chi\|_{X_1^{1,\frac{1}{2}}} \lesssim \|\chi\|_{\mathcal{Y}}, \tag{6.24}$$

$$2^k \|Q_{>k+10}\chi\|_{\underline{X}} \lesssim \|\chi\|_{\mathcal{Y}}. \tag{6.25}$$

Indeed, thanks to the spatial frequency localization $\chi = P_{[k-1,k+1]}\chi$, it follows that $\|\partial_x \chi\|_{S^1}$ is bounded by the sum of the left-hand sides of the preceding two inequalities. The first bound (6.24) is obtained as follows:

$$2^k \|Q_{\leq k+10}\chi\|_{X_1^{1,\frac{1}{2}}} \lesssim \sum_{j \leq k+10} 2^{2k} 2^{\frac{1}{2}j} \|Q_j \chi\|_{L^2_{t,x}} \lesssim \sum_{j \leq k+10} 2^{\frac{1}{2}(j-k)} (2^{\frac{5}{2}k} \|\chi\|_{L^2_{t,x}}) \lesssim \|\chi\|_{\mathcal{Y}}.$$

The second bound (6.25) follows from the time regularity of χ:

$$2^k \|Q_{>k+10}\chi\|_{\underline{X}} \lesssim 2^{\frac{1}{2}k} \|\Box \chi\|_{L^2_{t,x}} \lesssim 2^{\frac{1}{2}k} \|\partial_t^2 \chi\|_{L^2_{t,x}} + 2^{\frac{5}{2}k} \|\chi\|_{L^2_{t,x}} \lesssim \|\chi\|_{\mathcal{Y}}.$$

This completes the proof of (6.23). □

In order to estimate the action of a gauge transformation χ on the scalar field ϕ in the space S^1 it suffices to use the weaker norm $\widehat{\mathcal{Y}}$:

Lemma 6.10 *For $\chi^1, \chi^2 \in \widehat{\mathcal{Y}}(\mathbb{R}^{1+4})$, we have*

$$\|\chi^1 \chi^2\|_{\widehat{\mathcal{Y}}(\mathbb{R}^{1+4})} \lesssim \|\chi^1\|_{\widehat{\mathcal{Y}}(\mathbb{R}^{1+4})} \|\chi^2\|_{\widehat{\mathcal{Y}}(\mathbb{R}^{1+4})}. \tag{6.26}$$

Moreover, there exist functions $\Gamma_1 : [0, \infty) \to [1, \infty)$ and $\Gamma_2 : [0, \infty)^2 \to [1, \infty)$, which grow at most polynomially, such that the following estimates hold for every $\chi, \chi' \in \widehat{\mathcal{Y}}(\mathbb{R}^{1+4})$ and $\phi, \phi' \in S^1(\mathbb{R}^{1+4})$:

$$\|e^{i\chi}\phi\|_{S^1} \lesssim \Gamma_1(\|\chi\|_{\widehat{\mathcal{Y}}}) \|\phi\|_{S^1}, \tag{6.27}$$

$$\|e^{i\chi}\phi - e^{i\chi'}\phi'\|_{S^1} \lesssim \Gamma_1(\|\chi\|_{\widehat{\mathcal{Y}}}) \|\phi - \phi'\|_{S^1} + \Gamma_2(\|\chi\|_{\widehat{\mathcal{Y}}}, \|\chi'\|_{\widehat{\mathcal{Y}}}) \|\chi - \chi'\|_{\widehat{\mathcal{Y}}} \|\phi'\|_{S^1}. \tag{6.28}$$

Here, all norms are defined on the whole space-time $\mathbb{R}^{1+4}$.

The proof of this lemma requires further knowledge of the space S^1; we will defer this proof until Section 7.

The following simple lemma will be useful for patching up local solutions which satisfy certain compatibility conditions; see Proposition 6.16 and the first two steps in Section 6.5.

Lemma 6.11 *Let $\eta \in \dot{B}_1^{\frac{5}{2},2}(\mathbb{R}^{1+4})$. Then for $X = Y^1, S^1, \mathcal{Y}$ or $\widehat{\mathcal{Y}}$, we have $\eta X \subseteq X$. Furthermore, the following estimate holds:*

$$\|\eta\phi\|_X \lesssim \|\eta\|_{\dot{B}_1^{\frac{5}{2},2}} \|\chi\|_X. \tag{6.29}$$

The proof of this lemma will also be deferred until Section 7. The lemma should be interpreted as saying that the space X is stable under multiplication by a smooth rapidly decaying space-time cutoff η. In this sense, the choice of the space $\dot{B}_1^{\frac{5}{2},2}$ is not essential; it is simply a convenient space with a scale-invariant norm in which $\mathcal{S}(\mathbb{R}^{1+4})$ is dense.

Remark 6.12 In order to apply this lemma in an open set $\mathcal{O}$, we need to ensure that $\eta \in \dot{B}_1^{\frac{5}{2},2}(\mathcal{O})$, i.e., η is the restriction to $\mathcal{O}$ of an element in $\dot{B}_1^{\frac{5}{2},2}(\mathbb{R}^{1+4})$. A simple sufficient condition, which will be enough for almost all of our usage below, is if η is *smooth* on $\mathcal{O}$ and $\mathcal{O}$ is a *bounded open set with piecewise smooth boundary*.

We end this subsection with two lemmas, which will be useful for our proof below of the existence and continuous dependence statements of Theorem 6.1. The first lemma provides a criterion for a time-independent function χ to belong to $\mathcal{Y}[I]$ for a compact time interval I. The same will apply in sets of the form $\mathcal{O} = I \times O$, with $O \subset \mathbb{R}^4$, open.

Lemma 6.13 *Let $\underline{\chi} \in \dot{B}_{x;1}^{2,2} \cap \dot{B}_{x;1}^{\frac{5}{2},2}(\mathbb{R}^4)$, and I be a compact time interval containing* 0. *Extend $\underline{\chi}$ to $I \times \mathbb{R}^4$ by imposing $\partial_t \chi = 0$ and $\chi \restriction_{\{0\}\times\mathbb{R}^4} = \underline{\chi}$. Then $\chi \in \mathcal{Y}[I]$ and we have*

$$\|\chi\|_{\mathcal{Y}[I]} \lesssim \|\underline{\chi}\|_{\dot{B}_{x,1}^{2,2}} + |I|^{\frac{1}{2}} \|\underline{\chi}\|_{\dot{B}_{x,1}^{\frac{5}{2},2}}. \tag{6.30}$$

Proof By scaling and translation we can assume that $I = [0, 1]$. Due to the ℓ^1 dyadic summation in the spaces $\dot{B}_{x;1}^{2,2} \cap \dot{B}_{x;1}^{\frac{5}{2},2}(\mathbb{R}^4)$, we may assume that $\underline{\widetilde{\chi}}$ has dyadic frequency localization, i.e., $\underline{\widetilde{\chi}} = P_{[k-1,k+1]}\underline{\widetilde{\chi}}$ for some $k \in \mathbb{Z}$. To prove the lemma, it suffices to show that there exists an extension $\widetilde{\chi}$ of $\underline{\widetilde{\chi}}$ to $\mathbb{R}^{1+4}$ such that $\widetilde{\chi} \in \mathcal{Y}(\mathbb{R}^{1+4})$, $\partial_t \widetilde{\chi} = 0$ on $I \times \mathbb{R}^4$, $\widetilde{\chi} \restriction_{\{t=0\}} = \underline{\widetilde{\chi}}$ and satisfies

$$\|\widetilde{\chi}\|_{\mathcal{Y}[I]} \lesssim \|\underline{\widetilde{\chi}}\|_{\dot{B}_{x;1}^{2,2}(\mathbb{R}^4)} + \|\underline{\widetilde{\chi}}\|_{\dot{B}_{x;1}^{\frac{5}{2},2}(\mathbb{R}^4)}. \tag{6.31}$$

Let $\eta \in C_0^\infty(\mathbb{R})$ be a smooth compactly supported function such that $\eta = 1$ on I, and take $\widetilde{\chi}(t,x) = \eta_k(t)\underline{\widetilde{\chi}}(x)$, where

$$\eta_k(t) = \begin{cases} \eta(C^{-1}2^k t) & \text{for } k \le 0, \\ \eta(C^{-1}t) & \text{for } k \ge 0. \end{cases}$$

In Fourier space, $\widehat{\eta_k}$ decays rapidly away from $\{|\tau| \lesssim C^{-1}\min\{2^k, 1\}\}$, $\|\widehat{\eta_k}\|_{L^1_\tau} \lesssim 1$ and $\|\widehat{\eta_k}\|_{L^2_\tau} \lesssim C^{\frac{1}{2}} 2^{\frac{1}{2}\min\{k,0\}}$. Combining these facts with the assumption that $\underline{\widetilde{\chi}}$ is frequency localized at $\{|\xi| \approx 2^k\}$, (6.31) follows for C sufficiently large (independent of k). $\square$

The second lemma concerns solving a certain Poisson equation in $\widehat{\mathcal{Y}}[I]$, which arises when we attempt to gauge transform the solution obtained by patching to the global Coulomb gauge.

Lemma 6.14 *Let $I \subseteq \mathbb{R}$ be a time interval. Let $\eta \in \dot{B}^{\frac{5}{2},2}_1[I]$ and $\phi \in \widehat{\mathcal{Y}}[I]$. Consider the Poisson equation*

$$-\Delta\chi = \eta\Delta\phi.$$

Then the following statements hold.

(1) *The right-hand side belongs to $C_t\dot{B}^{0,2}_{x;1}$, and therefore we may define $\chi(t)$ for each $t \in I$ unambiguously as the convolution of $\eta\Delta\phi(t,x)$ with the Newton potential, i.e.,*

$$\chi(t,x) = \frac{3}{4\pi^2}\int_{\mathbb{R}^4} \frac{1}{|x-y|^2}\eta(t,y)\Delta\phi(t,y)\,\mathrm{d}y.$$

(2) *Moreover, $\chi \in \widehat{\mathcal{Y}}[I]$ and satisfies the estimate*

$$\|\chi\|_{\widehat{\mathcal{Y}}[I]} \lesssim \|\eta\|_{\dot{B}^{\frac{5}{2},2}_1[I]}\|\phi\|_{\widehat{\mathcal{Y}}[I]}. \tag{6.32}$$

The proof of Lemma 6.14 will be similar to that of Lemma 6.11. Hence it will be given in Section 7 as well.

6.4 Patching Compatible Pairs

In this subsection, we present a technical tool that will be used to quantitatively patch together local solutions, which are given by the small energy theorem (Theorem 1.2), to obtain a global solution with the desired properties.

We now introduce the notion of *compatible pairs*.

Definition 6.15 (Compatible $C_t\mathcal{H}^1$ pairs). Let $\mathcal{O} \subseteq \mathbb{R}^{1+4}$ be an open set and $\mathcal{Q} = \{Q_\alpha\}$ be a finite covering of $\mathcal{O}$. For each index α, consider a pair $(A_{[\alpha]}, \phi_{[\alpha]}) \in C_t\mathcal{H}^1(Q_\alpha)$ of a real-valued 1-form $A_{[\alpha]}$ and a $\mathbb{C}$-valued function $\phi_{[\alpha]}$ on Q_α. We say that the pairs $(A_{[\alpha]}, \phi_{[\alpha]})$ are *compatible* if for every α, β there exists a gauge transformation $\chi_{[\alpha\beta]} \in C_t\mathcal{G}^2 \cap C^0_{t,x}(Q_\alpha \cap Q_\beta)$ such that the following properties hold:

(1) For every α, we have $\chi_{[\alpha\alpha]} = 0$.
(2) For every α, β, we have

$$A_{[\beta]} = A_{[\alpha]} - \mathrm{d}\chi_{[\alpha\beta]}, \quad \phi_{[\beta]} = e^{i\chi_{[\alpha\beta]}}\phi_{[\alpha]} \quad \text{on } Q_\alpha \cap Q_\beta, \tag{6.33}$$

(3) For every α, β, γ, the following *cocycle condition* is satisfied:

$$\chi_{[\alpha\beta]} + \chi_{[\beta\gamma]} + \chi_{[\gamma\alpha]} \in 2\pi\mathbb{Z} \quad \text{on } Q_\alpha \cap Q_\beta \cap Q_\gamma. \tag{6.34}$$

The main result of this subsection is Proposition 6.16 below, whose formulation and proof were motivated by the classical result of Uhlenbeck [41] on weak compactness of connections with curvature bounded in L^p.

In order to state our result we need to specify the set $\mathcal{O}$ and the covering $\mathcal{Q}$. For this, we begin with the partition

$$\mathbb{R}^4 = \cup_\alpha R_\alpha \cup R_0^c$$

given in Proposition 6.7. Taking $I = [0, 1]$ and $r_c = 1$, (which suffices by scaling), we define

$$\mathcal{O} = I \times \mathbb{R}^4, \qquad Q_0 = I \times R_0^c, \qquad Q_\alpha = I \times 1.5R_\alpha.$$

The factor 1.5 above is what guarantees, in view of condition (4) in Proposition 6.7, that this covering is locally finite.

We also consider a smaller, subordinated subcovering $\mathcal{P} = \{P_\alpha\}$ given by

$$P_\alpha = I \times 1.25R_\alpha, \qquad P_0 = I \times (1.001R_0)^c, \qquad \mathcal{O} = \cup_\alpha P_\alpha$$

This is also locally finite. Using this notations we have:

Proposition 6.16 (Patching compatible pairs). *Let* $(A_{[\alpha]}, \phi_{[\alpha]})$ *on* Q_α *be* compatible pairs *associated to the above covering* $\mathcal{Q}$ *of* $\mathcal{O}$. *Suppose furthermore that for every* α, β, *the gauge transformation* $\chi_{[\alpha\beta]}$ *belongs to* $\mathcal{Y}(Q_\alpha \cap Q_\beta)$ *(defined in Section 6.3), which embeds into* $C_t\mathcal{G}^2 \cap C^0_{t,x}(Q_\alpha \cap Q_\beta)$.

