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Multi-criteria Group Decision-Making Portfolio Optimization Based on Variable Subscript Hesitant Fuzzy Linguistic Term Sets

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Abstract This paper applies the cumulative prospect theory to improve the traditional integer subscript hesitant fuzzy linguistic term set (IS-HFLTS). The proposed variable subscript hesitant fuzzy language set (VS-HFLTS) takes into account the subjectivity and differences of investors' behavior when investing in the securities market, and extends the application of behavioral finance to multi-criteria group decision-making portfolio. To evaluate the financial products, the VS-HFLTS multi-criteria group decision-making portfolio evaluation system is constructed in the research. The hesitant fuzzy linguistic value function and the purchase appetite weight function are proposed to convert the natural linguistic evaluation into a quantitative score for the judgment of future portfolio return in the securities market. Furthermore, a variable subscript hesitant fuzzy linguistic portfolio model is put forward for the risk-averse, risk-neutral, and risk-seeking investment decision-makers. The optimal portfolio strategy is obtained by solving the equivalent non-linear model. Meanwhile, an optimized group decision-making portfolio strategy is established to better achieve the goal of increasing group decision-making investors' returns or reducing risks. Finally, numerical simulations are performed to find effective frontiers for future portfolio selection, which verifies the validity and feasibility of the models and methods proposed in this paper.

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1 Introduction

Modern portfolio theory has been continuously enriched and expanded [1, 2] since put forward by Markowitz in 1952 [3]. Early studies on the mean–variance model mainly considered a single investor and single criteria, and adopted equivalent predictive mathematical models to estimate future security returns and losses [4,5]. For example, Young [6] offered the linear programming solution of the minimax portfolio selection model. Konno et al. [7] provided a linear solution for portfolio optimization based on the absolute mean-skewness variance model; Crama and Schyns [8] used the simulated annealing heuristic algorithm to solve the portfolio of mixed-integer quadratic programming. To make the portfolio more effective, scholars attempted to explore the portfolio selection from the perspective of a single investor and multiple criteria. Yanushevsky [9] measured the average trading volume of securities in the portfolio as the percentage of total outstanding shares over a certain period, thus quantifying the potential increase in stock prices and determining the maximum return of the multi-criteria portfolio; Patari et al. [10] proposed a multi-criteria decision-making method that integrates the Median-Scaling (MS), Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS), Analytic Hierarchy Process (AHP) and additive Data Envelopment Analysis (add. DEA), and empirical analysis on U.S. stocks to filter the best performing portfolio. With expanding research, the selection of the optimal investment strategy under the group decision-making and multi-criteria portfolio has become a hot topic. Glogger et al. [11] utilized the disagreements of different investors as an additional risk and adopted a non-linear multi-criteria approach based on emotions to study portfolio optimization. Given that the investment committee members may have conflicting investment standards, Xidonas et al. [12] constructed a standardized decision support business framework and conducted an empirical test on the 10-year Dow Jones Index, proving that the adjusted portfolio returns are better than other investment policies.

Most portfolio selections listed above were solved by linear optimization theory and related empirical analyses based on historical data securities markets. However, portfolio selection is a constantly changing dynamic process, and investment strategy also varies depending on the investor's risk preference and subjective judgment [13]. To address the uncertainty of risks and returns of financial products in portfolio optimization, scholars turned to fuzzy theory. Wang and Zhu [14] questioned the traditional method of considering probability theory to solve the portfolio and advocated adopting fuzzy sets to represent the risk and return of assets in a portfolio. Zhou and Xu [15] proposed score-hesitation trade-off rules based on an intuitionistic fuzzy environment and solved the optimization portfolio model. Aiming at the group decision-making portfolio, Zhou et al. [16] used hesitant fuzzy information to unify the opinions of different investors and obtained the optimal investment strategy. Traditional fuzzy sets employ numerical values to express investors' opinions in the decision-making process, while the HFLTS [17] uses natural linguistic evaluation that can better capture actual investment scenarios and reflect the risk preference of decision-makers [18]. Zhou and Xu [19] proposed a portfolio optimization model based on the HFLTS, and distinguished investors with different risk preferences by introducing the asymmetric sigmoid semantics. Although fuzzy theory effectively addresses the investor's uncertainty about decision-making information, in a real investment environment, decisions are affected not only by objective factors, such as personal knowledge and educational background but also by subtle psychological changes [20].

