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Towards an Efficient Approach for Mamdani Interval Type-3 Fuzzy Inference Systems

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Abstract This paper is part of the increasing interest regarding the application of interval type-3 fuzzy logic in real-world problems, where a better handling of uncertainty can be useful in achieving enhanced results. The main contribution of this paper is the proposal of new methods, such as Interval Type-3 Reduction and a practical way for modeling Interval Type-3 Membership Functions, based on the Footprint of Uncertainty (FOU) and Core of Uncertainty (COU) concepts, which reduce the gap between the theory and the practical implementation of Mamdani Interval Type-3 Fuzzy Systems. The main aim of the paper is not proving the superiority of Interval Type-3 Fuzzy Systems but providing a framework and a comprehensive illustration of the theory concepts to help future research work in developing optimization methodologies and new applications for this kind of systems, as well as finding their potential applicability, which can result from their ability in handling more complex uncertainty. Simulation results with two illustrative application examples show the potential of the presented approach in achieving an efficient implementation of Interval type-3 fuzzy systems.

Keywords Interval type-3 fuzzy logic · Type-2 fuzzy systems · Type reduction

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1 Introduction

Recently, there has been an increasing interest in Type-3 Fuzzy Inference Systems (T3 FISs), which are systems that can handle problems with higher complexity, due to multiple uncertainty sources, when compared with respect to Type-2 Fuzzy Inference Systems (T2 FISs). The main type-3 concepts are like type-2, but the utilization of these systems in real problems faces significant challenges. This is because these systems demand a higher computational effort and may not be suitable for many programming environments in achieving real-time execution, for example in control applications which is one of the most successful application areas of fuzzy logic.

Many authors have obtained interesting results and advantages with the implementation of T3 FISs. For example, in [1] the authors successfully implemented type-3 fuzzy controllers for non-linear plants, in [2] the authors propose an approach for forecasting the COVID-19 based on the principles of interval type-3 fuzzy sets and a mathematical tool known as the fractal dimension, in [3] the authors propose an computer-aided system for material surface quality control based on type-3 fuzzy logic (T3 FL), in [4] the authors propose a practical way to compute type-3 fuzzy systems in Simulink for robotics, in [5] the authors improve the performance of a metaheuristic algorithm aided with an expert system based on interval type-3 fuzzy logic, in [6] the authors propose a neuro-fuzzy controller which demonstrates robustness for non-linear plants, and finally in [7] the authors propose an approach of non-singleton system for fault detection in a flowmeter system applied in gas industry.

However, there is a need to enhance the computational efficiency and reduce the complexity of this kind of systems, which limit their implementation in a wider range of applications and in this way, we can truly explore the potential of T3 FISs.

Therefore, the focus of this paper is the contribution of a framework for the investigation and future applications of T3 FISs. The focus of this paper is on Mamdani Interval Type-3 Fuzzy Inference Systems (IT3 FISs), through the presentation of a novel approach for IT3 FISs, explaining the core concepts and illustrating with examples the suitability of the systems for problem solving. Some of the explained concepts are related to the IT3 FIS architecture, such as the mathematical expression of different kinds of IT3 Membership Functions and a way to propose new functions, the relation between GT2 FISs and IT3 FISs, and a proposed approach for Type-3 Reduction, which is a novelty for this kind of systems.

The paper starts by recalling the concepts about Type-2 Fuzzy Sets and T2 FISs in Sect. 2 and based on the recent advances in Type-3 Fuzzy Sets expands the concepts of T2 FISs to an understandable IT3 FIS approach (presented in Sect. 3). We are presenting in Sect. 4, for example, diverse kinds of Interval Type-3 Membership Functions (IT3 MFs), the structure of IT3 FISs including Interval Type-3 Reduction, and two examples of engineering applications. Finally, the conclusions are presented in Sect. 5.

2 Fuzzy Logic Background

Type-1 Fuzzy Logic was proposed as an alternative for achieving computing with words [8], offering the capability to create systems that contain the empirical knowledge of an expert and use them to compute decisions in the real world. However, this kind of systems are very versatile and can also be part of machine learning strategies, the most common are the neuro-fuzzy systems, for example the Adaptive Neuro-Fuzzy Inference System (ANFIS), which has been applied in different areas of engineering and science as [9–13].

On the other hand, Type-2 Fuzzy Logic emerged as an evolution of Type-1 Fuzzy Logic by handling not only vagueness but also the uncertainty in real problems [14]. This interesting concept of uncertainty allows to increase the robustness of Fuzzy Systems and to improve their performance for different applications. Some of these applications of Type-2 Fuzzy Logic are the following: in [15] the authors improve two metaheuristics by adapting their parameters based on expert knowledge implemented by Type-2 Fuzzy Systems, in [16] the authors implement an Interval Type-2 Fuzzy System in charge of controlling a hybrid autonomous vehicle, in [17] the authors proposed the application of controllers, based on Type-2 Fuzzy Logic, for robotics applied on agriculture, in [18] is presented the approach of fuzzy systems for cruise control applied on intelligent vehicles, in [19] the authors propose an interesting approach for time series prediction and chaotic synchronization called wavelet interval type-2 Takagi–Sugeno–Kang which is a derivation of neuro-fuzzy systems, in [20] the authors apply type-2 neuro-fuzzy systems in power management as a controllers for power-electronic systems, in [21] the authors apply interval type-2 fuzzy logic for fault diagnosis of gas turbines, in [22] the authors propose to perform the tracking control of hypersonic vehicle based on systems which contain type-2 fuzzy logic, and in [23] the authors propose a fuzzy logic model to evaluate the risk of earthquake hazard based on a rapid visual inspection.

In the following sections, the core concepts that are the basis of the proposals of this paper are presented, aiming to reduce the gap between the theoretical IT3 FISs and the real-world applications.

2.1 Footprint of Uncertainty and Core of Uncertainty

The Footprint of Uncertainty (FOU) and the Core of Uncertainty (COU) [24] are concepts directly related to Type-2 Fuzzy sets, but in the present paper are used to represent Interval Type-3 Membership Functions as are explained in following sections. The expressions of these concepts are illustrated in Eqs. (1) and (2), as follows:

$$\mathrm{FOU}(A^{ii}) = \{(x,u) \in \{X \times [0,1]\} | \mu_{A^{ii}}(x,u) > 0\} \quad (1)$$

$$\mathrm{COU}(A^{ii}) = \{(x,u) \in \{X \times [0,1]\} | \mu_{A^{ii}}(x,u) = 1\}. \quad (2)$$

Beyond the theoretical expression, the FOU and the COU are collections of points, the FOU is the collection of the bases of every embedded secondary membership function in Generalized Type-2 Membership functions, and the COU is the collection of the cores of the same functions. In the α-plane approximation of GT2 Membership Functions, both collections (of the FOU and COU) can be viewed as α-planes, with α = 0 and α = 1, respectively.

2.2 Mamdani Type-*n* Fuzzy Inference Systems

Probably one of the most applied kinds of Fuzzy Logic Systems are the Mamdani Fuzzy Inference Systems (FISs) [25], and this also applies for the different kinds of Fuzzy

Logic, including Type-2 Fuzzy Logic. Figure 1 illustrates the architecture of these systems.

This section presents an overview of the theoretical perspective applicable to the family of fuzzy set (type-n fuzzy sets).

The core of this kind of systems is the inference and the modus ponens logical structure is used; starting from this concept the mathematical perspective of Mamdani Fuzzy Inference Systems can be explained, and the general expression of this inference structure is expressed in (3):

$$\begin{array}{l} \mathbb{R}^1: \ IF\ x_1\ is\ \mathbb{F}_1^1\ \text{and}\ldots x_i\ is\ \mathbb{F}_i^1\ \text{and}\ldots x_n\ is\ \mathbb{F}_n^1 \\ \quad THEN\ y_1\ is\ \mathbb{G}_1^1\ldots y_j\ is\ \mathbb{G}_j^1\ldots, y_m\ is\ \mathbb{G}_m^1 \\ \qquad \ldots \\ \mathbb{R}^k: IFx_1\ is\ \mathbb{F}_1^k\ \text{and}\ \ldots x_i\ is\ \mathbb{F}_i^k\ \text{and}\ldots x_n\ is\ \mathbb{F}_n^k \\ \quad THEN\ y_1\ is\ \mathbb{G}_1^k\ldots y_j\ is\ \mathbb{G}_j^k\ldots, y_m\ is\ \mathbb{G}_m^k \\ \qquad \ldots \\ \mathbb{R}^r: \ IF\ x_1\ is\ \mathbb{F}_1^r\ \text{and}\ldots x_i\ is\ \mathbb{F}_i^r\ \text{and}\ldots x_n\ is\ \mathbb{F}_n^r \\ \quad THENy_1\ is\ \mathbb{G}_1^r\ldots y_j\ is\ \mathbb{G}_j^r\ldots, y_m\ is\ \mathbb{G}_m^r \end{array}, \tag{3}$$

where ($i = 1\ldots n$) is the number of inputs, ($j = 1\ldots m$) is the number of outputs, and ($k = 1\ldots r$) is the number of rules. Based on this, $\mathbb{F}_i^k$ is the type-n Fuzzy Set related to the kth rule and ith input, and $\mathbb{G}_j^k$ is the type-n Fuzzy Set related to the kth rule consequent and the jth output. According to [26], the rule's expression can be simplified as is expressed in (4):

$$\begin{array}{l} \mathbb{A}^k = \mathbb{F}_1^1 \times \ldots \mathbb{F}_i^1 \times \ldots \mathbb{F}_n^1 \\ \quad \mathbb{R}^k: \mathbb{A}^k \to \mathbb{G}_m^r \end{array}. \tag{4}$$

However, in practice, it is necessary to work with the membership functions of the Fuzzy Sets, as can be seen in Fig. 1; the first step is the fuzzification, and the resulting of this computation is $\mu_{\mathbb{F}_n}(x_n)$ which is the type-n membership degree of the nth fuzzy set.