Let $\{\underline{\chi}_{[\alpha\beta]}\}$ *be another collection of gauge transformations such that* $\underline{\chi}_{[\alpha\beta]} \in \mathcal{Y}(Q_\alpha \cap Q_\beta)$ *for every* α, β, *and satisfies the cocycle condition* (6.34). *Assume moreover that* $\{\chi_{[\alpha\beta]}\}$ *is* C^0 *close to* $\{\underline{\chi}_{[\alpha\beta]}\}$, *in the sense that*

$$\sup_{Q_\alpha \cap Q_\beta} |\chi_{[\alpha\beta]} - \underline{\chi}_{[\alpha\beta]}| < \epsilon_{**}, \tag{6.35}$$

where $\epsilon_{**} > 0$ *is a universal constant to be specified below.*

Then there exists gauge transformations $\chi_{[\alpha]} \in \mathcal{Y}(P_\alpha)$ *on each* P_α, *depending linearly on* $\chi_{[\alpha\beta]}$ *and* $\underline{\chi}_{[\alpha\beta]}$, *which satisfy*

$$-\chi_{[\alpha]} + \chi_{[\alpha\beta]} + \chi_{[\beta]} = \underline{\chi}_{[\alpha\beta]} \quad \textit{on } P_\alpha \cap P_\beta.$$

Moreover, $\chi_{[\alpha]}$ obey the following bounds with a universal implicit constant:

$$\sup_{\alpha} \|\chi_{[\alpha]}\|_{\mathcal{Y}(P_\alpha)} \lesssim \sup_{\alpha,\beta} \Big(\|\chi_{[\alpha\beta]}\|_{\mathcal{Y}(Q_\alpha \cap Q_\beta)} + \|\underline{\chi}_{[\alpha\beta]}\|_{\mathcal{Y}(Q_\alpha \cap Q_\beta)} \Big). \tag{6.36}$$

Remark 6.17 The role of the C^0 closeness condition (6.35) is to remove the $2\pi\mathbb{Z}$ ambiguity in the cocycle condition (6.34). More precisely, since both $\chi_{[\alpha\beta]}$ and $\underline{\chi}_{[\alpha\beta]}$ satisfy (6.34), we have

$$(\chi_{[\alpha\beta]} - \underline{\chi}_{[\alpha\beta]}) + (\chi_{[\beta\gamma]} - \underline{\chi}_{[\beta\gamma]}) + (\chi_{[\gamma\alpha]} - \underline{\chi}_{[\gamma\alpha]}) \in 2\pi\mathbb{Z}.$$

For a sufficiently small ϵ_{**} (say $\epsilon_{**} < \frac{2\pi}{3}$), the C^0 closeness condition (6.35) then implies that the absolute value of the left-hand side is bounded by $< 2\pi$; therefore, it follows that

$$(\chi_{[\alpha\beta]} - \underline{\chi}_{[\alpha\beta]}) + (\chi_{[\beta\gamma]} - \underline{\chi}_{[\beta\gamma]}) + (\chi_{[\gamma\alpha]} - \underline{\chi}_{[\gamma\alpha]}) = 0. \tag{6.37}$$

Proof Our $\{Q_\alpha\}$ covering is locally finite, so let $N_0 = N_0(d)$ (which we can take 4^4 in dimension $d = 4$) be so that each Q_α intersects at most N_0 neighbors. Then we define a reduction map $\mathfrak{R}$ which decreases the cube size by a fixed factor, so that $\mathfrak{R}^{N_0}(Q_\alpha) = P_\alpha$ for $\alpha \neq 0$, with the obvious adjustment $\mathfrak{R}^{-N_0}(Q_0^c) = P_0^c$ for $\alpha = 0$. For uniformity of notation, we write $\mathfrak{R}Q_0 := (\mathfrak{R}^{-1}(Q_0^c))^c$, so that $\mathfrak{R}^{N_0} Q_0 = P_0$.

Consider an enumeration of the elements in $\mathcal{Q}$ by positive integers $0, 1, \ldots, K$, in nonincreasing order of size, where we take Q_0 to be the first element. We proceed by induction on this enumeration.

For the induction step, suppose that we have constructed an open covering $\mathcal{Q}_{k-1} = \{Q_{\alpha,k-1}\}$, with $P_\alpha \subseteq Q_{\alpha,k-1} \subseteq Q_\alpha$, $\mathcal{O} = \cup_\alpha Q_{\alpha,k-1}$ and gauge transforms $\chi_{[\alpha]}$ on $Q_{\alpha,k-1}$ with $\alpha = 1, \ldots, k-1$ such that

(1) $Q_{\alpha,k-1} = \mathfrak{R}^{n(\alpha,k)} Q_\alpha$ where $n(\alpha,k)$ is between 0 and N_0, and is zero for $\alpha \geq k-1$,
(2) $-\chi_{[\alpha]} + \chi_{[\alpha\beta]} + \chi_{[\beta]} = \underline{\chi}_{[\alpha\beta]}$ for $1 \leq \alpha, \beta \leq k-1$ provided $Q_{\alpha,k-1} \cap Q_{\beta,k-1} \neq \emptyset$,
(3) $\|\chi_{[\alpha]}\|_{\mathcal{Y}(Q_{\alpha,k-1})} \lesssim X_\alpha$ for $1 \leq \alpha \leq k-1$,

where

$$X_\alpha = \sup_{Q_\alpha \cap Q_\beta \neq \emptyset} \Big(\|\chi_{[\alpha\beta]}\|_{\mathcal{Y}(Q_\alpha \cap Q_\beta)} + \|\underline{\chi}_{[\alpha\beta]}\|_{\mathcal{Y}(Q_\alpha \cap Q_\beta)} \Big).$$

Define the open covering $\mathcal{Q}_k$ so that $Q_{\alpha,k} = \mathfrak{R}Q_{\alpha,k-1}$ if $\alpha \leq k-1$ and Q_α is a neighbor of Q_k, and $Q_{\alpha,k} = Q_{\alpha,k-1}$ otherwise. We shall then construct a gauge transformation $\chi_{[k]}$ on $Q_{k,k} = Q_k$ such that the above properties hold with $k-1$ replaced by k, where $\chi_{[\alpha]}$ for $\alpha \leq k-1$ are defined by simply restricting to $Q_{\alpha,k} \subseteq Q_{\alpha,k-1}$. From this statement, the proposition will follow by induction, starting with $Q_{\alpha,0} = Q_\alpha$ and $\chi_{[0]} = 0$.

We remark that the uniformity in the estimate (3) is due to the fact that our covering of $\mathcal{O}$ is locally finite, and also that $\mathcal{Q}$ is slowly varying. Indeed, it is obvious in the proof below that the construction in the induction step only involves Q_k and its neighbors, whose side length is comparable to that of Q_k. Thus, for each α the sets $Q_{\alpha,k}$ are reduced in size only finitely many times, and the cutoff functions $\zeta_{[k]}$ below can be taken to be uniformly smooth with respect to the scale of Q_k.

We now proceed with the proof of the induction step. We begin by defining $\widetilde{\chi}_{[k]}$ on $Q_k \cap (\cup_{\alpha \le k-1} Q_{\alpha,k-1})$ to be

$$\widetilde{\chi}_{[k]} = \chi_{[k\alpha]} + \chi_{[\alpha]} + \underline{\chi}_{[\alpha k]} \quad \text{on } Q_k \cap Q_{\alpha,k-1} \text{ if it is nonempty.} \tag{6.38}$$

Observe that this definition is consistent on $Q_k \cap (\cup_{\alpha \le k-1} Q_{\alpha,k-1})$ thanks to property (2) in the induction hypothesis and the exact cocycle condition (6.37) for $\chi_{[\alpha\beta]} - \underline{\chi}_{[\alpha\beta]}$. Moreover, by considering a partition of unity subordinate to $\{Q_k \cap Q_{\alpha,k-1}\}_{\alpha=1,\dots,k-1}$ and using the induction hypothesis (3) and Lemma 6.11, we can derive the estimate

$$\|\widetilde{\chi}_{[k]}\|_{\mathcal{Y}(Q_k \cap Q_{\alpha,k})} \lesssim_{C_{k-1}} X_k \tag{6.39}$$

Now let $\zeta_{[k]} : \mathcal{O} \to [0, 1]$ be a smooth function that satisfies the following properties:

$$\zeta_{[k]} = 0 \quad \text{on } Q_k \setminus (\cup_{\alpha \le k-1} Q_{\alpha,k-1}), \tag{6.40}$$

$$\zeta_{[k]} = 1 \quad \text{on } \cup_{\alpha \le k-1} Q_{\alpha,k}. \tag{6.41}$$

We remark that such a ζ exists because by construction the two sets $Q_k \setminus (\cup_{\alpha \le k-1} Q_{\alpha,k-1})$ and $\cup_{\alpha \le k-1} Q_{\alpha,k}$ are separated by a distance which is proportional to the size of Q_k. This also allows us to choose the functions $\zeta_{[k]}$ uniformly smooth on Q_k.

Now we define

$$\chi_{[k]} := \zeta_{[k]} \widetilde{\chi}_{[k]} \quad \text{on } Q_{k,k} = Q_k. \tag{6.42}$$

Note that properties (1) and (2) are immediately consequences of the construction.

For the property (3), we observe that $\zeta_{[k]} \restriction_{Q_k}$ can be extended as an element in $C_0^\infty(\mathbb{R}^{1+4}) \subseteq \dot{B}_1^{\frac{5}{2},2}(\mathbb{R}^{1+4})$. Thus property (3) follows from Lemma 6.11 (in particular, stability of $\mathcal{Y}$ by cutoffs in $\dot{B}_1^{\frac{5}{2},2}$), Lemma 6.11, (6.38), (6.39) and (6.42). □

6.5 Proof of Existence and Continuous Dependence

Using the tools developed in the previous subsections, we are ready to prove the existence and continuous dependence statements of Theorem 6.1. In what follows, we will often use the shorthand $\mathcal{E} := \mathcal{E}[a, e, f, g]$.

Step 0. Preliminaries. Let (a, e, f, g) be an $\mathcal{H}^1$ initial data set satisfying the global Coulomb condition $\partial^\ell a_\ell = 0$ and $\mathcal{E}[a, e, f, g] < E$. It suffices to assume $r_c[a, e, f, g] < \infty$, since otherwise the small data result (Theorem 1.2) is applicable. By scaling, we may take

$$r_c[a, e, f, g] = 1. \tag{6.43}$$

By time reversal symmetry, it suffices to restrict to $t \geq 0$ and consider the unit time interval $I = [0, 1]$. Let $\{R_0^c\} \cup \{R_\alpha\}$ be the covering of $\mathbb{R}^4$ introduced in Section 6.2, such that the local small energy condition (6.15) holds, and let $\mathcal{Q} = \{Q_\alpha\}$, $\mathcal{P} = \{P_\alpha\}$ be the associated covering of $I \times \mathbb{R}^4$ defined in Section 6.4.

In what follows, we will construct a local-in-time solution (A, ϕ) in $I \times \mathbb{R}^4$, which obeys the S^1 a-priori regularity property (6.4). Moreover, we will show that our construction below also has the following two properties:

- Continuous dependence: the data-to-solution map is continuous as follows:

$$\mathcal{H}^1(\mathbb{R}^4) \ni (a, e, f, g) \to (A_0, A_x, \phi) \in Y^1(I \times \mathbb{R}^4) \times S^1(I \times \mathbb{R}^4) \times S^1(I \times \mathbb{R}^4).$$

- Regularity: If in addition $(a, e, f, g) \in \mathcal{H}^\infty(\mathbb{R}^4)$ then the solution (A, ϕ) belongs to $C_t^\infty \mathcal{H}^\infty(\mathbb{R}^4)$.

Theorem 6.1 will then follow by combining these statements with the uniqueness statement proved in Section 6.1.

Step 1. Construction of local Coulomb solutions. The goal of this step is to show that corresponding to the $\mathcal{Q} = \{Q_\alpha\}$ covering of $I \times \mathbb{R}^4$, introduced in Section 6.4, we can produce a compatible local solution $(A_{[\alpha]}, \phi_{[\alpha]})$ on each Q_α, each of which is the restriction of a small energy global Coulomb solution to (MKG) given by Theorem 1.2. We will in effect construct these solutions on the larger sets $I \times 3R_\alpha$, and then simply restrict them to Q_α.

Claim 1 The following statements hold for each Coulomb initial data (a, e, f, g) satisfying (6.15):

(1) On each set $I \times 3R_\alpha$ there exists an admissible $C_t \mathcal{H}^1(I \times 3R_\alpha)$ solution $(A_{[\alpha]}, \phi_{[\alpha]})$ and a gauge transformation $\underline{\chi}_{[\alpha]} \in \mathcal{G}^2(3R_\alpha)$ that satisfy the Coulomb gauge condition

$$\partial^\ell A_{[\alpha]\ell} = 0 \quad \text{on } I \times 3R_\alpha, \tag{6.44}$$

and the initial condition

$$(A_{[\alpha]j}, F_{[\alpha]0j}, \phi_{[\alpha]}, \mathbf{D}_{[\alpha]t}\phi_{[\alpha]}) = (a_j - \partial_j \underline{\chi}_{[\alpha]}, e_j, e^{i\underline{\chi}_{[\alpha]}} f, e^{i\underline{\chi}_{[\alpha]}} g) \quad \text{on } 3R_\alpha\,. \tag{6.45}$$

Moreover, $A_{[\alpha]x}, \phi_{[\alpha]} \in S^1(I \times 3R_\alpha)$, $A_{[\alpha]0} \in Y^1(I \times 3R_\alpha)$ depend continuously on the initial data in $\mathcal{H}^1$, and we have the smallness bound

$$\|A_{[\alpha]0}\|_{Y^1(I \times 3R_\alpha)} + \|A_{[\alpha]x}\|_{S^1(I \times 3R_\alpha)} + \|\phi_{[\alpha]}\|_{S^1(I \times 3R_\alpha)} \lesssim \epsilon_*, \tag{6.46}$$

(2) Extend $\underline{\chi}_{[\alpha]}$ to $I \times 3R_\alpha$ by requiring $\partial_t \underline{\chi}_{[\alpha]} = 0$; abusing the notation slightly, we shall denote the extension by $\underline{\chi}_{[\alpha]}$. Then

$$\Delta \underline{\chi}_{[\alpha]} = 0 \quad \text{on } I \times 3R_\alpha. \tag{6.47}$$

Moreover, $\underline{\chi}_{[\alpha]} \in \mathcal{Y}(I \times 3R_\alpha)$, depending continuously on the initial data, and obeys the estimate

$$\|\underline{\chi}_{[\alpha]}\|_{\mathcal{Y}(I\times 3R_\alpha)} \lesssim_E 1, \tag{6.48}$$

(3) For every α and β, there exists $\chi_{[\alpha\beta]} \in \mathcal{Y}(I \times (3R_\alpha \cap 3R_\beta))$ that connects $(A_{[\alpha]}, \phi_{[\alpha]})$ and $(A_{[\beta]}, \phi_{[\beta]})$ in the sense of Definition 6.15 and satisfies

$$\Delta \chi^{(n)}_{[\alpha\beta]} = 0 \quad \text{on } I \times (3R_\alpha \cap 3R_\beta). \tag{6.49}$$

Moreover, $\chi_{[\alpha\beta]}$ depends continuously on the initial data and obeys the estimate

$$\|\chi_{[\alpha\beta]}\|_{\mathcal{Y}(I\times(3R_\alpha\cap 3R_\beta))} \lesssim_E 1 \tag{6.50}$$

Finally, the following C^0 closeness condition holds:

$$\sup_{I\times 3R_\alpha \cap Q_\beta} |\chi_{[\alpha\beta]} - (\underline{\chi}_{[\alpha]} - \underline{\chi}_{[\beta]})| < \epsilon_{**}, \tag{6.51}$$

where $\epsilon_{**} > 0$ is the universal small constant that appeared in Proposition 6.16.