Prospect theory (PT) [21] better describes the psychological preferences of investors when facing decision-making. Shefrin and Statman [22] put forward the concept of behavior portfolio theory on the basis of prospect theory. However, an important factor that investment committees often ignore in a group decision-making process is that group decisions are prone to behavioral and psychological biases [23]. Prejudices may thus produce irrational decisions, affect portfolio returns, and increase credit risk exposure [24]. To better address the multi-criteria group decision-making portfolio, this paper introduces Cumulative Prospect Theory (CPT) [25] to improve the traditional HFLTS that considers only integer subscripts. CPT can describe the psychological sensitivity of investors to gains and losses (i.e., value function) as well as the differences in investor's psychological perception when the probability of a certain objective state changes (i.e., weight function). Using improved VS-HFLTS to evaluate financial products not only captures the investor's primary judgment on the future securities prices and their secondary interest in purchasing these securities but also reflects the investor's psychological sensitivity as the degree of evaluation deepens. The contributions of this study can be highlighted as follows:

1. We introduce the cumulative prospect theory into the hesitant fuzzy linguistic set, and the variable subscript hesitant fuzzy linguistic set is defined. The VS-HFLTS is used to describe different investors' judgments on financial products. Additionally, we defined the variable subscript hesitant fuzzy linguistic value function, which can transform the investor's natural linguistic evaluation into the corresponding semantic and thus quantitatively express the investor's opinions.
2. We propose a new two-stage VB-HFLTS portfolio evaluation system. The first-stage evaluation gives all investors' judgments on the ups and downs of the securities, and the secondary evaluation gives the judgments of the investor's purchasing desires. In each stage of evaluation, we also consider the preference evaluation between different investment criteria and the investor's decision-making evaluation under different investment criteria.
3. We develop fuzzy portfolio optimization through a quantitative portfolio evaluation system of VS-HFLTS that integrates preference evaluation and decision-making evaluation. The final score of each security through the system will represent the group decision-making investors' evaluation score of the security, which will be substituted into the portfolio optimization model to achieve the optimal allocation of the financial assets. The optimal portfolio model is solved for neutral, aggressive and conservative group decision-making investors, respectively, so as to help them to achieve higher-yielding or lower-risk portfolio strategies.

The remainder of this paper is organized as follows: Section 2 reviews some basic theories adopted in this paper. Section 3 introduces how to employ a hesitant fuzzy linguistic multi-criteria group decision-making portfolio evaluation system to assess financial products. Section 4 discusses how to score securities using a hesitant fuzzy linguistic multi-criteria group decision-making portfolio quantitative calculation system. Section 5 proposes a hesitant fuzzy linguistic multi-criteria group decision-making portfolio optimization model and solves the critical value of portfolio optimization. Section 6 offers numerical simulations and the analysis of simulation results. Section 7 concludes this paper.

2 Preliminary Knowledge

This section revises some basic knowledge and related extensions of the HFLTSs, in which we focus on our proposal.

2.1 HFLTS

Definition 1 [26] Let $a_i \in A$, $i = 1, 2, \ldots, N$ be fixed and $S = \{S_\alpha | \alpha = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a Linguistic Term Set (LTS). The mathematical form of the HFLTS can be shown as

$$H_S = \{\langle a_i, h_S(a_i)\rangle \mid a_i \in A\}, \tag{1}$$

where $h_S(a_i)$ indicates the possible membership degrees of element a_i that is mapped to set $X \subset A$. $h_S(a_i)$, named as the Hesitant Fuzzy Linguistic Element (HFLE),can be expressed as $h_S(a_i) = \{S\varphi_l(a_i) | S\varphi_l(a_i) \in S,\ l = 1, 2, \ldots, L(a_i)\}$, $\varphi_l \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}$, where L is the number of linguistic term in $h_S(a_i)$. H_S is the set of all HFLEs in the LTS S, that is, the HFLTS.

Definition 2 [17] Let S be a LTS, and $G_H$be a context-free grammar. The element of $G_H = (V_N, V_T, I, P)$ are defined as follows:

V_N = {<primary term>, <composite term>, <unary term>, <binary term>, <conjunction>};
V_T = {lower than, greater than, at least, at most, between, and, "$S_{-\tau}$", ⋯, "S_{-1}", "S_0", "S_1", ⋯,"S_τ"};
$I \in V_N$;
P = {I : : = <primary term> | <composite term>,
<composite term> : : = <unary relation> <primary term> | <binary relation> <primary term> <conjunction> <primary term> ,
<primary term> : : = "$S_{-\tau}$",⋯, "S_{-1}", "S_0", "S_1",⋯,"S_τ",
<unary relation> : : = lower than | greater than | at least | at most,
<binary relation> : : = between,
<conjunction> : : = and}.

Remark 1 The brackets in Definition 1 enclose optional elements and the symbol "|" indicates alternative elements.

