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Observer-Based Robust Adaptive TS Fuzzy Control of Uncertain Systems with High-Order Input Derivatives and Nonlinear Input–Output Relationships

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Abstract The conventional state-space form often leads to control design strategies and stability analysis techniques generally applicable to dynamical processes. Nevertheless, it may also lead to higher model complexity and loss of interpretability. Here, we skip this representation for nonlinear dynamical systems with high-order input derivatives and nonlinear input–output relationships. Specifically, we incorporate the principle of H_∞ design within an observer-based adaptive fuzzy controller to guarantee robust stabilization and trajectory tracking for such nonlinear systems. The proposed approach has four integral components. Firstly, zero-order Takagi–Sugeno fuzzy systems approximate nonlinear and uncertain functions by the estimated states of the observer. Secondly, the H_∞ control attenuates fuzzy approximation errors, observer errors, and environmental effects to a prescribed attenuation level. Thirdly, the adaptive laws and the H_∞ term are met with simple equations, avoiding the positive definite matrices in Lyapunov equations. Fourthly, a compensation term is added to ensure the stability of the closed-loop system. Fifthly, the Lyapunov theory guarantees the asymptotic stability of the overall system and the H_∞ tracking performance of the output. Finally, the proposed method is applied to two unknown nonlinear systems under disturbances, noises, packet loss, and asymmetric dead-zone. The first is a second-order spring-mass-damper trolley system, and the second is a third-order nonlinear system. Comparing the results with a recent competing controller reveals that the proposed approach improves transparency and lowers tunable parameters, fuzzy basis functions dimension, the observation and tracking errors, the consumed energies, and the settling times.

Keywords Nonlinear input–output relationships · Input derivatives · Robust control · Adaptive fuzzy control · Uncertainties

1 Introduction

Uncertainties and nonlinearities have been studied extensively in systems and control theory. Among them, models that involve the input's time derivatives present considerable challenge. Such dynamics appear in various applications such as cranes [1], cars with trailers [1], piezoelectric actuators [2], capacitor loops connected with voltage sources [3], inductor cut-sets connected with current sources [3], transducers in setting the reference inputs [3], active suspension systems with vibration control [3], descriptor variable systems [3], flat systems [1], and biological processes [4]. Controlling such models is challenging since they do not directly conform to the conventional state-space models, and inputs are not independent. Even if the input derivatives are alternated with state variables [3, 5, 6], this method faces two problems. First, these added states lead to the added controller complexity and its stability analysis. Second, as state variables describe the system behavior without external forces affecting the system, alternating the input derivatives with the states leads to less physical interpretation.

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Alternatively, directly handling such dynamics could be a valuable framework for such a problem, but few works have addressed it.

There are considerably fewer works on controlling the mentioned dynamics without converting to the conventional state-space form. Among the few, Zhang et al. offered indirect and direct adaptive first-order Takagi–Sugeno (TS) fuzzy controllers with state observers for models affine with the input and its first derivative [7, 8]. The fuzzy systems were used to estimate three unknown nonlinear functions. They later showed that converting the nonaffine nonlinear pure feedback dynamics to a form with the input derivative and input–output nonlinear relationships might also reach better tracking and avoid the backstepping structure with virtual inputs [9]. Furthermore, the work of [10] simplified the mentioned method through a low-pass filter driven by a control input. However, the mentioned papers only studied systems with the first-order derivative of the input.

TS fuzzy approaches have been developed over the past 30 years, particularly following the seminal works of Wang. As mentioned before, there are only four references on TS fuzzy approaches of uncertain nonlinear systems with the time derivative of the input. However, no fuzzy controller exists on unknown nonlinear systems, including higher-order input derivatives. In other words, to study a broader range of existing systems, we consider uncertain nonlinear systems, where the order of input derivatives is extended, and the model is nonaffine in the input and its derivatives. The aim is to design a simpler and more robust controller for handling different categories of uncertainties. As a result, we propose an observer-based H_∞ indirect adaptive zero-order TS fuzzy controller (OH$_\infty$IAZTSFC). In other words, the investigation's contributions are as follows:

- To study a broader range of existing systems, we consider uncertain nonlinear systems, where the input derivatives are extended to mth order and the model is nonaffine in not only u but also $\dot{u},\ldots,u^{(m-1)}$.
- We propose zero-order TS fuzzy systems, whose inputs are the states, the dynamical system input, and its derivatives, to improve transparency and simplicity and reduce the amount of online adjustable parameters.
- We propose a H_∞ controller and a compensation control for such augmented systems to guarantee asymptotic stability and improve robustness under uncertainties.
- We avoid solving the Riccati equation to define the $\boldsymbol{P}$ matrix and $\boldsymbol{B_c}$ vector in adaptive rules.

Note that the proposed method lowers tracking errors, observer errors, adjustable parameters, and consumed energies, besides improving the convergence speed under realistic uncertainties, such as unmeasured states, disturbances, measurement noise, data loss, and dead-zone.

This paper contains the following sections. Section 2 includes research in different categories. Section 3 describes the dynamics. Section 4 explains the proposed methodology. Section 5 investigates stability and convergence. Section 6 demonstrates the simulation results compared with a recent competing approach. Ultimately, Sect. 7 provides the conclusions.

Notation The superscripts "T" and "-1" show the transposition and inverse, respectively. R^n denotes n-dimensional Euclidean space. $\|.\|$ indicates the Euclidean norm, $\|A\|_B^2$ means A^TBA and $|\ .\ |$ shows the absolute value. For any derivable signal such as u, $\dot{u}$ is the first derivative, $\ddot{u}$ is the second derivative, and $u^{(m)}$ is the mth derivative. For any ideal C, C^* is the optimization, $\hat{C}$ is the estimation, and $\tilde{C} = C - \hat{C}$. Besides that, $D' = L^{-1}(s)D$ for any D.

2 Related Works

Fuzzy logic systems (FLSs) have universal approximation properties [11] to handle uncertainties. The literature is abundant, with excellent examples of applying fuzzy logic to control complex systems. One may refer to some seminal works, such as by Mamdani in 1975 [12], which was improved by online adaptive rules [13] and observer design [14] to handle model and external uncertainties. Recently, the observer-based adaptive fuzzy approach was developed to control different types of nonlinear systems, such as stochastic systems [15], fractional-order systems [16], large-scale systems [17], multi-agent systems [18], strict-feedback systems [19], nonstrict-feedback systems [20], rigorous feedback cascade systems [21], singular systems [22], stochastic systems [23], switched systems [24], non-triangular systems [25], and networked systems [26]. As a result, even the more recent works on fuzzy control generally assume the nonlinear systems in the standard and conventional state-space models.

H_∞ analysis [27] has been merged into the fuzzy controllers to improve robustness and semi-global stabilization. The fruitful results occur by attenuating the errors and the influence of environmental effects on an arbitrary level [28]. A combined observer-based H_∞ adaptive fuzzy approach is an excellent candidate for creating a comprehensible controller. For instance, an H_∞ controller with a PD form was employed in [29] for stable and robust tracking in the indirect adaptive fuzzy method. In [30], the composite learning based on a serial–parallel identifier promoted the mentioned combined method. The H_∞ controller compensated the residual errors of the fuzzy system and the observer to the desired level. In [31], an H_∞-H_2

strategy enhanced the trade-off between energy consumption and control performances. In [32], an adaptive event-triggered scheme reduced resource consumption and output variation. A networked TS fuzzy filter improved the design flexibility, guaranteeing stability with the desired H_∞ disturbance attenuation performance. Falla-Gh [33] considered the H_∞ controller to compensate uncertainties for different nonlinear systems in an adaptive fuzzy method. However, the strategies mentioned above focused on nonlinear systems with conventional state-space representation without any input derivates.