(4) Higher regularity: if in addition $(a, e, f, g) \in \mathcal{H}^\infty$, then for each α, β we have

$$\begin{gathered}(A_7[\alpha], \phi_{[\alpha]}) \in C_t^\infty \mathcal{H}^\infty(I \times 3R_\alpha), \quad \underline{\chi}_{[\alpha]} \in \mathcal{G}^\infty(I \times 3R_\alpha),\\ \chi_{[\alpha\beta]} \in C_t^\infty \mathcal{G}^\infty(I \times (3R_\alpha \cap 3R_\beta)).\end{gathered}$$

We proceed to the proof of this claim.

Step 1.1. Construction of $(A_{[\alpha]}, \phi_{[\alpha]})$ **and** $\underline{\chi}_{[\alpha]}$ **for** $\alpha \geq 1$. Our starting point here is the estimate (6.15). We insert a ball $3R_\alpha \subset B \subset 2B \subset 18R_\alpha$, which has radius $r_\alpha \approx \ell(R_\alpha)$. Applying Proposition 4.1 with $\sigma_1 = 4/3$ and $\sigma_0 = 2$ to (a, e, f, g) with respect to the ball B, we obtain an initial data set $(\widetilde{a}_{[\alpha]}, \widetilde{e}_{[\alpha]}, \widetilde{f}_{[\alpha]}, \widetilde{g}_{[\alpha]}) \in \mathcal{H}^1(\mathbb{R}^4)$, depending continuously on (a, e, f, g) in $\mathcal{H}^1$, such that we have the matching condition

$$(\widetilde{a}_{[\alpha]}, \widetilde{e}_{[\alpha]}, \widetilde{f}_{[\alpha]}, \widetilde{g}_{[\alpha]}) = (a, e, f, g) \quad \text{on } 4R_\alpha \tag{6.52}$$

and small energy

$$\mathcal{E}[\widetilde{a}_{[\alpha]}, \widetilde{e}_{[\alpha]}, \widetilde{f}_{[\alpha]}, \widetilde{g}_{[\alpha]}] \ll \epsilon_*^2. \tag{6.53}$$

However, our small localized data $(\widetilde{a}_{[\alpha]}, \widetilde{e}_{[\alpha]}, \widetilde{f}_{[\alpha]}, \widetilde{g}_{[\alpha]})$ is no longer in the Coulomb gauge. To rectify this we use a gauge transformation defined by

$$\underline{\chi}_{[\alpha]} := -(-\Delta)^{-1}\partial^\ell \widetilde{a}_{[\alpha]\ell},$$

where $(-\Delta)^{-1}$ on the right-hand side is defined as convolution with the Newtonian potential. In general, this expression may not be uniquely determined if we only knew $\partial^\ell \widetilde{a}_{[\alpha]\ell} \in L^2_x$. However, note that we have the support condition

$$\mathrm{supp}(\partial^\ell \widetilde{a}_{[\alpha]\ell}) \subseteq 9R_\alpha \setminus 4R_\alpha, \tag{6.54}$$

since $\widetilde{a} \equiv 0$ outside $9R_\alpha$. It follows that $\partial^\ell \widetilde{a}_{[\alpha]\ell} \in L^1_x \cap L^2_x(\mathbb{R}^4)$ and therefore the right-hand side is well-defined. The gauge transformed data set

$$(\check{a}_{[\alpha]}, \check{e}_{[\alpha]}, \check{f}_{[\alpha]}, \check{g}_{[\alpha]}) := (\widetilde{a}_{[\alpha]} - \mathrm{d}\underline{\chi}_{[\alpha]}, \widetilde{e}_{[\alpha]}, e^{i\underline{\chi}_{[\alpha]}} \widetilde{f}_{[\alpha]}, e^{i\underline{\chi}_{[\alpha]}} \widetilde{g}_{[\alpha]}), \tag{6.55}$$

is a small energy $\mathcal{H}^1(\mathbb{R}^4)$ Coulomb initial data set; hence Theorem 1.2 is applicable. Let $(A_{[\alpha]}, \phi_{[\alpha]})$ be the unique global small energy Coulomb solution to (MKG) given by Theorem 1.2. By construction, (6.44) and (6.45) hold; moreover, (6.46) and the continuous dependence property are consequences of Theorem 1.2.

We now verify (6.47) and (6.48) for $\underline{\chi}_{[\alpha]}$. Indeed, by the support condition (6.54) we directly get (6.47), as well as the uniform bounds

$$\|\partial_x^{(N)} \underline{\chi}_{[\alpha]}\|_{L^\infty_x(3.5R_\alpha)} \lesssim_N r_\alpha^{-N} \|\partial^\ell \widetilde{a}_{[\alpha]\ell}\|_{L^2_x} \lesssim r_\alpha^{-N} E^{\frac{1}{2}} \quad \text{for every } N \geq 0.$$

By Lemma 6.13 (see also Remark 6.12) this directly leads to (6.48). The continuous dependence similarly follows.

Step 1.3. Construction of $(A_{[\alpha]}, \phi_{[\alpha]})$ **and** $\underline{\chi}_{[\alpha]}$ **for** $\alpha = 0$. Again we start with (6.15) but with $\alpha = 0$. This time we insert the ball $\frac{1}{18}R_0 \subset B \subset 2B \subset \frac{1}{3}R_0$, which has radius $r_0 \approx \ell(R_0)$, and apply Proposition 4.2 with $\sigma_1 = \frac{4}{3}$ and $\sigma_0 = 2$. We obtain an initial data set $(\widetilde{a}_{[0]}, \widetilde{e}_{[0]}, \widetilde{f}_{[0]}, \widetilde{g}_{[0]}) \in \mathcal{H}^1(\mathbb{R}^4)$ such that

$$(\widetilde{a}_{[0]}, \widetilde{e}_{[0]}, \widetilde{f}_{[0]}, \widetilde{g}_{[0]}) = (a, e, f, g) \text{ on } (\tfrac{1}{4}R_0)^c, \tag{6.56}$$

$$\mathcal{E}[\widetilde{a}_{[0]}, \widetilde{e}_{[0]}, \widetilde{f}_{[0]}, \widetilde{g}_{[0]}] \ll \epsilon_*^2, \tag{6.57}$$

where the last line follows from (4.4) and our choice of R_0. As before, we define

$$\underline{\chi}_{[0]} := -(-\Delta)^{-1} \partial^\ell \widetilde{a}_{[0]\ell},$$

which is unambiguously defined due to the support condition

$$\mathrm{supp}(\partial^\ell \widetilde{a}_{[0]\ell}) \subseteq 1.5B \setminus B \tag{6.58}$$

as $\widetilde{a}_{[0]} = a$ on $(1.5B)^c$ is divergence-free. Again the gauge corrected data

$$(\check{a}_{[0]}, \check{e}_{[0]}, \check{f}_{[0]}, \check{g}_{[0]}) := (\widetilde{a}_{[0]} - \mathrm{d}\underline{\chi}_{[0]}, \widetilde{e}_{[0]}, e^{i\underline{\chi}_{[0]}} \widetilde{f}_{[0]}, e^{i\underline{\chi}_{[0]}} \widetilde{g}_{[0]}) \tag{6.59}$$

is an $\mathcal{H}^1(\mathbb{R}^4)$ Coulomb initial data set with energy $\ll \epsilon_*^2$. Hence we can apply Theorem 1.2 to define $(A_{[0]}, \phi_{[0]})$ as the unique global small energy Coulomb solution to (MKG) given by Theorem 1.2. Then (6.44), (6.45), (6.46) as well as the the continuous dependence property and the regularity property follow easily from construction.

As (6.47) is a direct consequence of (6.58), it remains to establish the bound (6.48) for $\underline{\chi}_{[0]}$. Using again the support condition (6.58) and the decay of the Newton potential we obtain

$$|\partial_x^N \underline{\chi}_{[0]}(x)| \lesssim r_0^{-N}(1 + r_0^{-1}|x|)^{-2} E^{\frac{1}{2}}, \qquad N \geq 0, \quad x \in (2B)^c$$

which suffices for (6.48).

Step 1.4. Properties of $\chi_{[\alpha\beta]}$. We now proceed to prove Statement (3). The existence of $\chi_{[\alpha\beta]}$ will be a consequence of Proposition 5.2 (local geometric uniqueness); the estimate (6.50) and the corresponding continuous dependence, on the other hand, will follow from the global Coulomb condition satisfied by each solution $(A_{[\alpha]}, \phi_{[\alpha]})$.

In what follows, we explain the details in the case $\alpha, \beta \neq 0$; the case $\alpha = 0$ is handled by an obvious modification. By construction, the initial data for $(A_{[\alpha]}, \phi_{[\alpha]})$ and $(A_{[\beta]}, \phi_{[\beta]})$ are gauge equivalent on $3R_\alpha \cap 3R_\beta$, with the gauge transformation given by $\underline{\chi}_{[\alpha]} - \underline{\chi}_{[\beta]}$. By scaling, each of these cubes has side length larger than 1, so their domains of dependence satisfy

$$I \times (2R_\alpha \cap 2R_\beta) \subset \mathcal{D}^+(3R_\alpha) \cap \mathcal{D}^+(3R_\beta)$$

Hence, Proposition 5.2 shows that the two solutions are gauge equivalent in $I \times (2R_\alpha \cap 2R_\beta)$. We denote by $\chi_{[\alpha\beta]} \in C_t \mathcal{G}^2(I \times (2R_\alpha \cap 2R_\beta))$ the transition map. A-priori this is only determined modulo 2π, but this ambiguity is easily fixed by requiring that

$$\chi_{[\alpha\beta]} = \underline{\chi}_{[\alpha]} - \underline{\chi}_{[\beta]} \qquad \text{on } \{0\} \times (2R_\alpha \cap 2R_\beta).$$

Moreover, this satisfies

$$\Delta \chi_{[\alpha\beta]} = 0 \quad \text{on } I \times (2R_\alpha \cap 2R_\beta), \tag{6.60}$$

thanks to the fact that $\Delta \chi_{[\alpha\beta]} = \partial^\ell A_{[\alpha]\ell} - \partial^\ell A_{[\beta]\ell} = 0$. Therefore, by the mean value property of harmonic functions,

$$\chi_{[\alpha\beta]}(t, x) = \int \chi_{[\alpha\beta]}(t, x - y) r_\alpha^{-4} \varphi(y/r_\alpha) \, \mathrm{d}y \quad \text{for } (t, x) \in Q_\alpha \cap Q_\beta, \tag{6.61}$$

where we recall that φ is a smooth radial function on $\mathbb{R}^4$ with $\int \varphi = 1$ and $\operatorname{supp} \varphi \subseteq \{|x| \leq 1\}$. Here we have also used the fact that $r_\alpha \approx r_\beta$, and that an $O(r_\alpha)$ spatial neighborhood of $Q_\alpha \cap Q_\beta$ is contained in $I \times (2R_\alpha \cap 2R_\beta)$.

It remains to prove (6.50) and (6.51). We begin with the following bounds for $\partial_t \chi_{[\alpha\beta]}$ and $\partial_t^2 \chi_{[\alpha\beta]}$: Differentiating (6.61) (in t, x), using Hölder's inequality and recalling the identity $\partial_\mu \chi_{[\alpha\beta]} = A_{[\alpha]\mu} - A_{[\beta]\mu}$, we have for $N \geq 0$

$$\|\partial_x^{(N)} \partial_{t,x} \chi_{[\alpha\beta]}\|_{L^\infty_{t,x}(Q_\alpha \cap Q_\beta)} \lesssim_N r_\alpha^{-1-N} \|A_{[\alpha]} - A_{[\beta]}\|_{L^\infty_t L^4_x(I \times (2R_\alpha \cap 2R_\beta))} \lesssim r_\alpha^{-1-N} \epsilon_*, \tag{6.62}$$

$$\begin{aligned}\|\partial_x^{(N)} \partial_t^2 \chi_{[\alpha\beta]}\|_{L^\infty_{t,x}(Q_\alpha \cap Q_\beta)} &\lesssim_N r_\alpha^{-2-N} \|\partial_t A_{[\alpha]0} - \partial_t A_{[\beta]0}\|_{L^\infty_t L^2_x(I \times (2R_\alpha \cap 2R_\beta))} \\ &\lesssim r_\alpha^{-2-N} \epsilon_*.\end{aligned} \tag{6.63}$$

Taking $N = 0$ and integrating (6.62), the C^0 closeness statement (6.51) follows. Moreover, we have

$$\|\chi_{[\alpha\beta]}\|_{L^\infty_{t,x}(Q_\alpha \cap Q_\beta)} \lesssim \epsilon_* + \|\underline{\chi}_{[\alpha]}\|_{L^\infty_x(1.5R_\alpha)} + \|\underline{\chi}_{[\beta]}\|_{L^\infty_x(1.5R_\beta)} \lesssim_E 1 \tag{6.64}$$

thanks to (6.48). Finally, observe that $Q_\alpha \cap Q_\beta$ is pre-compact for any pair α, β such that $\alpha \neq \beta$, since there is only one unbounded element in $\mathcal{Q}$, namely Q_0. From the bounds (6.62), (6.63) and (6.64), and the fact that $Q_\alpha \cap Q_\beta$ is pre-compact, we may easily construct an extension $\widetilde{\chi}_{[\alpha\beta]}$ of $\chi_{[\alpha\beta]}$ such that

$$\|\widetilde{\chi}_{[\alpha\beta]}\|_{\mathcal{Y}(\mathbb{R}^{1+4})} \lesssim_E 1 \tag{6.65}$$

Finally, we note that $\chi_{[\alpha\beta]}$ constructed above depend continuously on $(A_{[\alpha]}, \phi_{[\alpha]})$ and thus on the initial data (a, e, f, g) in $\mathcal{H}^1$.

