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Robust Variable Threshold Fuzzy Concept Lattice with Application to Medical Diagnosis

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Abstract Formal concept analysis is an effective tool for data analysis and visualization by means of concept lattice. Many concept lattice models have been studied in various settings. Variable threshold concept lattice is a fuzzy concept lattice constructed from fuzzy data. However, variable threshold concept lattice is not robust to noise because it employs a single threshold, instead of an interval to derive formal concepts. Thus, the paper introduces the tolerance threshold to variable threshold concept lattice, and forms the ROBust variable threshold fuzzy Concept Lattice (RobCL). By analyzing the properties of RobCL, we show that RobCL has some incremental characteristics and is able to model the incremental cognitive process, which makes RobCL distinctive from other concept lattice models. A comparative study shows that variable threshold concept lattice is just a special case of RobCL; in other words, when two thresholds coincide with each other, RobCL degenerates to variable threshold concept lattice and the incremental characteristics vanish. In addition, the proposed model is also applied to medical diagnosis and shows its superiority over the previous model.

Keywords Fuzzy formal context · Fuzzy concept lattice · Variable threshold concept lattice · Robust variable threshold fuzzy concept lattice

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1 Introduction

Formal concept analysis (FCA) [1] is a mathematical tool for analyzing data and representing knowledge by means of concept lattice [2] and implication [3–6]. A concept in FCA is a pair of extent and intent, where the extent refers to the objects covered by the concept and the intent refers to the common features of the objects in the extent. In FCA, the intents and extents of concepts are uniquely determined by each other, and the complete lattice formed by all concepts is called concept lattice. Concept lattice has been successfully applied in data mining [7, 8], machine learning [9–11], feature selection [12, 13] and concept cognitive learning [14–17].

In classical FCA, the relations between objects and attributes are precise. However, in practical applications, most of the relations are fuzzy and uncertain. Thus, Burusco [18] developed the theory of *L* fuzzy concept lattice and presented a construction method of fuzzy concept lattice. The concept operators in [18] are based on t-conorm, and *L* fuzzy concepts can be obtained by calculating the fixed point sets of the concept operators. Such concept operators, however, do not form Galois connection, and some basic properties of classical formal concepts in [1] do not hold in the fuzzy case. Therefore, Bělohlávek [19] generalized Galois connection from the perspective of fuzzy logic and extended FCA to the fuzzy case [20]. Since the number of formal concepts extracted from fuzzy data can be very large, Bělohlávek [21] introduced true-stressing hedge and constructed true-stressing hedge based fuzzy concept lattice. It was shown [21] that the stronger the hedge, the smaller the fuzzy concept lattice. In addition, in order to efficiently build fuzzy concept lattice, various methods and algorithms have been proposed. For example, Mao [22] utilized weighted graph to generate all crisp-fuzzy concepts. Singh [23] proposed a method to generate fuzzy concepts based on maximal acceptance of fuzzy attributes. Shemis [24] employed Shannon entropy to update fuzzy concept lattice in massive data.

On the other hand, Zhang [25] introduced threshold based fuzzy concepts and constructed variable threshold concept lattice. Thus, one may adjust threshold to control the scale of variable threshold concept lattice. In addition, the number of formal concepts in a variable threshold concept lattice is far less than that in a fuzzy concept lattice. Since threshold was used as a different strategy from hedge, Bělohlávek analyzed the relationship between true-stressing hedge and threshold in [26], and showed that the thresholdbased approach can be considered as a special case of the true-stressing hedge approaches.

From the perspective of theory, it seems that the fuzzy concept lattice model proposed in [26] is the most general one for visualizing fuzzy data. From the perspective of application, however, the more general the models, the more time-consuming the algorithms of constructing the models, because the general model will contain more concepts than other models. It may be an issue of application to choose a proper model. Therefore, from the perspective of application, variable threshold concept lattice may be of more advantages in visualizing fuzzy data. Thus, in order to control the number of fuzzy concepts within a proper scale, the paper chooses variable threshold concept lattice as a baseline.

In variable threshold model proposed by [25], whether an attribute belongs to an object is determined by the degree of the object having the attribute, which should be greater than or equal to a given threshold. However, in the real setting, since data may fluctuate within a range, relying only on a threshold may be susceptible to noise and reduce the robustness of the model. For example, setting the threshold to 0.60, then only when the degree of an object having an attribute is greater than or equal to 0.60, one may regard the object having the attribute; if the degree decreases to 0.59, the object will never have the attribute. Since the values 0.59 and 0.60 are very close, it may be inappropriate to diagnose that a patient has some disease if the possibility is greater than or equal to 0.60, instead of 0.59. In other words, variable threshold concept lattice is not robust to noise.

In order to solve the problems of variable threshold concept lattice, the paper introduces another threshold, called the tolerance threshold, to variable threshold concept lattice and proposes the ROBust variable threshold fuzzy Concept Lattice (RobCL). The properties of RobCL show that, distinctive from other concept lattice models (including variable threshold concept lattice), RobCL exhibits the incremental characteristics when identifying attributes for objects or recognizing objects for attributes. The incremental characteristics reflect the deepening of cognition and can be expressed as the so-called incremental sequences. We show that such sequences will end in finite steps and are able to mark the final concepts with their incremental sequences. Such final concepts form the robust variable threshold concepts. A further comparison with variable threshold concept lattice shows that when the two thresholds coincide with each other, RobCL degenerates to variable threshold concept lattice and the incremental characteristics vanish. In other words, variable threshold model is actually a special case of RobCL. Therefore, our model is an improvement and development of Zhang's thought.

The main contributions of the paper can be summarized as follows: (1) we introduced the tolerance threshold to variable threshold concept lattice and proposed RobCL to improve the robustness of variable threshold concept lattice; (2) we studied the properties of RobCL and showed that RobCL exhibits the incremental characteristics in constructing robust concepts; (3) we applied RobCL to the heart disease data set in UCI and showed its robustness in medical diagnosis.

This paper is organized as follows. Some related notions and properties in FCA and variable threshold concept lattice will be reviewed in Sect. 2, and the limitations of variable threshold concept lattice will be discussed in Sect. 3. In Sect. 4, we will introduce double threshold operators and analyze their properties. Then fuzzy concepts in RobCL can be captured and RobCL related properties will be proved. This paper is concluded in Sect. 5.

2 Preliminaries

2.1 Basic notions of FCA

In FCA [27], data are represented by formal context.

Definition 1 [27] A formal context is a triple $K = (G, M, I)$, where G and M are the set of objects and the set of attributes respectively, and $I \subseteq G \times M$ is a binary relation between G and M. For $g \in G$ and $m \in M$, $(g, m) \in I$ expresses that the object g has the attribute m.

Definition 2 [27] Let $K = (G, M, I)$ be a formal context. For $A \subseteq G$, $B \subseteq M$, one can define two operators:

$$A^I = \{m \in M \mid \forall g \in A, gIm\}$$
$$B^I = \{g \in G \mid \forall m \in B, gIm\}.$$

If $A^I = B$ and $B^I = A$ then the pair (A, B) is called a formal concept, where A is the extent of the concept and B is the intent of the concept. For two concepts (A_1, B_1) and (A_2, B_2), defining the order

$$(A_1, B_1) \le (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \subseteq B_2)$$

will lead to a complete lattice, called the concept lattice of K.

Concept lattice can be regarded as a visual representation of formal context. Another representation of formal context, which can be regarded as knowledge representation, is attribute implication; one may refer to [27] for the details of attribute implication and [3–6, 28] for the details of decision implication.

2.2 Variable threshold concept lattice

Variable threshold concept lattice is constructed on fuzzy formal context.

Definition 3 [25] A fuzzy formal context is a triple $K = (G, M, I)$, where G is a non-empty finite set of objects, M is a non-empty finite set of attributes, and $I : G \times M \to [0, 1]$ is a fuzzy relation between G and M. For $g \in G$, $m \in M$, $I(g, m)$ denotes the degree at which the object g has the attribute m.

Definition 4 [25] Let $K = (G, M, I)$ be a fuzzy formal context and $\alpha \in (0, 1]$. For $X \subseteq G$, $B \subseteq M$, we define the variable threshold operators as:

$$X^\alpha = \{m \in M \mid I(x, m) \ge \alpha, \forall x \in X\}$$
$$B^\alpha = \{x \in G \mid I(x, m) \ge \alpha, \forall m \in B\}.$$

If $X^\alpha = B$ and $B^\alpha = X$, the pair (X, B) is called a variable threshold (formal) concept, where X is the extent and B is the intent. The set of all the variable threshold concepts of K constitutes a complete lattice, denoted by $L^\alpha(K)$ and called the variable threshold concept lattice of K, under the partial order "$\le_\alpha$": $(X_1, B_1) \le_\alpha (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \subseteq B_2)$. The infimums and supremums are given by:

$$(X_1, B_1) \wedge (X_2, B_2) = (X_1 \cap X_2, (B_1 \cup B_2)^{\alpha\alpha})$$
$$(X_1, B_1) \vee (X_2, B_2) = ((X_1 \cup X_2)^{\alpha\alpha}, B_1 \cap B_2).$$

Proposition 1 [25] Let (G, M, I) be a fuzzy formal context. For X, X_1, $X_2 \subseteq G$, B, B_1, $B_2 \subseteq M$, and $\alpha \in (0, 1]$, we have:

1. $X_1 \subseteq X_2 \Rightarrow X_2^\alpha \subseteq X_1^\alpha$, $B_1 \subseteq B_2 \Rightarrow B_2^\alpha \subseteq B_1^\alpha$
2. $X \subseteq X^{\alpha\alpha}$, $B \subseteq B^{\alpha\alpha}$
3. $X^\alpha = X^{\alpha\alpha\alpha}$, $B^\alpha = B^{\alpha\alpha\alpha}$
4. $X \subseteq B^\alpha \Leftrightarrow B \subseteq X^\alpha$
5. $(X_1 \cup X_2)^\alpha = X_1{}^\alpha \cap X_2{}^\alpha$, $(B_1 \cup B_2)^\alpha = B_1{}^\alpha \cap B_2{}^\alpha$
6. $(X_1 \cap X_2)^\alpha \supseteq X_1{}^\alpha \cup X_2{}^\alpha$, $(B_1 \cap B_2)^\alpha \supseteq B_1{}^\alpha \cup B_2{}^\alpha$.