3 Problem Description

Consider nth-order unknown nonlinear systems with mth-order input derivatives and nonlinear input–output relationships as

$$\begin{cases} \dot{x}_i = x_{i+1}, \quad 1 \le i \le n-1, \\ \dot{x}_n = f(\boldsymbol{x}, u, \dot{u}, \ldots, u^{(m-1)}) + g(\boldsymbol{x}, u, \dot{u}, \ldots, \\ \qquad u^{(m-1)})u^{(m)} + d(t), \\ y = x_1, \end{cases} . \quad (1)$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T = (x, \dot{x}, \ldots, x^{n-1})^T \in R^n$ is the unavailable state vector of the plant, $f(.)$ and $g(.)$ are the unknown nonlinear functions, $y \in R$ is the output of the system, $u \in R$ is the system input to which its derivatives also contribute, and $d(t)$ is the unknown external disturbance. Because of the appearance of the derivatives of u in the dynamic equations, (1) does not have the conventional state-space representation. $f(.)$ and $g(.)$ relate to $\boldsymbol{u} = (u, \dot{u}, \ldots, u^{(m-1)})$; which causes the nonlinearity of the input–output relationship.

Equation (1) is represented in the following form by

$$\boldsymbol{A_0} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \text{ and } \boldsymbol{C} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}:$$

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A_0}\boldsymbol{x} + \boldsymbol{B}\big(f(\boldsymbol{x}, \boldsymbol{u}) + g(\boldsymbol{x}, \boldsymbol{u})u^{(m)} + d\big), \\ y = \boldsymbol{C}^T. \end{cases} \quad (2)$$

Assumption 1 $f(.)$ and $g(.)$ are unknown, nonlinear, bounded, smooth, and continuous functions for $\boldsymbol{x} \in R^n$ and $\boldsymbol{u} \in R^m$. It is assumed that $0 < |f(.)| \le F^U$, where F^U is an unknown upper bound of $f(.)$. As the system (1) should be controllable, $g^{-1}(.)$ exists which means $g(.) \ne 0$. There are unknown constants G_L and G^U such that $0 < G_L \le |g(.)| \le G^U$, where G_L and G^U show the lower and upper bound, respectively.

Assumption 2 $u(t)$ is a continuous input which is m-times differentiable.

Assumption 3 $y_d(t)$ is the reference signal which is n-times differentiable, bounded, smooth, and known.

Assumption 4 $d(t)$ is the unknown bounded disturbance, i.e., $|d| \le D$ and D is the unknown upper bound.

Control Objectives This paper's main goals are to track the reference trajectory and to achieve high robustness against uncertainties, whereas all signals involved are bounded.

4 Proposed Controller Design

Figure 1 shows the proposed method of this section.

4.1 Proposed Adaptive Fuzzy Controller

By knowing $f(.)$, $g(.)$, and all the elements of $\boldsymbol{x}$ with $d(t) = 0$, we propose the control law for the maximal order of the input derivatives as

$$u^{*(m)}(t) = \frac{1}{g(\boldsymbol{x}, \boldsymbol{u}^*)}\Big[-f(\boldsymbol{x}, \boldsymbol{u}^*) + y_m^{(n)}(t) + \boldsymbol{K}^T\boldsymbol{e}\Big], \quad (3)$$

where $\boldsymbol{u}^* = \Big(u^*, \dot{u}^*, \ldots, u^{*(m-1)}\Big)$, $\boldsymbol{e} = [e, \dot{e}, \ldots, e^{n-1}]^T$, $e = y_d - y$ is the tracking error, $\boldsymbol{K} = [k_n, k_{n-1}, \ldots, k_1]^T$ which yields all roots of $s^n + k_1 s^{n-1} + \cdots + k_n = 0$ in the open left half of the complex s-plane and guarantees $\boldsymbol{A_1} = \boldsymbol{A_0} - \boldsymbol{B}\boldsymbol{K}^T$ to be strictly Hurwitz in Lyapunov equation as

$$(\boldsymbol{A_0} - \boldsymbol{B}\boldsymbol{K}^T)^T\boldsymbol{P_0} + \boldsymbol{P_0}(\boldsymbol{A_0} - \boldsymbol{B}\boldsymbol{K}^T) = -\boldsymbol{Q_0}, \quad (4)$$

where $\boldsymbol{P_0}$ and $\boldsymbol{Q_0}$ are positive definite matrices.

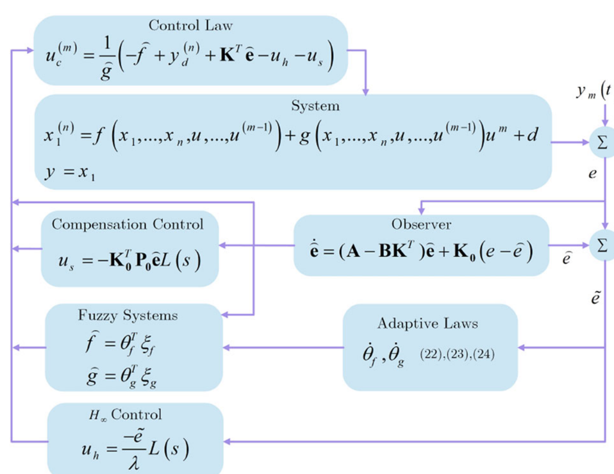

Fig. 1 Scheme of the proposed OH$_\infty$IAZTSFC method

Substituting (3) into (1), the closed-loop control system is given as

$$e^{(n)} + k_1 e^{(n-1)} + \cdots + k_n e = 0, \tag{5}$$

where it means e converges to zero as $t \to \infty$.

Unfortunately, $f(.)$ and $g(.)$ are unknown, all the states are not available, and the disturbances exist in real applications. Therefore, FLSs are used for the estimation.

Lemma 1 [11] *FLS can approximate any continuous function on a compact set.*

We use zero-order TS FLSs to identify the unknown functions in the dynamical system such that the adaptive laws adjust the fuzzy controller parameters based on the Lyapunov synthesis approach. We describe the lth rule as

IF $\hat{x}_1$ is A_1^l and $\hat{x}_2$ is A_2^l and $\cdots$ and $\hat{x}_n$ is A_n^l and u is B_1^l and $\dot{u}$ is B_2^l and $\cdots$ and $u^{(m-1)}$ is B_m^l;
THEN $y_l = a^l$,

where $l = (1,\ldots,M)$, M is the number of rules; $\hat{\boldsymbol{x}} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n]^T$ is the estimation of $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$ based on the observer design; $\hat{\boldsymbol{x}}$ and $\boldsymbol{u}$ are the fuzzy inputs; A_i^l is the fuzzy set of $\hat{x}_i$, $i = 1,\ldots,n$; $B_{i'}^l$ is the fuzzy set of $u^{(i'-1)}$, $i' = 1,\ldots,m$; y_l is the output of lth rule; and a^l is the constant coefficient of the lth output function.

The FLS's output is obtained as

$$y = \frac{\sum_{l=1}^{M} y_l \prod_{i=1}^{n} \exp\left(-\left(\frac{\hat{x}_i - \bar{x}_i^l}{\delta_{1,i}^l}\right)^2\right) \prod_{i'=1}^{m} \exp\left(-\left(\frac{u^{(i'-1)} - \bar{u}_{i'}^l}{\delta_{2,i'}^l}\right)^2\right)}{\sum_{l=1}^{M} \prod_{i=1}^{n} \exp\left(-\left(\frac{\hat{x}_i - \bar{x}_i^l}{\delta_i^l}\right)^2\right) \prod_{i'=1}^{m} \exp\left(-\left(\frac{u^{(i'-1)} - \bar{u}_{i'}^l}{\delta_{2,i'}^l}\right)^2\right)} = \theta^T \xi(\hat{\boldsymbol{x}}, \boldsymbol{u}), \tag{6}$$

and

$$\theta = \left(a^1, a^2, \ldots, a^M\right)^T,$$
$$\xi(\hat{\boldsymbol{x}}, \boldsymbol{u}) = [\xi_1, \xi_2, \ldots, \xi_M]^T,$$
$$\xi_l = \frac{\prod_{i=1}^{n} \exp\left(-\left(\frac{\hat{x}_i - \bar{x}_i^l}{\delta_{1,i}^l}\right)^2\right) \prod_{i'=1}^{m} \exp\left(-\left(\frac{u^{(i'-1)} - \bar{u}_{i'}^l}{\delta_{2,i'}^l}\right)^2\right)}{\sum_{l=1}^{M} \prod_{i=1}^{n} \exp\left(-\left(\frac{\hat{x}_i - \bar{x}_i^l}{\delta_{1,i}^l}\right)^2\right) \prod_{i'=1}^{m} \exp\left(-\left(\frac{u^{(i'-1)} - \bar{u}_{i'}^l}{\delta_{2,i'}^l}\right)^2\right)}, \tag{7}$$

where the fuzzifier is singleton, the membership functions are Gaussian with the center $\bar{x}_i^l$ and the width $\delta_{1,i}^l$ for $\hat{x}_i$ and also with the center $\bar{u}_{i'}^l$ and the width $\delta_{2,i'}^l$ for $u^{(i'-1)}$, the inference engine is product, and the defuzzifier is center-average.