Definition 3 [17] Given S being a LTS and S_{ll} being the expression domain generated by G_H, let $E_{G_H} : S_{ll} \to H_S$ be a function that transforms the linguistic expressions S_{ll} to the HFLTS H_S. *The linguistic expression* $ll \in S_{ll}$ is converted into the HFLE by means of the following transformation:

1. $E_{G_H}(S_\alpha) = \{S_\alpha | S_\alpha \in S\}$;
2. $E_{G_H}(at\ most\ S_t) = \{S_\alpha | S_\alpha \in S\ and\ S_\alpha \le S_t\}$;
3. $E_{G_H}(lower\ than\ S_t) = \{S_\alpha | S_\alpha \in S\ and\ S_\alpha < S_t\}$;
4. $E_{G_H}(at\ least\ S_t) = \{S_\alpha | S_\alpha \in S\ and\ S_\alpha \ge S_t\}$;
5. $E_{G_H}(great\ than\ S_t) = \{S_\alpha | S_\alpha \in S\ and\ S_\alpha > S_t\}$;
6. $$E_{G_H}(between\ S_t\ and\ S_{t'}) = \{S_\alpha | S_\alpha \in S\ and\ S_t \le S_\alpha \le S_{t'}\}.$$

Definition 4 [27] Let $S = \{S_\alpha | \alpha = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be an HFLTS and θ_α be the semantic of a hesitant fuzzy linguistic term S_α. The hesitant fuzzy linguistic scale function (HFLSF) to translate a linguistic term into its corresponding semantic is defined as f: $S_\alpha \to \theta_\alpha$, and g is a strictly monotonically increasing function, and $f(S_\alpha) \in [0, 1]$.

2.2 Cumulative Prospect Theory

Introduced by Tversky and Kahneman [25], CPT is presented as a pioneer decision theory under risk and uncertainty. Its major attraction lies not only in its exclusive properties to capture loss aversion, risk-seeking, non-linear preferences, and source dependence, but also in its consistency with the stochastic dominance axiom, thus allowing prospects with a large number of outcomes.

In its parametric form, CPT preferences are jointly determined by the value function $V(x)$ and the probability weighting function $W(p)$. The value function captures four risk profiles of investors: (1) risk seeking for gains, (2) risk aversion for loss, (3) low probability of risk aversion for gains, and (4) high probability associated with risk seeking for losses. In addition, the value function exhibits the following properties: reference dependent, diminishing sensitivity, and loss aversion. Therefore, $V(x)$ is both concave (above the reference point) and convex (below the reference point) to ensure a decreasing impact of changes in gains and losses as the distance from the reference point increases (diminishing sensitivity). Furthermore, given that losses are considered to loom longer than gains, $V(x)$ is steeper for losses than for gains. Formally, $V(x)$ is represented by the classical power function as follows:

$$V(x_i) = \begin{cases} x^\alpha, & if\ x \ge 0\ ; \\ -\lambda(-x)^\beta & , if\ x < 0\ , \end{cases} \tag{2}$$

where $\lambda \ge 1$ represents the loss aversion parameter; α, β $(0 < \alpha, \beta \le 1)$ are parameters of risk aversion in gains and risk preference in losses, respectively.

On the other hand, the decision weight takes the form of a cumulative probability weighted in a non-linear way. Thus, it incorporates non-linear preferences and the four risk profiles outlined above. Similar to $V(x)$, the diminishing sensitivity property also applies to the weighting function, with a different reference. The response to changes in probability decreases as probability deviates from the frontiers of impossibility and certainty [28]. The decision weighting functions for gains and losses are both *S*-shaped with reference to the identity line. In addition to diminishing sensitivity, the weighting function also captures the attractiveness property. So, the higher the curve, the greater the attractiveness of the prospect for the investor. The parametric form of the weighting function proposed by Tversky and Kahneman [25] is the following:

$$\begin{cases} W^{+}(p) = \dfrac{p^{\gamma}}{(p^{\gamma} + (1-p)^{\gamma})^{\frac{1}{\gamma}}} \ for\ gains\,; \\ W^{-}(p) = \dfrac{p^{\delta}}{(p^{\delta} + (1-p)^{\delta})^{\frac{1}{\delta}}} \ for\ losses\,, \end{cases} \quad (3)$$

where γ and δ are the respective curvature of $W^{+}(p)$ and $W^{-}(p)$ and the point at which they cross the identity line.

Finally, the prospect value, CPT, is obtained from combining $V(x)$ and decision weights $\pi(p)$ as follows.

$$\mathrm{CPT}(x,p) = \pi^{-}(p)\,V(x^{-}) + \pi^{+}(p)\,V(x^{+})\,, \quad (4)$$

where

$$\begin{cases} \pi^{-}(p) = W^{-}(\sum_{-m}^{i} p_i) - W^{-}(\sum_{-m}^{i-1} p_i),\ for\ 1-m \le i \le 0\,; \\ \pi^{+}(p) = W^{+}(\sum_{i}^{n} p_i) - W^{+}(\sum_{i+1}^{n} p_i),\ for\ 0 \le i \le n-1\,. \end{cases} \quad (5)$$

2.3 VS-HFLTS

The subscripts of a traditional LTS S, $S = \{S_{\alpha}|\alpha = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$, are equally spaced integers. When investors make natural linguistic evaluations of financial products in the securities market, investors' psychology may undergo subtle changes as the degree of evaluation deepens [29]. It is manifested in the LTS, that is, as the degree of semantics increases, the absolute value of linguistic term subscript increases, and the absolute value of the difference between adjacent linguistic term subscripts gradually decreases. However, behavioral finance demonstrated that most investors become more cautious and sensitive when facing high risks and returns. The CPT can explain the differences in the psychology and behavior of various investors. Using the adjusting risk attitude parameters, the CPT shows that different types of investors have different investment preferences and risk attitudes when facing uncertain investment risks and returns. Inspired by the CPT, this paper improves the uniform LTS's subscript [30]. By introducing the CPT's value function into the subscript of HFLTS, a variable subscript hesitant fuzzy linguistic set is constructed, and the corresponding hesitant fuzzy linguistic value function (HFLVF) is proposed.