Step 1.5. Completion of proof of Claim 1. Restricting $(A_{[\alpha]}, \phi_{[\alpha]})$ and $\underline{\chi}_{[\alpha]}$ to Q_α, and $\chi_{[\alpha\beta]}$ to $Q_\alpha \cap Q_\beta$, Statements (1)–(3) follow from the previous steps. On the other hand, Statement 4 (persistence of regularity) can be quickly read off from the above construction, using the corresponding statements in Propositions 4.1, 4.2 and Theorem 1.2. We omit the details.

Step 2. Construction of global almost Coulomb solution. We now construct a global solution (A', ϕ') on $I \times \mathbb{R}^4$ such that $A'_x, \phi' \in S^1[I]$ and $A'_0 \in Y^1[I]$ by patching together the compatible pairs obtained in the previous step. This solution will *not* satisfy the global Coulomb condition (6.3) in general. Nevertheless, it will have the redeeming feature that the spatial divergence $\partial^\ell A'_\ell$ obeys an improved bound compared to a general derivative of a A'. This feature will be a consequence of the fact that (A', ϕ') will be constructed by patching together local *Coulomb* solutions $(A_{[\alpha]}, \phi_{[\alpha]})$.

The above statements are made precise in the following claim.

Claim 2 For any initial data (a, e, f, g) of energy at most E, with $r_c \geq 1$ and satisfying[6] (6.15) there exists an admissible $C_t\mathcal{H}^1$ solution (A', ϕ') to (MKG) on $I \times \mathbb{R}^4$ such that the following statements hold.

(1) The data for (A', ϕ') on $\{t = 0\}$ coincide with (a, e, f, g), i.e.,

$$(A'_j, F'_{0j}, \phi', \mathbf{D}'_t\phi') \restriction_{\{t=0\}} = (a_j, e_j, f, g). \tag{6.66}$$

[6] The only reason for this requirement is to ensure a uniform construction of (A', ϕ'), which guarantees its continuous dependence on the initial data.

(2) The solution (A', ϕ') satisfies $A'_x, \phi' \in S^1[I]$, $A'_0 \in Y^1[I]$, depends continuously on the initial data, and obeys

$$\|A'_0\|_{Y^1[I]} + \|A'_x\|_{S^1[I]} + \|\phi'\|_{S^1[I]} \lesssim_{E,K} 1 \tag{6.67}$$

where K is the total number of cubes in the set $\{R_\alpha\}$ constructed in Section 6.2. In our case, $K \lesssim (r_0/r_c)^4$.

(3) The spatial divergence of A' satisfies $\partial^\ell A'_\ell \in C^0_t \dot{B}^{0,2}_{x;1}(I \times \mathbb{R}^4)$. Therefore, the convolution with the Newtonian potential

$$\chi := -(-\Delta)^{-1}\partial^\ell A'_\ell = -\frac{3}{4\pi^2}\int_{\mathbb{R}^4} \frac{1}{|x-y|^2}\partial^\ell A'_\ell(t,y)\,\mathrm{d}y$$

is unambiguously defined and belongs to $C^0_t \dot{B}^{2,2}_{x;1} \subseteq C^0_{t,x}$. Moreover, it satisfies the additional estimates

$$\|\chi\|_{\widehat{\mathcal{Y}}[I]} \lesssim_{E,K} 1 \tag{6.68}$$

$$\|\partial_x \chi\|_{S^1[I]} \lesssim_{E,K} 1 \tag{6.69}$$

(4) If additionally $(a, e, f, g) \in \mathcal{H}^\infty$, then we have

$$(A', \phi') \in C^\infty_t \mathcal{H}^\infty(I \times \mathbb{R}^4) \quad \text{and} \quad \chi \in C^\infty_t \mathcal{G}^\infty(I \times \mathbb{R}^4).$$

To prove the claim, we begin by applying Proposition 6.16 to the covering $\mathcal{Q}$ of $I \times \mathbb{R}^4$, the compatible pairs $(A_{[\alpha]}, \phi_{[\alpha]})$ and the gauge transformations $\chi_{[\alpha\beta]}$ and $\underline{\chi}_{[\alpha\beta]} := \underline{\chi}_{[\alpha]} - \underline{\chi}_{[\beta]}$; note that the C^0 closeness condition has been established in (6.51). Then for the sub-covering $\mathcal{P} = \{P_\alpha\}$, we obtain gauge transformations $\chi_{[\alpha]} \in \mathcal{Y}(P_\alpha)$ such that

$$\|\chi_{[\alpha]}\|_{\mathcal{Y}(P_\alpha)} \lesssim_E 1 \tag{6.70}$$

$$\chi_{[\alpha\beta]} = \chi_{[\alpha]} + \underline{\chi}_{[\alpha]} - \chi_{[\beta]} - \underline{\chi}_{[\beta]}. \tag{6.71}$$

This identity motivates the following definition of the desired global solution (A', ϕ'). Let η_α be a smooth partition of unity adapted to the covering $\{P_\alpha\}$. Since $\mathcal{P}$ is a locally finite covering where intersecting cubes have comparable sizes, we can choose this partition of unity so that the η_α's are uniformly smooth on the scale of their respective cubes. We define the global solution (A', ϕ') as follows:

$$\begin{aligned} A'_\mu &:= \sum_\alpha \eta_\alpha (A_{[\alpha]\mu} - \partial_\mu \chi_{[\alpha]} - \partial_\mu \underline{\chi}_{[\alpha]}), \\ \phi' &:= \sum_\alpha \eta_\alpha e^{i(\chi_{[\alpha]} + \underline{\chi}_{[\alpha]})}\phi_{[\alpha]}. \end{aligned} \tag{6.72}$$

Such a definition makes sense, since (6.71) implies that on every $P_\alpha \cap P_\beta \neq \emptyset$, we have

$$A_{[\alpha]\mu} - \partial_\mu \chi_{[\alpha]} - \partial_\mu \underline{\chi}_{[\alpha]} = A_{[\beta]\mu} - \partial_\mu \chi_{[\beta]} - \partial_\mu \underline{\chi}_{[\beta]}, \tag{6.73}$$

$$e^{i(\chi_{[\alpha]}+\underline{\chi}_{[\alpha]})}\phi_{[\alpha]} = e^{i(\chi_{[\beta]}+\underline{\chi}_{[\beta]})}\phi_{[\beta]}. \tag{6.74}$$

For every $\alpha \neq 0$, $\eta_\alpha \in \dot{B}_1^{\frac{5}{2},2}(P_\alpha)$ since η_α is smooth and P_α is pre-compact. On the other hand for $\alpha = 0$ we have $1 - \eta_0 \in \dot{B}_1^{\frac{5}{2},2}(P_0)$. By Lemmas 6.9, 6.10, 6.11 and estimates (6.46), (6.48), (6.70), we have

$$\|\eta_\alpha(A_{[\alpha]0} - \partial_t \chi_{[\alpha]} - \partial_t \underline{\chi}_{[\alpha]})\|_{Y^1[I]} \lesssim_E 1$$

$$\|\eta_\alpha(A_{[\alpha]x} - \partial_x \chi_{[\alpha]} - \partial_x \underline{\chi}_{[\alpha]})\|_{S^1[I]} \lesssim_E 1$$

$$\|\eta_\alpha e^{i(\chi_{[\alpha]}+\underline{\chi}_{[\alpha]})}\phi_{[\alpha]}\|_{S^1[I]} \lesssim_E 1$$

Adding up the preceding estimates, (6.67) follows. The continuous dependence on the initial data and the persistence of regularity also follow directly from our construction.

It remains to establish Statement (3) and the bounds (6.68), (6.69). This part depends crucially on the special cancellation that occurs only for $\partial^\ell A'_\ell$. Indeed, thanks to (6.44), (6.47) and (6.73) on each $P_\alpha \cap P_\beta \neq \emptyset$, we have

$$\partial^\ell A'_\ell = \partial^\ell \sum_\alpha \eta_\alpha(A_{[\alpha]\ell} - \partial_\ell \chi_{[\alpha]} - \partial_\ell \underline{\chi}_{[\alpha]}) = -\sum_\alpha \eta_\alpha \Delta \chi_{[\alpha]},$$

$$\partial^\ell (A'_\ell - A'_\ell) = -\sum_\alpha \eta_\alpha \Delta(\chi_{[\alpha]} - \chi_{[\alpha]}).$$

Equipped with these formulae, we are ready to establish (6.68) and (6.69). Since η_α extends naturally to $\dot{B}_1^{\frac{5}{2},2}(I \times \mathbb{R}^4)$ and $\chi_{[\alpha]} \in \mathcal{Y}[I] \subseteq \widehat{\mathcal{Y}}[I]$, we are in position to apply Lemma 6.14 to each summand $\eta_\alpha \Delta \chi_{[\alpha]}$. Then (6.68) follows. To estimate the $S^1[I]$ norm of $(-\Delta)^{-1}\partial_j \partial^\ell A'_\ell$, simply observe that

$$\|(-\Delta)^{-1}\partial_j \partial^\ell A'_\ell\|_{S^1[I]} \lesssim \|A'_x\|_{S^1[I]} \lesssim_{E,K} 1$$

Thus (6.69) follows.

Step 3. Gauge transformation to Coulomb solution. In this final step of the proof of existence and continuous dependence, we perform a gauge transformation to (A', ϕ') in order to impose the global Coulomb condition $\partial^\ell A_\ell = 0$. The gauge transformation cannot be put directly into $\mathcal{Y}[I]$, but this difficulty can be circumvented using the elliptic equations of (MKG) in the global Coulomb gauge.

From the previous step, recall the definition

$$\chi = -(-\Delta)^{-1}\partial^\ell A'_\ell \quad \text{on } I \times \mathbb{R}^4,$$

where the first term on the right-hand side is defined as in Statement (3) in Claim 2. As $\partial^\ell A'_\ell \restriction_{\{t=0\}} = 0$, it follows that

$$\chi \restriction_{\{t=0\}} = 0. \tag{6.75}$$

Directly taking the ∂_t derivative of χ twice and using the fact that (A', ϕ') satisfies (MKG), we see that $\partial_t \chi$ and $\partial_t^2 \chi$ are given by

$$\begin{aligned}
\partial_t \chi &= -(-\Delta)^{-1} \partial^\ell \partial_t A'_\ell = -(-\Delta)^{-1} \Big(\mathrm{Im}[\phi' \overline{\mathbf{D}'_t \phi'}] + \Delta A'_0 \Big), \\
\partial_t^2 \chi &= -(-\Delta)^{-1} (\partial_t \partial^\ell F_{0\ell} + \partial_t A'_0) = -(-\Delta)^{-1} \Big(\partial^\ell \mathrm{Im}[\phi' \overline{\mathbf{D}'_\ell \phi'}] + \Delta \partial_t A'_0 \Big).
\end{aligned}$$

Since $\phi', A'_0 \in C^0_t \dot{H}^1_x$ and $\mathbf{D}'_{t,x} \phi', \partial_t A'_0 \in C^0_t L^2_x$, we have $\mathrm{Im}[\phi' \overline{\mathbf{D}'_{t,x} \phi'}] \in C^0_t \dot{H}^{-1}_x$. Therefore, $(-\Delta)^{-1} \mathrm{Im}[\phi' \overline{\mathbf{D}'_t \phi}]$ and $(-\Delta)^{-1} \partial^\ell \mathrm{Im}[\phi' \overline{\mathbf{D}'_\ell \phi}]$ are well-defined as convolution with the Newtonian potential. By the non-existence of non-trivial entire harmonic functions in L^2_x and $\dot{H}^1_x \subseteq L^4_x$, it follows that

$$\partial_t \chi = -\, (-\Delta)^{-1} \mathrm{Im}[\phi' \overline{\mathbf{D}'_t \phi'}] + A'_0 \in C^0_t \dot{H}^1_x \tag{6.76}$$

$$\partial_t^2 \chi = -\, (-\Delta)^{-1} \partial^\ell \mathrm{Im}[\phi' \overline{\mathbf{D}'_\ell \phi'}] + \partial_t A'_0 \in C^0_t L^2_x. \tag{6.77}$$

Let (A, ϕ) be defined by applying the gauge transformation χ to (A', ϕ'), i.e.,

$$(A, \phi) = (A' - \mathrm{d}\chi, e^{i\chi} \phi').$$

By (6.75), we have

$$(A_j, F_{0j}, \phi, \mathbf{D}_t \phi) \restriction_{t=0} = (A'_j, F'_{0j}, \phi', \mathbf{D}'_t \phi') \restriction_{t=0} = (a_j, e_j, f, g).$$

Furthermore, thanks to the equation $\Delta \chi = \partial^\ell A'_\ell$, it follows that (A, ϕ) satisfies the global Coulomb condition (6.3) on $I \times \mathbb{R}^4$. By (6.68), (6.69), (6.76), (6.77) and Lemma 6.10, we have $A_0 \in Y^1[I]$ and $A_x, \phi \in S^1[I]$ with

$$\|A_0\|_{Y^1[I]} + \|A_x\|_{S^1[I]} + \|\phi\|_{S^1[I]} \lesssim_{E,K} 1$$

Combining these statements, we conclude that (A, ϕ) is an admissible $C_t \mathcal{H}^1$ solution to (MKG) in the global Coulomb gauge on $I \times \mathbb{R}^4$ with the initial data (a, e, f, g), which satisfies the conditions in Theorem 6.1. Further, from the previous step, it follows that (A, ϕ) is uniformly approximated by $\mathcal{H}^\infty$ solutions, thereby finishing the proof of Theorem 6.1. We conclude the proof with two remarks:

Remark 6.18 Our construction yields a solution operator that depends continuously on the initial data for a class of $\mathcal{H}^1$ data which satisfy the uniform bounds (6.15). However the final result does not depend on the choice of the partition $\{R_\alpha\}$.