Therefore, $f(.)$ and $g(.)$ are approximated as

$$\begin{aligned} \hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f) &= \theta_f^T \xi_f(\hat{\boldsymbol{x}}, \boldsymbol{u}), \\ \hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_g) &= \theta_g^T \xi_g(\hat{\boldsymbol{x}}, \boldsymbol{u}), \end{aligned} \tag{8}$$

where ξ_f, ξ_g are the fuzzy basis function vectors, and θ_f, θ_g are the adjustable parameters vectors.

Definition 1 The optimal parameters' vectors are described as

$$\begin{aligned} \theta_f^* &= \arg\min_{\theta_f \in \Omega_f} \left[\sup\left|\hat{f} - f\right|\right], \\ \theta_g^* &= \arg\min_{\theta_g \in \Omega_g} \left[\sup\left|\hat{g} - g\right|\right], \end{aligned} \tag{9}$$

where $\Omega_f \triangleq \left\{\theta_f \in R^{N_f} \middle| |\theta_f| \le M_f\right\}$ and $\Omega_g \triangleq \left\{\theta_g \in R^{N_g} \middle| 0 < \varepsilon < |\theta_g| \le M_g\right\}$ are the constraint sets, and the constants of M_f, M_g , and ε are unknown.

Definition 2 The minimum approximation error is defined as follows:

$$\omega_1 \triangleq \hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f^*) - f(\boldsymbol{x}, \boldsymbol{u}) + (\hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_g^*) - g(\boldsymbol{x}, \boldsymbol{u}))u_c^{(m)}. \tag{10}$$

Definition 3 The observer error vector is written as

$$\hat{\boldsymbol{e}} = \left(\hat{e}, \dot{\hat{e}}, \ldots, \hat{e}^{(n-1)}\right)^T, \tag{11}$$

where $\hat{e} = y_d - \hat{x}$.

The proposed fuzzy control is given by feedback linearization method as

$$u_c^{(m)} = \frac{1}{\hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_g)}\left[-\hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f) + y_m^{(n)}(t) + \boldsymbol{K}^T\hat{\boldsymbol{e}} - u_h - u_s\right], \tag{12}$$

where $\hat{f}(.)$ and $\hat{g}(.)$ are the FLSs' outputs; u_h is an H_∞ control term to attenuate the uncertainties effect; u_s is a linear combination of the error estimates; u_h and u_s ensure the stability of the closed-loop system.

By (12) and (1), the dynamics is rewritten as

$$\begin{cases} \dot{\boldsymbol{e}} = \boldsymbol{A_1 e} + \boldsymbol{B}[(\hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f) - \hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f^*)) + \\ \quad (\hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_g) - \hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_g^*))u_c^{(m)} + u_h + u_s + \omega], \\ e = \boldsymbol{C}^T\boldsymbol{e}, \end{cases} \tag{13}$$

where $\omega = \omega_1 - d$.

4.2 Designing the Observer

The concerns over the cost of sensors lead to using observers for estimating unavailable states. The observer error equation is written as follows [34]:

$$\begin{cases} \dot{\hat{\boldsymbol{e}}} = \boldsymbol{A}_1\hat{\boldsymbol{e}} - \boldsymbol{BK}^T\hat{\boldsymbol{e}} + \boldsymbol{K}_0(e - \hat{e}), \\ \hat{e} = \boldsymbol{C}^T\hat{\boldsymbol{e}}, \end{cases} \tag{14}$$

where $\boldsymbol{K_0} = [k_{01}, k_{02}, \ldots, k_{0n}]^T$ is the observer gain vector satisfying $\boldsymbol{A_2} = \boldsymbol{A_1} - \boldsymbol{K_0}\boldsymbol{C}^T$ to be a strict Hurwitz matrices.

By defining $\tilde{\boldsymbol{e}} = \boldsymbol{e} - \hat{\boldsymbol{e}} = \hat{\boldsymbol{x}} - \boldsymbol{x} \in R^n$, the observation state-space equation is calculated as

$$\begin{cases} \dot{\tilde{\boldsymbol{e}}} = \boldsymbol{A_2}\tilde{\boldsymbol{e}} + \boldsymbol{B}[(\hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f) - \hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f^*)) + \\ \qquad (\hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_g) - \hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_g^*))u_c^{(m)} + u_h + u_s + \omega], \\ \tilde{e} = \boldsymbol{C}^T\tilde{\boldsymbol{e}}. \end{cases} \tag{15}$$

As only y is measurable, $\tilde{e}$ is available. Therefore, the strictly positive real (SPR) Lyapunov design method is employed to analyze the system stability and one has

$$\begin{aligned} \dot{\tilde{\boldsymbol{e}}} &= H(s)L(s)L^{-1}(s)\Big[(\hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f) - \hat{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_f^*)) + \\ &(\hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta g) - \hat{g}(\hat{\boldsymbol{x}}, \boldsymbol{u}|\theta_g^*))u_c^{(m)} + u_h + u_s + \omega\Big], \end{aligned} \tag{16}$$

where $H(s) = \boldsymbol{C}^T(sI - \boldsymbol{A_2})^{-1}\boldsymbol{B}$ is in stable; $L(s) = s^{(n-1)} + C_{n-1}^1 \alpha s^{n-2} + \cdots + C_{n-1}^{n-2}\ \alpha^{n-2}s + \alpha^{n-1} = (s+\alpha)^{n-1}$, $\alpha > 0$, where $L^{-1}(s)$ is a proper stable transfer function and $H(s)L(s)$ is a proper SPR transfer function.

By defining $\tilde{\theta}_f = \theta_f - \theta_f^*$ and $\tilde{\theta}_g = \theta_g - \theta_g^*$, (15) is represented as follows:

$$\begin{cases} \dot{\tilde{\boldsymbol{e}}} = \boldsymbol{A_2}\tilde{\boldsymbol{e}} + \boldsymbol{B_c}(\tilde{\theta}_f\xi_f' + \tilde{\theta}_g\xi_g' u_c^{(m)} + u_h' + u_s' + \omega') \\ \tilde{e} = \boldsymbol{C}^T\tilde{\boldsymbol{e}}, \end{cases}. \tag{17}$$

where $\boldsymbol{B_c} = \left[1, C_{n-1}^1\alpha, \ldots, C_{n-1}^{n-2}\alpha^{n-2}, \alpha^{n-1}\right]^T$; $\xi_f'(.) = L^{-1}(s)\xi_f(.)$; $\xi_g'(.) = L^{-1}(s)\xi_g(.)$; $u_h' = L^{-1}(s)u_h$; $u_s' = L^{-1}(s)u_s$; and $\omega' = L^{-1}(s)\omega$.

4.3 Proposed Robust Controllers and Adaptive Laws

The H_∞ control term, the compensation control term, and adaptive laws are described in this section. The Lyapunov function proves the validity of these equations.

As $H(s)L(s)$ is a proper SPR transfer function, there are $\boldsymbol{P} > 0$ and $\boldsymbol{Q} > 0$ in the Riccati equation such that [35]

$$\begin{aligned} &\boldsymbol{A_2}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A_2} - \boldsymbol{P}\boldsymbol{B_c}\left(\frac{2}{\lambda} - \frac{1}{\rho^2}\right)\boldsymbol{B_c}^T\boldsymbol{P} = -\boldsymbol{Q}, \\ &\boldsymbol{P}\boldsymbol{B_c} = \boldsymbol{C}, \end{aligned} \tag{18}$$

where $(0 < \lambda \leq 2\rho^2)$.

The H_∞ controller compensates for the approximation error. Lowering ρ improves the tracking performance while it could saturate the control input and cause higher frequency chattering.