Definition 5 Let $S = \{S_{v(l)}|l = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a Variable Subscript Hesitant Fuzzy Linguistic Term Set (VS-HFLTS). $S_{v(l)}$ is termed the Variable Subscript Hesitant Fuzzy Linguistic Element (VS-HFLE), $l = -\tau, \ldots, -1, 0, 1, \ldots, \tau$ is the subscript of the original uniformly spaced linguistic term, and $v(l)$ is the improved Variable Subscript of Linguistic Term that can be expressed as follows:

$$v(l) = \begin{cases} l^{\alpha},\ l = 0, \cdots, \tau; \\ -\gamma \cdot (-l)^{\beta},\ l = -\tau, \cdots, 0\,, \end{cases} \quad (6)$$

where α and β are the investor's risk attitude parameters when making investment decisions and thus can reflect the investor's sensitivity to gains and losses, respectively; γ indicates the degree of investor's loss aversion, and $\gamma > 1$. The greater the value of γ, the higher the degree of loss aversion. As the research is based on the same investor's psychological judgment on securities' risks and returns, this paper assumes that $\alpha = \beta$. The greater the value of α and β, the smaller the investor's sensitivity to gains and losses, which means that the investor is more willing to take risks to make profits [12], specifically:

(i) If $0 < \alpha = \beta < 1$, the investor's corresponding risk attitude is risk-averse;
(ii) If $\alpha = \beta = 1$, the investor's corresponding risk attitude is risk-neutral;
(iii) If $\alpha = \beta > 1$, the investor's corresponding risk attitude is risk-seeking.

Example 1 Suppose a risk-averse investor uses the traditional HFLTS to evaluate the rise and fall of stocks, e.g., $\{S_{-4}$ = **limit-down**, S_{-3} = **slump**, S_{-2} = **medium down**, S_{-1} = **marginally lower**, S_0 = **unchanged**, S_{1} = **marginally higher**, S_2 = **medium up**, S_3 = **jump**, S_{4} = **limit-up**$\}$.

If $\alpha = \beta = 0.88$, $\gamma = 2.25$, the improved VS-HFLTS can be obtained using the formula (2), i.e., $\{S_{-7.62}$ = **limit-down**, $S_{-5.92}$ = **slump**, $S_{-4.14}$ = **medium down**, $S_{-2.25}$ = **marginally lower**, S_0 = **unchanged**, S_1 = **marginally higher**, $S_{1.84}$ = **medium up**, $S_{2.63}$ = **jump**, $S_{3.39}$ = **limit-up**$\}$. It can be seen that, in the case of a risk-averse investor, the absolute value of the difference between the VS-HFLTS subscripts exhibits a decreasing trend, that is, the difference between the subscripts of $S_{1.84}$ and S_1 is greater than $S_{2.63}$ and $S_{1.84.}$ This explains that the sensitivity of the difference between the evaluation "**medium up**" and "**marginally higher**" must be greater than the "**jump**" and "**medium up.**" It also reflects that the risk-averse investor is thus more nervous and cautious when appraising stocks that are rising and falling sharply than fluctuating moderately. In other words, the risk-averse investor has a more sensitive perception of gains and losses. Moreover, a comparison of subscripts corresponding to the evaluations "**limit-up**" and "**limit-down**" implies that the VS-HFLTS considers the degree of investor's risk aversion, so the subscript of "limit-down" is more negative.

If $\alpha = \beta = 1.12$, $\gamma = 2.08$, the corresponding investor is risk-seeking and the obtained subscripts of the improved VS-HFLTS are $\{S_{-9.83}, S_{-7.12}, S_{-4.52}, S_{-2.08}, S_0, S_1, S_{2.17}, S_{3.42}, S_{4.72}\}$. The absolute value of the difference between the subscripts thus exhibits an increasing trend. Specifically, the difference between the evaluations "**limit-up**" and evaluation "**jump**" is greater than to the difference between evaluations "**jump**" and "**medium up**." Analogously, this illustrates that a risk-seeking investor is more willing to pursue risks when evaluating stocks with large fluctuations (high-risk or high-return stocks). Risk-seeking investors are less sensitive to gains and losses than risk-averse investors, and their degree of loss aversion is lower when facing losses.