Remark 6.19 Our proof gives an a-priori bound on the S^1 norm of (A_x, ϕ) (as well as the Y^1 norm of A_0) of the form $\lesssim (r_0/r_c)^4 C_E$, where the dependence on the energy E of C_E is polynomial. By comparison with the gauge-free nonlinear wave equation, one would conjecture that the bound should be independent of r_0/r_c, and that $C_E \approx E^{1/2} + E$ by (6.7). However, our present argument is very far from that.

7 Proof of Gauge Transformation and Cutoff Estimates

The purpose of this section is to provide proofs of Lemmas 6.10, 6.11 and 6.14, which were used in Section 6 in the proof of Theorem 6.1. In Section 7.1, we recall some properties of the space S^1 needed for establishing these statements. In Section 7.2, we give a proof of Lemma 6.10 concerning gauge transformation with $\chi \in \widehat{\mathcal{Y}}$. Finally, in Section 7.3, we prove Lemmas 6.11 and 6.14.

In this section, when we omit writing the domain on which a norm is defined, it is to be understood that the norm is defined globally on $\mathbb{R}^{1+4}$. All functions considered in this section will be assumed to be $\mathcal{S}(\mathbb{R}^{1+4})$, unless otherwise stated. Furthermore, we will follow the common abuse of terminology and refer to semi-norms as simply *norms*.

7.1 Further Structure of S^1

We recall the structure of the S^1 norm from [19]. The S^1 norm takes the form (see also Remark 6.8)

$$\|\varphi\|_{S^1} := \Big(\sum_k \|\partial_{t,x} P_k \varphi\|_{S_k}^2\Big)^{\frac{1}{2}} + \|\varphi\|_{\underline{X}}.$$

The $\underline{X}$ norm was defined in (6.19). For every $k \in \mathbb{Z}$, we define the S_k norm as

$$\|\varphi\|_{S_k} := \|\varphi\|_{S_k^{\mathrm{str}}} + \|\varphi\|_{X_\infty^{0,\frac{1}{2}}} + \|\varphi\|_{S_k^{\mathrm{ang}}}$$

where the $X_\infty^{0,\frac{1}{2}}$ norm was defined in (6.17), (6.18), and we define

$$\|\varphi\|_{S_k^{\mathrm{str}}} := \sup_{(q,r):\frac{1}{q}+\frac{3}{2}\frac{1}{r}\leq\frac{3}{4}} 2^{(\frac{1}{q}+\frac{4}{r}-2)k} \|\varphi\|_{L_t^q L_x^r}, \quad \|\varphi\|_{S_k^{\mathrm{ang}}} := \sup_{\ell<0} \|\varphi\|_{S_{k,k+2\ell}^{\mathrm{ang}}},$$

$$\|\varphi\|_{S_{k,j}^{\mathrm{ang}}} := \Big(\sum_{\omega\in\Omega_\ell} \|P_\ell^\omega Q_{<k+2\ell}\varphi\|_{S_k^\omega(\ell)}^2\Big)^{\frac{1}{2}}, \quad \text{where } \ell = \lceil \frac{j-k}{2} \rceil.$$

The preceding square sum runs over $\Omega_\ell := \{\omega\}$ consisting of finitely overlapping covering of $\mathbb{S}^3$ by caps ω of diameter 2^ℓ, and the symbols of the multipliers P_ℓ^ω form a smooth partition of unity associated to this covering. The *angular sector norm* $S_k^\omega(\ell)$ contains the square-summed $L_t^2 L_x^\infty$ norm with gain in the radial dimension in Fourier space (essentially as in [16]) and the null frame space (first introduced in the wave map context [31,38]). Fortunately, for most of our argument, we need not use the fine structure of this norm. Hence we omit the precise definition, and refer the reader to [19, Eq. (8)]. The following stability property for $S^{\mathrm{ang}}_{k_0,j_0}$ is our only necessity.

Lemma 7.1 *Let $k_0, j_0, k_2 \in \mathbb{Z}$ be such that $j_0 < k_0$. Then for $\eta, \varphi \in H^{\infty}_{t,x}(\mathbb{R}^{1+4})$, we have*

$$\| P_{k_0}(S_{\leq j_0-30}\eta\, P_{k_2}\varphi)\|_{S^{\mathrm{ang}}_{k_0,j_0}} \lesssim \|\eta\|_{L^{\infty}_{t,x}} \|P_{k_2}\varphi\|_{S_{k_2}} \tag{7.1}$$

Moreover, the left-hand side is vacuous unless $k_2 \in [k_0-5, k_0+5]$.

Proof This lemma is essentially [31, Section 16:Case 2(b).3.(b).2(b)] and [39, Lemma 9.1]. We sketch the proof, following the notation in [19, Section 3].

We may assume that $k_2 \in [k_0-5, k_0+5]$, as the left-hand side is clearly vacuous otherwise. Moreover, using the embedding $X_1^{0,\frac{1}{2}} \subseteq S^{\mathrm{ang}}_{k_0,j_0}$, the case $j_0 > k_0 - C$ for any constant $C > 0$ is easy to handle. Hence we may assume that $j_0 \leq k_0 - 20$, and in particular $j_0 < k_2$.

Let $\ell_0 = \lceil \frac{j_0-k_0}{2} \rceil$ and fix $\omega \in \Omega_{\ell_0}$. Thanks to the small space-time Fourier support of $S_{\leq j_0-30}\eta$, we have

$$\begin{aligned} &P_{k_0} P^{\omega}_{\ell_0} Q_{<k_0+2\ell_0}(S_{\leq j_0-30}\eta P_{k_2}\varphi) \\ &\quad = P_{k_0} P^{\omega}_{\ell_0} Q_{<k_0+2\ell_0}\Big(S_{\leq j_0-30}\eta \sum_{\omega'\subseteq\omega} P_{k_2} P^{\omega'}_{\ell_0-5} Q_{<k_2+2\ell_0+10}\varphi\Big) \end{aligned}$$

where we sum over caps $\omega' \in \Omega_{\ell_0-5}$ such that $\omega' \subseteq \omega$. Similarly, given a radially directed rectangular block $\mathcal{C}_k(\ell) \subseteq \{2^{k_0-5} \leq |\xi| \leq 2^{k_0+5}\}$ of dimensions $2^k \times (2^{k+\ell})^3$ with $k \leq k_0$, $\ell \leq 0$ and $k+\ell \geq k_0 + 2\ell_0$, we have

$$\begin{aligned} &P_{\mathcal{C}_k(\ell)} P_{k_0} P^{\omega}_{\ell_0} Q_{<k_0+2\ell}(S_{\leq j_0-30}\eta P_{k_2}\varphi) \\ &\quad = P_{\mathcal{C}_k(\ell)} P_{k_0} P^{\omega}_{\ell_0} Q_{<k_0+2\ell_0}\Big(S_{\leq j_0-30}\eta \sum_{\omega'\subseteq\omega}\sum_{\mathcal{C}'_k(\ell)} P_{\mathcal{C}'_k(\ell)} P_{k_2} P^{\omega'}_{\ell_0-5} Q_{<k_2+2\ell_0+10}\varphi\Big) \end{aligned}$$

where ω' is summed over the same set and we sum over $\mathcal{C}'_k(\ell)$ which is either equal to or adjacent to $\mathcal{C}_k(\ell)$. The projections $P_{\mathcal{C}_k(\ell)}$, $P_{\mathcal{C}'_k(\ell)}$ and $P_{k_0} P^{\omega}_{\ell_0} Q_{<k_0+2\ell}$ are disposable (i.e., has a Schwarz kernel of $L^1_{t,x}$ norm $\lesssim 1$), hence they are bounded in all functions spaces under consideration. Moreover, from the definitions in [19, Section 3], it is clear that

$$\|\eta\varphi\|_X \leq \|\eta\|_{L^{\infty}_{t,x}} \|\varphi\|_X,$$

for $X = S^{\mathrm{str}}_k, L^2_t L^{\infty}_x, NE$, and $PW^{\pm}_{\omega}(\ell)$. For every sign $\pm$ and cap $\omega' \in \Omega_{\ell_0-5}$ with $\omega' \subseteq \omega$, we also have

$$\|\varphi\|_{PW^{\pm}_{\omega}(\ell_0)} \leq \|\varphi\|_{PW^{\pm}_{\omega'}(\ell_0-5)}.$$

Recalling the definition of the $S_k^{\omega}(\ell)$ norm [19, Eq. (8)], we see that

$$\begin{aligned}&\|P_{k_0} P_\ell^{\omega} Q_{<k+2\ell}(S_{\le j_0-30}\eta P_{k_2}\varphi)\|_{S_{k_0}^{\omega}(\ell_0)}\\ &\lesssim \|\eta\|_{L^\infty_{t,x}} \sum_{\omega'\subseteq\omega} \|P_{k_2} P_{\ell_0-5}^{\omega'} Q_{<k_2+2\ell_0+10}\varphi\|_{S_{k_2}^{\omega'}(\ell_0-5)}.\end{aligned}$$

We square sum this bound in $\omega \in \Omega_{\ell_0}$. Note that if we replace $Q_{<k_2+2\ell_0+10}$ by $Q_{<k_2+2\ell_0-10}$, then the last factor is controlled by the $S^{\mathrm{ang}}_{k_2,k_2+\ell_0-5}$ norm of $P_{k_2}\varphi$. For the resulting error, we use the embedding $X_1^{0,\frac{1}{2}} \subseteq S_k^{\omega'}(\ell)$ and estimate

$$\begin{aligned}\Big(\sum_{\omega\in\Omega_\ell} \|P_k P_\ell^{\omega} Q_{k+2\ell-C\le\cdot<k+2\ell+C}\varphi\|^2_{S_k^{\omega'}(\ell)}\Big)^{\frac{1}{2}} &\lesssim_C \|P_k Q_{k+2\ell-C\le\cdot<k+2\ell+C}\varphi\|_{X_1^{0,\frac{1}{2}}}\\ &\lesssim_C \|P_k\varphi\|_{X_\infty^{0,\frac{1}{2}}},\end{aligned}$$

and apply this inequality to $k = k_2$, $\ell = \ell_0 - 5$ and $C = 10$. The lemma follows. □

7.2 Gauge Transformation Estimate

Here we establish Lemma 6.10. This is carried out in two steps. The first one deals with the algebra type property for the space $\widehat{\mathcal{Y}}$:

Lemma 7.2 *The space $\widehat{\mathcal{Y}}$ is an algebra,*

$$\|\chi^1\chi^2\|_{\widehat{\mathcal{Y}}} \lesssim \|\chi^1\|_{\widehat{\mathcal{Y}}}\|\chi^2\|_{\widehat{\mathcal{Y}}}, \tag{7.2}$$

Further, for any F of class $C^6(\mathbb{R})$ with $F(0) = 0$ we have the Moser type estimate

$$\|F(\chi)\|_{\widehat{\mathcal{Y}}} \lesssim (\|\chi\|_{\widehat{\mathcal{Y}}} + \|\chi\|^2_{\widehat{\mathcal{Y}}})(1 + \|\chi\|^4_{L^\infty_{t,x}}), \tag{7.3}$$

Proof The main step of the proof is to establish the result for a component of the $\widehat{\mathcal{Y}}$ norm, namely the $\ell^1 L_t^\infty \dot{H}_x^2$ norm. We begin with a simple observation, namely that by Bernstein's inequality we have

$$\|\chi\|_{L^\infty_{t,x}} \lesssim \|\chi\|_{\ell^1 L_t^\infty \dot{H}_x^2}$$

This is the only place where the ℓ^1 summation is used. The bound (7.2) for the $\ell^1 L_t^\infty \dot{H}_x^2$ norm is now an application of the standard Littlewood–Paley trichotomy, which in effect yields the stronger bound

$$\|\chi^1\chi^2\|_{\ell^1 L_t^\infty \dot{H}_x^2} \lesssim \|\chi^1\|_{\ell^1 L_t^\infty \dot{H}_x^2}\|\chi^2\|_{L^\infty_{t,x}} + \|\chi^1\|_{L^\infty_{t,x}}\|\chi^2\|_{\ell^1 L_t^\infty \dot{H}_x^2}$$

A similar bound can be proved for the $Y^{2,2}$ norm in an analogous manner.