This concept can be directly related to the inference observing the expression to estimate the membership degree of the system rules; this expression can be seen in (5) as follows:

$$\mu_{\mathbb{R}_j^k}(\overline{x}) = \overbrace{\left[\prod_{i=0}^{n} \mu_{\mathbb{F}_n^k}(x_n)\right]}^{\phi^k(\overline{x})} \prod \mu_{\mathbb{G}_j^k}(y_j), \tag{5}$$

where $\phi^k(\overline{x})$ is known as the firing strength of the rule and $\mu_{\mathbb{R}^k}(\overline{x}, y_j)$ is also known as the knowledge of the kth rule for the jth output. This rule knowledge is a type-n membership function, and according to modus ponens, the knowledge of the system can be aggregated with s-norm operators; this expression can be found in (6) as follows:

$$\mu_{\mathbb{B}_j}(\overline{x}, y_j) = \coprod_{k=0}^{r} \mu_{\mathbb{R}_j^k}(\overline{x}, y_j), \tag{6}$$

where $\mu_{\mathbb{B}_j}(\overline{x}, y_j)$ is the aggregated knowledge of the system and is a type-n membership function.

So, from (4) to (6) the knowledge base and inference are covered , and finally for the defuzzifier stage, considering a type-n membership function, it is necessary to perform a type reduction; this reduction variate depends on the type of the system but can be expressed as a family of functions which transform a type-n membership function into crisp outputs (7).

$$\widehat{y}_j = typeReduction(y_j, \mu_{\mathbb{B}_j}(\overline{x}, y_j)). \tag{7}$$

2.3 Advances in Type-3 Fuzzy Logic

On the other hand, the recent proposal of fuzzy logic is T3 FL that supplies a better model of uncertainty. Some key concepts which allow the development of this approach were introduced in [27], specifically the concept of type-n, and in recent years gained notoriety as can be noted, for example, in [6], where the authors apply successfully the IT3 Fuzzy Logic in adaptive controllers with high complexity. In addition, in [28] the authors propose a new methodology for design IT3 Non-Singleton Fuzzy Systems and demonstrate the capability of their model in a complex application, and in [29] the authors prove the effectivity of IT3 Fuzzy Logic Controllers in power management in a solar energy system.

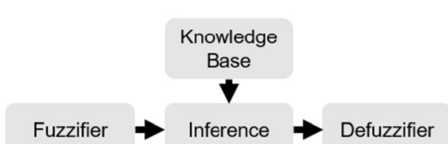


Fig. 1 Mamdani Fuzzy Inference System architecture

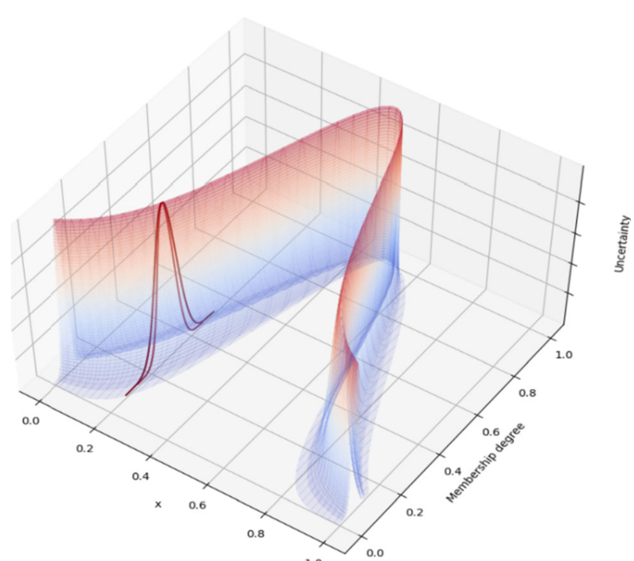


Fig. 2 Type-3 Membership Function

Many details of T3 FL theory are explained in a deeper fashion in [26], where the authors summarize the most recent advances.

The definition of Type-3 Fuzzy Set is expressed in (8) and an example can be seen in Fig. 2.

$$A^{(3)} = \left\{ \left((x,u), \mu_{A^{(3)}}(x,u,v) \right) \middle| x \in X, u \in U \subseteq [0,1], v \in V \subseteq [0,1] \right\}. \tag{8}$$

Another way to explain of Type-3 Fuzzy Sets (T3 FSs) is by seeing the evolution of Fuzzy Numbers, remembering that these Fuzzy Numbers can also be represented by membership degree and are the result of evaluating a membership function (MF). So, the fuzzy numbers are the core of the inference and finally the computation of the FISs. Figure 3 summarizes the evolution of Fuzzy Numbers according to their uncertainty modeling.

As can be seen, the membership degree of a Type-1 MF is a singleton value; however, for an IT2 MF the membership degree is an interval, for GT2 MFs the membership degree is Type-1 MFs, and finally for IT3 MFs the membership degree is an IT2 MF.

This implies some considerations for the construction of a Fuzzy Inference System architecture based on IT3 Fuzzy Logic. In the next section, a method for implementing IT3 Fuzzy Inference Systems (IT3 FISs) is presented.

3 Interval Type-3 Fuzzy Inference Systems

In this section, a comprehensive approach to Interval Type-3 Mamdani Fuzzy Inference Systems (IT3 FISs) is presented, which includes an approach for building Interval Type-3 Membership Functions (IT3 MFs) from the FOU and COU, and additionally an approach for Interval Type-3 Reduction was proposed.

However, starting from the architecture of the systems, in an analogous form to IT2 FISs [30] that are approximated with two type-1 fuzzy systems, IT3FISs can be

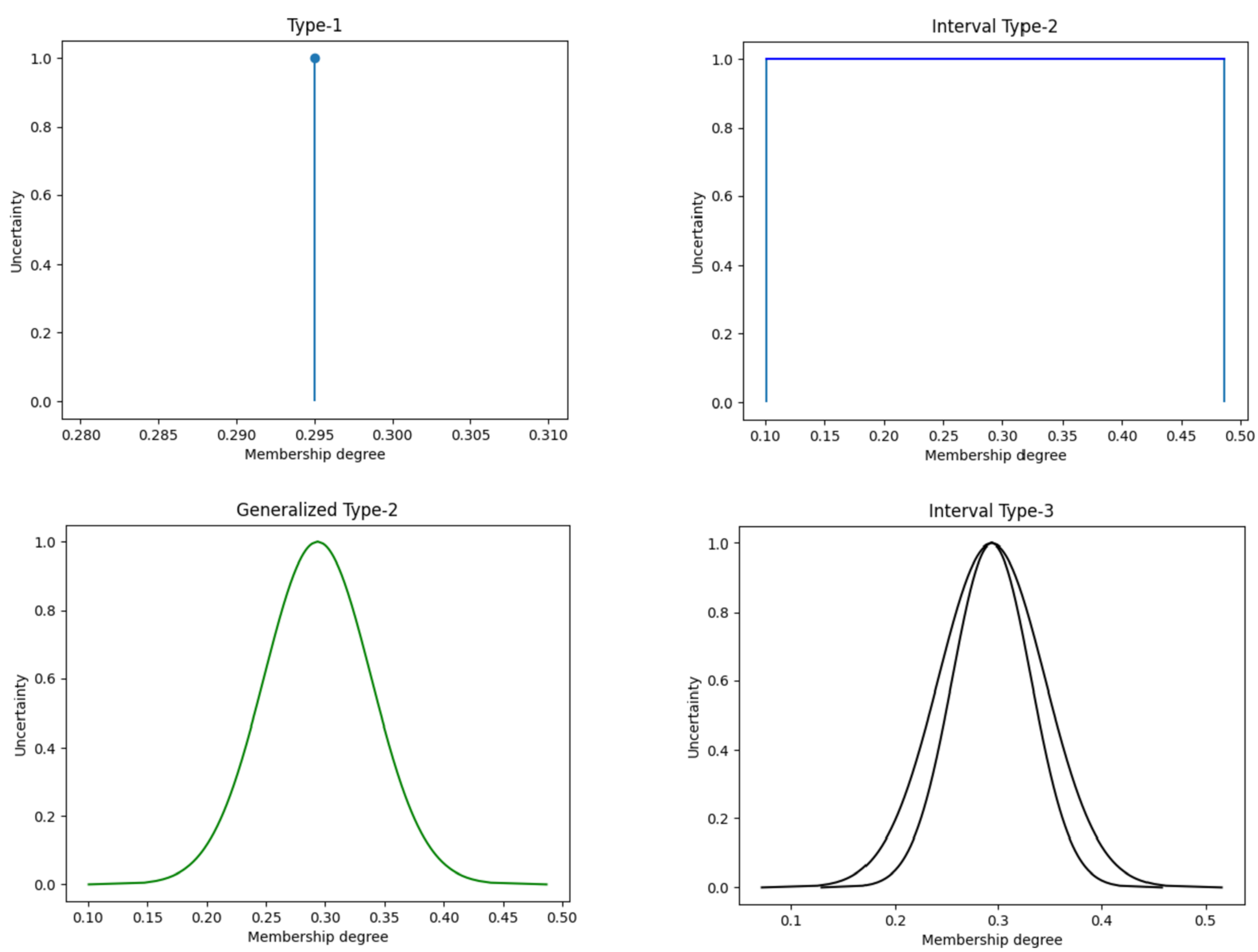


Fig. 3 The evolution of fuzzy numbers

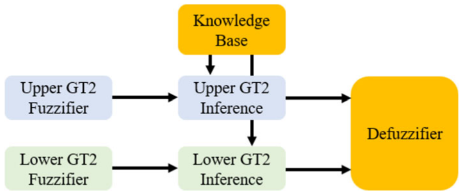


Fig. 4 Interval Type-3 Fuzzy Inference System architecture

approximated by two separate GT2 FISs, and this is illustrated in Fig. 4.