Definition 6 Let $S = \{ S_{v(l)} \,|\, l = -\tau, \cdots, -1, 0, 1, \cdots, \tau.\}$ be a VS-HFLTS, $\theta_{v(l)} \in R^+$ be the semantic value of a variable subscript fuzzy hesitant linguistic term $S_{v(l)}$. The hesitant fuzzy linguistic value function (HFLVF) transforming a variable subscript hesitant fuzzy linguistic term into its corresponding semantic is defined as g: $S_{v(l)} \rightarrow \theta_{v(l)}$, and has the following properties:

(i) g is a strictly monotonically increasing function with regard to the variable subscript $v(l)$;
(ii) The negative operators for VS-HFLTS are defined as $neg\ (S_{v(l)}) = S_{-v(l)}$.

3 VS-HFLTS Portfolio Evaluation System

Based on the CPT, this paper establishes an HFLTS multi-criteria group decision-making two-stage portfolio evaluation system that adopts both the integer and variable subscripts HFLTS to capture investor psychology and behavior in transaction markets. Suppose that the financial market consists of T securities, $x_t(t = 1, 2, \ldots, T)$, and Q investors, $e_q(q = 1, 2, \ldots, Q)$. The primary (first-stage) evaluation requires each investor to evaluate all securities' rise or fall trends with respect to the primary criteria C_i($i = 1, 2, \ldots, I$). Specifically, the pairwise comparisons between each pair of the primary criteria are performed to obtain the investors' primary preference evaluation. In addition, the decision-making evaluations of the securities are conducted to be judged under different primary criteria. The primary security score (the prospect value of a security) is then obtained through the quantitative aggregation of the primary preference and decision-making evaluation. With the obtained pre-judgment of the rise and fall evaluation, the secondary (second-stage) evaluation requires each investor to assess the appetite to purchase securities with respect to the secondary criteria $\tilde{C}_j\ (j = 1, 2, \cdots, n)$. Similar to the primary evaluation, the secondary security score (the purchase appetite weight function value) is obtained through the quantitative aggregation of the secondary preference and decision evaluation. The specific steps of the two-stage portfolio evaluation of HFLTS multi-criteria group decision-making are as follows:

Step 1 Set up the HFLTS for the preference and decision-making evaluations of multi-criteria group decision-making portfolio primary evaluation process. The same operations are performed in the case of the secondary evaluation process. It is noteworthy that the investors should use rational and objective attitudes to compare the criteria pairwise in the preference evaluation. Therefore, the preference evaluation term set in this paper adopts IS-HFLTS. The decision-making evaluation of securities considering different evaluation criteria is closely related to the investor's personality, psychology, knowledge and cultural background. Therefore, the decision-making evaluation term set in this paper utilizes the VS-HFLTS. Table 1 introduces the types of HFLTS used in each step of the evaluation process. To distinguish between the primary and the secondary evaluations, this paper uses " $\sim$ " to identify the elements involved in the secondary evaluation process.

Step 2 The investor e_q performs pairwise comparisons of the primary criteria (i.e., criteria $C_m$and C_n, $m, n = 1, \ldots, I$) and gives the natural language evaluations corresponding to the primary preference evaluation. Adopt the context-free grammar G_H to convert the obtained natural language into linguistic expressions for primary preference evaluation.

Step 3 Convert the linguistic expressions of primary preference evaluation into an IS-HFLE $h_S^{(q)}(C_m, C_n)$ using the transformation function in Definition 3. Then, the investor e_q's primary Integer Subscript Hesitant Fuzzy Linguistic Preference Matrix (IS-HFLPM) $\mathbf{P}_S^{(q)}$ for the primary criteria C_i can be obtained. The primary IS-HFLPM can be expressed as follows:

Table 1 Two-stage preference and decision-making evaluation HFLTS

	Preference evaluation	Decision-making evaluation
Primary	$S_p = \{S_\alpha \,|\alpha = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$	$S_d = \{S_{v(l)} \| l = -\tau, \cdots, -1, 0, 1, \cdots, \tau\}$
Secondary	$\tilde{S_p} = \{ \tilde{S_\alpha} \,|\, \alpha = -\tilde{\tau}, \cdots, -1, 0, 1, \cdots, \tilde{\tau}\}$	$\tilde{S_d} = \{ \tilde{S}_{v(\tilde{l})} \,|\, \tilde{l} = -\tilde{\tau}, \cdots, -1, 0, 1, \cdots, \tilde{\tau}\}$

$$\mathbf{P}_S^{(q)} = \begin{pmatrix} H_S^{(q)}(C_1,C_1) & \cdots & H_S^{(q)}(C_1,C_i) & \cdots & H_S^{(q)}(C_1,C_I) \\ \vdots & \ddots & \vdots & \ddots & \cdots \\ H_S^{(q)}(C_i,C_1) & \cdots & H_S^{(q)}(C_i,C_i) & \cdots & H_S^{(q)}(C_i,C_I) \\ \vdots & \ddots & \vdots & \ddots & \cdots \\ H_S^{(q)}(C_I,C_1) & \cdots & H_S^{(q)}(C_I,C_i) & \cdots & H_S^{(q)}(C_I,C_I) \end{pmatrix}. \quad (7)$$

Step 4 The investor e_q gives the natural language evaluations corresponding to the primary decision-making evaluation of T securities with respect to the primary criteria C_i. Adopt the context-free grammar G_H to convert the obtained natural language into linguistic expressions for primary decision-making evaluation.