To estimate $F(\chi)$ we use a continuous Littlewood–Paley theory decomposition,

$$1 = \int_{-\infty}^{\infty} P_k \, \mathrm{d}k, \qquad P_{<j} = \int_{-\infty}^{j} P_k \, \mathrm{d}k$$

where χ is a continuous dyadic frequency parameter. See e.g. [39] for a similar argument. Representing χ as

$$\chi = \int_{-\infty}^{\infty} P_k \chi \, \mathrm{d}k,$$

for $F(\chi)$ we have the similar representation

$$F(\chi) = F(0) + \int_{-\infty}^{\infty} F'(P_{<k}\chi) P_k \chi \, \mathrm{d}k$$

which is easily seen to converge in $L^\infty_{t,x}$. Now it suffices to estimate the nonlinear term in $L^\infty_{t,x}$,

$$\|\partial_x^N F'(P_{<k}\chi)\|_{L^\infty_{t,x}} \lesssim 2^{-Nk}(1 + \|\chi\|^3_{L^\infty_{t,x}}), \qquad N = 0, 1, 2, 3.$$

Then the integrand satisfies the bound

$$\|\partial_x^N F'(P_{<k}\chi) P_k \chi\|_{L^\infty_t L^2_x} \lesssim 2^{(2-N)k} \|P_k \chi\|_{L^\infty_t \dot{H}^2_x}$$

After dyadic integration in k this yields the bound

$$\|F(\chi)\|_{\ell^1 L^\infty_t \dot{H}^2_x} \lesssim \|\chi\|_{\ell^1 L^\infty_t \dot{H}^2_x}(1 + \|\chi\|^3_{L^\infty_{t,x}}) \tag{7.4}$$

which is the $\ell^1 L^\infty_t \dot{H}^2_x$ counterpart of (7.3).

To also estimate the $Y^{2,2}$ norm of $F(\chi)$ we differentiate twice,

$$\partial^2_{x,t} F(\chi) = \partial^2_{x,t}\chi \, F'(\chi) + \partial_{x,t}\chi \, \partial_{x,t}\chi \, F''(\chi) \tag{7.5}$$

We need to estimate the terms on the right in $L^2_t \dot{H}^{\frac{1}{2}}_x$. We have the Bernstein type bounds

$$\|F'(\chi) - F'(0)\|_{L^\infty_{t,x} \cap L^\infty_t \dot{W}^{1,4}_x} \lesssim \|F(\chi)\|_{\ell^1 L^\infty_t \dot{H}^2_x}$$

and similarly for $F''(\chi)$, where the norm on the right is further estimated as in (7.4). Also we control $\partial^2_{x,t}\chi$ in $L^2_t \dot{H}^{\frac{1}{2}}_x$, as well as

$$\|\partial_{x,t}\chi\|_{L^4 \dot{W}^{\frac{3}{4},4}_x} \lesssim \|\partial^2_{x,t}\chi\|^{\frac{1}{2}}_{L^2_t \dot{H}^{\frac{1}{2}}_x} \|\chi\|_{L^\infty_{t,x}}$$

Hence for the first term on the right in (7.5) it remains to establish the bound

$$\|fG\|_{L^2_t\dot H^s_x}\lesssim\|f\|_{L^2_t\dot H^s_x}\|G\|_{L^\infty_{t,x}\cap L^\infty_t\dot W^{1,4}_x},\qquad s=\frac{1}{2}.$$

But this follows by interpolation from the $s=0$ and $s=1$ cases, which are straightforward.

Similarly, for the second term on the right in (7.5) we need to establish the bound

$$\|f_1f_2G\|_{L^2_t\dot H^{\frac{1}{2}}_x}\lesssim\|f_1\|_{L^4\dot W^{\frac{3}{4},4}_x}\|f_1\|_{L^4\dot W^{\frac{3}{4},4}_x}\|G\|_{L^\infty_{t,x}\cap L^\infty_t\dot W^{1,4}_x}.$$

which is again a simple exercise which is left for the reader. □

The second step deals with the stability of the S^1 space with respect to multiplication by $\widehat{\mathcal{Y}}$. Before we state it, we begin with a dyadic decomposition of the $Y^{n,2}$ norms which will be used repeatedly in the sequel. Precisely, for $N=0,1,2,\dots$, the following square summability estimate holds:

$$\Big(\sum_\ell 2^{2N\ell}\|S_\ell\chi\|^2_{Y^{0,2}}\Big)^{\frac{1}{2}}+\Big(\sum_{k,j}(2^{Nk}+2^{Nj})^2\|P_kT_j\chi\|^2_{Y^{0,2}}\Big)^{\frac{1}{2}}\lesssim\|\chi\|_{Y^{N,2}}.\tag{7.6}$$

Lemma 7.3 *The following estimate holds:*

$$\|\chi\varphi\|_{S^1}\lesssim\|\chi\|_{\widehat{\mathcal{Y}}}\|\varphi\|_{S^1}.\tag{7.7}$$

Proof We begin by splitting

$$\chi\varphi=\sum_{k_0}P_{k_0}Q_{\le k_0+25}(\chi\varphi)+\sum_{k_0}P_{k_0}Q_{>k_0+25}(\chi\varphi)\tag{7.8}$$

Step 1. Contribution of $\sum_{k_0}P_{k_0}Q_{\le k_0+25}(\chi\varphi)$. In this step, we will show

$$\|\sum_{k_0}P_{k_0}Q_{\le k_0+25}(\chi\varphi)\|_{S^1}\lesssim\|\chi\|_{Y^{2,2}\cap L^\infty_{t,x}}\|\varphi\|_{S^1}.\tag{7.9}$$

We need different arguments for different parts of the S^1 norm. The common strategy, however, is to divide into two cases, one in which χ has a high space-time frequency and the other in which χ has very low space-time frequency.

In the former case, we will rely on the following simple lemma:

Lemma 7.4 *Let $j_0 \le k_0 + 30$.*

(1) *For $\ell > k_0 - 5$, we have*

$$\begin{aligned} &2^{k_0} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_\ell \chi \, P_{k_2} \varphi) \|_{L^2_{t,x}} \\ &\lesssim 2^{\frac{1}{2}(j_0 - k_0)} 2^{\frac{3}{2}(k_0 - \ell)} 2^{\frac{1}{2}(k_2 - \ell)} (2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}}) (2^{k_2} \| P_{k_2} \varphi \|_{S_{k_2}}), \end{aligned} \tag{7.10}$$

and the left-hand side of (7.11) is vacuous unless $k_2 \le \ell + 10$.

(2) *If $\ell \le k_0 - 5$, we have instead*

$$2^{k_0} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_\ell \chi \, P_{k_2} \varphi) \|_{L^2_{t,x}} \lesssim 2^{\frac{1}{2}(j_0 - \ell)} (2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}}) (2^{k_2} \| P_{k_2} \varphi \|_{S_{k_2}}). \tag{7.11}$$

Moreover, the left-hand side of (7.11) is vacuous unless $k_2 \in [k_0 - 5, k_0 + 5]$.

Proof The claims regarding the range of k_2 are clear. We estimate the left-hand side of (7.10) by

$$\begin{aligned} &\lesssim 2^{k_0} 2^{\frac{1}{2} j_0} \| S_\ell \chi \|_{L^2_t L^{\frac{8}{3}}_x} \| P_{k_2} \varphi \|_{L^\infty_t L^8_x} \\ &\lesssim 2^{\frac{1}{2}(k_2 - \ell)} 2^{\frac{1}{2}(j_0 - \ell)} 2^{k_0 - \ell} (2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}}) (2^{k_2} \| P_{k_2} \varphi \|_{S_{k_2}}). \end{aligned}$$

For (7.11), we estimate

$$\lesssim 2^{k_0} 2^{\frac{1}{2} j_0} \| S_\ell \chi \|_{L^2_t L^\infty_x} \| P_{k_2} \varphi \|_{L^\infty_t L^2_x} \lesssim 2^{\frac{1}{2}(j_0 - \ell)} (2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}}) (2^{k_2} \| P_{k_2} \varphi \|_{S_{k_2}}).$$

□

We now proceed to treat each constituent of the S^1 norm.

Case 1.1. S^{str}_k part of S^1. Here we prove

$$\Big(\sum_{k_0} 2^{2k_0} \| P_{k_0} Q_{\le k_0 + 25} (\chi \varphi) \|^2_{S^{\mathrm{str}}_{k_0}} \Big)^{\frac{1}{2}} \lesssim \| \chi \|_{Y^{2,2} \cap L^\infty_{t,x}} \| \varphi \|_{S^1}. \tag{7.12}$$

We split the summand on the left-hand side as follows:

$$\begin{aligned} 2^{k_0} \| P_{k_0} Q_{\le k_0 + 25} (\chi \varphi) \|_{S^{\mathrm{str}}_{k_0}} \lesssim &\sum_{\ell > k_0 - 5} 2^{k_0} \| P_{k_0} Q_{\le k_0 + 25} (S_\ell \chi \varphi) \|_{S^{\mathrm{str}}_{k_0}} \\ &+ 2^{k_0} \| P_{k_0} Q_{\le k_0 + 25} (S_{\le k_0 - 5} \chi \varphi) \|_{S^{\mathrm{str}}_{k_0}}. \end{aligned} \tag{7.13}$$

For the first term on the right-hand side, we use the embedding $P_{k_0}(X_1^{0,\frac{1}{2}}) \subseteq S_{k_0}^{\mathrm{str}}$ and Lemma 7.4 to estimate

$$
\begin{aligned}
&\lesssim \sum_{\ell > k_0 - 5} \sum_{j_0 \le k_0 + 25} 2^{k_0} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_\ell \chi \varphi) \|_{L^2_{t,x}} \\
&\lesssim \sum_{\ell > k_0 - 5} \sum_{j_0 \le k_0 + 25} \sum_{k_2 \le \ell + 10} 2^{\frac{1}{2}(j_0 - k_0)} 2^{\frac{3}{2}(k_0 - \ell)} 2^{\frac{1}{2}(k_2 - \ell)} (2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}}) (2^{k_2} \| P_{k_2} \varphi \|_{S_{k_2}}) \\
&\lesssim \| \varphi \|_{S^1} \sum_{\ell > k_0 - 5} 2^{\frac{3}{2}(k_0 - \ell)} \, 2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}}
\end{aligned}
$$

which is square summable in k_0, thanks to (7.6).

For the second term in (7.13), we can freely replace φ by $P_{[k_0-5,k_0+5]}\varphi$. Then removing $P_{k_0} Q_{\le k_0+25}$, which is disposable, and using Hölder with $S_{\le k_0 - 5}\chi \in L^\infty_{t,x}$, we see that

$$
2^{k_0} \| P_{k_0} Q_{\le k_0 + 25} (S_{\le k_0 - 5} \chi \varphi) \|_{S^{\mathrm{str}}_{k_0}} \lesssim \sum_{k_2 \in [k_0 - 5, k_0 + 5]} \| \chi \|_{L^\infty_{t,x}} 2^{k_2} \| P_{k_2} \varphi \|_{S^{\mathrm{str}}_{k_2}},
$$

which is acceptable.

Case 1.2. $X_\infty^{0,\frac{1}{2}}$ **part of** S^1. We prove

$$
\left(\sum_{k_0} 2^{2k_0} \| P_{k_0} Q_{\le k_0 + 25} (\chi \varphi) \|^2_{X_\infty^{0,\frac{1}{2}}} \right)^{\frac{1}{2}} \lesssim \| \chi \|_{Y^{2,2} \cap L^\infty_{t,x}} \| \varphi \|_{S^1}. \tag{7.14}
$$

The summand on the left-hand side is bounded by

$$
\begin{aligned}
&\sup_{j_0 \le k_0 + 25} 2^{k_0} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (\chi \varphi) \|_{L^2_{t,x}} \\
&\quad \lesssim \sup_{j_0 \le k_0 + 25} 2^{k_0} \sum_{\ell > k_0 - 5} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_\ell \chi \varphi) \|_{L^2_{t,x}} \\
&\qquad + \sup_{j_0 \le k_0 + 25} 2^{k_0} \sum_{\ell \in [j_0 - 30, k_0 - 5]} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_\ell \chi \varphi) \|_{L^2_{t,x}} \\
&\qquad + \sup_{j_0 \le k_0 + 25} 2^{k_0} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_{\le j_0 - 30} \chi \varphi) \|_{L^2_{t,x}}
\end{aligned} \tag{7.15}
$$

Let $j_0 \le k_0 + 25$. Using Lemma 7.4 and proceeding as in Case 1.1, the first term can be bounded by

$$
\begin{aligned}
&2^{k_0} \sum_{\ell > k_0 - 5} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_\ell \chi \varphi) \|_{L^2_{t,x}} \\
&\lesssim \sum_{\ell > k_0 - 5} \sum_{k_2 \leq \ell + 10} 2^{\frac{1}{2}(j_0 - k_0)} 2^{\frac{3}{2}(k_0 - \ell)} 2^{\frac{1}{2}(k_2 - \ell)} (2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}}) (2^{k_2} \| P_{k_2} \varphi \|_{S_{k_2}}) \\
&\lesssim 2^{\frac{1}{2}(j_0 - k_0)} \| \varphi \|_{S^1} \sum_{\ell > k_0 - 5} 2^{\frac{3}{2}(k_0 - \ell)} \, 2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}},
\end{aligned}
$$

which is ℓ^2 summable in k_0 thanks to (7.6).

For the second term in (7.15), we can replace φ by $P_{[k_0-5,k_0-5]}\varphi$. Then we estimate

$$
\begin{aligned}
&2^{k_0} \sum_{\ell \in [j_0 - 30, k_0 - 5]} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_\ell \chi \varphi) \|_{L^2_{t,x}} \\
&\lesssim \sum_{\ell \in [j_0 - 30, k_0 - 5]} \sum_{k_2 \in [k_0 - 5, k_0 + 5]} 2^{\frac{1}{2}(j_0 - \ell)} (2^{2\ell} \| S_\ell \chi \|_{Y^{0,2}}) (2^{k_2} \| P_{k_2} \varphi \|_{S_{k_2}}) \\
&\lesssim \| \chi \|_{Y^{2,2}} \sum_{k_2 \in [k_0 - 5, k_0 + 5]} (2^{k_2} \| P_{k_2} \varphi \|_{S_{k_2}}),
\end{aligned}
$$

which is acceptable.

For the third term in (7.15), we can replace φ by $P_{[k_0-5,k_0-5]} Q_{[j_0-5, j_0+5]}\varphi$. Therefore

$$
\begin{aligned}
&2^{k_0} 2^{\frac{1}{2} j_0} \| P_{k_0} Q_{j_0} (S_{\leq j_0 - 5} \chi \varphi) \|_{L^2_{t,x}} \\
&\lesssim \| \chi \|_{L^\infty_{t,x}} \sum_{k_2 \in [k_0 - 5, k_0 + 5]} \sum_{j_2 \in [j_0 - 5, j_0 + 5]} 2^{k_2} 2^{\frac{1}{2} j_2} \| P_{k_2} Q_{j_2} \varphi \|_{L^2_{t,x}}
\end{aligned}
$$

which is acceptable.