3 Limitations of variable threshold concept lattice

As stated in the Introduction section, variable threshold concept lattice is not robust to noise because it employs a single threshold, instead of an interval to derive formal concepts. In order to reveal the limitations of variable threshold concept lattice, we take the heart disease data set in UCI as an example to illustrate the generation process of concepts on a single threshold.

After normalization, the corresponding fuzzy formal context is obtained and shown in Table 1, where $G = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ represents 10 patients, and $M = \{a, b, c, d, e\}$ represents the age level, the degree of chest pain, the maximum heart rate, the ST inhibition and the serum cholesterol level. In Table 1, patients $x_1 - x_5$ do not have heart disease and $x_6 - x_{10}$ have heart disease.

When setting $\alpha = 0.60$, we can obtain all the variable threshold concepts of Table 1 as follows: $(x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}, \emptyset)$, $(x_2x_4x_6x_7x_8x_9x_{10}, c)$, $(x_1x_6x_9x_{10}, e)$, $(x_6x_7x_8x_9x_{10}, bcd)$, $(x_6x_8x_9, bcde)$, $(x_7x_8x_9x_{10}, abcd)$, $(x_2x_6x_7x_8x_9x_{10}, bc)$, $(x_9x_{10}, abcde)$. All the concepts are meaningful because they capture specific groups of patients with their common symptoms. For example, $(x_6x_7x_8x_9x_{10}, bcd)$ implies that at the threshold level 0.60, all the possible common symptoms that the patients $\{x_6x_7x_8x_9x_{10}\}$ are suffering from are $\{bcd\}$ and conversely, all the patients in G who are suffering from $\{bcd\}$ are $\{x_6x_7x_8x_9x_{10}\}$. In other words, at the threshold level 0.60, the pair $(x_6x_7x_8x_9x_{10}, bcd)$ represents a disease with $\{bcd\}$ as its salient symptoms and $\{x_6x_7x_8x_9x_{10}\}$ as its patients.

Table 1 A fuzzy formal context

	a	b	c	d	e
x_1	0.43	0.00	0.24	0.20	0.80
x_2	0.48	0.60	0.62	0.50	0.58
x_3	0.45	0.52	0.53	0.40	0.50
x_4	0.56	0.58	0.60	0.20	0.40
x_5	0.35	0.00	0.50	0.30	0.50
x_6	0.47	0.75	0.72	0.70	0.60
x_7	0.72	0.75	0.75	0.71	0.50
x_8	0.62	1.00	0.80	0.82	0.48
x_9	0.66	1.00	0.90	0.71	0.80
x_{10}	0.63	0.70	0.70	0.68	0.60

Variable threshold concept lattice is not robust to noise because any threshold, regardless of whether it is determined by experts or obtained adaptively from data, is a concise number and is not robust to slight disturbance. For example, if setting $\alpha = 0.60$, 0.60 will be regarded as a threshold at which an attribute belongs to a set of objects. If an attribute, however, belongs to some object in the set at the threshold less than 0.60, say 0.58, the object is regarded as not having the attribute, which may cause the object to be excluded from the set of objects having the attribute.

For example, at the threshold level 0.60, the variable threshold concept $(x_2x_6x_7x_8x_9x_{10}, bc)$ represents a disease with $\{bc\}$ as its salient symptoms and $\{x_2x_6x_7x_8x_9x_{10}\}$ as its patients. Thus, all the patients in G who are suffering from $\{bc\}$ at the threshold level 0.60 are $\{x_2x_6x_7x_8x_9x_{10}\}$. For the patient x_4, however, since $I(x_4, b) = 0.58$ and $I(x_4, c) = 0.60$, x_4 is excluded from $\{x_2x_6x_7x_8x_9x_{10}\}$ because x_4 has b (the degree of chest pain) at the degree 0.58. It is well known that human body is a changing system and any unexpected factors such as personal emotions may cause the measures of symptoms to fluctuate in a range. Therefore, a tolerance threshold β may help to reduce the impact of noise and make variable threshold concept lattice more robust.

At the threshold level 0.60, i.e., $\alpha = 0.60$, set the tolerance threshold to be 0.58, i.e., $\beta = 0.58$. For $X = \{x_2x_6x_7x_8x_9x_{10}\}$, we can compute $X^\alpha = \{bc\}$, i.e., the salient symptoms of $\{x_2x_6x_7x_8x_9x_{10}\}$ are $\{bc\}$; in this case, we compute $X^{\alpha\beta}$ instead of $X^{\alpha\alpha}$, where $X^{\alpha\beta} = \{g \in G \mid I(g, m) \geq \beta, \forall m \in X^\alpha\}$, i.e., we collect all the objects that have the salient symptoms in X^α at the degree greater than or equal to β. Since $\beta \leq \alpha$, all the objects having the salient symptoms in X^α at the degree greater than or equal to α will be included in $X^{\alpha\beta}$; at the same time, all the objects having the salient symptoms in X^α at the degree in $[\beta, \alpha)$ will be also included in $X^{\alpha\beta}$. For $X = \{x_2x_6x_7x_8x_9x_{10}\}$, we have $X^{\alpha\beta} = \{x_2x_4x_6x_7x_8x_9x_{10}\} \supseteq \{x_2x_6x_7x_8x_9x_{10}\} = X^{\alpha\alpha}$. Thus, the patient x_4 is also included in $X^{\alpha\beta}$, because $I(x_4, b) = 0.58$ falls in [0.58, 0.60).

4 Robust variable threshold fuzzy concept lattice

In this section, we will construct the robust version of variable threshold concept lattice, called ROBust variable threshold fuzzy Concept Lattice (RobCL).

4.1 Double threshold operators

First, we extends the operators in Definition 4 to the double threshold operators.

Definition 5 Let (G, M, I) be a fuzzy formal context, $\alpha \in (0, 1]$ and $\beta \in (0, 1]$. For $X \subseteq G$, $B \subseteq M$, we can define double threshold operators as follows:

$$X^\alpha = \{m \in M \mid I(g, m) \geq \alpha, \forall g \in X\}$$
$$B^\beta = \{g \in G \mid I(g, m) \geq \beta, \forall m \in B\}.$$

According to FCA, X^α in Definition 5 represents the common attributes that X has at the threshold α, and B^β represents the objects having B at the threshold β. In order to further clarify the implication of Definition 5, let us analyze the case of $\alpha \geq \beta$. In the case of $\alpha \geq \beta$, α can be regarded as the threshold of objects having attributes and β can be regarded as the tolerance threshold of objects having attributes, or equivalently, α and β can be regarded as the thresholds of objects having salient or general features respectively. Similarly, when $\beta \geq \alpha$, β can be regarded as the threshold of objects having attributes and α can be regarded as the tolerance threshold.

Specifically, for $g \in G$ and $m \in M$, if $I(g, m) < \beta$, it can be considered that the object g does not have the attribute m; if $I(g, m) \geq \beta$, the object g has the attribute m, where if $I(g, m) \geq \alpha$, the object g has the attribute m as its salient feature, and if $\beta \leq I(g, m) < \alpha$, the object g has the attributes m as its general feature. Thus, given a set X of objects, the salient features of X can be obtained by X^α; for the salient features in X^α, the objects that have X^α as their general features can be obtained by $X^{\alpha\beta}$. Since the values in $[\beta, \alpha)$ are tolerance values of objects having attributes, the objects having X^α at the degree greater than or equal to β should be included in $X^{\alpha\beta}$. More implications of double threshold operators will be explored in the following.

For brevity, we write $\{x\}^\alpha$ as x^α for $x \in G$ and $\{m\}^\beta$ as m^β for $m \in M$. Operating $\alpha\beta$ on X or $\beta\alpha$ on B with n times will lead to $X^{(\alpha\beta)^n}$ and $B^{(\beta\alpha)^n}$ respectively.

The operators in Definition 5 have the following properties.

Proposition 2 Let (G, M, I) be a fuzzy formal context. For X, X_1, $X_2 \subseteq G$, B, B_1, $B_2 \subseteq M$, $\alpha \in (0, 1]$ and $\beta \in (0, 1]$, the following conclusions hold.

1. $X_1 \subseteq X_2 \Rightarrow X_2^\alpha \subseteq X_1^\alpha$, $B_1 \subseteq B_2 \Rightarrow B_2^\beta \subseteq B_1^\beta$
2. If $\alpha \geq \beta$ then $X \subseteq X^{\alpha\beta}$
3. If $\alpha \leq \beta$ then $B \subseteq B^{\beta\alpha}$.

Proof (1) Let $m \in X_2^\alpha$. Then we have $I(g, m) \geq \alpha$ for any $g \in X_2$, implying that $I(x, m) \geq \alpha$ for any $x \in X_1$ since $X_1 \subseteq X_2$. Thus we have $m \in X_1^\alpha$ and $X_2^\alpha \subseteq X_1^\alpha$. Similarly, we have $B_2^\beta \subseteq B_1^\beta$.

(2) Suppose $\alpha \ge \beta$. For $g \in X$, we have $I(g,m) \ge \alpha$ for any $m \in X^{\alpha}$, implying that $I(g,m) \ge \alpha \ge \beta$ for any $m \in X^{\alpha}$, since $\alpha \ge \beta$. Thus, we have $g \in X^{\alpha}$ and $X \subseteq X^{\alpha}$.

(3) Similar to the proof of (2). □

Proposition 2 (1) implies that the more objects a set contains, the fewer attributes the objects have; similarly, the more attributes a set contains, the fewer objects the attributes pertain to. Proposition 2 (2) implies that when $\alpha \ge \beta$, all the objects in X have the attributes in X^{α} at the threshold β. As shown in Fig. 1, this is obvious, because $X^{\alpha\beta}$ contains the objects that have the attributes in X^{α} at the threshold β and the objects in X have the attributes in X^{α} at the threshold $\alpha \ge \beta$. From the perspective of robustness, Proposition 2 (2) implies that introducing tolerance threshold allows more objects having attributes X^{α} to be considered. The similar explanation applies to Proposition 2 (3), as shown in Fig. 2.