Step 5 Convert the linguistic expressions of primary decision-making evaluation into a VS-HFLE $h_S^{(q)}(x_t^i)$ using the transformation function in Definition 3. Then, the investor e_q's primary Variable Subscript Hesitant Fuzzy Linguistic Decision-Making Matrix (VS-HFLDMM) $\mathbf{D}_S^{(q)}$ for the T securities with respect to the primary criteria C_i can be obtained. The primary VS-HFLDMM can be expressed as follows:

$$\mathbf{D}_S^{(q)} = \begin{pmatrix} H_S^{(q)}(x_1^1) & \cdots & H_S^{(q)}(x_1^i) & \cdots & H_S^{(q)}(x_1^I) \\ \vdots & \ddots & \vdots & \ddots & \cdots \\ H_S^{(q)}(x_t^1) & \cdots & H_S^{(q)}(x_t^i) & \cdots & H_S^{(q)}(x_t^I) \\ \vdots & \ddots & \vdots & \ddots & \cdots \\ H_S^{(q)}(x_T^1) & \cdots & H_S^{(q)}(x_T^i) & \cdots & H_S^{(q)}(x_T^I) \end{pmatrix}. \quad (8)$$

Step 6 Similar to the primary evaluation process, the investor e_q performs pairwise comparisons of the secondary criteria (i.e., criteria $\tilde{C}_m$ and $\tilde{C}_n$, *m, n* = 1, 2, …, *J*), and the secondary IS-HFLPM $\tilde{\mathbf{P}}_S^{(q)}$ is obtained as follows:

$$\tilde{\mathbf{P}}_S^{(q)} = \begin{pmatrix} \tilde{H}_S^{(q)}(\tilde{C}_1,\tilde{C}_1) & \cdots & \tilde{H}_S^{(q)}(\tilde{C}_1,\tilde{C}_j) & \cdots & \tilde{H}_S^{(q)}(\tilde{C}_1,\tilde{C}_J) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{H}_S^{(q)}(\tilde{C}_j,\tilde{C}_1) & \cdots & \tilde{H}_S^{(q)}(\tilde{C}_j,\tilde{C}_j) & \cdots & \tilde{H}_S^{(q)}(\tilde{C}_j,\tilde{C}_J) \\ \vdots & \ddots & \vdots & \ddots & \cdots \\ \tilde{H}_S^{(q)}(\tilde{C}_J,\tilde{C}_1) & \cdots & \tilde{H}_S^{(q)}(\tilde{C}_J,\tilde{C}_j) & \cdots & \tilde{H}_S^{(q)}(\tilde{C}_J,\tilde{C}_J) \end{pmatrix}. \quad (9)$$

Step 7 The investor e_q evaluates T securities with respect to the secondary criteria $\tilde{C}_j$, and the secondary VS-HFLDMM $\tilde{\mathbf{D}}_S^{(q)}$ is obtained as follows:

$$\tilde{\mathbf{D}}_S^{(q)} = \begin{pmatrix} \tilde{H}_S^{(q)}(x_1^1) & \cdots & \tilde{H}_S^{(q)}(x_1^i) & \cdots & \tilde{H}_S^{(q)}(x_1^J) \\ \cdots & \ddots & \cdots & \ddots & \cdots \\ \tilde{H}_S^{(q)}(x_t^1) & \cdots & \tilde{H}_S^{(q)}(x_t^i) & \cdots & \tilde{H}_S^{(q)}(x_t^J) \\ \cdots & \ddots & \cdots & \ddots & \cdots \\ \tilde{H}_S^{(q)}(x_T^1) & \cdots & \tilde{H}_S^{(q)}(x_T^i) & \cdots & \tilde{H}_S^{(q)}(x_T^J) \end{pmatrix}. \quad (10)$$

4 VS-HFLTS Portfolio Quantitative Calculation System

The previous section presents the process of obtaining the preference matrix and decision-making matrix of different investors for evaluating securities considering the primary and secondary criteria. This section quantitatively aggregates the obtained IS-HFLPM and VS-HFLDMM to calculate the preference scores of criteria, the decision scores of securities with respect to criteria, and the final score of each security that can be fitted into the portfolio model.

4.1 VS-HFLTS Multi-criteria Group Decision-Making Portfolio Primary Quantitative Score

The primary quantitative scores of T securities are obtained through the quantitative aggregation of the primary IS-HFLPM $\mathbf{P}_S^{(q)}$ and the primary VS-HFLDMM $\mathbf{D}_S^{(q)}$. The specific steps of the primary scoring process are as follows:

IS-HFLPM $\mathbf{P}_S^{(q)}$

Step 1 Use the linguistic scale function to compute the HFLE (i.e.,$h_S^{(q)}(C_m, C_n)$) in each HFLTS (i.e.,$H_S^{(q)}(C_m, C_n)$) of the preference matrix, given that $E_p\left(h_S^{(q)}(C_m, C_n)\right) = f(S_\alpha)$, where f (■) represents the HFLSF.