In the following sections, we are presenting some details about this architecture based on concepts of Type-1 and Type-2 Fuzzy Logic.

3.1 Interval Type-3 Membership Functions

This section presents the proposed approach for building IT3 MFs from the COU and FOU concepts, and the objective of this approach is to provide an efficient and clear strategy for expressing this kind of functions.

The approach consists of handling families of IT3 MFs based on the kind of secondary membership function; this is from the perspective of zSlices [31].

There are some possible families which are an example of the proposal of this journal.

3.1.1 Based on Gaussian MFs with uncertainty in σ

The first family of IT3 MFs proposed is composed of embedded IT2 Gaussian MF with uncertainty in σ, this function is widely used in the realm of IT2 FISs, and its graphical representation can be seen in Fig. 5.

As can be noted, this function is symmetric, and the core of the function is a single point, meaning that this family of IT3 MFs requires that the FOU and the COU fit the following requisites; the FOU must be an interval and the COU the midpoint of the FOU.

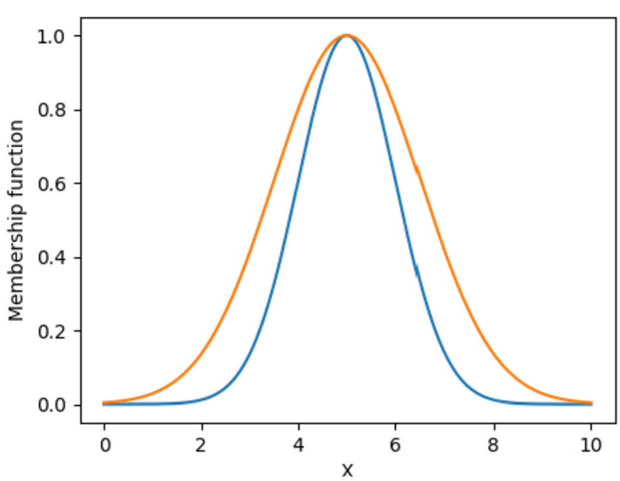


Fig. 5 Gaussian with uncertainty in sigma

The general equation for this family of IT3 MFs is expressed in (9) as follows:

$$\mathrm{IT3\,GaussS2}(\mathrm{u},\mathrm{x}) = \begin{cases} \overline{\mu}_{t3}(u,x) = e^{-\frac{1}{2}\left(\frac{u-\mu_{COU}(\mathrm{x})}{\sqrt{\left(\frac{\left(-\frac{1}{2}\right)\left(\underline{\mu}_{FOU}(x)-\mu_{COU}(\mathrm{x})\right)^2}{\ln(0.001)}\right)(1+v)}}\right)^2} \\ \underline{\mu}_{t3}(u,x) = e^{-\frac{1}{2}\left(\frac{u-\mu_{COU}(\mathrm{x})}{\sqrt{\left(\frac{\left(-\frac{1}{2}\right)\left(\underline{\mu}_{FOU}(x)-\mu_{COU}(\mathrm{x})\right)^2}{\ln(0.001)}\right)(1-v)}}\right)^2} \end{cases}, \tag{9}$$

where $\overline{\mu}_{t3}(u,x)$ and $\underline{\mu}_{t3}(u,x)$ composed the upper and lower boundaries of the embedded IT2 MFs which conform the IT3 MF, and $v \in [0,1]$ is the uncertainty in σ.

Based on these concepts, two examples of IT3 MFs members of this family are presented. Both examples are summarized in Table 1.

As can be noted, it is very simple to propose IT3 MFs functions by the definition of FOU and COU, in this case following the restrictions for this family of functions.

3.1.2 Based on Triangular MFs with Uncertainty in the Spread

This family of IT3 MFs is composed of embedded triangular IT2 MFs with uncertainty in the spread; the graphical illustration of a zSlide of this family can be seen in Fig. 6.

As can be noted, this function requires that the FOU be an interval, but the COU not necessarily must be the midpoint of the FOU, so, this family of IT3 MFs is more flexible than the first family presented.

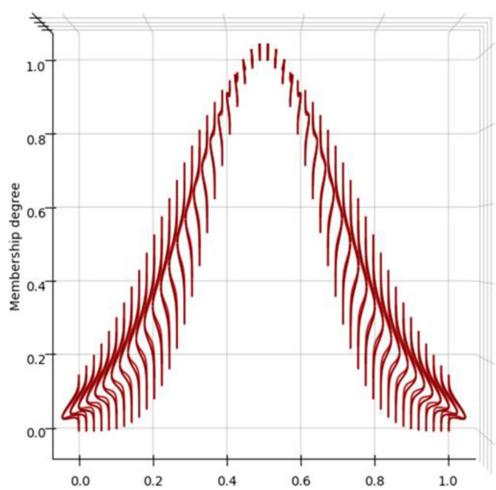
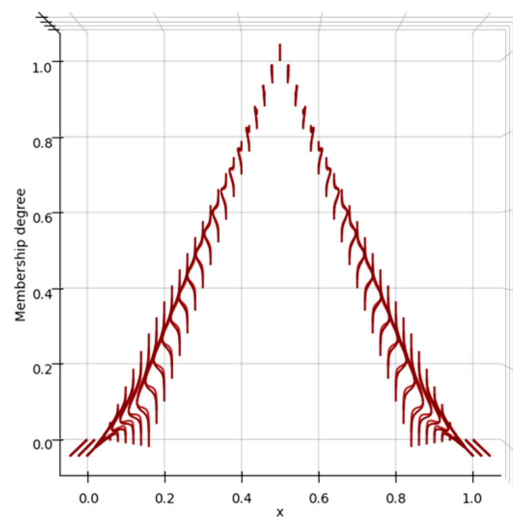
On the other hand, the general expression of this family of IT3 MF is (10).

$$IT3\,TriangularS(x) = \begin{cases} \overline{\mu}_{t3}(u,x) = \begin{cases} \frac{u-\mu_{COU}(\mathrm{x})}{\mu_{COU}(\mathrm{x})-(\underline{\mu}_{FOU}(x)-s_1)}; & (\underline{\mu}_{FOU}(x)-s_1)<u\leq\mu_{COU}(x); \\ \frac{(\overline{\mu}_{FOU}(x)+s_2)-u}{(\overline{\mu}_{FOU}(x)+s_2)-\mu_{COU}(\mathrm{x})}; & \mu_{COU}(x)<u<(\overline{\mu}_{FOU}(x)+s_2) \end{cases} \\ \underline{\mu}_{t3}(u,x) = \begin{cases} \frac{u-\mu_{COU}(\mathrm{x})}{\mu_{COU}(\mathrm{x})-(\underline{\mu}_{FOU}(x)+s_1)}; & (\underline{\mu}_{FOU}(x)+s_1)<u\leq\mu_{COU}(x); \\ \frac{(\overline{\mu}_{FOU}(x)-s_2)-u}{(\overline{\mu}_{FOU}(x)-s_2)-\mu_{COU}(\mathrm{x})}; & \mu_{COU}(x)<u<(\overline{\mu}_{FOU}(x)-s_2) \end{cases} \end{cases} \tag{10}$$

where $\overline{\mu}_{t3}(u,x)$ and $\underline{\mu}_{t3}(u,x)$ composed the upper and lower boundaries of the embedded IT2 MFs which conform the IT3 MF, and $[s_1, s_2]$ generate the uncertainty in the spread.

Based on these concepts, two examples of IT3 MFs members of this family are presented. Both examples are summarized in Table 2.

Table 1 Examples of IT3 GaussianS2 MFs

Example 1	Example 2
$\overline{\mu}_{FOU}(x) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma_1}\right)^2}$ $\underline{\mu}_{FOU}(x) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma_2}\right)^2}$ $\mu_{COU}(\mathrm{x}) = \frac{\overline{\mu}_{FOU}(x) + \underline{\mu}_{FOU}(x)}{2}$	$\overline{\mu}_{FOU}(x) = \begin{cases} \frac{x-a_1}{b-a_1}; a_1 < x < b \\ \frac{c_1-x}{c_1-b}; b < x < c_1 \end{cases}$ $\underline{\mu}_{FOU}(x) = \begin{cases} \frac{x-a_2}{b-a_2}; a_2 < x < b \\ \frac{c_2-x}{c_2-b}; b < x < c_1 \end{cases}$ $\mu_{COU}(\mathrm{x}) = \frac{\overline{\mu}_{FOU}(x) + \underline{\mu}_{FOU}(x)}{2}$
	

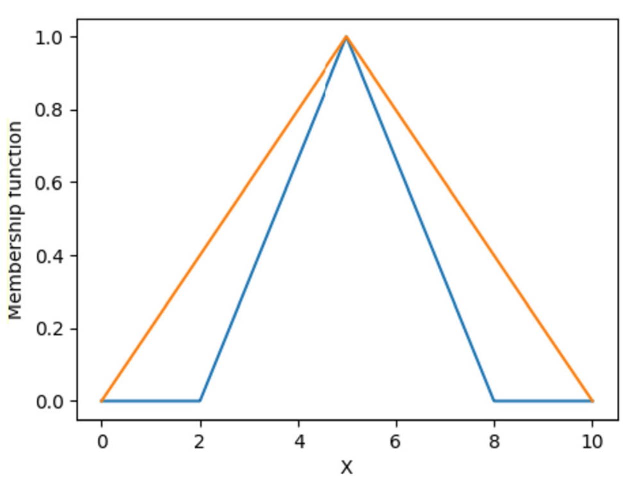


Fig. 6 Triangular membership function with uncertainty in the spread

As can be noted, the example proposed for this family of IT3 MFs has the same FOU and COU as the first family, but the resultant MF is significantly different; this is because the uncertainty is handled in a different way.