Since $\tilde{e}$ is measurable, we define the H_∞ control term as follows:

$$u_h' = -\frac{1}{\lambda}\boldsymbol{B_c}^T\boldsymbol{P}\tilde{\boldsymbol{e}}. \tag{19}$$

By $\boldsymbol{P}\boldsymbol{B}_c = \boldsymbol{C}$ (18), we can simplify (19) as

$$u_h' = -\frac{1}{\lambda}\boldsymbol{C}^T\tilde{\boldsymbol{e}} = -\frac{\tilde{e}}{\lambda}. \tag{20}$$

The above equation is considered without needing for finding $n \times n$-matrix $\boldsymbol{P}$ and $n \times 1$-vector $\boldsymbol{B_c}$. Therefore, we describe the robust term by only one designing parameter as λ.

The compensation control term is defined as

$$u_s' = -\boldsymbol{K_0}^T\boldsymbol{P_0}\hat{\boldsymbol{e}}. \tag{21}$$

Lemma 2 *The adaptive laws are defined as*

$$\begin{aligned} \dot{\theta}_f &= -\gamma_1\tilde{\boldsymbol{e}}^T\boldsymbol{P}\boldsymbol{B_c}\xi_f'(\hat{\boldsymbol{x}}, \boldsymbol{u}) = -\gamma_1\tilde{e}\xi_f'(\hat{\boldsymbol{x}}, \boldsymbol{u}), \\ \dot{\theta}_g &= -\gamma_2\tilde{\boldsymbol{e}}^T\boldsymbol{P}\boldsymbol{B_c}\xi_g'(\hat{\boldsymbol{x}}, \boldsymbol{u})u_c^{(m)} = -\gamma_2\tilde{e}\xi_g'(\hat{\boldsymbol{x}}, \boldsymbol{u})u_c^{(m)}, \end{aligned} \tag{22}$$

where $\gamma = [\gamma_1, \gamma_2] > 0$ regulate the convergence of the adaptive parameters. According to $\boldsymbol{P}\boldsymbol{B_c} = \boldsymbol{C}$ (18), the control law and adaptive laws are simplified to be free of designing $\boldsymbol{P}$ matrix and $\boldsymbol{B_c}$ vector.

Lemma 3 [13] *The projection method restricts θ_f and θ_g to be in the sets of Ω_f and Ω_g, respectively. As a result, $\dot{\theta}_f$ is modified as*

$$\dot{\theta}_f = \begin{cases} -\gamma_1\tilde{e}\xi_f'(\hat{\boldsymbol{x}}, \boldsymbol{u}) \\ \text{if } \left(\|\theta_f\| < M_f\right) \text{ or } (\|\theta_f\| = M_f \text{ and } \tilde{e}\theta_f^T\xi_f' \geq 0), \\ -\gamma_1\tilde{e}\xi_f'(\hat{\boldsymbol{x}}, \boldsymbol{u}) + \gamma_1\tilde{e}\dfrac{\theta_f\theta_f^T\xi_f'(\hat{\boldsymbol{x}}, \boldsymbol{u})}{\|\theta_f\|^2}, \\ \text{if } (\|\theta_f\| = M_f \text{ and } \tilde{e}\theta_f^T\xi_f' < 0), \end{cases}. \tag{23}$$

and $\dot{\theta}_g$ is rewritten as follows:

a. For an element $\theta_{gi} = \varepsilon$ of θ_g, one has

$$\dot{\theta}_{gi} = \begin{cases} -\gamma_2\tilde{e}\xi_g'(\hat{\boldsymbol{x}}, \boldsymbol{u})u_c^{(m)}, & \text{if } \tilde{e}\xi_{gi}'u_c^m < 0, \\ 0, & \text{if } \tilde{e}\xi_{gi}'u_c^m \geq 0, \end{cases}. \tag{24}$$

to guarantee $|\theta_g| \geq \varepsilon$.

b. Otherwise,

$$\dot{\theta}_g = \begin{cases} -\gamma_2\tilde{e}\xi_g'(\hat{\boldsymbol{x}}, \boldsymbol{u})u_c^{(m)}, \\ \text{if } \left(\|\theta_g\| < M_g\right) \text{ or } \left(\|\theta_g\| = M_g \text{ and } \tilde{e}\theta_g^T\xi_g'u_c^{(m)} \geq 0\right). \\ -\gamma_2\tilde{e}\xi_g'(\hat{\boldsymbol{x}}, \boldsymbol{u})u_c^{(m)} + \gamma_2\tilde{e}\dfrac{\theta_g\theta_g^T\xi_g'(\hat{\boldsymbol{x}}, \boldsymbol{u})u_c^{(m)}}{\|\theta_g\|^2}, \\ \text{if } \left(\|\theta_g\| = M_g \text{ and } \tilde{e}\theta_g^T\xi_g'u_c^{(m)} < 0\right), \end{cases}. \tag{25}$$

which guarantees $|\theta_g| \leq M_g$.

By the above Lemma, it is proved that θ_f and θ_g are bounded and also $\tilde{\theta}_f$ and $\tilde{\theta}_g$ ultimately converge to compact residual sets around zero.

5 Stability Analysis

The stability of the closed-loop system in the proposed scheme is proven by the following theorem.

Theorem 1 *By considering Assumptions* 1–4 *for the plant* (1) *with the control law* (12), *the robust controllers* (20), (21), *and the adaptive laws* (23)–(25), *the following results are obtained*:

- *The asymptotic stability is proved, and the system output y follows the reference signal y_d.*
- *All the signals of the closed-loop system, i.e., $\boldsymbol{x}(t)$, $\hat{\boldsymbol{x}}(t)$, $\theta_f(t)$, $\theta_g(t)$, $u_c(t)$, $\dot{u}_c(t)$,..., and $u_c^{(m)}$ are bounded for all* $t \geq 0$.
- *The H_∞ tracking criterion is achieved for a prescribed attenuation level* $\rho > 0$.

Proof The Lyapunov function is considered as

$$V = \frac{1}{2}\hat{\boldsymbol{e}}^T\boldsymbol{P_0}\hat{\boldsymbol{e}} + \frac{1}{2}\tilde{\boldsymbol{e}}^T\boldsymbol{P}\tilde{\boldsymbol{e}} + \frac{1}{2\gamma_1}\tilde{\theta}_f^T\tilde{\theta}_f + \frac{1}{2\gamma_2}\tilde{\theta}_g^T\tilde{\theta}_g, \tag{26}$$

The time derivative of V is computed as

$$\begin{aligned}\dot{V} = &-\frac{1}{2}\hat{\boldsymbol{e}}^T\boldsymbol{Q_0}\hat{\boldsymbol{e}} - \frac{1}{2}\tilde{\boldsymbol{e}}^T\boldsymbol{Q}\tilde{\boldsymbol{e}} + \frac{1}{2}\tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\left(\frac{2}{\lambda} - \frac{1}{\rho^2}\right)\\ &\boldsymbol{B_c}^T\boldsymbol{P}\tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}\boldsymbol{C}\boldsymbol{K_0}^T\boldsymbol{P_0}\hat{\boldsymbol{e}} + \boldsymbol{e}\boldsymbol{PB_c}\left(\boldsymbol{\theta_f^T}\xi' + \boldsymbol{\theta_g^T}\xi_g'\boldsymbol{u_c^{(m)}} + \boldsymbol{u_h'} + \boldsymbol{u_s'} + \boldsymbol{\omega'}\right)\\ &+\frac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\theta}_f + \frac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\theta}_g.\end{aligned} \tag{27}$$

According to $\boldsymbol{C} = \boldsymbol{PB_c}$ in the third line of Eq. (18), we have

$$\begin{aligned}\dot{V} = &-\frac{1}{2}\hat{\boldsymbol{e}}^T\boldsymbol{Q_0}\hat{\boldsymbol{e}} - \frac{1}{2}\tilde{\boldsymbol{e}}^T\boldsymbol{Q}\tilde{\boldsymbol{e}} + \frac{1}{2}\tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\left(-\frac{1}{\rho^2}\right)\boldsymbol{B_c}^T\boldsymbol{P}\tilde{\boldsymbol{e}}\\ &+\tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\left(\frac{1}{\lambda}\boldsymbol{B_c}^T\boldsymbol{P}\tilde{\boldsymbol{e}} + \boldsymbol{K_0}^T\boldsymbol{P_0}\hat{e} + u_h' + u_s' + \omega'\right)\\ &+\frac{1}{\gamma_1}\tilde{\theta}_f^T\left(\dot{\theta}_f + \gamma_1\tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\xi_f'\right)\\ &+\frac{1}{\gamma_2}\tilde{\theta}_g^T\left(\dot{\theta}_g + \gamma_2\tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\boldsymbol{\xi_g'}\boldsymbol{u_c^{(m)}}\right).\end{aligned} \tag{28}$$