It should be noted that if $\alpha \ge \beta$, Proposition 2 (2) cannot apply to $B \subseteq M$, and that if $\alpha \le \beta$, Proposition 2 (3) cannot apply to $X \subseteq G$, as shown in Example 1. However, if $\alpha = \beta$, both Proposition 2 (2) and (3) hold for any X and B,

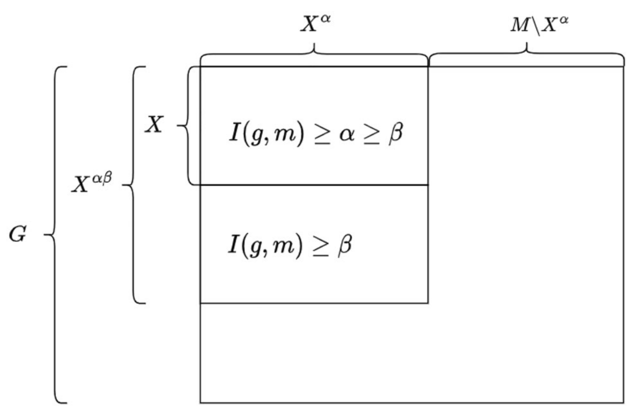

Fig. 1 The relationship between X and $X^{\alpha\beta}$ when $\alpha \ge \beta$

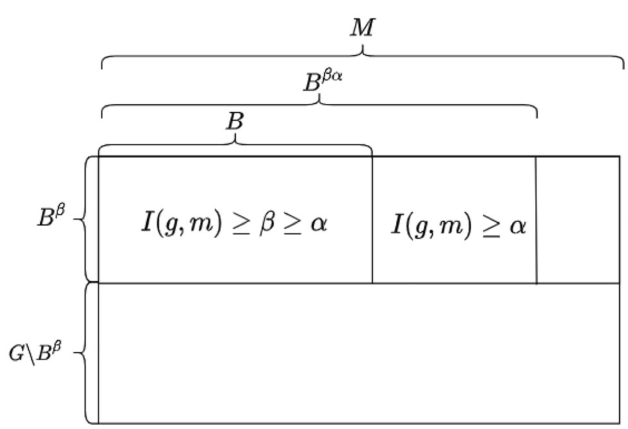

Fig. 2 The relationship between B and $B^{\beta\alpha}$ when $\beta \ge \alpha$

i.e., $X \subseteq X^{\alpha\beta}$ and $B \subseteq B^{\beta\alpha}$. In this case, double threshold operators degenerate to variable threshold operators.

Example 1 Set $\alpha = 0.50$ and $\beta = 0.40$. In Table 1, for $X = \{x_6x_7x_8x_{10}\}$, we can compute $X^{\alpha} = \{bcd\}$ and $X^{\alpha\beta} = \{x_2x_3x_6x_7x_8x_9x_{10}\}$, i.e., $X \subseteq X^{\alpha\beta}$.

If $B = \{bcde\}$, we have $B^{\beta} = \{x_2x_3x_6x_7x_8x_9x_{10}\}$ and $B^{\beta\alpha} = \{bc\}$, i.e., $B \nsubseteq B^{\beta\alpha}$. This is because for $e \in B$, there exists the object $x_8 \in G$ satisfying $0.40 \le I(x_8, e) < 0.50$ and thus $e \notin B^{\beta\alpha}$.

Similarly, if setting $\alpha = 0.30$ and $\beta = 0.40$ and letting $X = \{x_2x_4x_5x_7\}$, we have $\{x_2x_4x_5x_7\}^{\alpha} = \{ace\}$ and $\{x_2x_4x_5x_7\}^{\alpha\beta} = \{x_2x_3x_4x_6x_7x_8x_9x_{10}\}$, i.e., $X \nsubseteq X^{\alpha\beta}$. This is because for $x_5 \in X$, there exists the attribute $a \in M$ satisfying $0.30 \le I(x_5, a) < 0.40$ and thus $x_5 \notin X^{\alpha\beta}$.

If setting $\alpha = \beta = 0.50$, the double threshold operators will degenerate to variable threshold operators and both Proposition 2 (2) and (3) hold for any X and B. For example, for $X = \{x_6x_7x_8x_{10}\}$, we have $X^{\alpha\beta} = \{x_2x_6x_7x_8x_9x_{10}\}$ and for $B = \{bcde\}$, we have $B^{\beta\alpha} = \{bcde\}$, satisfying Proposition 2 (2) and (3).

In addition, set $\alpha = 0.50$ and $\beta = 0.40$, and let $X = \{x_6x_7x_8x_{10}\}$. Then, we have $X^{\alpha\beta\alpha} = \{bc\}$ and $X^{\alpha\beta\alpha\beta} = \{x_2x_3x_4x_6x_7x_8x_9x_{10}\}$, i.e., $X^{\alpha\beta} \subset X^{\alpha\beta\alpha\beta}$. In the case of variable threshold concept lattice, however, setting $\alpha = \beta = 0.50$, we have $X^{\alpha\beta} = X^{\alpha\beta\alpha\beta} = \{x_2x_6x_7x_8x_9x_{10}\}$.

By Example 1, if $\alpha \ge \beta$, then $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$ may hold, a distinctive characteristic of double threshold operators from variable threshold operators, as verified by Proposition 3.

Proposition 3 Let (G, M, I) be a fuzzy formal context and $X \subseteq G$. If $\alpha \ge \beta$, then

1. $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$ if and only if there exists $g \in X^{(\alpha\beta)^{n+1}}$ such that $I(g,m) \ge \beta$ for any $m \in X^{(\alpha\beta)^n\alpha}$ and $\beta > I(g, m_1)$ for some attribute $m_1 \in X^{(\alpha\beta)^{n-1}\alpha}$.
2. If $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$, all attributes m_1 in $X^{(\alpha\beta)^{n-1}\alpha}\backslash X^{(\alpha\beta)^n\alpha}$ satisfy $\alpha > I(g_1, m_1) \ge \beta$ for some object $g_1 \in X^{(\alpha\beta)^n}\backslash X^{(\alpha\beta)^{n-1}}$.

Proof (1) By Proposition 2 (2), we know that $X^{(\alpha\beta)^n} \subseteq X^{(\alpha\beta)^{n+1}}$, and thus $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$ if and only if there exists $g \in X^{(\alpha\beta)^{n+1}}\backslash X^{(\alpha\beta)^n}$. By definition of $X^{(\alpha\beta)^{n+1}}$, $g \in X^{(\alpha\beta)^{n+1}}$ if and only if $I(g,m) \ge \beta$ for any $m \in X^{(\alpha\beta)^n\alpha}$. By definition of $X^{(\alpha\beta)^n}$, $g \notin X^{(\alpha\beta)^n}$ if and only if $g \notin X^{((\alpha\beta)^{n-1}\alpha)\beta}$, if and only if there exists $m_1 \in X^{(\alpha\beta)^{n-1}\alpha}$ such that $\beta > I(g, m_1)$. Thus, the conclusion holds.

(2) For each $m_1 \in X^{(\alpha\beta)^{n-1}\alpha}\backslash X^{(\alpha\beta)^n\alpha}$, since $m_1 \notin X^{(\alpha\beta)^n\alpha}$, there exists some object $g_1 \in X^{(\alpha\beta)^n}$ such that

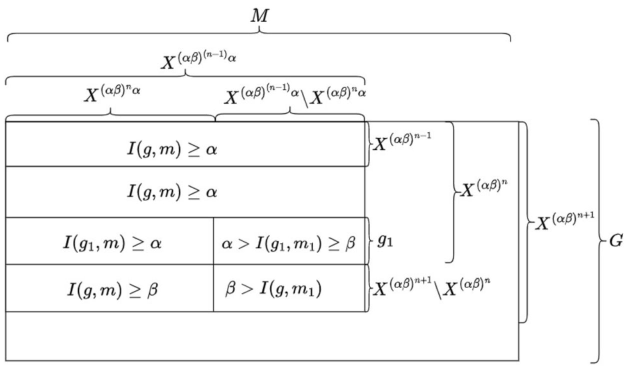

Fig. 3 The change from $X^{(\alpha\beta)^n}$ to $X^{(\alpha\beta)^{n+1}}$

$\alpha > I(g_1, m_1)$. Because $m_1 \in X^{(\alpha\beta)^{n-1}\alpha}$ and $g_1 \in X^{(\alpha\beta)^n} = X^{((\alpha\beta)^{n-1}\alpha)\beta}$, we have $I(g_1, m_1) \geq \beta$ and thus $\alpha > I(g_1, m_1) \geq \beta$. □

Compared with variable threshold concept lattice, Proposition 3 (1) says that the concept generation of RobCL is not a one-step process. This is because there may exist an object $g \in X^{(\alpha\beta)^{n+1}} \backslash X^{(\alpha\beta)^n}$ such that $I(g, m) \geq \beta$ for any $m \in X^{(\alpha\beta)^n\alpha}$ and $\beta > I(g, m_1)$ for some $m_1 \in X^{(\alpha\beta)^{n-1}\alpha}$ (see Fig. 3).

If $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$, we have $X^{(\alpha\beta)^{n-1}\alpha} \neq X^{(\alpha\beta)^n\alpha}$ and $X^{(\alpha\beta)^n} \neq X^{(\alpha\beta)^{n-1}}$[1] and thus, by Proposition 3 (2), there must exist some attribute $m_1 \in X^{(\alpha\beta)^{n-1}\alpha} \backslash X^{(\alpha\beta)^n\alpha}$ such that $\alpha > I(g_1, m_1) \geq \beta$ for some $g_1 \in X^{(\alpha\beta)^n} \backslash X^{(\alpha\beta)^{n-1}}$ (see Fig. 3). In other words, if $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$, there must exist $g_1 \in X^{(\alpha\beta)^n} \backslash X^{(\alpha\beta)^{n-1}}$ with $\alpha > I(g_1, m_1) \geq \beta$ for $m_1 \in X^{(\alpha\beta)^{n-1}\alpha} \backslash X^{(\alpha\beta)^n\alpha}$.