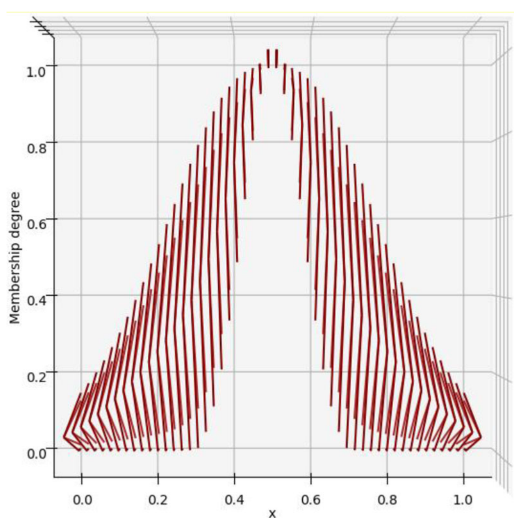
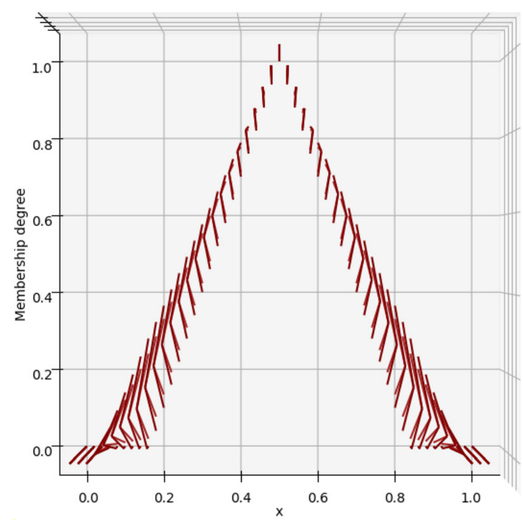
Regarding the inference, the process is not complex because the same concepts explained in Sect. 2 in Eqs. (4) to (6) are applied. However, the main challenge is now the Type-3 Reduction process.

3.2 Interval Type-3 Reduction and Defuzzification

In this section, the proposed approach for Interval Type-3 reduction is addressed. To explain the proposed approach clearly, the following example (Fig. 7) of this kind of systems is presented. It is important to note that for this proposed approach the α-plane representation is used.

Here the color of the membership functions (Fig. 7) stands for the relationship between antecedent and consequents (The system rules). The process for computing fuzzy inference starts with the fuzzification, and for this

Table 2 Examples of IT3 Triangular MFs

Example 1	Example 2
$\overline{\mu}_{FOU}(x) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma_1}\right)^2}$ $\underline{\mu}_{FOU}(x) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma_2}\right)^2}$ $\mu_{COU}(x) = \frac{\overline{\mu}_{FOU}(x) + \underline{\mu}_{FOU}(x)}{2}$	$\overline{\mu}_{FOU}(x) = \begin{cases} \frac{x-a_1}{b-a_1}; a_1 < x < b \\ \frac{c_1-x}{c_1-b}; b < x < c_1 \end{cases}$ $\underline{\mu}_{FOU}(x) = \begin{cases} \frac{x-a_2}{b-a_2}; a_2 < x < b \\ \frac{c_2-x}{c_2-b}; b < x < c_1 \end{cases}$ $\mu_{COU}(\mathrm{x}) = \frac{\overline{\mu}_{FOU}(x) + \underline{\mu}_{FOU}(x)}{2}$
	

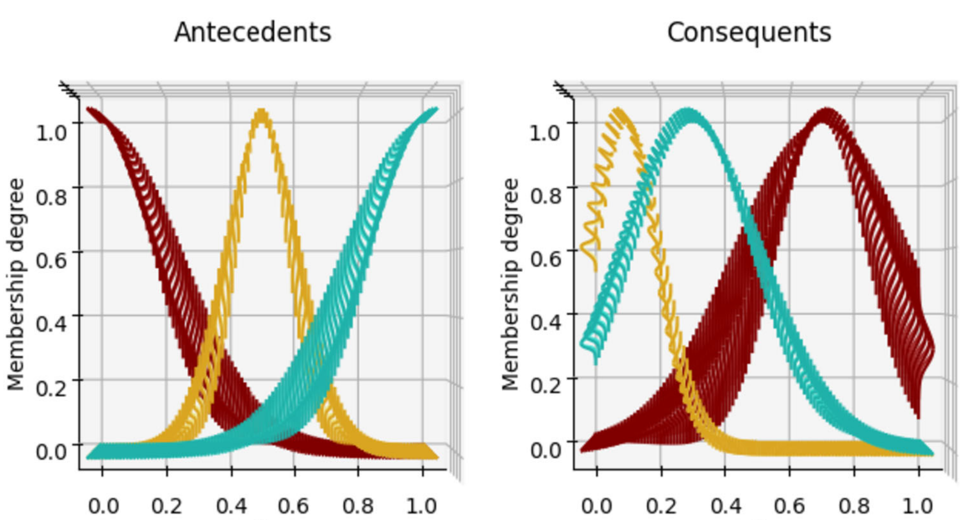


Fig. 7 IT3 FISs example

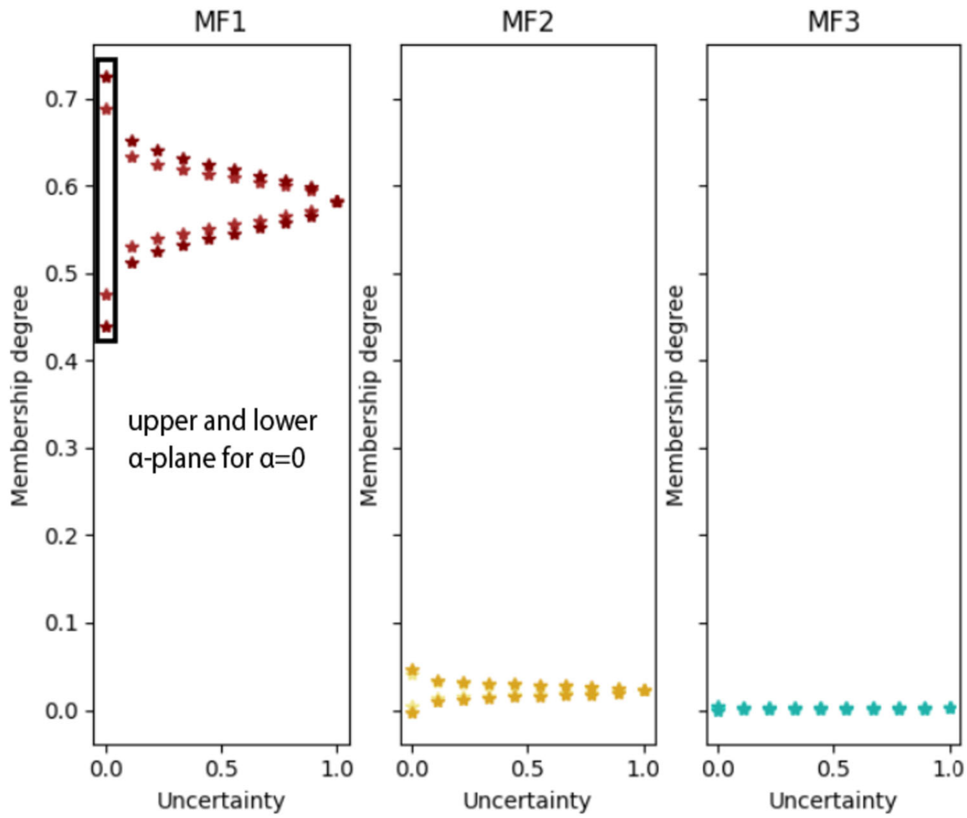


Fig. 8 IT3 Fuzzification results (10 α-planes)

example the system is evaluated with a value of 0.2, and the result can be seen in Fig. 8.

As can be seen in Fig. 8, the result of fuzzification in IT3 FISs is IT2 MFs, and in this figure the number of α-plane is 10 for a better appreciation; however, this number can be increased for a better approximation. An example with 100 α-planes is illustrated in Fig. 9.

Based on this, membership degrees can be performed (6) obtaining $\mu_{\mathbb{B}_j}(\overline{x}, y_j)$ which is the aggregated knowledge of the system for the jth output and is the input for the type-reduction algorithm.