By replacing $u_s' = -\boldsymbol{K_0^T}\boldsymbol{P_0}\hat{\boldsymbol{e}}$, $u_h' = -\frac{1}{\lambda}\boldsymbol{B_c^T}\boldsymbol{P}\tilde{\boldsymbol{e}}$, $\dot{\theta}_f = -\gamma_1\tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\xi_f'$ and $\dot{\theta}_g = -\gamma_2\tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\xi_g' u_c^{(m)}$, the result is realized as

$$\dot{V} = -\frac{1}{2}\hat{\boldsymbol{e}}^T\boldsymbol{Q_0}\hat{\boldsymbol{e}} - \frac{1}{2}\tilde{\boldsymbol{e}}^T\boldsymbol{Q}\tilde{\boldsymbol{e}} - \frac{1}{2\rho^2}\tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\boldsymbol{B_c}^T\boldsymbol{P}\tilde{\boldsymbol{e}} + \tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}\omega'. \tag{29}$$

By considering $a = \omega'$ and $b = \tilde{\boldsymbol{e}}^T\boldsymbol{PB_c}$ in Young's inequality, which holds $ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2 a^2$, the asymptotic stability is proved as follows:

$$\dot{V} \leq -\frac{1}{2}\hat{\boldsymbol{e}}^T\boldsymbol{Q_0}\hat{\boldsymbol{e}} - \frac{1}{2}\tilde{\boldsymbol{e}}^T\boldsymbol{Q}\tilde{\boldsymbol{e}} + \frac{1}{2}\rho^2\omega'^T\omega'. \tag{30}$$

Hence $\dot{V}$ is negative semi-definite when ω' is too small.

The integration of the above equation from 0 to T gives

$$\begin{aligned}\int_0^T \dot{V}(t) \leq &-\frac{1}{2}\int_0^T \hat{\boldsymbol{e}}^T\boldsymbol{Q_0}\hat{\boldsymbol{e}}\,\mathrm{d}t - \frac{1}{2}\int_0^T \tilde{\boldsymbol{e}}^T\boldsymbol{Q}\tilde{\boldsymbol{e}}\,\mathrm{d}t +\\ &\frac{1}{2}\rho^2\int_0^T \omega'^T\omega'\,\mathrm{d}t.\end{aligned} \tag{31}$$

By the Euclidean norm and $\|A\|_B^2 = A^TBA$, we have

$$\begin{aligned}\int_0^T \dot{V}(t) \leq &-\frac{1}{2}\int_0^T \|\hat{\boldsymbol{e}}\|_{\boldsymbol{Q_0}}^2\,\mathrm{d}t - \frac{1}{2}\int_0^T \|\tilde{\boldsymbol{e}}\|_{\boldsymbol{Q}}^2\,\mathrm{d}t +\\ &\frac{1}{2}\rho^2\int_0^T \|\omega'\|^2\,\mathrm{d}t.\end{aligned} \tag{32}$$

By considering $\boldsymbol{Q_1} = diag[\boldsymbol{Q_0}, \boldsymbol{Q}]$ and $\boldsymbol{e_1} = [\hat{\boldsymbol{e}}, \tilde{\boldsymbol{e}}]^T$, we have

$$2(V(T) - V(0)) \leq -\int_0^T \|\boldsymbol{e_1}\|_{\boldsymbol{Q_1}}^2\,\mathrm{d}t + \rho^2\int_0^T \|\omega'\|^2\,\mathrm{d}t. \tag{33}$$

Assuming a positive constant $M_\omega > 0$ such that $\int_0^\infty \|\omega'\|^2 dt \leq M_\omega$, we have

$$2(V(T) - V(0)) \leq -\int_0^T \|\boldsymbol{e_1}\|_{\boldsymbol{Q_1}}^2\,\mathrm{d}t + \rho^2 M_\omega. \tag{34}$$

As $V(T) \geq 0$, we get

$$\int_0^\infty \|\boldsymbol{e_1}\|_{\boldsymbol{Q_1}}^2\,\mathrm{d}t \leq 2V(0) + \rho^2 M_\omega. \tag{35}$$

Therefore, the integral $\int_0^T \|\bar{\boldsymbol{e}}\|_{\boldsymbol{Q_1}}^2\,dt$ is bounded. By Barbalat's lemma, it is proved that $\lim_{t\to\infty}\tilde{\boldsymbol{e}} = 0$, and $\lim_{t\to\infty}\hat{\boldsymbol{e}} = 0$, which yield $\lim_{t\to\infty}\boldsymbol{e} = 0$. It shows that y asymptotically tracks y_d. The boundedness of the errors also guarantees the boundedness of $\hat{\boldsymbol{x}}$, $\boldsymbol{x}$, and $u_c^{(m)}$. As the signals of $u, \dot{u}, \ldots$, and $u^{(m-1)}$ are obtained through the integration of $u_c^{(m)}$ signal, their boundedness is proved. Moreover, by Lemma 3, the parameters of θ_f and θ_g can reach their optimal values.

By defining $\boldsymbol{P_1} = diag[\boldsymbol{P_0}, \boldsymbol{P}]$, we have

$$\begin{aligned}\int_0^\infty \|\bar{\boldsymbol{e}}\|_{\boldsymbol{Q_1}}^2\,\mathrm{d}t \leq &\bar{\boldsymbol{e}}^T(0)\boldsymbol{P_1}\bar{\boldsymbol{e}}(0) + \frac{1}{\gamma_1}\tilde{\theta}_f^T(0)\tilde{\theta}_f(0)\\ &+\frac{1}{\gamma_2}\tilde{\theta}_g^T(0)\tilde{\theta}_g(0) + \rho^2 M_\omega.\end{aligned} \tag{36}$$

The above shows the following H_∞ tracking performance is met. □

6 Simulation Results

We apply two uncertain nonlinear systems with inputs derivative under various uncertainties for evaluation.

6.1 Second-Order System

Example 1 The spring-mass-damper system is used in different architectures, e.g., cars, trains, screw presses in forging processes, seismically excited multi-story buildings, and human structure [36–40]. The spring-mass-damper trolley system is expressed as [7, 41]

$$m\ddot{y}_2 + f(\dot{y}_2 - \dot{y}_1) + k\exp^{-(y_2-y_1)}(y_2 - y_1) = 0, \quad (37)$$

where y_1 is the displacement input, y_2 is the mass displacement output, m is the mass, and f and k are the coefficients of the viscous friction and the stiffness spring, respectively (see Fig. 2).

By defining $x_1 = y_2$, $x_2 = \dot{y}_2$, $u = y_1$, $\dot{u} = \dot{y}_1$ [7, 8], (37) is described as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \dfrac{-fx_2 - k\exp^{-(x_1-u)}(x_1 - u) - bx_1^3}{m} + \dfrac{f}{m}\dot{u} + d, \cdot \\ y \ = x_1 + v, \end{cases} \quad (38)$$

where d and v are a disturbance and a measurement noise, respectively.