From the perspective of cognition, starting from $X^{(\alpha\beta)^{n-1}}$, one can obtain the salient features of $X^{(\alpha\beta)^{n-1}}$, i.e., $X^{(\alpha\beta)^{n-1}\alpha}$; in order to collect the objects that also have $X^{(\alpha\beta)^{n-1}\alpha}$, one should employ the tolerance threshold β and thus include g_1 in $X^{(\alpha\beta)^n}$. Since $g_1 \in X^{(\alpha\beta)^n}$ and m_1 is not a salient feature of g_1, m_1 is excluded from $X^{(\alpha\beta)^n\alpha}$ and is not a salient feature of $X^{(\alpha\beta)^n}$. Thus, Proposition 3 (2) says that if $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$, there exists attribute $m_1 \in X^{(\alpha\beta)^{n-1}\alpha} \backslash X^{(\alpha\beta)^n\alpha}$ that is a general but not salient feature of $X^{(\alpha\beta)^n}$. The same conclusion applies to $X^{(\alpha\beta)^n}$. Since $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$, there may exist some attribute in $X^{(\alpha\beta)^n\alpha}$ that is not a salient feature of $X^{(\alpha\beta)^{(n+1)}\alpha}$.

Such incremental generating processes can be visually represented by the symbol "→". For example, "$X \to B$" denotes the process from X to B (i.e. X^α), and "$B \to X$" denotes the process from B to X (i.e. B^β).

Example 2 For the fuzzy formal context in Table 1, set $\alpha = 0.70$ and $\beta = 0.65$. Let $X = \{x_7\}$. Since $X^\alpha = \{abcd\}$, $X^{\alpha\beta} = \{x_7x_9\}$, $X^{\alpha\beta\alpha} = \{bcd\}$, $X^{\alpha\beta\alpha\beta} = \{x_6x_7x_8x_9x_{10}\}$ and $X^{\alpha\beta\alpha\beta\alpha} = \{bc\}$, the incremental sequence of X is:

$$\{x_7\} \to \{abcd\} \to \{x_7x_9\} \to \{bcd\} \to \{x_6x_7x_8x_9x_{10}\} \to \{bc\} \to \{x_6x_7x_8x_9x_{10}\}.$$

In this process, since $\{x_7\}^{\alpha\beta} = \{x_7x_9\} \subset \{x_6x_7x_8x_9x_{10}\} = \{x_7\}^{(\alpha\beta)^2}$, there exists $x_6 \in \{x_7\}^{(\alpha\beta)^2} \backslash \{x_7\}^{\alpha\beta}$ such that $I(x_6, m) \geq \beta$ for any $m \in X^{\alpha\beta\alpha}$ and $\beta > I(x_6, a)$ for $a \in \{x_7\}^\alpha$. Moreover, for each attribute m_1 in $X^\alpha \backslash X^{(\alpha\beta)\alpha}$, i.e., the attribute a, there exists an object x_9 such that $\alpha > I(x_9, a) \geq \beta$.

From the perspective of cognition, for X, $X^\alpha = \{abcd\}$ captures all the salient symptoms of X at the threshold 0.70, and $X^{\alpha\beta} = \{x_7x_9\}$ collected all the patients who have the symptoms $\{abcd\}$ at the threshold 0.65. It should be noted that in the process, the patient x_9 having the symptom a with $\alpha = 0.70 > I(x_9, a) = 0.66 \geq 0.65 = \beta$ has been included in $X^{\alpha\beta}$, because although x_9 has the symptom a as its general feature it should not be excluded from $X^{\alpha\beta}$, since the value 0.66 is very close to 0.70. From the perspective of medical science, the operation can avoid missed diagnoses. Since x_9 is included in $X^{\alpha\beta}$, the salient features of $X^{\alpha\beta}$ should be updated to $\{bcd\}$, where the attribute a has been removed from X^α because a is not a salient feature of x_9. In order to collect all the objects that have $\{bcd\}$ as their symptoms at the tolerance threshold 0.65, one should compute $X^{\alpha\beta\alpha\beta}$ and include x_6, x_8 and x_{10} as its members. Finally, one can obtain the salient features $\{bc\}$ of the patients $\{x_6x_7x_8x_9x_{10}\}$ and conversely, no patient can be added to $\{x_6x_7x_8x_9x_{10}\}$ because no patient has the symptoms $\{bc\}$ at the degree greater than or equal to $\beta = 0.65$. In fact, since $\{x_6x_7x_8x_9x_{10}\}$ are the patients with heart disease, the pair $(x_6x_7x_8x_9x_{10}, bc)$ is the concept of heart disease that contains $\{bc\}$ as its salient symptoms and $\{x_6x_7x_8x_9x_{10}\}$ as its patients.

Similarly, if starting from the sets $\{x_6x_7\}$ and $\{x_7x_8\}$, one can obtain the following incremental sequences:

$$\{x_6x_7\} \to \{bcd\} \to \{x_6x_7x_8x_9x_{10}\} \to \{bc\} \to \{x_6x_7x_8x_9x_{10}\}$$

$$\{x_7x_8\} \to \{bcd\} \to \{x_6x_7x_8x_9x_{10}\} \to \{bc\} \to \{x_6x_7x_8x_9x_{10}\}.$$

Both of the processes lead to the concept of heart disease.

[1] Otherwise, if $X^{(\alpha\beta)^{n-1}\alpha} = X^{(\alpha\beta)^n\alpha}$, then $X^{(\alpha\beta)^n} = X^{((\alpha\beta)^{n-1}\alpha)\beta} = X^{((\alpha\beta)^n\alpha)\beta} = X^{(\alpha\beta)^{n+1}}$, a contradiction with $X^{(\alpha\beta)^n} \subset X^{(\alpha\beta)^{n+1}}$. Similarly, we have $X^{(\alpha\beta)^n} \neq X^{(\alpha\beta)^{n-1}}$.

4.2 Incremental Sequence

By Example 2, one may find that the incremental sequences may end at some point. Specifically, if $\alpha \geq \beta$, for the set X of objects, we may have the following incremental sequence:

$$X \to X^{(\alpha)} \to X^{(\alpha\beta)} \to X^{(\alpha\beta)\alpha} \to X^{(\alpha\beta)^2} \to X^{(\alpha\beta)^2\alpha} \to \cdots \to X^{(\alpha\beta)^n}$$

satisfying $X \subseteq X^{(\alpha\beta)} \subseteq X^{(\alpha\beta)^2} \cdots \subseteq X^{(\alpha\beta)^n}$ and $X^{\alpha} \supseteq X^{(\alpha\beta)\alpha} \supseteq X^{(\alpha\beta)^2\alpha} \cdots \supseteq X^{(\alpha\beta)^n\alpha}$. Thus, in the incremental sequence, X will collect more and more objects that have the same attributes with X and capture less and less common attributes of them. In the paper, when $\alpha \geq \beta$, we refer to the incremental sequence from X to $X^{(\alpha\beta)^n}$ as the object incremental sequence of X.

Similarly, if $\beta \geq \alpha$, the attribute incremental sequence of B is as follows:

$$B \to B^{(\beta)} \to B^{(\beta\alpha)} \to B^{(\beta\alpha)\beta} \to B^{(\beta\alpha)^2} \to B^{(\beta\alpha)^2\beta} \to \cdots \to B^{(\beta\alpha)^n}.$$

Obviously, if both G and M are finite sets, both object and attribute incremental sequences will end at some points. Even G or M is infinite, we have the following results.

Proposition 4 Let (G, M, I) be a fuzzy formal context, $X \subseteq G$, and $B \subseteq M$. Then, the following conclusions hold.

1. If there exists a non-negative integer n such that $X^{(\alpha\beta)^n} = X^{(\alpha\beta)^{n+1}}$, then we have $X^{(\alpha\beta)^n} = X^{(\alpha\beta)^{n+k}}$ for any non-negative integer $k \geq 0$.
2. If there exists a non-negative integer m such that $B^{(\beta\alpha)^m} = B^{(\beta\alpha)^{m+1}}$, then we have $B^{(\beta\alpha)^m} = B^{(\beta\alpha)^{m+k}}$ for any non-negative integer $k \geq 0$.

In order to further analyze incremental sequence, we will discuss the relationships between incremental sequences. To this end, we will take object incremental sequences as examples and classify object incremental sequences into two types of relationships, convergence or parallel.

Definition 6 For $X_1, X_2 \subseteq G$, we say that (the incremental sequences of) X_1 and X_2 converge in the object-object way at (i, j), if (i, j) satisfies $X_1^{(\alpha\beta)^i} = X_2^{(\alpha\beta)^j}$; if (i, j) further satisfies $X_1^{(\alpha\beta)^{i-1}\alpha} \neq X_2^{(\alpha\beta)^{j-1}\alpha}$, $X_1^{(\alpha\beta)^i} \neq X_2^{(\alpha\beta)^{j-1}}$ and $X_1^{(\alpha\beta)^{i-1}} \neq X_2^{(\alpha\beta)^j}$, (i, j) is the smallest object-object convergent unit of X_1 and X_2. Similarly, for $X_1, X_2 \subseteq G$, if there exists (i, j) satisfying $X_1^{(\alpha\beta)^i\alpha} = X_2^{(\alpha\beta)^j\alpha}$, X_1 and X_2 converge in the object-attribute way at (i, j); if (i, j) further satisfies $X_1^{(\alpha\beta)^i} \neq X_2^{(\alpha\beta)^j}$, $X_1^{(\alpha\beta)^i\alpha} \neq X_2^{(\alpha\beta)^{j-1}\alpha}$ and $X_1^{(\alpha\beta)^{i-1}\alpha} \neq X_2^{(\alpha\beta)^j\alpha}$, (i, j) the smallest object-attribute convergent unit of X_1 and X_2.