Figure 10 illustrates the result of the inference ($\mu_{\mathbb{B}_j}(\overline{x}, y_j)$) and the result of the inference of the analogous system based on Generalized Type-2 Fuzzy Sets.

As can be seen, the difference between IT3 FIS and GT2 FIS $\mu_{\mathbb{B}_j}(\overline{x}, y_j)$) is the type of embedded secondary membership functions; for IT3 FIS the embedded functions are IT2 MFs and for GT2 FIS the embedded functions are T1 MFs.

On the other hand, considering the perspective of IT3 MFS as a combination of two upper and lower GT2 MFs, the GT2 type reduction and interval type-3 reduction are illustrated in Fig. 11.

Based on these methodologies, Fig. 12 illustrates the result of the earlier example after the computation of the Interval Type-2 Reduction of every α-plane.

In this example, the difference between both type reductions can be seen; the IT3 FIS requires an additional

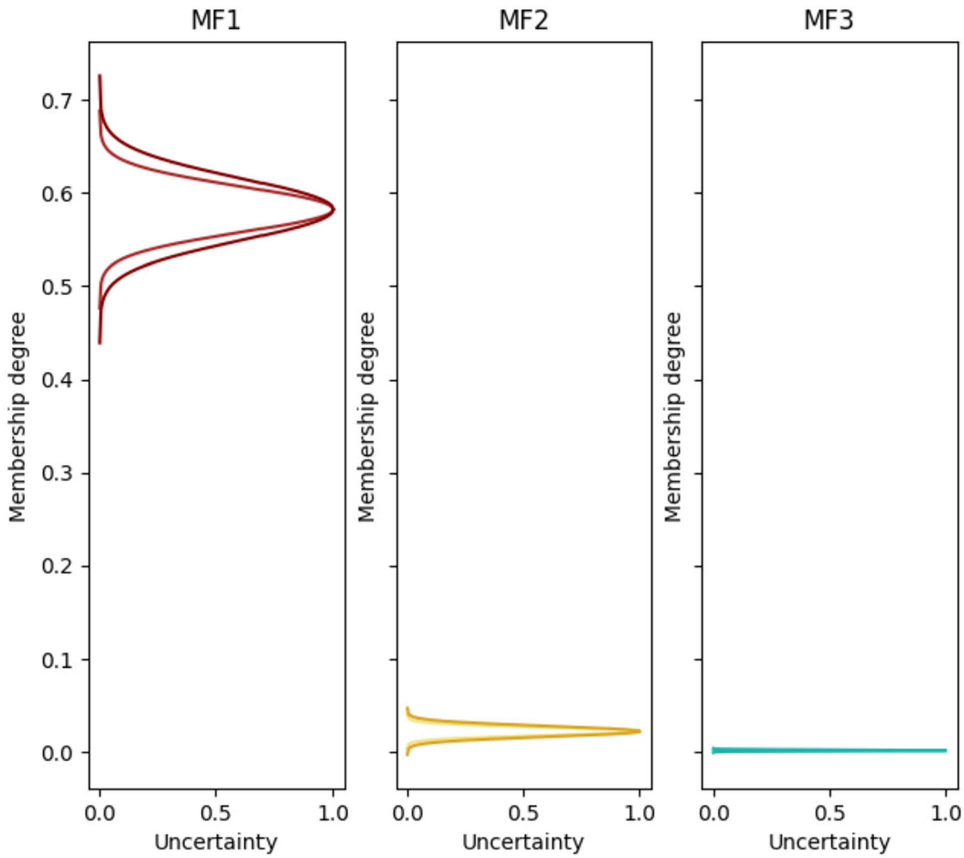


Fig. 9 IT3 Fuzzification results (100 α-planes)