Physical parameters are $m = 1\,\mathrm{kg}$, $k = 10\,\mathrm{N/m}$, $f = 200\,\mathrm{Ns/m}$ [7, 8]. The reference trajectory is a unit step function. 4^{th}-order Runge–Kutta method is taken to solve equations in the numerical simulations in which the step time of the simulation experiments is 0.0001s. The initial values are $\boldsymbol{x}(0) = [2, 1.3]^T$, $\hat{\boldsymbol{x}}(0) = [1.5, 1.2]^T$, $u(0) = 0$. The control parameters are assigned by trial and error to attain reasonable tracking error and control energy consumption. Therefore, $\boldsymbol{K} = [1000, 60]^T$, $\boldsymbol{K_0} = [1, 2]^T$, $\gamma_1 = 1, \gamma_2 = 2$, $L(s) = (s + 20)$, and $\lambda = 0.01$. The Gaussian functions are used in the estimation of $f(.)$ and $g(.)$ with $\delta = 0.5$ for all membership functions. Furthermore, the centers of the fuzzy membership functions are equally spaced in the range of $\hat{x}_1$, $\hat{x}_2$, and u. The universe of discourse of each FLS's input variable is defined by three fuzzy sets; therefore, the number of fuzzy rules equals 27 ($3 \times 3 \times 3 = 27$). Hence, each of ξ_f, ξ_g, θ_f, and θ_g includes

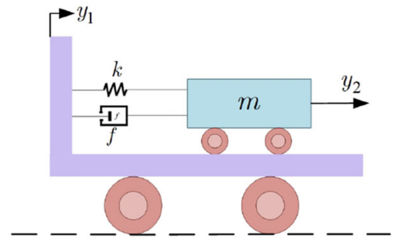

Fig. 2 The spring-mass-damper trolley system

27 parameters. For comparison results, the method of [7] is considered, which employs observer-based indirect adaptive first-order TS fuzzy controllers (OIAFTSFC). By defining three fuzzy sets for each input in the first-order TS, the dimension of the fuzzy basis functions and the adaptive parameters increases to 36 elements ($3 \times 3 \times 4 = 36$) in OIAFTSFC. The simulation time equals 1 s in Fig. 3 and Table 1. We investigate different conditions in the following case studies (A-G).

Case A. Ideal Condition Here, we consider the trolley when there are no environmental effects. Figure 3 shows the tracking of the reference signal and the estimated state, the boundedness of the system inputs, the estimations of the unknown nonlinear functions, and the better transient response. Table 1 shows that the proposed method lowers the mean squared tracking error e_1 (MSE), the mean squared observer error $\hat{e}_1$ ($MS\hat{E}$), the consumed control energy measurement criteria by u (J_1), the consumed control energy measurement criteria by $\dot{u}$ (J_2), and the settling time (T_s) by 60.3%, 32.7%, 2.9%, 75%, and 48.6%, respectively.

Case B. Sinusoidal Disturbance The trolley is considered in the presence of a fast external sinusoidal disturbance $d(t) = 3\sin(20t)$. The comparison results between the two methods are shown in Table 1. OH$_\infty$IAZTSFC lowers MSE by 60.3%, $MS\hat{E}$ by 32.7%, J_1 by 2.9%, J_2 by 74.9%, and T_s by 48.6%.

Case C. Pulse Disturbance Here, we apply an external pulse disturbance $d(t) = \begin{cases} 1, & 0.3 \le t \le 0.6 \\ 0, & otherwise \end{cases}$.. Table 1 shows that OH$_\infty$IAZTSFC lowers MSE by 60.3%, $MS\hat{E}$ by 32.7%, J_1 by 2.8%, J_2 by 75%, and T_s by 48.1%.

Case D. Noise Under the measurement noise ($SNR = 40\,dB$), MSE, $MS\hat{E}$, J_1, J_2, and T_s are improved in the proposed method by 60.2%, 32.7%, 2.9%, 79.3%, and 48.1%, respectively (see Table 1).

Case E. Data Loss Data loss is in networked control systems with negative effects on system stability. The stochastic data losses effect in the trolley system is shown as [42]

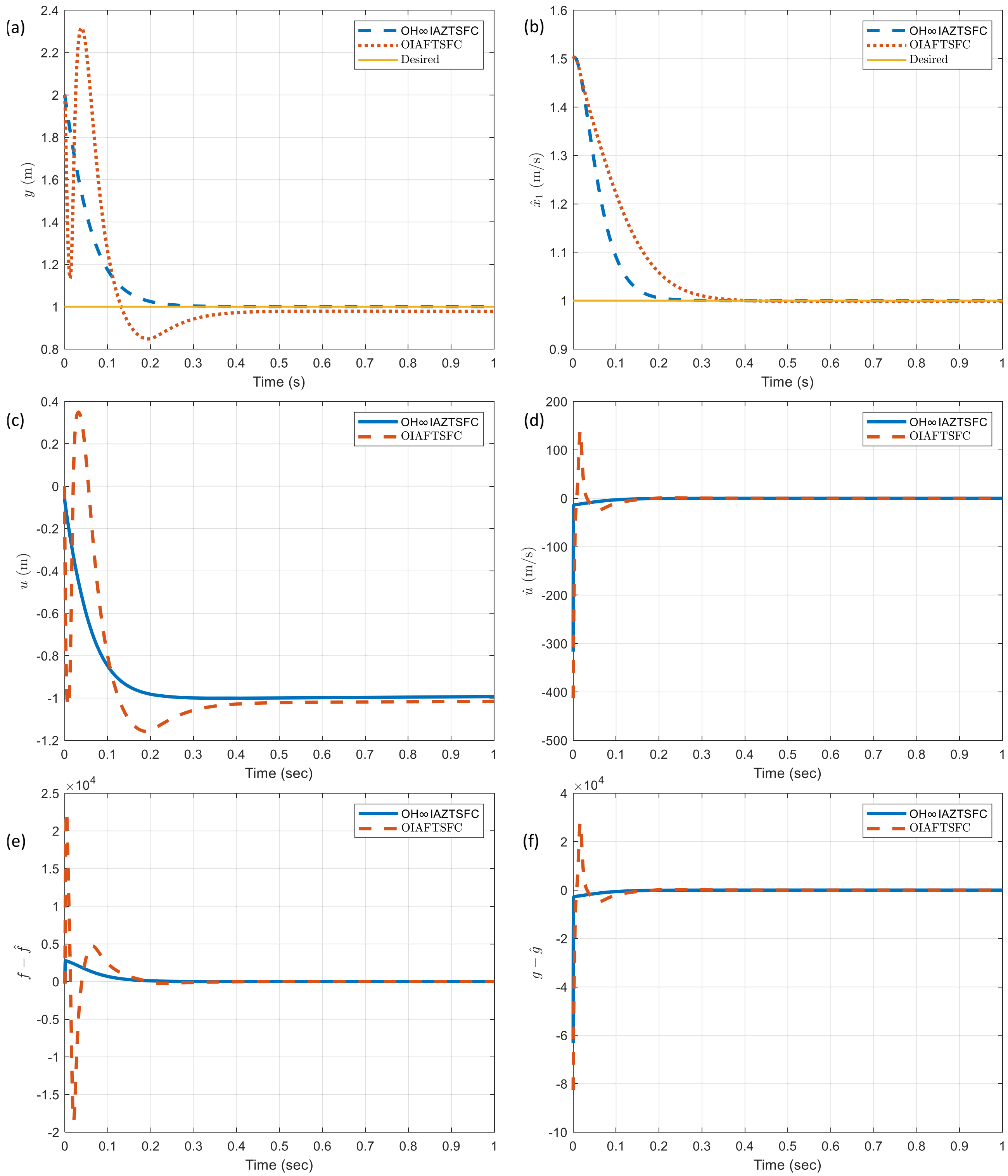

Fig. 3 Comparison of the proposed method (OH$_\infty$IAZTSFC) with OIAFTSFC [7] in Case A. **a** Trajectory of y. **b** $\hat{x}_1$. **c** $\dot{u}$. **d** u. **e** $f-\hat{f}$. **f** $g-\hat{g}$

$$x_{ic}(t) = \alpha x_i(t) + (1-\alpha)x_i(t-1), \quad (39)$$

where $x_{ic}(t)$ is the ith states used in the control, $x_i(t)$ is the ith state to be transmitted through the communication network, and $0 < \alpha < 1$ is the constant data loss probability. By supposing $\alpha = 0.65$, Table 1 shows that OH$_\infty$IAZTSFC lowers MSE (by 98.2%), $MS\hat{E}$ (by 29.6%), J_1 (by 62.7%), J_2 (by 88.1%), and T_s (by 29%).