Step 2 Compute the score of each HFLTS (i.e.,$H_S^{(q)}(C_m, C_n)$) in the preference matrix. The HFLTS score indicates the degree of investor's preference for different primary criteria obtained through the pairwise comparison and can be calculated as follows:

$$E_p^{(q)}\left(H_s^{(q)}(C_m, C_n)\right) = \frac{1}{L}\left(\sum_L f(S_\alpha)\right), \quad (11)$$

where L represents the number of linguistic terms in $H_S^{(q)}(C_m, C_n)$, i.e., the number of the IS-HFLEs contained in S_α.

Step 3 The variance of $H_S^{(q)}(C_m, C_n)$ can describe the investor e_q's hesitant degree in conducting the preference evaluation of different primary criteria. The calculation formula can be expressed as follows:

$$V_p^{(q)}\left(H_s^{(q)}(C_m, C_n)\right) = \sqrt{\frac{1}{L}\sum_L \left(f(S_\alpha) - E_p^{(q)}\left(H_s^{(q)}(C_m, C_n)\right)\right)^2}. \quad (12)$$

Step 4 The preference weight of each criterion refers to the weight of each criterion in the IS-HFLPM obtained by the pairwise comparison of criteria. Specifically, investors' hesitant degrees in evaluating different primary criteria are compared to obtain the preference weight of each primary criterion C_i. The preference weight calculation formula is as follows:

$$w_p^{(q)}(C_i) = \frac{\left(0.5 - v_p^{(q)}(C_i)\right)}{\sum_{i=1}^{I}\left(0.5 - v_p^{(q)}(C_i)\right)}, \quad (13)$$

where $V_p^{(q)}(C_i) = \frac{\sum_{n=1}^{I} V_p^{(q)}\left(H_S^{(q)}(C_i, C_n)\right)}{\sum_{i=1}^{I}\sum_{n=1}^{I} V_p^{(q)}\left(H_S^{(q)}(C_i, C_n)\right)}$ indicates the degree of investor e_q's hesitation in evaluating the primary criterion C_i.

VS-HFLDMM $\mathbf{D}_S^{(q)}$

The HFLE (i.e.,$h_S^{(q)}(x_t^i)$) in each HFLTS (i.e.,$H_S^{(q)}(x_t^i)$) of the decision-making matrix is computed using the HFLVF, given that $E_d\left(h_S^{(q)}(x_t^i)\right) = g\left(S_{v(l)}\right)$, where $g(\cdot)$ represents the HFLVF. Similar to the quantitative scoring process of the IS-HFLPM $\mathbf{P}_S^{(q)}$, the score of each HFLTS (i.e.,$H_S^{(q)}(x_t^i)$) in the decision-making matrix indicates the investor e_q's evaluation on the security x_t considering the primary criteria C_i, and can be calculated as follows:

$$E_d^{(q)}\left(H_S^{(q)}(x_t^i)\right) = \frac{1}{L}\left(\sum_L g\left(S_{v(l)}\right)\right). \quad (14)$$

The degree of investor e_q's hesitation in conducting the evaluation of securities with respect to the primary criteria C_i and can be obtained as follows:

$$V_d^{(q)}\left(H_S^{(q)}(x_t^i)\right) = \sqrt{\frac{1}{L}\sum_L \left(g\left(S_{v(l)}\right) - E_d^{(q)}\left(H_S^q(x_t^i)\right)\right)^2}. \quad (15)$$

The decision weight of each criterion refers to the weight for each investor to evaluate securities under different criteria weights in the VS-HFLDMM. The decision weight of each primary criterion C_i in evaluating securities can be expressed as follows:

$$w_d^{(q)}(C_i) = \frac{\left(0.5 - V_d^{(q)}(C_i)\right)}{\sum_{i=1}^{I}\left(0.5 - V_d^{(q)}(C_i)\right)}. \quad (16)$$

where $V_d^{(q)}(C_i) = \frac{\sum_{n=1}^{I} V_d^{(q)}\left(H_S^{(q)}(x_t^i)\right)}{\sum_{n=1}^{I}\sum_{t=1}^{T} V_d^{(q)}\left(H_S^{(q)}(x_t^i)\right)}$ indicates the degree of investor e_q's hesitation in evaluating the primary criterion C_i.

(3) Calculate the primary quantitative score of each security.