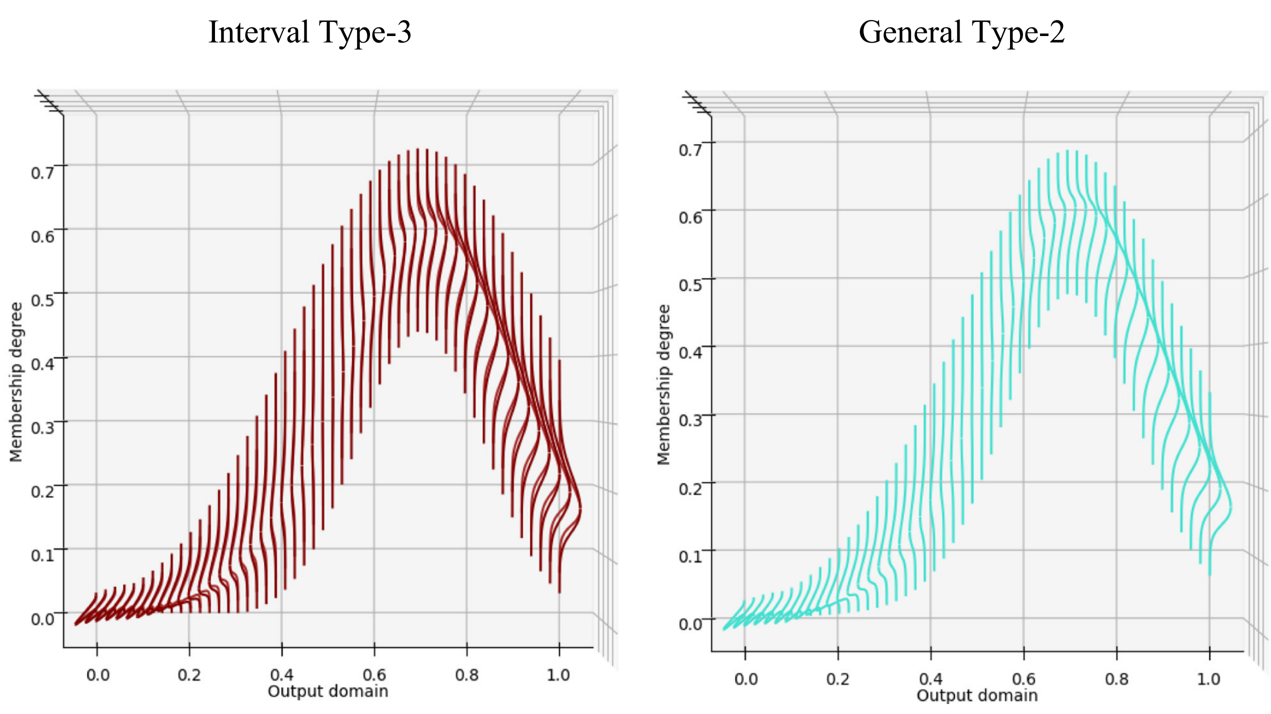


Fig. 10 Examples of output fuzzy sets

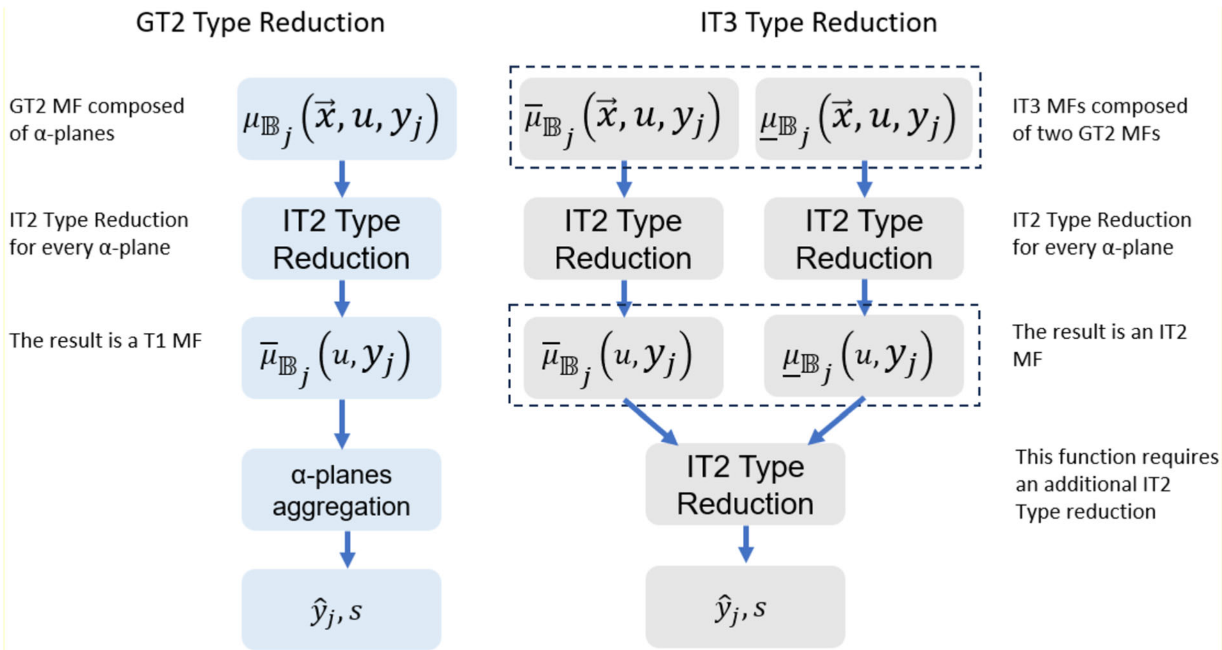


Fig. 11 α-Plane type reduction

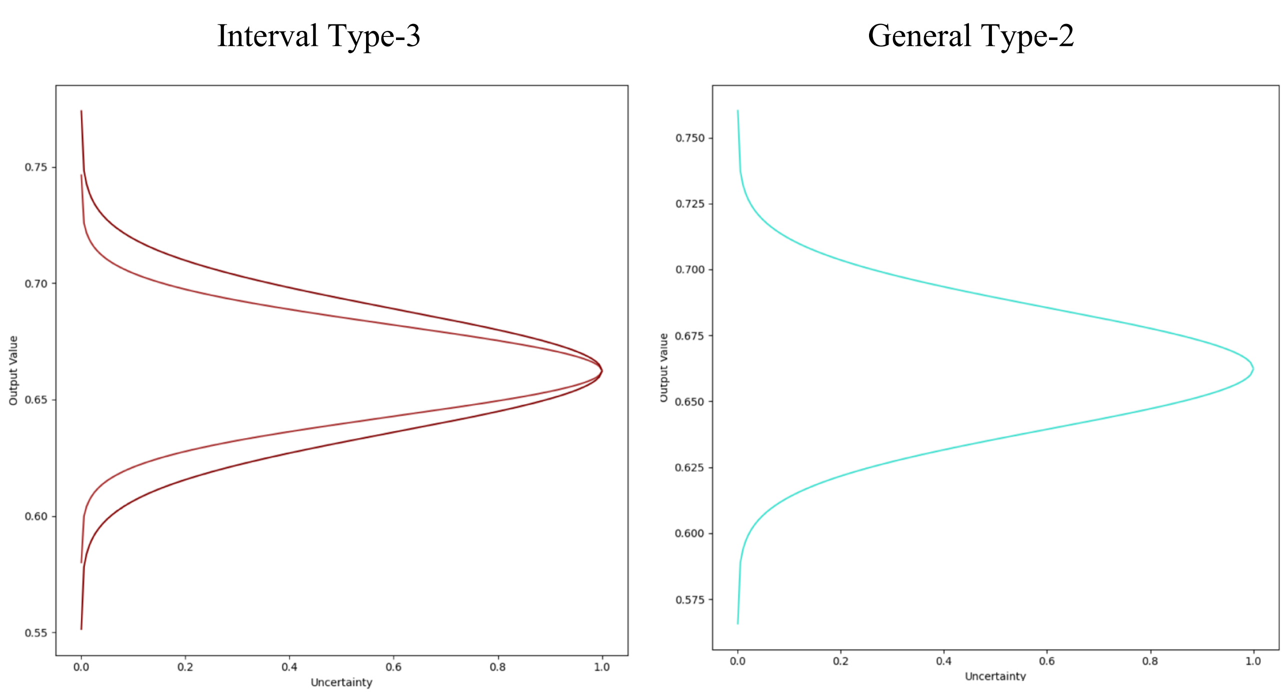


Fig. 12 Output before α-plane type reduction

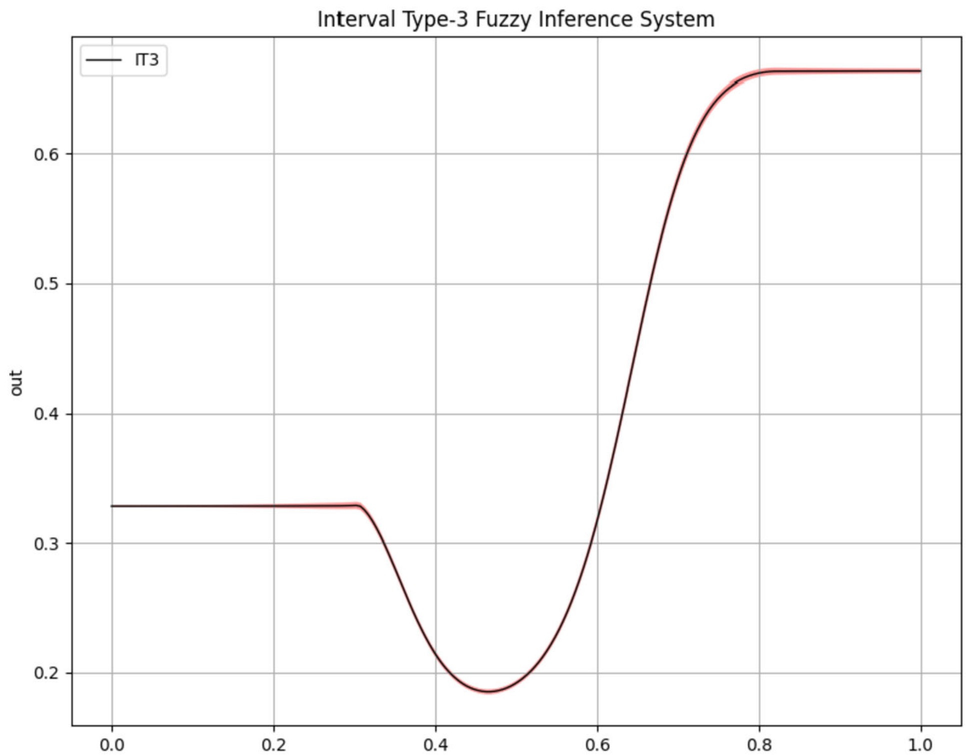


Fig. 13 IT3 FISs control surface

Table 3 Computational complexity comparison of types of FISs

Step	Interval type-2	Generalized type-2	Interval type-3
Fuzzification	$2T_1$	$\alpha(2T_1)$	$2\alpha(2T_1)$
Inference	$2T_1$	$\alpha(2T_1)$	$2\alpha(2T_1)$
Defuzzification	$2(N \times T_1)$	$2\alpha(N \times T_1) + T_1$	$4\alpha(N \times T_1) + 2\alpha(N \times T_1)$

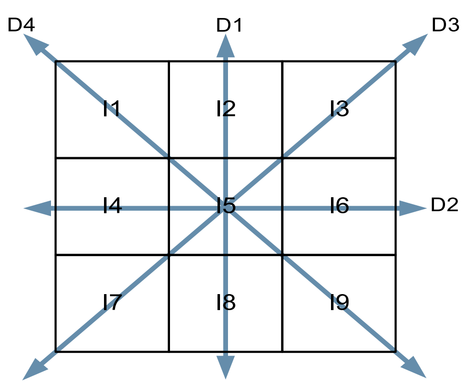


Fig. 14 Pixel neighborhood

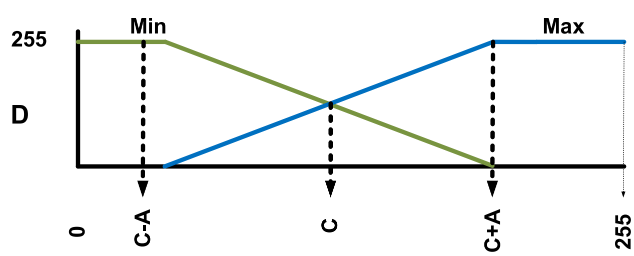


Fig. 15 Fuzzy sets

IT2 type reduction and GT2 FIS requires the α-plane aggregation.

Computing the proposed methodology of IT3 FISs for every point in the input domain, the result can be appreciated in Fig. 13.

3.3 Computational Complexity

This section reviews the variations of the different types of FISs regarding the computational complexity of these systems using as unit the computational complexity of the simpler FIS which is the Type-1 FISs.

Table 3 summarizes this analysis separating the variations of FISs in the common steps.

Considering that the number of iterations of the type-reduction algorithm depends on the algorithm and the aimed relative error, the evolution of the FISs implies more and more computational effort for achieving a better approximation to the theoretical model, for example, more α-planes. The process whose complexity is more affected by the evolution of FISs is the defuzzification; while the increase in other steps is simple, the increase of computational effort for defuzzification is affected by more variables.