By Definition 6, if X_1 and X_2 converge in the object-attribute way at (i, j), then we have $X_1^{(\alpha\beta)^i\alpha} = X_2^{(\alpha\beta)^j\alpha}$ and thus $X_1^{(\alpha\beta)^i\alpha\beta} = X_2^{(\alpha\beta)^j\alpha\beta}$, i.e., $X_1^{(\alpha\beta)^{i+1}} = X_2^{(\alpha\beta)^{j+1}}$. Therefore, X_1 and X_2 converge in the object-object way at $(i+1, j+1)$. Similarly, if X_1 and X_2 converge in the object-object way at (i, j), X_1 and X_2 will also converge in the object-attribute way at $(i+1, j+1)$. Thus, for brevity, we will not distinguish the two types of convergence of object incremental sequences in the following.

Example 3 In Table 1, let $X_1 = \{x_6x_7x_9x_{10}\}$ and $X_2 = \{x_7x_8x_9\}$, and set $\alpha = 0.50$ and $\beta = 0.40$. The object incremental sequences of X_1 and X_2 are as follows:

$$\{x_6x_7x_9x_{10}\} \to \{bcde\} \to \{x_2x_3x_6x_7x_8x_9x_{10}\} \to \{bc\} \to \{x_2x_3x_4x_6x_7x_8x_9x_{10}\}$$

$$\{x_7x_8x_9\} \to \{abcd\} \to \{x_2x_3x_6x_7x_8x_9x_{10}\} \to \{bc\} \to \{x_2x_3x_4x_6x_7x_8x_9x_{10}\}.$$

Because there exists (1, 1) such that $X_1^{\alpha} \neq X_2^{\alpha}$, $X_1^{\alpha\beta} = X_2^{\alpha\beta}$, $X_1 \neq X_2^{\alpha\beta}$ and $X_1^{\alpha\beta} \neq X_2$, (1, 1) is the smallest object-object convergent unit of X_1 and X_2.

The convergence of attribute incremental sequences can be defined similarly.

Definition 7 For $B_1, B_2 \subseteq G$, we say that B_1 and B_2 converge in the attribute-attribute way at (i, j), if (i, j) satisfies $B_1^{(\beta\alpha)^i} = B_2^{(\beta\alpha)^j}$; if (i, j) further satisfies $B_1^{(\beta\alpha)^{i-1}\beta} \neq B_2^{(\beta\alpha)^{j-1}\beta}$, $B_1^{(\beta\alpha)^{i-1}} \neq B_2^{(\beta\alpha)^j}$ and $B_1^{(\beta\alpha)^i} \neq B_2^{(\beta\alpha)^{j-1}}$, (i, j) is said to be the smallest attribute-attribute convergent unit of B_1 and B_2. Similarly, if (i, j) satisfies $B_1^{(\beta\alpha)^i\beta} = B_2^{(\beta\alpha)^j\beta}$, B_1 and B_2 converge in the attribute-object way at (i, j); if (i, j) further satisfies $B_1^{(\beta\alpha)^{i-1}} \neq B_2^{(\beta\alpha)^{j-1}}$, $B_1^{(\beta\alpha)^{i-1}\beta} \neq B_2^{(\beta\alpha)^j\beta}$ and $B_1^{(\beta\alpha)^i\beta} \neq B_2^{(\beta\alpha)^{j-1}\beta}$, (i, j) is the smallest attribute-object convergent unit of B_1 and B_2.

Next, we will explore the properties of incremental sequence. First, we need the following results.

Lemma 1 *Let $X_1, X_2 \subseteq G$ be two sets of objects such that $X_1 \subset X_1^{\alpha\beta}$ and $X_2 \subset X_2^{\alpha\beta}$. If $\alpha \geq \beta$ and (i, j) is the smallest object-object convergent unit of X_1 and X_2, then we have $X_1 \subset X_1^{(\alpha\beta)} \subset X_1^{(\alpha\beta)^2} \cdots \subset X_1^{(\alpha\beta)^i}$ and $X_2 \subset X_2^{(\alpha\beta)} \subset X_2^{(\alpha\beta)^2} \cdots \subset X_2^{(\alpha\beta)^j}$.*

Proof If $\alpha \geq \beta$, by Proposition 2 (2) we have $X_1 \subseteq X_1^{(\alpha\beta)} \subseteq X_1^{(\alpha\beta)^2} \cdots \subseteq X_1^{(\alpha\beta)^{i-1}} \subseteq X_1^{(\alpha\beta)^i}$. Since $X_1 \subset X_1^{\alpha\beta}$, suppose that there exists $1 \leq k < i$ such that

$X_1^{(\alpha\beta)^k} = X_1^{(\alpha\beta)^{k+1}}$. Since $k < i$, by Proposition 4 we have $X_1^{(\alpha\beta)^k} = X_1^{(\alpha\beta)^{k+(i-k-1)}} = X_1^{(\alpha\beta)^{i-1}} = X_1^{(\alpha\beta)^{k+(i-k)}} = X_1^{(\alpha\beta)^i}$. However, because (i, j) is smallest object-object convergent unit of X_1 and X_2, we have $X_1^{(\alpha\beta)^i} = X_2^{(\alpha\beta)^j}$ and $X_1^{(\alpha\beta)^{i-1}} \neq X_2^{(\alpha\beta)^j}$, a contradiction with $X_1^{(\alpha\beta)^{i-1}} = X_2^{(\alpha\beta)^i} = X_2^{(\alpha\beta)^j}$. Similarly, we have $X_2 \subset X_2^{(\alpha\beta)} \subset X_2^{(\alpha\beta)^2} \cdots \subset X_2^{(\alpha\beta)^j}$. □

Similarly, the smallest attribute-attribute convergent unit has the following property.

Lemma 2 *Let B_1, $B_2 \subseteq M$ be two subsets of attributes such that $B_1 \subset B_1^{\beta\alpha}$ and $B_2 \subset B_2^{\beta\alpha}$. If $\alpha \leq \beta$ and (i, j) is the smallest object-object convergent unit of B_1 and B_2, then we have $B_1 \subset B_1^{(\beta\alpha)} \subset B_1^{(\beta\alpha)^2} \cdots \subset B_1^{(\beta\alpha)^i}$ and $B_2 \subset B_2^{(\beta\alpha)} \subset B_2^{(\beta\alpha)^2} \cdots \subset B_2^{(\beta\alpha)^j}$.*

Now, we can present the conditions of determining the convergence of object incremental sequences.

Theorem 1 *If $\alpha \geq \beta$ and two sets X_1, $X_2 \subseteq G$ converge, then there exist integers $m, n, k \geq 0$ such that $X_1^{(\alpha\beta)^m} \subseteq X_2^{(\alpha\beta)^k} \subseteq X_1^{(\alpha\beta)^n}$. Furthermore, if both G and M are finite, the converse also hold, i.e., X_1 and X_2 converge if and only if there exist integer $m, n, k \geq 0$ such that $X_1^{(\alpha\beta)^m} \subseteq X_2^{(\alpha\beta)^k} \subseteq X_1^{(\alpha\beta)^n}$.*

Proof If X_1 and X_2 converge, then there exist $i, j \geq 0$ satisfying $X_1^{(\alpha\beta)^i} = X_2^{(\alpha\beta)^j}$. Setting $m = n = i$, $k = j$, then we have $X_1^{(\alpha\beta)^m} \subseteq X_2^{(\alpha\beta)^k} \subseteq X_1^{(\alpha\beta)^n}$.

Conversely, suppose that both G and M are finite. Since G is finite, there exists a non-negative integer i such that $X_1^{(\alpha\beta)^i} = X_1^{(\alpha\beta)^{i+1}}$. In this case, by Proposition 4 (1) we have $X_1^{(\alpha\beta)^i} = X_1^{(\alpha\beta)^{i+m}} = X_1^{(\alpha\beta)^{i+n}}$, where $m, n \geq 0$. From the condition $X_1^{(\alpha\beta)^m} \subseteq X_2^{(\alpha\beta)^k} \subseteq X_1^{(\alpha\beta)^n}$ and Proposition 2 (2), we can derive $X_1^{(\alpha\beta)^{m+i}} \subseteq X_2^{(\alpha\beta)^{k+i}} \subseteq X_1^{(\alpha\beta)^{n+i}}$. Thus, we have $X_1^{(\alpha\beta)^i} = X_1^{(\alpha\beta)^{i+m}} \subseteq X_2^{(\alpha\beta)^{i+k}} \subseteq X_1^{(\alpha\beta)^{i+n}} = X_1^{(\alpha\beta)^i}$, which yields $X_2^{(\alpha\beta)^{i+k}} = X_1^{(\alpha\beta)^i}$, i.e., X_1 and X_2 converge. □

Theorem 1 presents a sufficient and necessary condition for determining the convergence of X_1 and X_2. In fact, even if X_1 and X_2 will converge, there may be no any relation between X_1 and X_2. However, Theorem 1 says that if, after a certain number of double threshold operations, the incremental results of X_1 and X_2 have the inclusion relation $X_1^{(\alpha\beta)^m} \subseteq X_2^{(\alpha\beta)^k} \subseteq X_1^{(\alpha\beta)^n}$, X_1 and X_2 will converge. Intuitively, if after a certain number of double threshold operations, X_1 and X_2 satisfy $X_1^{(\alpha\beta)^m} \subseteq X_2^{(\alpha\beta)^k}$, the inclusion relation will be maintained in the subsequent incremental results; in this case, however, X_1 and X_2 may not converge even if both G and M are finite. This is because the final incremental results of X_1 and X_2 may have a proper inclusion relation (see Example 4). Thus, the condition $X_2^{(\alpha\beta)^k} \subseteq X_1^{(\alpha\beta)^n}$ further ensures that X_1, after a certain number of double threshold operations, will collect more objects than $X_2^{(\alpha\beta)^k}$, excluding the proper inclusion relation between X_1 and X_2.