Table 1 Comparison of the proposed method (OH_∞IAZTSFC) with OIAFTSFC [7] for Example 1 by mean squared errors (MSE and $\hat{MSE}$), the consumed control energies (J_1 and J_2), the settling time (T_s), and the number of the adaptive parameters

Method	OIAFTSFC [7]							OH_∞IAZTSFC						
Case studies	A.	B.	C.	D.	E.	F.	G.	A.	B.	C.	D.	E.	F.	G.
$MSE = \frac{1}{T}\int_0^T e_1^2 \mathrm{d}t$	0.0783	0.0784	0.0784	0.0784	0.2262	0.2701	0.0823	**0.0311**	**0.0311**	**0.0312**	**0.0311**	**0.0040**	**0.0311**	**0.0332**
$\hat{MSE} = \frac{1}{T}\int_0^T \hat{e}_1^2 \mathrm{d}t$	0.0159	0.0159	0.0159	0.0159	0.0142	**0.0105**	0.0156	**0.0107**	**0.0107**	**0.0107**	**0.0107**	**0.0100**	0.0107	**0.0110**
$J_1 = \frac{1}{T}\int_0^T \lvert u(t)\rvert \mathrm{d}t$	0.9746	0.9755	0.9743	0.9746	1.1536	1.3383	1.0894	**0.9464**	**0.9471**	**0.9464**	**0.9464**	**0.4303**	**0.9441**	**0.9436**
$J_2 = \frac{1}{T}\int_0^T \lvert \dot{u}(t)\rvert \mathrm{d}t$	4.0937	4.0947	4.0940	4.0944	3.9658	7.4908	3.8763	**1.0253**	**1.0269**	**1.0246**	**2.6350**	**0.4722**	**1.0267**	**1.0242**
T_s	0.317	0.319	0.316	0.317	5.660	3.871	6.006	**0.163**	**0.164**	**0.164**	**0.163**	**4.020**	**0.165**	**0.167**
Adaptive param. no.	36	36	36	36	36	36	36	**27**	**27**	**27**	**27**	**27**	**27**	**27**

The cases of A, B, C, D, E, F, and G show the system under ideal condition, sinusoidal disturbance, pulse disturbance, noise, data loss, dead-zone, and sensitivity analysis, respectively. Bold numbers indicate better performance

Case F. Dead-Zone The dead-zone nonlinearities of the actuators negatively affect the control systems. The output of the dead-zone with input $u(t)$ is considered as

$$\phi(u(t)) = \begin{cases} m_r(u(t) - b_r), & \text{if } \ u(t) \geq b_r, \\ 0, & \text{if } \ -b_l < u(t) < b_r, \\ m_l(u(t) - b_l), & \text{if } \ u(t) \leq -b_l, \end{cases} . \quad (40)$$

where m_r and m_l are the slope of the dead-zone, and b_r and b_l are the right and left dead-zone breakpoints, respectively.

We consider the asymmetric dead-zone by $m_r = m_l = 0.4$, $b_r = 0.5$, and $b_l = -0.9$ (see Table 1). OH_∞IAZTSFC improves all of the comparison criteria, e.g., MSE, J_1, and J_2 by 88.5%, 29.5%, and 86.3%, respectively. The proposed method also shows a fast convergence rate, as T_s improves by 95.7% compared with OIAFTSFC, whereas the mean squared observer error has no significant change (1.9%).

Case G. Sensitivity Analysis The sensitivity of the proposed method to variations of the controller's parameters is studied here. We calculate performance change with respect to lowering the controller's parameters by 10% ($\boldsymbol{K_0}$, $\boldsymbol{K}$, γ, α, and λ) from the designed values. Table 1 shows that the proposed method maintains its advantage in terms of the comparison criteria, e.g., MSE (by 59.7%), $\hat{MSE}$ (by 29.5%), J_1 (by 13.4%), J_2 (by 73.6%), and T_s (by 97.2%).

6.2 Third-Order System

Example 2 Consider the following third-order system as [9]:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = F(x_1, x_2, u) + G(x_1, x_2, u)\dot{u} + d \\ y = x_1 + v, \end{cases} . \quad (41)$$

and

$$F(x_1, x_2, u) = 2\left(1 - x_1^2\right)\left(-x_1 + 2x_2 + \frac{u}{\sqrt{|u| + 0.1}} - 2x_1^2 x_2\right) - x_2 - \frac{1}{2u(|u| + 0.1)\sqrt{|u| + 0.1}},$$
$$G(x_1, x_2, u) = \frac{\sqrt{|u| + 0.1}}{|u| + 0.1}, \quad (42)$$

where $\boldsymbol{x} = [x_1, x_2, x_3]^T$ is the state vector, y is the system output, u and $\dot{u}$ are the system inputs, d is an external disturbance, and v is a measurement noise.

In this example, the reference trajectory y_d is a step function. The numerical simulation method is a fourth-order Rung-Kutta with the simulation step time 0.001 sec. The initial states are supposed to be $\boldsymbol{x}(0) = [0.2, 0, 0]^T$ and $\hat{\boldsymbol{x}}(0) = [0.1, 0.1, 0.1]^T$. The control parameters are assigned by trial and error to attain reasonable tracking error and control energy consumption, e.g., $\boldsymbol{K} = [600, 400, 100]^T$, $\boldsymbol{K_0} = [50, 300, 1500]^T$, $\gamma_1 = 1$, $\gamma_2 = 1$, $L(s) = (s^2 + 14s + 49)$, and $\lambda = 0.00004$. The basic fuzzy functions are the Gaussian membership function with $\delta = 0.5$. Furthermore, the centers of the fuzzy membership functions are equally spaced in the range of the inputs of the fuzzy system $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, and u. The universe of discourse of each input variable consists of three fuzzy sets, so there are 81 fuzzy rules ($3 \times 3 \times 3 \times 3 = 81$) and 81 adaptive parameters. But, in OIAFTSFC, ξ_f, ξ_g, θ_f, and θ_g include 135 ($27 \times 5 = 135$) parameters because first-order TS increases the dimension burden. The following seven cases (A–G) express the better performance of the proposed method compared with OIAFTSFC in all the evaluation criteria.

Case A. Ideal Condition By considering the system (38) without any environmental effects, the better results of the tracking performance and the transient response are shown in Fig. 4. Table 2 also expresses the proposed approach lowers MSE by 36.6%, $MS\hat{E}$ by 35.9%, J_1 by 70%, J_2 by 85.4%, and T_s by 36.3% compared to OIAFTSFC.

Case B. Sinusoidal Disturbance The effect of a disturbance signal $d(t) = 3\sin(20t)$ is discussed. OH$_\infty$IAZTSFC lowers MSE by 37.9%, $MS\hat{E}$ by 37%, J_1 by 23.7%, J_2 by 68.4%, and T_s by 40.5% in comparison with OIAFTSFC (see Table 2).

Case C. Pulse Disturbance In the presence of $d(t) = \begin{cases} 1 & 0.5 \le t \le 2.5 \\ 0 & otherwise \end{cases}$., OH$_\infty$IAZTSFC lowers MSE by 36.4%, $MS\hat{E}$ by 35.5%, J_1 by 79.4%, J_2 by 85.4%, and T_s by 53.1% in comparison with OIAFTSFC (see Table 2).

Case D. Measurement Noise Under noise with $SNR = 45\,\mathrm{d}B$, Table 2 shows that OH$_\infty$IAZTSFC lowers MSE by 30.5%, $MS\hat{E}$ by 30.1%, J_1 by 68.6%, J_2 by 53.3%, and T_s by 49.8%.

Case E. Data Loss By supposing $\alpha = 0.98$ in (39), Table 2 shows that OH$_\infty$IAZTSFC lowers MSE by 33.3%, $MS\hat{E}$ by 32.7%, J_1 by 42.8%, J_2 by 78.5%, and T_s by 48.3%.

Case F. Dead-Zone Under the asymmetric dead-zone with $m_r = m_l = 0.3$, $b_r = 0.5$, $b_l = -0.8$, Table 2 reveals that the proposed method lowers MSE by 23.4%, $MS\hat{E}$ by 23.1%, J_1 by 18.7%, J_2 by 36%, and T_s by 53.5% compared with OIAFTSFC.

Case G. Sensitivity Analysis The sensitivity of the introduced method to variations of the controller's parameters is analyzed here. We calculate performance with respect to lowering the controller's parameters by 10% ($\boldsymbol{K_0}$, $\boldsymbol{K}$, γ, α, and λ) from the designed values. Table 2 shows that OH$_\infty$IAZTSFC improves MSE (by 35.2%), $MS\hat{E}$ (by 34.4%), J_1 (by 69.5%), J_2 (by 91.7%), and T_s (by 48.9%).The results show the better performance of the proposed method in all criteria.