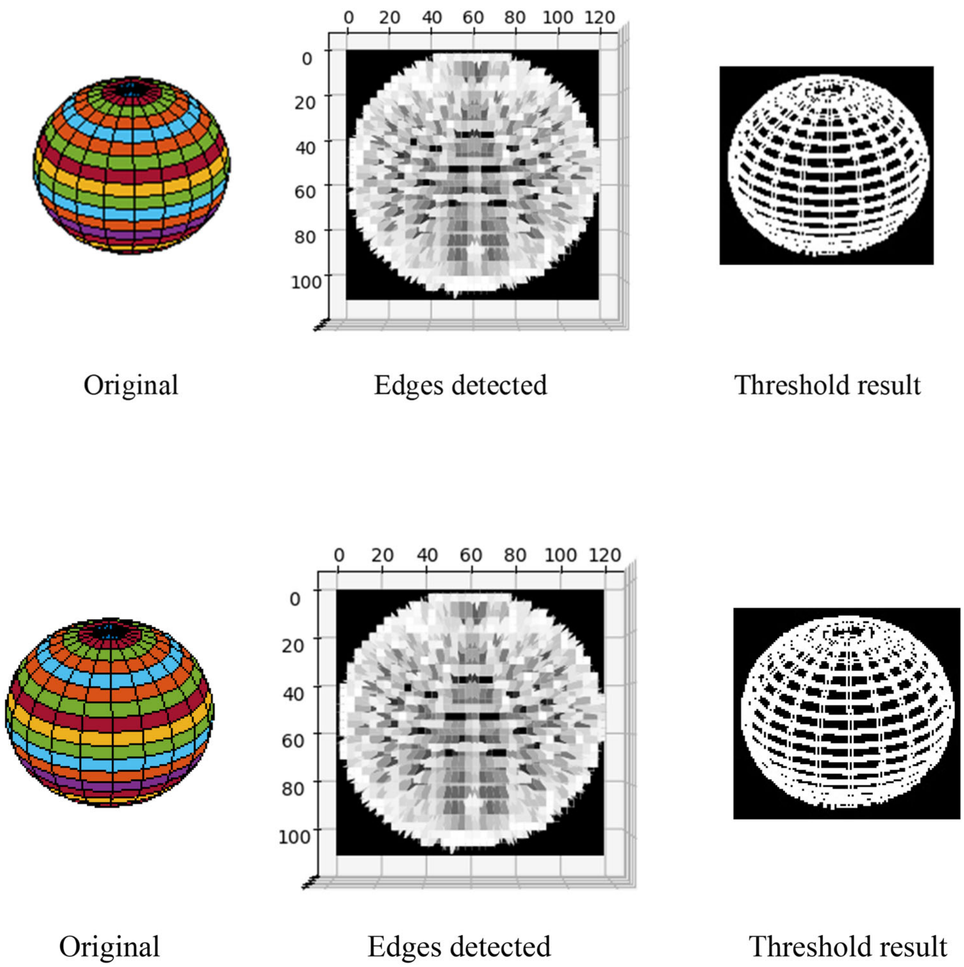


Fig. 16 Sphere edge results

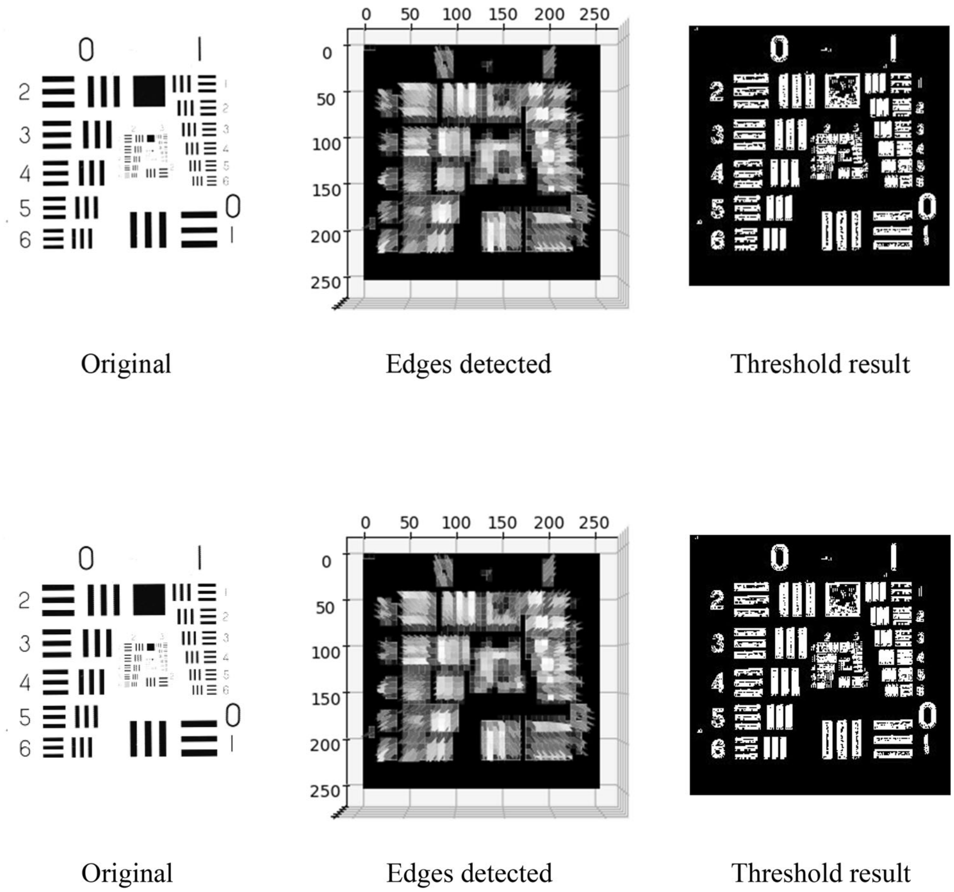


Fig. 17 Chart edge results

4 Illustrative Application Examples

In this section, two illustrative application examples of Interval Type-3 Mamdani Fuzzy Inference System are presented. In this case, the aim is to demonstrate the applicability of this kind of systems, but currently without performing parameter optimization.

4.1 Fuzzy Edge Detection

Based on the type-2 fuzzy system developed by the authors in [32], an Interval Type-3 Fuzzy Edge detector version is proposed as follows. The fuzzy edge detector must find the edges of a given image. In this case, an image is represented by the pixels.

Based on a pixel neighborhood, the inputs for the inference system and the fuzzy sets are proposed. Figure 14 illustrates the features selected from the pixel neighborhood.

The approximate gradients are expressed in Eq. (10):

$$\begin{matrix} D1 = |I2 - I8| & D2 = |I4 - I6| \\ D3 = |I3 - I7| & D4 = |I1 - I9| \end{matrix}. \tag{10}$$

Figure 15 illustrates how to build the corresponding fuzzy sets.

For the inference, six fuzzy rules related to the previous IT2 FS are proposed; Eq. (11) expresses these rules.

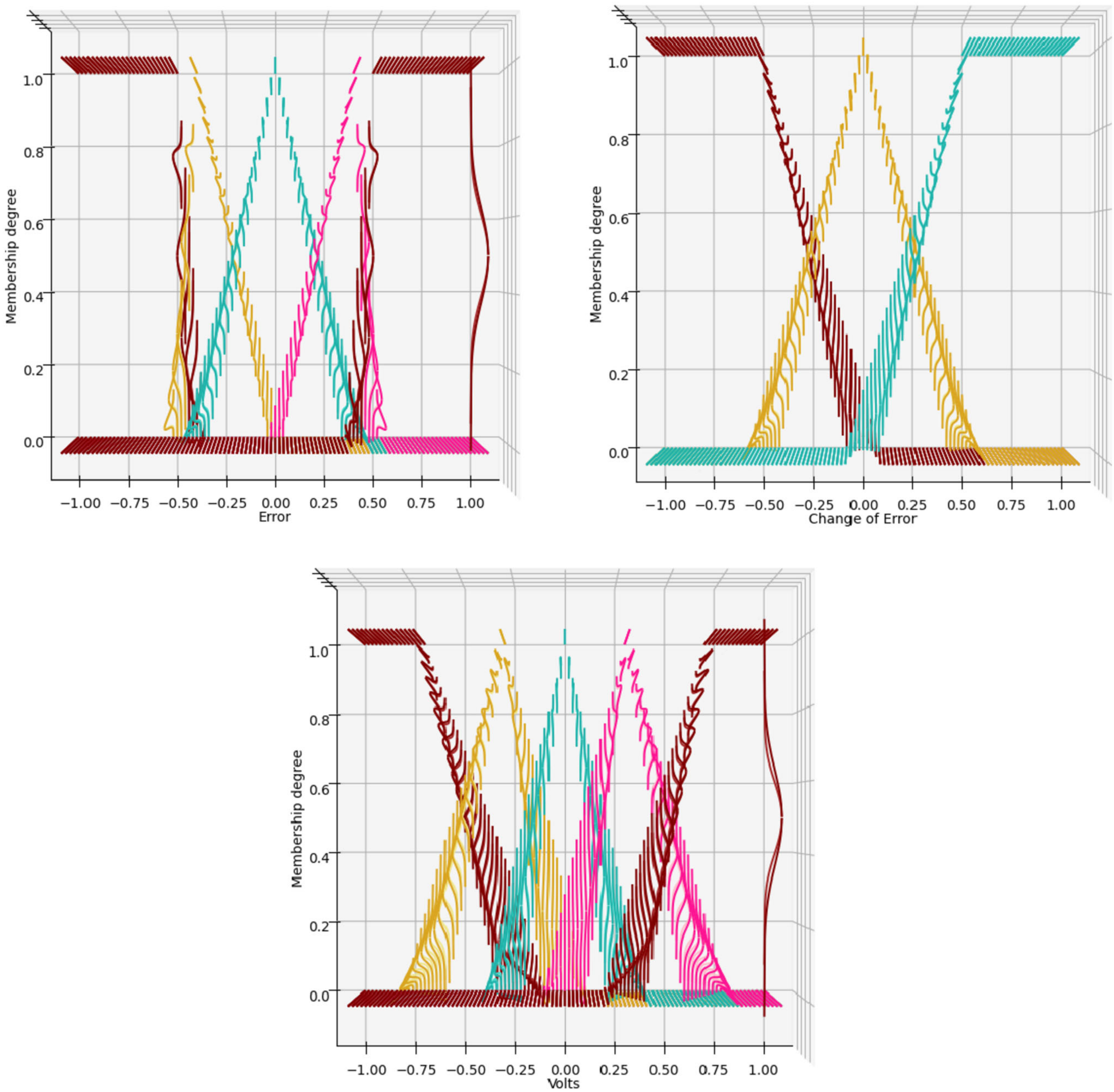


Fig. 18 Controller IT3 fuzzy sets

Table 4 Base of rules

	Error		Change of error	Delta voltage	
If	Negative	&	Very negative	Then	Very negative
If	Negative	&	Zero	Then	Very negative
If	Negative	&	Very positive	Then	Negative
If	Zero	&	Very negative	Then	Positive
If	Zero	&	Very positive	Then	Negative
If	Zero	&	Very positive	Then	Negative
If	Positive	&	Very negative	Then	Positive
If	Positive	&	Zero	Then	Very positive
If	Positive	&	Very positive	Then	Very positive
If	Zero	&	Zero	Then	Zero
If	Negative	&	Negative	Then	Very negative
If	Zero	&	Negative	Then	Positive
If	Positive	&	Negative	Then	Very positive
If	Positive	&	Positive	Then	Very positive
If	Zero	&	Positive	Then	Negative
If	Negative	&	Positive	Then	Very negative