Example 4 For Table 1, set $\alpha = 0.50$ and $\beta = 0.40$. Let $X_1 = \{x_7\}$ and $X_2 = \{x_5\}$. The object incremental sequences of X_1 and X_2 are as follows:

$$\{x_7\} \to \{abcde\} \to \{x_2x_3x_6x_7x_8x_9x_{10}\} \to \{bc\}$$
$$\to \{x_2x_3x_4x_6x_7x_8x_9x_{10}\} \to \{bc\}\{x_5\} \to \{ce\}$$
$$\to \{x_2x_3x_4x_5x_6x_7x_8x_9x_{10}\} \to \{c\}$$
$$\to \{x_2x_3x_4x_5x_6x_7x_8x_9x_{10}\}.$$

If setting $m = k = 1$, we have $X_1^{\alpha\beta} \subseteq X_2^{\alpha\beta}$. When cognition ends, however, we also have $X_1^{(\alpha\beta)^3} = X_1^{(\alpha\beta)^2} \subset X_2^{\alpha\beta} = X_2^{(\alpha\beta)^2}$, i.e., the condition $X_1^{(\alpha\beta)^m} \subseteq X_2^{(\alpha\beta)^k}$ in Theorem 1 cannot ensure the convergence of X_1 and X_2.

Similarly, one can determine the convergence of attribute incremental sequences by the following conclusion.

Theorem 2 *If $\beta \geq \alpha$ and two sets B_1, $B_2 \subseteq M$ converge, then there exist integers $m, n, k \geq 0$ such that $B_1^{(\beta\alpha)^m} \subseteq B_2^{(\beta\alpha)^k} \subseteq B_1^{(\beta\alpha)^n}$. Furthermore, if both G and M are finite, the converse also hold, i.e., B_1 and B_2 converge if and only if there exist integers $m, n, k \geq 0$ such that $B_1^{(\beta\alpha)^m} \subseteq B_2^{(\beta\alpha)^k} \subseteq B_1^{(\beta\alpha)^n}$.*

We have discussed the convergence of object and attribute incremental sequences and presented some results to judge the convergence of object and attribute incremental sequences. The results are useful especially when both G and M are finite, a general case in data mining. If some incremental sequences do not meet the definitions of convergence, the incremental sequences are considered to be parallel to each other.

4.3 Robust variable threshold fuzzy concept lattice

If two sets converge, the two sets can be considered to be with the same information. In other words, after a certain number of double threshold operations, the incremental results of the two sets tend to be stable and coincide. Next, we will analyze the relationships of object and attribute incremental sequences and then construct the robust variable threshold fuzzy concept lattice based on double threshold operators.

In the following, we also suppose $\alpha \geq \beta$ and take object incremental sequence as example to establish the equivalence relation R. Similar discussions can be applied to attribute incremental sequences.

Definition 8 For the power set $P(G)$ of G, we can define the relation R on $P(G)$ as follows: for X_1, $X_2 \in P(G)$, $\langle X_1, X_2 \rangle \in R$ if and only if X_1 and X_2 converge.

Example 5 For Table 1, set $\alpha = 0.50$ and $\beta = 0.40$. Let $X_1 = \{x_2x_8x_{10}\}$ and $X_2 = \{x_2x_7x_8x_9\}$. The object incremental sequences of X_1 and X_2 can be computed as follows:

$$\{x_2x_8x_{10}\} \to \{bcd\} \to \{x_2x_3x_6x_7x_8x_9x_{10}\} \to \{bc\} \to \{x_2x_3x_4x_6x_7x_8x_9x_{10}\} \to \{bc\}$$

$$\{x_2x_7x_8x_9\} \to \{bcd\} \to \{x_2x_3x_6x_7x_8x_9x_{10}\} \to \{bc\} \to \{x_2x_3x_4x_6x_7x_8x_9x_{10}\} \to \{bc\}.$$

Thus, we have $\langle X_1, X_2 \rangle \in R$.

The relation R in Definition 8 has the following properties.

Proposition 5 Let $K = (G, M, I)$ be a fuzzy formal context. The relation R in Definition 8 is an equivalence relation, i.e., for any X_1, $X_2 \in P(G)$

1. $\langle X_1, X_1 \rangle \in R$
2. If $\langle X_1, X_2 \rangle \in R$, then $\langle X_2, X_1 \rangle \in R$
3. If $\langle X_1, X_2 \rangle \in R$ and $\langle X_2, X_3 \rangle \in R$, then $\langle X_1, X_3 \rangle \in R$.

Proof The proofs of self-reflexivity and symmetry are obvious.

If $\langle X_1, X_2 \rangle \in R$ and $\langle X_2, X_3 \rangle \in R$, there exists (i, j) satisfying $X_1^{(\alpha\beta)^i} = X_2^{(\alpha\beta)^j}$ and (m, n) satisfying $X_2^{(\alpha\beta)^m} = X_3^{(\alpha\beta)^n}$. Setting $k = \max(i, j, m, n)$, by Proposition 4 we have $X_1^{(\alpha\beta)^k} = X_2^{(\alpha\beta)^k}$ and $X_2^{(\alpha\beta)^k} = X_3^{(\alpha\beta)^k}$, and hence $X_1^{(\alpha\beta)^k} = X_3^{(\alpha\beta)^k}$, i.e., X_1 and X_3 converge. Thus, we have $\langle X_1, X_3 \rangle \in R$. □

According to Definition 8, the equivalence class of $X \in P(G)$ can be denoted as $[X]_R = \{X_1 \mid \langle X, X_1 \rangle \in R\}$, i.e., the sets that converge with X.

In the following, we suppose that G is finite. Thus, for $X \in P(G)$, there exists a non-negative integer n satisfying $X^{(\alpha\beta)^n} = X^{(\alpha\beta)^{n+1}}$, denoted as $[X]_{\max} = X^{(\alpha\beta)^n}$. Clearly, for any non-negative integer k, we have $[X]_{\max} = ([X]_{\max})^{(\alpha\beta)^k}$. Furthermore, for any $X_1 \in [X]_R$, we have $[X_1]_{\max} = [X]_{\max}$, meaning that all the sets in $[X]_R$ end at $[X]_{\max}$.

Definition 9 Let $K = (G, M, I)$ be a fuzzy formal context. If $\alpha \geq \beta$, for $X \subseteq G$, denote $\overline{X} = [X]_{\max}$ and $\overline{X}^{\alpha} = [X]_{\max}^{\alpha}$ and call $(\overline{X}, \overline{X}^{\alpha})$ a robust variable threshold concept of (G, M, I), where $\overline{X}$ is the extent and $\overline{X}^{\alpha}$ is the intent. Denote by $L^{\alpha\beta}(K)$ all the robust variable threshold concepts of K of thresholds α and β.

By Definition 9, the extent $\overline{X}$ of $(\overline{X}, \overline{X}^{\alpha})$ collects all the objects that have the salient features in $\overline{X}^{\alpha}$, and the intent $\overline{X}^{\alpha}$ captures the salient features common to all the objects in $\overline{X}$. In this case, the pair $(\overline{X}, \overline{X}^{\alpha})$ is stable because they can be identified by each other, i.e., we have $\overline{X}^{\alpha} = (\overline{X})^{\alpha}$ and $\overline{X} = (\overline{X}^{\alpha})^{\beta}$.

Robust variable threshold concepts in Definition 9 have the following properties.

Proposition 6 Let $K = (G, M, I)$ be a fuzzy formal context with G and M being finite, and $X, X_1, X_2 \subseteq G$. If $\alpha \geq \beta$, then the following conclusions hold.

1. $X \subseteq \overline{X}$
2. $X_1 \subseteq X_2 \Rightarrow \overline{X_1} \subseteq \overline{X_2}$
3. $\overline{X} = \overline{X}^{\alpha\beta} = \overline{\overline{X}}$
4. $\overline{X_1} \cup \overline{X_2} \subseteq \overline{\overline{X_1} \cup \overline{X_2}}$
5. $\overline{X_1} \cap \overline{X_2} = \overline{\overline{X_1} \cap \overline{X_2}}$
6. $\overline{X}^{\alpha} = \overline{X}^{\alpha\beta\alpha}$
7. $X_1 \subseteq X_2 \Rightarrow \overline{X_2}^{\alpha} \subseteq \overline{X_1}^{\alpha}$
8. $\overline{X_1} \subseteq \overline{X_2} \Leftrightarrow \overline{X_2}^{\alpha} \subseteq \overline{X_1}^{\alpha}$
9. $(\overline{X_1} \cup \overline{X_2})^{\alpha} = \overline{X_1}^{\alpha} \cap \overline{X_2}^{\alpha}$
10. $(\overline{X_1}^{\alpha} \cup \overline{X_2}^{\alpha})^{\beta} = \overline{X_1}^{\alpha\beta} \cap \overline{X_2}^{\alpha\beta}$.

Proof The conclusions of (1)–(3) and (6) follow from the definitions of $\overline{X}$ and $[X]_{\max}$ and Proposition 2. The conclusion of (4) follows from (1). The conclusion of (7) follows from (2) and Proposition 2.

(5) Since $\overline{X_1} \cap \overline{X_2} \subseteq \overline{X_1}$ and $\overline{X_1} \cap \overline{X_2} \subseteq \overline{X_2}$, by (2) and (3) we have $\overline{\overline{X_1} \cap \overline{X_2}} \subseteq \overline{\overline{X_1}} = \overline{X_1}$ and $\overline{\overline{X_1} \cap \overline{X_2}} \subseteq \overline{\overline{X_2}} = \overline{X_2}$, and thus $\overline{\overline{X_1} \cap \overline{X_2}} \subseteq \overline{X_1} \cap \overline{X_2}$. Conversely, by (1), we have $\overline{X_1} \cap \overline{X_2} \subseteq \overline{\overline{X_1} \cap \overline{X_2}}$ and thus $\overline{X_1} \cap \overline{X_2} = \overline{(\overline{X_1} \cap \overline{X_2})}$.

(8) If $\overline{X_1} \subseteq \overline{X_2} = \overline{X_2}^{\alpha\beta}$, by Proposition 2 we have $\overline{X_2}^{\alpha\beta\alpha} \subseteq \overline{X_1}^{\alpha}$. Since by (6) we have $\overline{X_2}^{\alpha\beta\alpha} = \overline{X_2}^{\alpha}$, we obtain $\overline{X_2}^{\alpha} \subseteq \overline{X_1}^{\alpha}$. Conversely, by $\overline{X_2}^{\alpha} \subseteq \overline{X_1}^{\alpha}$ and Proposition 2 we have $\overline{X_1}^{\alpha\beta} \subseteq \overline{X_2}^{\alpha\beta}$. Since $\overline{X_1} = [X_1]_{\max}$ and $\overline{X_2} = [X_2]_{\max}$, we have $\overline{X_1} = \overline{X_1}^{\alpha\beta}$ and $\overline{X_2} = \overline{X_2}^{\alpha\beta}$, and thus $\overline{X_1} \subseteq \overline{X_2}$.