Case 1.3. $S^{\mathrm{ang}}_{k,j}$ **part of** S^1. Here we prove

$$
\Big(\sum_{k_0} 2^{2k_0} \sup_{j_0 < k_0} \| P_{k_0} Q_{<j_0} (\chi \varphi) \|^2_{S^{\mathrm{ang}}_{k_0, j_0}} \Big)^{\frac{1}{2}} \lesssim \| \chi \|_{Y^{2,2} \cap L^\infty_{t,x}} \| \varphi \|_{S^1}. \tag{7.16}
$$

Fix k_0 and $j_0 < k_0$. As before, we split

$$
\begin{aligned}
2^{k_0} \| P_{k_0} Q_{<j_0} (\chi \varphi) \|_{S^{\mathrm{ang}}_{k_0, j_0}} \lesssim & \sum_{\ell > k_0 + 5} 2^{k_0} \| P_{k_0} Q_{<j_0} (S_\ell \chi \varphi) \|_{S^{\mathrm{ang}}_{k_0, j_0}} \\
& + \sum_{\ell \in [j_0 - 30, k_0 - 5]} 2^{k_0} \| P_{k_0} Q_{<j_0} (S_\ell \chi \varphi) \|_{S^{\mathrm{ang}}_{k_0, j_0}} \\
& + 2^{k_0} \| P_{k_0} Q_{<j_0} (S_{<j_0 - 30} \chi \varphi) \|_{S^{\mathrm{ang}}_{k_0, j_0}}
\end{aligned}
$$

Using the embedding $P_{k_0} Q_{<j_0}(X_1^{0,\frac{1}{2}}) \subseteq S^{\mathrm{ang}}_{k_0,j_0}$, the first two terms can be treated by proceeding as in Case 1.2. On the other hand, for the third term, we use Lemma 7.1 to estimate

$$\sup_{j_0} 2^{k_0} \| P_{k_0} Q_{<j_0}(S_{<j_0-20}\chi\varphi)\|_{S^{\mathrm{ang}}_{k_0,j_0}} \lesssim \|\chi\|_{L^\infty_{t,x}} \sum_{k_2 \in [k_0-5,k_0+5]} 2^{k_2} \| P_{k_2}\varphi\|_{S_{k_2}},$$

which is square summable in k_0, proving (7.16).

Step 2. Contribution of $\sum_{k_0} P_{k_0} Q_{>k_0+25}(\chi\varphi)$. When the output is away from the cone, the $\underline{X}$ norm dominates the whole S^1 norm. To see this, let $k_0 \in \mathbb{Z}$. As $P_{k_0}(X_1^{0,\frac{1}{2}}) \subseteq S_{k_0}$, we have

$$\begin{aligned}
\|\partial_{t,x} P_{k_0} Q_{>k_0+25}(\eta\varphi)\|_{S_{k_0}} &\lesssim \sum_{j_0>k_0+25} 2^{\frac{3}{2}j_0} \| P_{k_0} Q_{j_0}(\eta\varphi)\|_{L^2_{t,x}} \\
&\lesssim \sum_{j_0>k_0+25} 2^{\frac{1}{2}(k_0-j_0)} \| P_{k_0} Q_{j_0}(\eta\varphi)\|_{\underline{X}} \\
&\lesssim \| P_{k_0} Q_{>k_0+20}(\eta\varphi)\|_{\underline{X}}.
\end{aligned}$$

Thus by L^2 almost orthogonality,

$$\Big\| \sum_{k_0} P_{k_0} Q_{>k_0+25}(\eta\varphi) \Big\|^2_{S^1} \lesssim \sum_{k_0} \| P_{k_0} Q_{>k_0+20}(\eta\varphi)\|^2_{\underline{X}}. \tag{7.17}$$

To conclude the proof of (7.7), it remains to estimate the right-hand side of (7.17). This is the content of Lemma 7.5 below. □

Lemma 7.5 *The following estimate holds.*

$$\Big(\sum_{k_0} \| P_{k_0} Q_{>k_0+20}(\chi\varphi)\|^2_{\underline{X}} \Big)^{\frac{1}{2}} \lesssim \|\chi\|_{\mathfrak{Y}} \|\varphi\|_{S^1}. \tag{7.18}$$

Proof Since the spaces have different regularity in space and time, we will need to divide into cases depending on both the space and time frequency configurations. We begin with the standard Littlewood–Paley trichotomy in the spatial Fourier variable:

$$\begin{aligned}
P_{k_0} Q_{>k_0+20}(\chi\varphi) &= P_{k_0} Q_{>k_0+20}(\chi_{<k_0+10}\varphi_{[k_0-5,k_0+20]}) \\
&\quad + P_{k_0} Q_{>k_0+20}(\chi_{[k_0-5,k_0+5]}\varphi_{<k_0-5}) \\
&\quad + \sum_{k_1 \geq k_0+10} \sum_{k_2 \in [k_1-5,k_1+5]} P_{k_0} Q_{>k_0+20}(\chi_{k_1}\varphi_{k_2}).
\end{aligned}$$

In each case we will further divide into cases, which will essentially correspond to doing another round of Littlewood–Paley trichotomy in the temporal Fourier variable.

Case 1. (LH) **interaction**. Here we treat the contribution of

$$P_{k_0} Q_{j_0} (\chi_{\leq k_0+10} \varphi_{[k_0-5,k_0+20]})$$

We divide further into two sub-cases, depending on the temporal frequency of $\chi_{\leq k_0+10}$.

Case 1.1 χ **has high temporal frequency,** $j_1 > j_0 - 20$. Recalling that $\underline{X}$ is an $L^2_{t,x}$ based norm, by orthogonality it suffices to estimate

$$\Big\| \sum_{k_2 \in [k_0-5,k_0+20]} \| P_{k_0} Q_{j_0} (T_{>j_0-20} \chi_{\leq k_0+10} \varphi_{k_2}) \|_{\underline{X}} \Big\|_{\ell^2_{k_0,j_0}(j_0>k_0+20)} \tag{7.19}$$

We estimate each summand as follows:

$$\begin{aligned}
&\| P_{k_0} Q_{j_0} (T_{>j_0-20} \chi_{\leq k_0+10} \, \varphi_{k_2}) \|_{\underline{X}} \\
&\quad \lesssim \sum_{k_1 \leq k_0+10} \sum_{j_1 > j_0-20} 2^{2j_0} 2^{-\frac{1}{2}k_0} \| T_{j_1} \chi_{k_1} \|_{L^2_t L^\infty_x} \| \varphi_{k_2} \|_{L^\infty_t L^2_x} \\
&\quad \lesssim \sum_{k_1 \leq k_0+10} \sum_{j_1 > j_0-20} 2^{2(j_0-j_1)} 2^{\frac{3}{2}(k_1-k_0)} (2^{2j_1} \| T_{j_1} \chi_{k_1} \|_{Y^{0,2}}) (2^{k_2} \| \varphi_{k_2} \|_{S_{k_2}})
\end{aligned}$$

We now sum up $k_2 \in [k_0-5, k_0+20]$ and take the $\ell^2_{k_0,j_0}(j_0 > k_0+20)$ summation. Then (7.19) is estimated by

$$\lesssim \| \varphi \|_{S^1} \Big\| \sum_{k_1 \leq k_0+10} \sum_{j_1 \geq j_0-20} 2^{2(j_0-j_1)} 2^{\frac{3}{2}(k_1-k_0)} (2^{2j_1} \| T_{j_1} \chi_{k_1} \|_{Y^{0,2}}) \Big\|_{\ell^2_{k_0,j_0}(j_0>k_0+20)}$$

which in turn is bounded by $\lesssim \| \varphi \|_{S^1} \| \chi \|_{Y^{2,2}}$ thanks to (7.6).

Case 1.2 χ **has low temporal frequency,** $j_1 \leq j_0 - 20$. It suffices to bound

$$\Big\| \sum_{k_2 \in [k_0-5,k_0+20]} \| P_{k_0} Q_{j_0} (T_{\leq j_0-20} \chi_{\leq k_0+10} \varphi_{k_2}) \|_{\underline{X}} \Big\|_{\ell^2_{k_0,j_0}(j_0>k_0+20)} \tag{7.20}$$

By the restrictions on the Fourier supports of inputs and the output, we can freely replace φ_{k_2} by $\sum_{j_2 \in [j_0-C, j_0+C]} Q_{j_2} \varphi_{k_2}$. Thus throwing away $P_{k_0} Q_{j_0}$, estimating $T_{\leq j_0-20} \chi_{\leq k_0+10}$ in $L^\infty_{t,x}$ and $Q_{j_2} \varphi_{k_2}$ in $L^2_{t,x}$, we can estimate the summand in (7.20) by

$$\| P_{k_0} Q_{j_0} (T_{\leq j_0-20} \chi_{\leq k_0+10} \varphi_{k_2}) \|_{\underline{X}} \lesssim \sum_{j_2 \in [j_0-C, j_0+C]} \| \chi \|_{L^\infty_{t,x}} \| Q_{j_2} \varphi_{k_2} \|_{\underline{X}}$$

Summing it up, we obtain (7.20) $\lesssim \| \chi \|_{L^\infty_{t,x}} \| \varphi \|_{\underline{X}}$ as desired.

Case 2. (HL) **interaction.** Here we treat the contribution of

$$P_{k_0} Q_{j_0} (\chi_{[k_0-5,k_0+5]} \varphi_{<k_0-5})$$

As in the previous case, we divide into two sub-cases.

Case 2.1 χ **has high temporal frequency,** $j_1 > j_0 - 20$. As in the previous case, we need to consider

$$\Big\| \sum_{k_1 \in [k_0-5, k_0+5]} \sum_{k_2 < k_0 - 5} \| P_{k_0} Q_{j_0} (T_{> j_0 - 20} \chi_{k_1} \varphi_{k_2}) \|_{\underline{X}} \Big\|_{\ell^2_{k_0, j_0}(j_0 > k_0 + 20)} \tag{7.21}$$

Disposing $P_{k_0} Q_{j_0}$ and using Hölder, we estimate each summand as

$$\begin{aligned} \| P_{k_0} Q_{j_0} (T_{> j_0-20} \chi_{k_1} \varphi_{k_2}) \|_{\underline{X}} &\lesssim \sum_{j_1 > j_0 - 20} 2^{2 j_0} 2^{-\frac{1}{2} k_0} \| T_{j_1} \chi_{k_1} \|_{L^2_{t,x}} \| \varphi_{k_2} \|_{L^\infty_{t,x}} \\ &\lesssim \sum_{j_1 > j_0 - 20} 2^{2(j_0 - j_1)} 2^{k_2 - k_0} (2^{2 j_1} \| T_{j_1} \chi_{k_1} \|_{Y^{0,2}}) (2^{k_2} \| \varphi_{k_2} \|_{S_{k_2}}) \end{aligned}$$

Thanks to the high-low gain $2^{k_2 - k_0}$, this can be summed up in $\ell^2_{k_0, j_0}(j_0 > k_0 + 20)$ using (7.6). We conclude (7.21) $\lesssim \| \chi \|_{Y^{2,2}} \| \varphi \|_{S^1}$, as desired.

Case 2.2 χ **has low temporal frequency,** $j_1 \leq j_0 - 20$. We consider

$$\Big\| \sum_{k_1 \in [k_0-5, k_0+5]} \sum_{k_2 < k_0 - 5} \| P_{k_0} Q_{j_0} (T_{\leq j_0 - 20} \chi_{k_1} \varphi_{k_2}) \|_{\underline{X}} \Big\|_{\ell^2_{k_0, j_0}(j_0 > k_0 + 20)} \tag{7.22}$$

In this case, we can replace φ_{k_2} by $\sum_{j_2 \in [j_0 - C, j_0 + C]} Q_{j_2} \varphi_{k_2}$, thanks to the restrictions on the Fourier supports. Then as before, we estimate

$$\begin{aligned} &\| P_{k_0} Q_{j_0} (T_{\leq j_0 - 20} \chi_{k_1} \varphi_{k_2}) \|_{\underline{X}} \\ &\lesssim \sum_{j_2 \in [j_0 - C, j_0 + C]} 2^{2 j_0} 2^{-\frac{1}{2} k_0} \| T_{\leq j_0 - 20} \chi_{k_1} \|_{L^\infty_t L^2_x} \| Q_{j_2} \varphi_{k_2} \|_{L^2_t L^\infty_x} \\ &\lesssim \sum_{j_2 \in [j_0 - C, j_0 + C]} 2^{\frac{5}{2}(k_2 - k_0)} (2^{2 k_1} \| \chi_{k_1} \|_{L^\infty_t L^2_x}) \| Q_{j_2} \varphi_{k_2} \|_{\underline{X}} \end{aligned}$$

Thanks again to the high-low gain $2^{\frac{5}{2}(k_2 - k_0)}$ this is again summable, and we obtain (7.22) $\lesssim \| \chi \|_{L^\infty_t \dot{H}^2_x} \| \varphi \|_{\underline{X}}$.

Case 3. (HH) **interaction.** Here we treat the contribution of

$$P_{k_0} Q_{j_0} (\chi_{k_1} \varphi_{k_2})$$

where $|k_1 - k_2| \leq 5$, $k_1 \geq k_0 + 10$.