7 Conclusion

This paper studies nonlinear systems with high-order input derivatives and nonlinear input–output relationships. The dynamics are investigated without converting them to standard state-space forms to increase simplicity and interpretability. The proposed controller guarantees robust stability and trajectory tracking under uncertainties. The proposed method utilizes the simple structure of the zero-order TS fuzzy systems to increase transparency and estimate unknown nonlinear functions. The state observer approximates the unmeasured states to decrease the cost. The H_∞ controller tries to compensate for the residual errors. The adaptive rules and the H_∞ controller have simplified which their equations are expressed without determining $\boldsymbol{P_0}$, $\boldsymbol{Q_0}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ matrixes, and $\boldsymbol{B_c}$ vector. Compensation control is appended to ensure stability. The asymptotic stability of the closed-loop system and the parameters' convergence are proved with the Lyapunov theory. Finally, the effectiveness of the proposed method is revealed via simulations of the second-order trolley system and the third-order SISO system in the presence of unmeasured states, disturbances, measurement noises, data loss, and asymmetric dead-zone. The results show that the proposed approach has better tracking behavior and lower energy consumption with a fast response. Our future research directions in this research will be as follows:

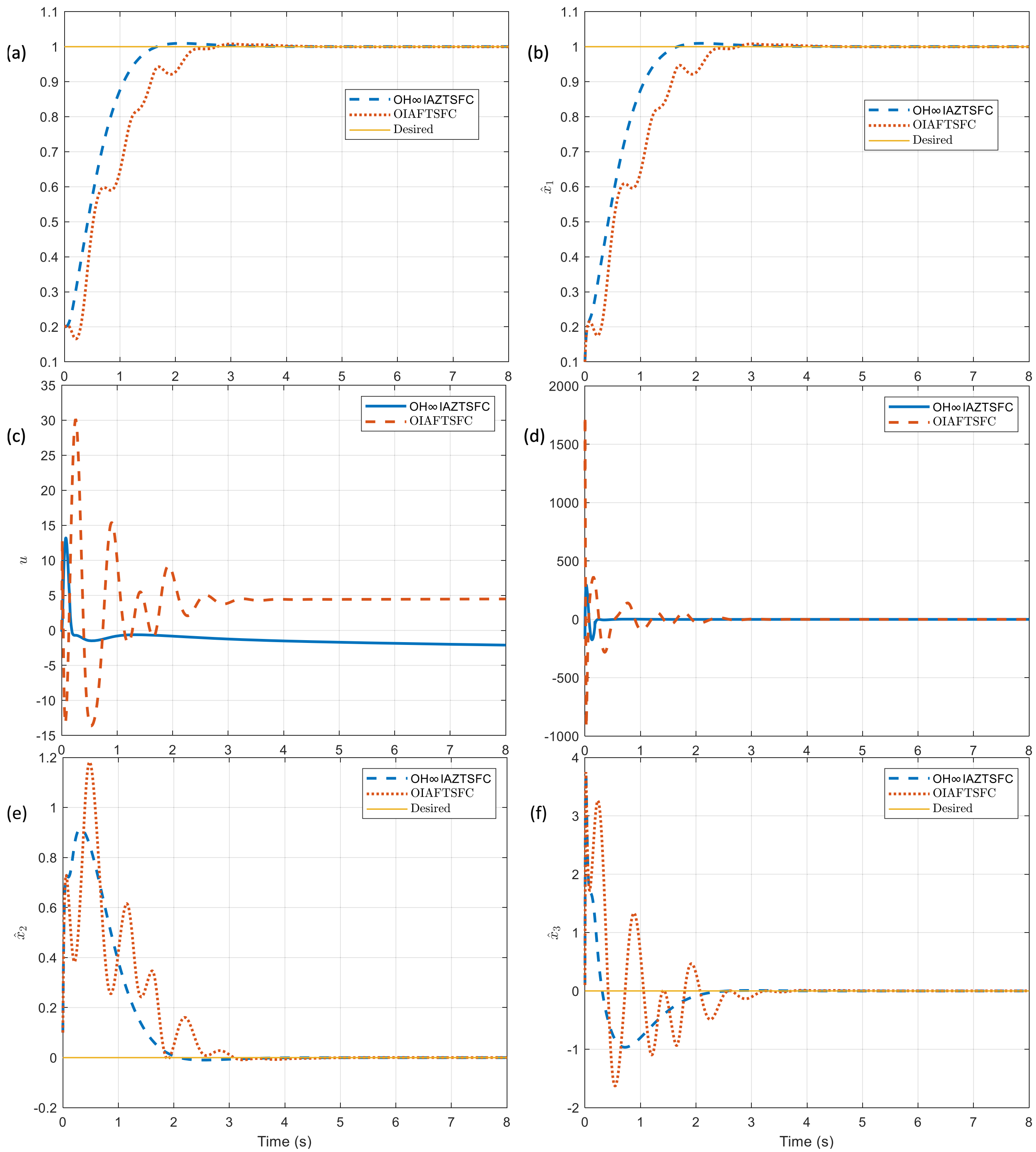

Fig. 4 Comparison of OH$_\infty$IAZTSFC with (OIAFTSFC) [7] in Case A. of Example 2. **a** Trajectory of output. **b** Trajectory of $\hat{x}_1$. **c** Designed Controller ($\dot{u}$). **d** Input (u). **e** Trajectory of $\hat{x}_2$. **f** Trajectory of $\hat{x}_3$

Table 2 Comparison of the proposed method (OH_∞IAZTSFC) with OIAFTSFC [7] for Example 2 by mean squared errors (MSE and $\hat{MSE}$), the consumed control energies (J_1 and J_2), the settling time (T_s), and the number of the adaptive parameters

Method	OIAFTSFC [7]							OH_∞IAZTSFC						
Case studies	A.	B.	C.	D.	E.	F.	G.	A.	B.	C.	D.	E.	F.	G.
$MSE = \frac{1}{T}\int_0^T e_1^2 \mathrm{d}t$	0.0503	0.0494	0.0500	0.0455	0.0492	0.0431	0.0489	**0.0319**	**0.0307**	**0.0318**	**0.0316**	**0.0328**	**0.0330**	**0.0317**
$\hat{MSE} = \frac{1}{T}\int_0^T \hat{e}_1^2 \mathrm{d}t$	0.0501	0.0492	0.0498	0.0455	0.0490	0.0432	0.0488	**0.0321**	**0.0310**	**0.0321**	**0.0318**	**0.0330**	**0.0332**	**0.0320**
$J_1 = \frac{1}{T}\int_0^T \lvert u(t)\rvert \, \mathrm{d}t$	5.2632	4.3726	11.1402	5.7127	4.6284	2.1039	6.7028	**1.5812**	**3.3374**	**2.2975**	**1.7941**	**2.6496**	**1.7097**	**2.0448**
$J_2 = \frac{1}{T}\int_0^T \lvert \dot{u}(t)\rvert \, \mathrm{d}t$	26.1439	27.6450	26.2925	30.2042	26.0688	5.4155	30.3064	**3.8269**	**8.7237**	**3.8259**	**14.1021**	**5.6117**	**3.4670**	**2.5118**
T_s	2.245	2.436	3.060	2.890	2.571	3.046	2.493	**1.430**	**1.450**	**1.436**	**1.450**	**1.330**	**1.415**	**1.273**
Adaptive Param. No.	135	135	135	135	135	135	135	***81***	***81***	***81***	***81***	***81***	***81***	***81***

The cases of A, B, C, D, E, F, and G show controlling under ideal conditions, sinusoidal disturbance, pulse disturbance, noise, data loss, dead-zone, and sensitivity analysis, respectively. Bold numbers indicate better performance

- Use optimal controllers to adjust the parameters we obtain with trial and error.
- Study various uncertainties, such as hysteresis, delay, and saturation in the model, to check the robustness.
- Study more expressions of uncertainties in fuzzy systems, such as in type-2 fuzzy systems, to reach higher performance levels.

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