(9) Since $\overline{X_1} \subseteq \overline{X_1} \cup \overline{X_2}$ and $\overline{X_2} \subseteq \overline{X_1} \cup \overline{X_2}$, by Proposition 2 we have $(\overline{X_1} \cup \overline{X_2})^{\alpha} \subseteq \overline{X_1}^{\alpha}$, $(\overline{X_1} \cup \overline{X_2})^{\alpha} \subseteq \overline{X_2}^{\alpha}$ and $(\overline{X_1} \cup \overline{X_2})^{\alpha} \subseteq \overline{X_1}^{\alpha} \cap \overline{X_2}^{\alpha}$. Suppose that there exists $m \in (\overline{X_1}^{\alpha} \cap \overline{X_2}^{\alpha}) \setminus (\overline{X_1} \cup \overline{X_2})^{\alpha}$. Since $m \notin (\overline{X_1} \cup \overline{X_2})^{\alpha}$, there exists $x \in (\overline{X_1} \cup \overline{X_2})$ satisfying $I(x, m) < \alpha$. By $m \in (\overline{X_1}^{\alpha} \cap \overline{X_2}^{\alpha})$, all elements $x_1 \in \overline{X_1}$ and $x_2 \in \overline{X_2}$ satisfy $I(x_1, m) \geq \alpha$ and $I(x_2, m) \geq \alpha$, implying that for any $x' \in (\overline{X_1} \cup \overline{X_2})$, we have

$I(x',m)<\alpha$, a contradiction with $I(x,m)<\alpha$ for $x\in(\overline{X_1}\cup\overline{X_2})$.

(10) Since $\overline{X_1}^{\alpha}\subseteq(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})$ and $\overline{X_2}^{\alpha}\subseteq(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})$, by Proposition 2 we have $\overline{X_1}^{\alpha\beta}\subseteq(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})^{\beta}$, $(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})^{\beta}\subseteq\overline{X_2}^{\alpha\beta}$ and $(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})^{\beta}\subseteq\overline{X_1}^{\alpha\beta}\cap\overline{X_2}^{\alpha\beta}$. Suppose that there exists $g\in(\overline{X_1}^{\alpha\beta}\cap\overline{X_2}^{\alpha\beta})\backslash\overline{(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})^{\beta}}$. Since $g\in\overline{X_1}^{\alpha\beta}\cap\overline{X_2}^{\alpha\beta}$, by definition of $\overline{X}^{\alpha\beta}$, all elements $m_1\in\overline{X_1}^{\alpha}$ and $m_2\in\overline{X_2}^{\alpha}$ satisfy $I(g,m_1)\geq\beta$ and $I(g,m_2)\geq\beta$. Therefore, for $g\in\overline{X_1}^{\alpha\beta}\cap\overline{X_2}^{\alpha\beta}$, any $m\in(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})$ satisfies $I(g,m)\geq\beta$, i.e., $g\in(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})^{\beta}$, a contradiction with $g\notin(\overline{X_1}^{\alpha}\cup\overline{X_2}^{\alpha})^{\beta}$. □

By Proposition 6 (1), (2) and (3), the operator $\overline{(\cdot)}$ is a closure operator on $P(G)$, which is similar to the closure operators $(\cdot)^{II}$ in FCA and $(\cdot)^{\alpha\alpha}$ in variable threshold concept lattice. Basically, the properties enable the system of robust variable concepts to be a closure system and thus a complete lattice, as shown in Theorem 3. The property (5) in Proposition 6 says that the meet of two extents is also an extent, a property also similar to FCA and variable threshold concept lattice. The property simplifies the definition of infimum in $L^{\alpha\beta}(K)$ of Theorem 3 from $\overline{\overline{X_1}\cap\overline{X_2}}$ to $\overline{X_1}\cap\overline{X_2}$. The property (8) in Proposition 6 shows the equivalence between the two inequalities, enabling the two equivalent definitions of the partial order defined in Eq. (1). The properties (5) and (9) in Proposition 6 indicate the applicability of the distributive property, similar to the property (5) in Proposition 1.

One distinctive characteristic of robust variable threshold fuzzy concept lattice from FCA and variable threshold concept lattice is its lack of symmetry between object and attribute. This is because in Proposition 2 the operator $(\cdot)^{\alpha\beta}$ does not satisfy $X\subseteq X^{\alpha\beta}$ if $\alpha\leq\beta$, and the operator $(\cdot)^{\beta\alpha}$ does not satisfy $B\subseteq B^{\beta\alpha}$ if $\alpha\geq\beta$. Thus, in robust variable threshold fuzzy concept lattice, we construct our model from sets of objects for $\alpha\leq\beta$ and from sets of attributes for $\alpha\geq\beta$, and all the properties in Proposition 6 concerning intents are also starting from sets of objects.

Thus, by Proposition 6, for two robust variable threshold concepts $(\overline{X_1},\overline{X_1}^{\alpha})$, $(\overline{X_2},\overline{X_2}^{\alpha})$, we can define

$$(\overline{X_1},\overline{X_1}^{\alpha})\leq(\overline{X_2},\overline{X_2}^{\alpha})\Leftrightarrow\overline{X_1}\subseteq\overline{X_2}\quad(\Leftrightarrow\overline{X_2}^{\alpha}\subseteq\overline{X_1}^{\alpha})\tag{1}$$

where $(\overline{X_1},\overline{X_1}^{\alpha})$ is called a sub-concept of $(\overline{X_2},\overline{X_2}^{\alpha})$ and $(\overline{X_2},\overline{X_2}^{\alpha})$ is a super-concept of $(\overline{X_1},\overline{X_1}^{\alpha})$.

Theorem 3 shows that the above defined partial order leads to a complete lattice.

Theorem 3 *Let $K=(G,M,I)$ be a fuzzy formal context with G and M being finite. For two robust variable threshold concepts $(\overline{X_1},\overline{X_1}^{\alpha})$, $(\overline{X_2},\overline{X_2}^{\alpha})\in L^{\alpha\beta}(K)$, define the infimum and the supremum by*

$$(\overline{X_1},\overline{X_1}^{\alpha})\wedge(\overline{X_2},\overline{X_2}^{\alpha})=(\overline{X_1}\cap\overline{X_2},(\overline{X_1}\cap\overline{X_2})^{\alpha})$$
$$(\overline{X_1},\overline{X_1}^{\alpha})\vee(\overline{X_2},\overline{X_2}^{\alpha})=(\overline{\overline{X_1}\cup\overline{X_2}},(\overline{\overline{X_1}\cup\overline{X_2}})^{\alpha}).$$

Then, $(L^{\alpha\beta}(K),\wedge,\vee)$ forms a complete lattice, called the ROBust variable threshold fuzzy Concept Lattice (RobCL) of K.

Proof By Proposition 6 (5), we have $(\overline{X_1}\cap\overline{X_2},(\overline{X_1}\cap\overline{X_2})^{\alpha})\in L^{\alpha\beta}(K)$ and $(\overline{\overline{X_1}\cup\overline{X_2}},\left(\overline{\overline{X_1}\cup\overline{X_2}}\right)^{\alpha})\in L^{\alpha\beta}(K)$. Furthermore, it is obvious that $(\overline{X_1}\cap\overline{X_2},(\overline{X_1}\cap\overline{X_2})^{\alpha})$ is a sub-concept of $(\overline{X_1},\overline{X_1}^{\alpha})$ and $(\overline{X_2},\overline{X_2}^{\alpha})$, and that $(\overline{\overline{X_1}\cup\overline{X_2}},\left(\overline{\overline{X_1}\cup\overline{X_2}}\right)^{\alpha})$ is a super-concept of $(\overline{X_1},\overline{X_1}^{\alpha})$ and $(\overline{X_2},\overline{X_2}^{\alpha})$.

Next, we prove that any sub-concept of $(\overline{X_1},\overline{X_1}^{\alpha})$ and $(\overline{X_2},\overline{X_2}^{\alpha})$ is a sub-concept of $(\overline{X_1}\cap\overline{X_2},(\overline{X_1}\cap\overline{X_2})^{\alpha})$, i.e., $(\overline{X_1}\cap\overline{X_2},(\overline{X_1}\cap\overline{X_2})^{\alpha})$ is the infimum of $(\overline{X_1},\overline{X_1}^{\alpha})$ and $(\overline{X_2},\overline{X_2}^{\alpha})$. For any sub-concept $(\overline{X_3},\overline{X_3}^{\alpha})$ of $(\overline{X_1},\overline{X_1}^{\alpha})$ and $(\overline{X_2},\overline{X_2}^{\alpha})$, since $\overline{X_3}\subseteq\overline{X_1}$ and $\overline{X_3}\subseteq\overline{X_2}$, we have $\overline{X_3}\subseteq\overline{X_1}\cap\overline{X_2}$ and thus $(\overline{X_3},\overline{X_3}^{\alpha})$ is a sub-concept of $(\overline{X_1}\cap\overline{X_2},(\overline{X_1}\cap\overline{X_2})^{\alpha})$.

Similarly, for any super-concept $(\overline{X_4},(\overline{X_4})^{\alpha})$ of $(\overline{X_1},\overline{X_1}^{\alpha})$ and $(\overline{X_2},\overline{X_2}^{\alpha})$, since $\overline{X_1}\cup\overline{X_2}\subseteq\overline{X_4}$, by Proposition 6 (2) and (3), we have $\overline{(\overline{X_1}\cup\overline{X_2})}\subseteq\overline{\overline{X_4}}=\overline{X_4}$, i.e., $(\overline{X_1}\cup\overline{X_2},(\overline{\overline{X_1}\cup\overline{X_2}})^{\alpha})$ is the supermum of $(\overline{X_1},\overline{X_1}^{\alpha})$ and $(\overline{X_2},\overline{X_2}^{\alpha})$. □

Definition 10 Let $K=(G,M,I)$ be a fuzzy formal context. If $\beta\geq\alpha$, for $B\subseteq M$, denote $\overline{B}=[B]_{\max}$ and $\overline{B}^{\beta}=[B]^{\beta}_{\max}$ and call $(\overline{B}^{\beta},\overline{B})$ a robust variable threshold concept of (*G*, *M*, *I*), where $\overline{B}^{\beta}$ is the extent and $\overline{B}$ is the intent.