Case 3.1 χ **has high spatial frequency,** $k_1 > j_0 - 20$. We first consider

$$\Big\| \sum_{k_1 > j_0 - 20} \sum_{k_2 \in [k_1 - 5, k_1 + 5]} \| P_{k_0} Q_{j_0} (\chi_{k_1} \varphi_{k_2}) \|_{\underline{X}} \Big\|_{\ell^2_{k_0, j_0}(j_0 > k_0 + 20)} \tag{7.23}$$

Throwing away Q_{j_0}, applying Bernstein in space and using Hölder, we estimate each summand by

$$\begin{aligned}\|P_{k_0} Q_{j_0}(\chi_{k_1}\varphi_{k_2})\|_{\underline{X}} &\lesssim 2^{2j_0} 2^{\frac{3}{2}k_0} \|\chi_{k_1}\varphi_{k_2}\|_{L^2_t L^1_x} \\ &\lesssim 2^{2(j_0-k_1)} 2^{\frac{3}{2}(k_0-k_1)} (2^{2k_1}\|\chi_{k_1}\|_{Y^{0,2}})(2^{k_2}\|\varphi_{k_2}\|_{S_{k_2}})\end{aligned}$$

Using (7.6) and the square summability of $2^{k_2}\|\varphi_{k_2}\|_{S_{k_2}}$, the last expression can be summed up in the ℓ^1 sense over $\{(k_0, j_0, k_1, k_2) : j_0 > k_0 + 20, k_1 > j_0 - 20, |k_1 - k_2| \leq 5\}$ and be estimated by $\lesssim \|\chi\|_{Y^{2,2}}\|\varphi\|_{S^1}$.

Case 3.2 χ **has high temporal frequency,** $k_1 \leq j_0 - 20$, $j_1 > j_0 - 20$. Next, we estimate

$$\Big\| \sum_{k_1 \in [k_0-5, j_0-20]} \sum_{k_2 \in [k_1-5, k_1+5]} \|P_{k_0} Q_{j_0}(T_{>j_0-20}\chi_{k_1}\varphi_{k_2})\|_{\underline{X}} \Big\|_{\ell^2_{k_0,j_0}(j_0>k_0+20)} \tag{7.24}$$

Throwing away Q_{j_0}, applying Bernstein in space and using Hölder, we have

$$\begin{aligned}&\|P_{k_0} Q_{j_0}(T_{>j_0-20}\chi_{k_1}\varphi_{k_2})\|_{\underline{X}} \\ &\quad\lesssim \sum_{j_1>j_0-20} 2^{2j_0} 2^{\frac{3}{2}k_0} \|T_{j_1}\chi_{k_1}\varphi_{k_2}\|_{L^2_t L^1_x} \\ &\quad\lesssim \sum_{j_1>j_0-20} 2^{2(j_0-j_1)} 2^{\frac{3}{2}(k_0-k_1)} (2^{2j_1}\|T_{j_1}\chi_{k_1}\|_{Y^{0,2}})(2^{k_2}\|\varphi_{k_2}\|_{S_{k_2}})\end{aligned}$$

Using the triangle inequality to pull out k_1, k_2 summations out of $\ell^2_{k_0,j_0}(j_0 > k_0 + 20)$ and performing the latter summation, we estimate (7.24) by

$$\lesssim \sum_{k_1} \sum_{k_2 \in [k_1-5, k_1+5]} \Big(\sum_{j_1 > k_1 - 20} 2^{4j_1} \|T_{j_1}\chi_{k_1}\|^2_{Y^{0,2}} \Big)^{\frac{1}{2}} (2^{k_2}\|\varphi_{k_2}\|_{S_{k_2}}),$$

which is estimated by $\lesssim \|\chi\|_{Y^{2,2}}\|\varphi\|_{S^1}$ using (7.6) and the square summability of $2^{k_2}\|\varphi_{k_2}\|_{S_{k_2}}$.

Case 3.3 χ **is close to frequency origin,** $k_1 \leq j_0 - 20$, $j_1 \leq j_0 - 20$. In this case, we estimate

$$\Big\| \sum_{k_1 \in [k_0-5, j_0-20]} \sum_{k_2 \in [k_1-5, k_1+5]} \|P_{k_0} Q_{j_0}(T_{\leq j_0-20}\chi_{k_1}\varphi_{k_2})\|_{\underline{X}} \Big\|_{\ell^2_{k_0,j_0}(j_0>k_0+20)} \tag{7.25}$$

As before, the restrictions on the Fourier supports allow us to replace φ_{k_2} by the expression $\sum_{j_2 \in [j_0-C, j_0+C]} Q_{j_2}\varphi_{k_2}$. Throwing away Q_{j_0}, applying Bernstein and using Hölder (and furthermore the fact that $T_{\leq j_0-20}$ is bounded in $L^\infty_t L^2_x$), the summand in (7.25) is estimated by

$$\| P_{k_0} Q_{j_0} (T_{\leq j_0 - 20} \chi_{k_1} \varphi_{k_2}) \|_{\underline{X}} \lesssim \sum_{j_2 \in [j_0 - C, j_0 + C]} 2^{\frac{3}{2}(k_0 - k_1)} (2^{2k_1} \| \chi_{k_1} \|_{L^\infty_t L^2_x}) \| Q_{j_2} \varphi_{k_2} \|_{\underline{X}}.$$

The last expression can be summed up using (7.6) and $\ell^2_{k_2, j_2}$ summability of $\| Q_{j_2} \varphi_{k_2} \|_{\underline{X}}$, leading to (7.25) $\lesssim \| \chi \|_{L^\infty_t \dot{H}^2_x} \| \varphi \|_{\underline{X}}$ as desired.

7.3 Cutoff Estimates

In this subsection, we prove Lemmas 6.11 and 6.14.

We begin with a brief discussion on $\dot{B}^{\frac{5}{2}, 2}_1$, which basically plays the role of the space of smooth cutoffs. Recall that $\dot{B}^{\frac{5}{2}, 2}_1$ is an atomic space, whose atoms satisfy $\eta = S_{[\ell - 1, \ell + 1]} \eta$ and $2^{\frac{5}{2}\ell} \| \eta \|_{L^2_{t,x}} \leq 1$ for some $\ell \in \mathbb{Z}$. Note that the following ℓ^1 summability estimate holds:

$$\sum_{\ell} 2^{\frac{5}{2}\ell} \| S_\ell \eta \|_{L^2_{t,x}} + \sum_{\ell} 2^{2\ell} \| S_\ell \eta \|_{L^\infty_t L^2_x} + \| \eta \|_{L^\infty_{t,x}} \lesssim \| \eta \|_{\dot{B}^{\frac{5}{2}, 2}_1} \tag{7.26}$$

Note furthermore that

$$\dot{B}^{\frac{5}{2}, 2}_1 \subseteq \dot{H}^{\frac{5}{2}}_{t,x} \cap \ell^1 C^0_t \dot{H}^2_x \subseteq \widehat{\mathcal{Y}} \tag{7.27}$$

which follows easily from Bernstein's inequality.

We first establish Lemma 6.11. By the definition of restriction spaces, it suffices to prove the following global statement.

Lemma 7.6 *The following estimate holds for $X = Y^1, S^1, \widehat{\mathcal{Y}}$ or $\mathcal{Y}$.*

$$\| \eta \varphi \|_{X(\mathbb{R}^{1+4})} \lesssim \| \eta \|_{\dot{B}^{\frac{5}{2}, 2}_1 (\mathbb{R}^{1+4})} \| \varphi \|_{X(\mathbb{R}^{1+4})}. \tag{7.28}$$

Proof Before we begin, note that the following cutoff estimates hold:

$$\| \eta \varphi \|_{Y^{1,2}} \lesssim \| \eta \|_{\dot{B}^{\frac{5}{2}, 2}_1} \| \varphi \|_{Y^{1,2}}, \tag{7.29}$$

$$\| \eta \varphi \|_{\ell^1 Y^{2,2}} \lesssim \| \eta \|_{\dot{B}^{\frac{5}{2}, 2}_1} \| \varphi \|_{\ell^1 Y^{2,2}}. \tag{7.30}$$

Indeed, both estimates can be proved in a similar manner as (7.2); we omit the details. With (7.29) and (7.30) in our hand, we proceed to the proof of (7.28).

Case 1: $X = Y^1$. Recall that $Y^1 = Y^{1,2} \cap Y^{1,\infty}$. The desired estimate for the $Y^{1,2}$ norm of $\eta \varphi$ follows from (7.29); thus it remains to bound $\| \eta \varphi \|_{Y^{1,\infty}}$. By the Leibniz rule, Hölder, $\dot{H}^1_x \subseteq L^4_x$ Sobolev and (7.26), we have

$$\begin{aligned}\|\partial_{t,x}(\eta\varphi)\|_{Y^{0,\infty}} &\lesssim \|\eta\partial_{t,x}\varphi\|_{L^\infty_t L^2_x} + \|\partial_{t,x}\eta\varphi\|_{L^\infty_t L^2_x} \\ &\lesssim (\|\partial_{t,x}\eta\|_{L^\infty_t L^4_x} + \|\eta\|_{L^\infty_{t,x}})(\|\partial_{t,x}\varphi\|_{L^\infty_t L^2_x} + \|\varphi\|_{L^\infty_t L^4_x}) \\ &\lesssim \|\eta\|_{B^{\frac{5}{2},2}_1}\|\varphi\|_{Y^1},\end{aligned}$$

which completes the proof in this case.

Cases 2 & 3: $X = S^1$ **or** $\widehat{\mathcal{Y}}$. These cases are immediate consequences of (7.2), (7.7) and the embedding (7.27).

Case 4: $X = \mathcal{Y}$. Recall that $\mathcal{Y} = \ell^1 Y^{2,2} \cap \ell^1 Y^{2,\infty}$. For the $\ell^1 Y^{2,2}$ norm of $\eta\varphi$, we use (7.30). In order to bound the $\ell^1 Y^{2,\infty}$ norm of $\eta\varphi$, we first use the Leibniz rule to compute

$$\partial_t(\eta\varphi) = \partial_t\eta\varphi + \eta\partial_t\varphi, \quad \partial_t^2(\eta\varphi) = \partial_t^2\eta\varphi + 2\partial_t\eta\partial_t\varphi + \eta\partial_t^2\varphi.$$

By the embedding $\dot{B}^{N+\frac{1}{2},2}_1 \subseteq \ell^1 C^0 \dot{H}^N_x$ and the definition of the space $\ell^1 Y^{2,\infty}$, we have $\partial_t^{(N)}\eta, \partial_t^{(N)}\varphi \in \ell^1 C^0_t \dot{H}^{2-N}_x$ for $N = 0, 1, 2$. Thus the desired estimate is easily obtained using the standard Littlewood–Paley trichotomy; we leave the details to the reader. □

Finally, we give a proof of Lemma 6.14. Extending η and φ to the whole space in such a way that $\eta \in \dot{B}^{\frac{5}{2},2}_1(\mathbb{R}\times\mathbb{R}^4)$ and $\varphi \in \widehat{\mathcal{Y}}(\mathbb{R}\times\mathbb{R}^4)$, it suffices to consider the case $I = \mathbb{R}$. Thus Lemma 6.14 would follow once we establish the following statement.

Lemma 7.7 *Let $\eta \in \dot{B}^{\frac{5}{2},2}_1(\mathbb{R}^{1+4})$ and $\varphi \in \widehat{\mathcal{Y}}(\mathbb{R}^{1+4})$. Let $\chi := (-\Delta)^{-1}(\eta\Delta\varphi)(t)$ be given as convolution with the Newton potential. Then we have*

$$\|\chi\|_{\widehat{\mathcal{Y}}} \lesssim \|\eta\|_{\dot{B}^{\frac{5}{2},2}_1}\|\varphi\|_{\widehat{\mathcal{Y}}}. \tag{7.31}$$

Proof From the embedding (7.27), it easily follows that $\eta\Delta\varphi \in \ell^1 C^0_t L^2_x(\mathbb{R}\times\mathbb{R}^4)$ with

$$\|\eta\Delta\varphi\|_{\ell^1 L^\infty_t L^2_x} \lesssim \|\eta\|_{\ell^1 L^\infty_t \dot{H}^2_x}\|\Delta\varphi\|_{\ell^1 L^\infty_t \dot{H}^2_x} \lesssim \|\eta\|_{\dot{B}^{\frac{5}{2},2}_1}\|\varphi\|_{\widehat{\mathcal{Y}}}.$$

Therefore, the estimate for $\|\chi\|_{\ell^1 L^\infty_t \dot{H}^2_x}$ in the $\widehat{\mathcal{Y}}$ norm in (7.31) follows. It remains to establish the estimate for the $Y^{2,2}$ norm in (7.31); for this we will show that

$$\|\chi\|_{Y^{2,2}} \lesssim \|\eta\|_{\dot{B}^{\frac{5}{2},2}_1}\|\varphi\|_{Y^{2,2}} \tag{7.32}$$

The left-hand side is equivalent to $\|\partial^2_{x,t}(\eta\Delta\varphi)\|_{L^2_t \dot{H}^{-\frac{3}{2}}_x}$. We apply the Leibniz rule to write

$$\partial^2_{x,t}(\eta\Delta\varphi) = \partial^2_{x,t}\eta\Delta\varphi + \partial_{x,t}\eta\partial_{x,t}\Delta\varphi + \eta\partial^2_{x,t}\Delta\varphi$$

We can estimate

$$\|\Delta\varphi\|_{L_t^2 \dot{H}_x^{\frac{1}{2}}} + \|\partial_{x,t}\Delta\varphi\|_{L_t^2 \dot{H}_x^{-\frac{1}{2}}} + \|\partial_{x,t}^2\Delta\varphi\|_{L_t^2 \dot{H}_x^{-\frac{3}{2}}} \lesssim \|\varphi\|_{Y^{2,2}}$$

and, by the trace theorem,

$$\|\partial_{x,t}^2\eta\|_{L_t^\infty L_x^2} + \|\partial_{x,t}\eta\|_{L_t^\infty \dot{H}_x^1} + \|\eta\|_{L_t^\infty \ell^1 \dot{H}_x^2} \lesssim \|\eta\|_{\dot{B}_1^{\frac{5}{2},2}}$$

Hence it remains to establish the fixed time multiplicative estimates

$$\dot{H}_x^{\frac{1}{2}} \times L_x^2 \to \dot{H}_x^{-\frac{3}{2}}, \qquad \dot{H}_x^{-\frac{1}{2}} \times \dot{H}_x^1 \to \dot{H}_x^{-\frac{3}{2}}, \qquad \dot{H}_x^{-\frac{3}{2}} \times \ell^1 \dot{H}_x^2 \to \dot{H}_x^{-\frac{3}{2}}$$

These in turn are easily obtained using the standard Littlewood–Paley trichotomy. □

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