$$
\begin{array}{c}
if\ (D1\ is\ \min)\ and\ (D2\ is\ \max)\ and\ (D3\ is\ \max)\ and \\
(D4\ is\ \max)\ then\ Y\ is\ Edge \\
if\ (D1\ is\ \max)\ and\ (D2\ is\ \min)\ and\ (D3\ is\ \max)\ and \\
(D4\ is\ \max)\ then\ Y\ is\ Edge \\
if\ (D1\ is\ \max)\ and\ (D2\ is\ \max)\ and\ (D3\ is\ \min)\ and \\
(D4\ is\ \min)\ then\ Y\ is\ E\mathrm{d}ge \\
if\ (D1\ is\ \max)\ and\ (D2\ is\ \max)\ and\ (D3\ is\ \min)\ and \\
(D4\ is\ \max)\ then\ Y\ is\ Edge \\
if\ (D1\ is\ \max)\ and\ (D2\ is\ \max)\ and \\
(D3\ is\ \max)\ and\ (D4\ is\ \min)\ then\ Y\ is\ Edge \\
if\ (D1\ is\ \max)\ and\ (D2\ is\ \max)\ and\ (D3\ is\ \max)\ and \\
(D4\ is\ \max)\ then\ Y\ is\ Edge
\end{array}
\tag{11}
$$

Some examples of the results of the implementation of the resulting system are shown as follows. Figure 16 illustrates the edges found for the sphere benchmark image. Figure 17 illustrates the edge results for the chart image.

As can be noted, the proposed fuzzy system performs the edge detection successfully for both examples; however, the system can be optimized for applying on noisy images.

4.2 DC Motor Controller

Another natural application for the explained systems is in fuzzy control. In this case, we are presenting an example based on the controller presented in [33]. The controlled plant in this example is a DC motor; in the reference, the authors analyze the effect of the uncertainty in the performance. However, in this paper, the focus is on showing the usability of these type-3 systems on known problems.

The graphical representation of the fuzzy sets is illustrated in Fig. 18.

The rules proposed for this fuzzy system are expressed in Table 4.

Based on the proposed fuzzy sets and rules, the control surface generated can be appreciated in Fig. 19.

The result of this controller in the plant is plotted in Fig. 20 with a reference of 1.0 for 1 s of simulation.

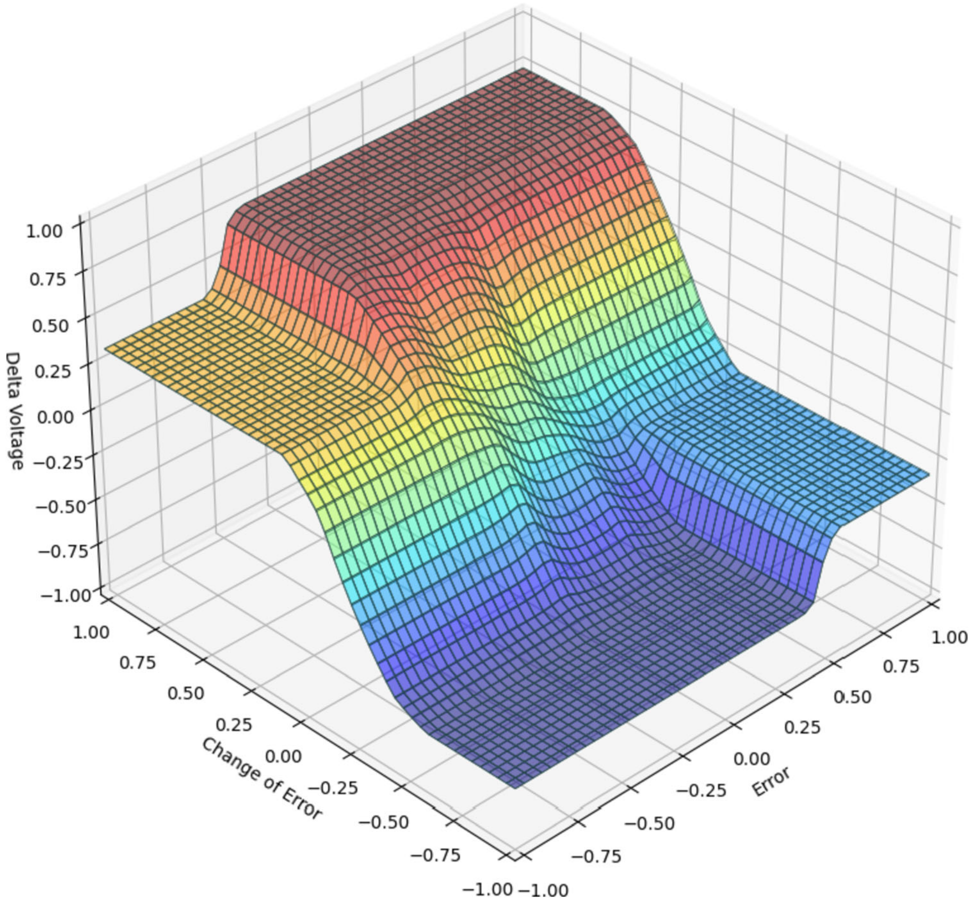


Fig. 19 Control surface

The plant response in Fig. 20 is good when compared with the ones obtained in [33].

5 Conclusions

In conclusion, this paper has presented the core concepts about Mamdani IT3 FISs based on the previous concepts of T2 FIS and the recent advances in IT3 Fuzzy Sets. This kind of systems requires significantly more computational effort when compared with respect to their predecessors, but in the presented illustrative application examples we can witness the applicability of the theory in real problems.

The paper did not have the intention, at this moment, of demonstrating the superiority of the T3 FL over the Type-2, but the intention was to clarify and propose new concepts that provide a framework for allowing more investigation in the area. New type-3 MFs have been proposed that can be defined and parameterized in a simple way. In addition, a new type-3 reduction method was proposed that can reduce computational cost for real applications. As can be inferred from the theory, the uncertainty handling is more complex, but it is more powerful and would provide an efficient tool for exploring their potential for real-world applications. For future work, we are planning to test the proposed concepts in diverse decision-making problems in

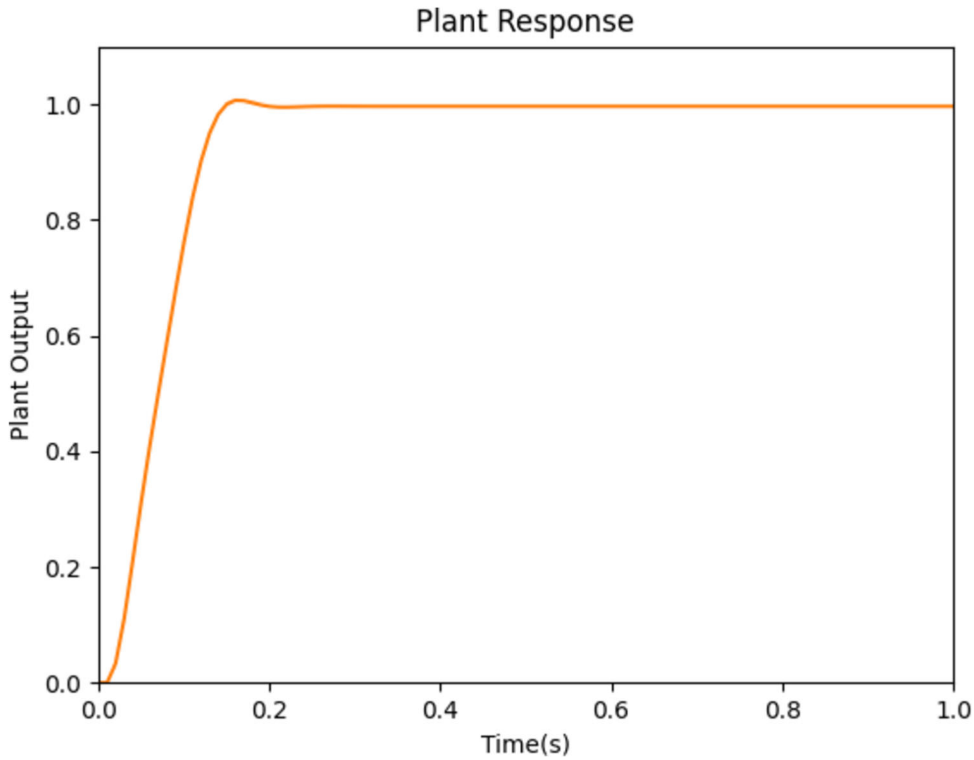


Fig. 20 Plant response with the controller

areas such as control, robotics, monitoring and diagnosis, and others.

Data availability Not applicable.

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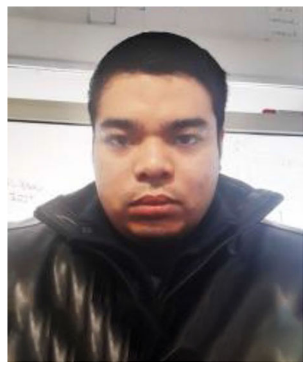

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