By Definition 10, the extent $\overline{B}^{\beta}$ of $(\overline{B}^{\beta},\overline{B})$ collects all the objects that possess the salient features in $\overline{B}$, and the intent $\overline{B}$ captures the salient features common to all the objects in $\overline{B}^{\beta}$. In this case, the pair $(\overline{B}^{\beta},\overline{B})$ is stable because they can be identified by each other. Obviously, if we change the set X of objects in Proposition 6 with the set B of attributes and the operator $\alpha\beta$ with $\beta\alpha$, the equations in Proposition 6 still hold.

Example 6 Setting $\alpha=0.54$ and $\beta=0.51$, we can obtain the robust variable threshold fuzzy concept lattice of Table 1, as shown in Fig. 4. For comparison, the variable threshold concept lattices with $\alpha=\beta=0.54$ and $\alpha=\beta=0.51$ of Table 1 are shown in Fig. 5 and Fig. 6.

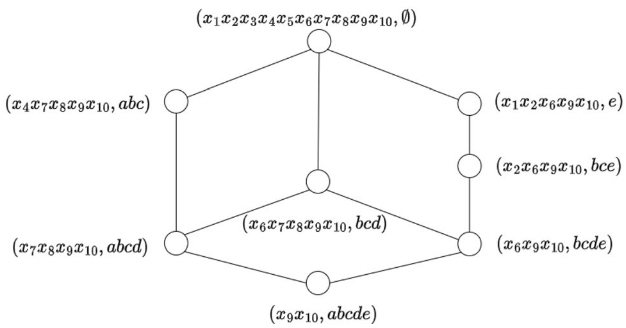

Fig. 4 RobCL with $\alpha = 0.54$ and $\beta = 0.51$

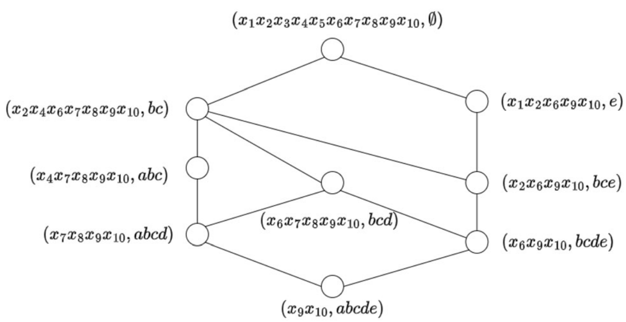

Fig. 5 Variable threshold concept lattice with $\alpha = \beta = 0.54$

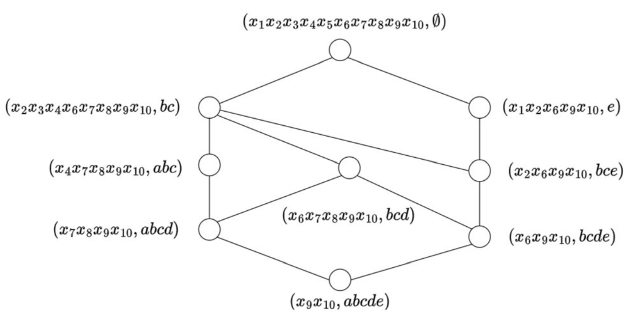

Fig. 6 Variable threshold concept lattice with $\alpha = \beta = 0.51$

In Fig. 4, all the meets of extents are also extents. For example, the meet of the extents of $(x_7x_8x_9x_{10}, abcd)$ and $(x_6x_9x_{10}, bcde)$ is the extent of $(x_9x_{10}, abcde)$. Comparing Fig. 4 with Fig. 5, we can find that the variable threshold concept $(x_2x_4x_6x_7x_8x_9x_{10}, bc)$ has been removed from Fig. 4. This is because the object incremental sequence of $\{x_2x_4x_6x_7x_8x_9x_{10}\}$ is

$$\{x_2x_4x_6x_7x_8x_9x_{10}\} \rightarrow \{bc\} \rightarrow \{x_2x_3x_4x_6x_7x_8x_9x_{10}\} \rightarrow \emptyset \rightarrow \{x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}\}$$

i.e., the corresponding robust variable threshold concept of $\{x_2x_4x_6x_7x_8x_9x_{10}\}$ is $(x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}, \emptyset)$. In the incremental sequence of $\{x_2x_4x_6x_7x_8x_9x_{10}\}$, $\{x_2x_4x_6x_7x_8x_9x_{10}\}^{\alpha} = \{bc\}$ captures all the salient symptoms of $\{x_2x_4x_6x_7x_8x_9x_{10}\}$, and $\{x_2x_4x_6x_7x_8x_9x_{10}\}^{\alpha\beta} = \{x_2x_3x_4x_6x_7x_8x_9x_{10}\}$ collects all the patients who have the symptoms $\{bc\}$ at the degree greater than or equal 0.51. Compared with variable threshold concept lattice, x_3 has been included in $\{x_2x_4x_6x_7x_8x_9x_{10}\}^{\alpha\beta}$, whereas $\{x_2x_4x_6x_7x_8x_9x_{10}\}^{\alpha\alpha} = \{x_2x_4x_6x_7x_8x_9x_{10}\}$ does not take x_3 as its member. This is because for the symptoms $\{x_2x_4x_6x_7x_8x_9x_{10}\}^{\alpha} = \{bc\}$, as stated above, if one employs α for the following computation, i.e. $\{x_2x_4x_6x_7x_8x_9x_{10}\}$, one may collect only the patients who have the symptoms $\{bc\}$ at the degree greater than or equal to 0.54, missing the patients who have the symptoms $\{bc\}$ less than but close to 0.54, i.e. x_3, where $\beta \leq I(x_3, b) = 0.52 \leq \alpha$ and $\beta \leq I(x_3, c) = 0.53 \leq \alpha$. Continuing the process, we can obtain the robust variable threshold concept $(x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}, \emptyset)$ for $\{x_2x_4x_6x_7x_8x_9x_{10}\}$, i.e., there is no common salient symptoms for $\{x_2x_4x_6x_7x_8x_9x_{10}\}$ at the threshold $\alpha = 0.54$ and the tolerance threshold $\beta = 0.51$.

Similarly, comparing Fig. 4 with Fig. 6, one can find that the variable threshold concept $(x_2x_3x_4x_6x_7x_8x_9x_{10}, bc)$ has been removed from Fig. 6. In Fig. 6, for $\{x_2x_3x_4x_6x_7x_8x_9x_{10}\}$, since $\alpha = 0.51$, we have $\{x_2x_3x_4x_6x_7x_8x_9x_{10}\}^{\alpha} = \{bc\}$. In Fig. 4, however, because $0.51 < I(x_3, b) = 0.52 < 0.54$ and $0.51 < I(x_3, c) = 0.53 < 0.54$, we have $\{x_2x_3x_4x_6x_7x_8x_9x_{10}\}^{\alpha} = \emptyset$, where $\alpha = 0.54$. Thus, $(x_2x_3x_4x_6x_7x_8x_9x_{10}, bc)$ is a variable threshold concept of Fig. 6 with 0.51 being the threshold of objects having attributes, whereas $(x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}, \emptyset)$ is a robust variable threshold concept of Fig. 4 with 0.54 being the threshold of objects having attributes and 0.51 being the tolerance threshold.

5 Conclusion and Further Work

In this paper, we introduced the tolerance threshold to variable threshold concept lattice, solving the problem that variable threshold concept lattice is easily disturbed by noise. Then, we constructed RobCL and proved that RobCL is a complete lattice. In addition, we further made a comparison with variable threshold concept lattice and the results show that when the two thresholds coincide with each other, RobCL degenerates to variable threshold concept lattice and thus variable threshold concept lattice should be regarded as a special case of RobCL.

In the process of constructing RobCL, by introducing the tolerance threshold, an object may have a salient feature, a general feature or no feature; in other words, there are three possible states between an object and an attribute, considering with the idea of three-way decision [29–32]. It is easy to find the difference between RobCL and three-way concept lattice [33–36], because the latter is constructed on a binary formal context by identifying the common attributes not shared by extents, whereas the former is constructed on a fuzzy formal context by taking into account the general attributes shared by extents. Certainly, the relationship between RobCL and three-way concept lattice may be further clarified after determining whether the three states can be replaced by three values, say 0, 0.5 and 1, without altering the structure of RobCL. The latter, however, seems not straightforward because the incremental characteristics of incremental sequence make it difficult to capture the underlying principle. For example, Proposition 3 still holds for this case, with substituting $I(g,m) \geq \beta$ by $I(g,m) \geq 0$ and $\alpha > I(g_1,m_1) \geq \beta$ by $I(g_1,m_1) = 0.5$.

Another further work concerns the relationship between RobCL and variable threshold concept lattice. Although it is clear that RobCL is a general case of variable threshold concept lattice, the preliminary results show that the robust version of variable threshold concept lattice, i.e. RobCL may reduce the fuzzy concepts in variable threshold concept lattice, a surprising result. The basic explanation for this is that the robustness reduce the fuzzy concepts that are not robust; however, further clarification is also needed.

It should be noted that although RobCL was applied only in medical diagnosis in the paper, RobCL can be applied in all the fields whose data can be represented as fuzzy two-dimensional tables. For example, in uncertain group decision-making, the set of objects usually consists of different experts, the set of attributes consists of different solutions, and the values between objects and attributes are the evaluating results of solutions by experts. In this case, if only a simple threshold is chosen, the experts' attitudes towards the solutions will be either 'agree' or 'disagree'. Thus, a tolerance threshold may allow the evaluating results to fluctuate within a certain small range and improve the robustness of group decision-making in the uncertain case.

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Data availability All relevant data are within the paper.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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