After converting investor e_q's evaluations to the primary HFLEs, scoring the preference matrix and the decision-making matrix, and obtaining investor e_q's preference weights for different primary criteria and the decision weights of T securities under the primary criteria. The primary quantitative score of the security x_t representing the prospect value for investors to judge the rise and fall of securities can be calculated as follows:

$$E^{(q)}(x_t) = \sum_{i=1}^{I} w_p^{(q)}(C_i) \cdot w_d^{(q)}(C_i) \cdot E_d^{(q)}\left(H_S^{(q)}(x_t^i)\right). \quad (17)$$

4.2 VS-HFLTS Multi-criteria Group Decision-making Portfolio Secondary Quantitative Score

Securities' secondary quantitative scores can be obtained by scoring the IS-HFLPM and the VS-HFLDMM with respect to the secondary criteria $\tilde{C}_j$. The specific steps of the scoring process are similar to the primary quantitative scoring process and do not need to be repeated here. The secondary quantitative score of the security x_t representing represents the investor e_q's psychological perception to gains and losses regarding, and can be calculated as follows:

$$\theta = \tilde{E}^{(q)}(x_t) = \sum_{j=1}^{J} \tilde{w}_p^{(q)}(C_j) \cdot \tilde{w}_d^{(q)}(C_j) \cdot \tilde{E}_d^{(q)}\left(\tilde{H}_S^{(q)}(x_t^j)\right). \quad (18)$$

The degree of investor e_q's hesitation in evaluating the security x_t with respect to the secondary criteria $\tilde{C}_j$ and can be obtained as follows:

$$HD^{(q)}(x_t) = \sum_{j=1}^{J} V_d^{(q)}\left(\tilde{H}_S^{(q)}(x_t^j)\right). \quad (19)$$

The normalized hesitation degree can be obtained by normalizing as follows:

$$NHD^{(q)}(x_t) = \frac{HD^{(q)}(x_t)}{\sum_{t=1}^{T} HD^{(q)}(x_t)}. \quad (20)$$

Following the definition of the weight function in the CPT, this paper defines the weight function of the investor e_q's purchase appetite for security x_t with respect to the secondary criteria $\tilde{C}_j$ as follows:

$$\Pi^{(q)}(x_t) = \frac{\left(1 - NHD^{(q)}(x_t)\right)^{\theta}}{\left[\left(1 - NHD^{(q)}(x_t)\right)^{\theta} + \left(NHD^{(q)}(x_t)\right)^{\theta}\right]}. \quad (21)$$

4.3 Collective Group Decision Score Aggregation Process

Final quantitative scores can be obtained using each investor's prospect value of securities (i.e., primary quantitative scores) and each investor's purchase appetite weight for securities (i.e., secondary quantitative scores). The investor e_q's final quantitative score of the security x_t can be expressed as follows:

$$\overline{E}^{(q)}(x_t) = E^{(q)}(x_t) \cdot \Pi^{(q)}(x_t). \tag{22}$$

Consequently, the final score matrix of Q investors for T securities can be obtained through the HFLTS portfolio quantitative calculation as follows:

$$\overline{\mathbf{E}} = \begin{pmatrix} \overline{E}^{(1)}(x_1) & \cdots & \overline{E}^{(1)}(x_t) & \cdots & \overline{E}^{(1)}(x_T) \\ \cdots & \ddots & \cdots & \ddots & \cdots \\ \overline{E}^{(q)}(x_1) & \cdots & \overline{E}^{(q)}(x_t) & \cdots & \overline{E}^{(q)}(x_T) \\ \cdots & \ddots & \cdots & \ddots & \cdots \\ \overline{E}^{(Q)}(x_1) & \cdots & \overline{E}^{(Q)}(x_t) & \cdots & \overline{E}^{(Q)}(x_T) \end{pmatrix}. \tag{23}$$

The investor's weight refers to the weight of different investors in the final score matrix. The weights of different investors can be calculated as follows:

$$w(e_q) = \frac{\sum_{t=1}^{T} \overline{E}^{(q)}(x_t)}{\sum\limits_{q=1}^{Q} \sum\limits_{t=1}^{T} \overline{E}^{(q)}(x_t)}. \tag{24}$$

As a result, the final score of each security after aggregating the decision-making opinions of all investors in a group can be obtained as follows:

$$\overline{E}(x_t) = \sum_{q=1}^{Q} w(e_q) \cdot \overline{E}^{(q)}(x_t). \tag{25}$$

The evaluation, quantitative calculation, and score aggregation processes are summarized in the VS-HFLTS multi-criteria group decision-making portfolio, as shown in Fig. 1.

5 VS-HFLTS Multi-criteria Group Decision-making Portfolio Optimization Model

Assuming that after group decision-making aggregation, it is determined that the investment ratio in security x_t is $\eta_t (t = 1, 2, \ldots, T)$, $-1 \le \eta_t \le 1$, the multi-criteria group decision-making portfolio based on VS-HFLTS can be described as

$$\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t.$$

The expected return of the portfolio can be expressed as

$$E\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right) = \frac{1}{T}\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right), \tag{26}$$

where n is the number of securities in the portfolio.

The risk of the portfolio can be expressed as

$$D\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left[\overline{E}(x_t) \cdot \eta_t - E\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right)\right]^2}. \tag{27}$$

5.1 Portfolio Optimization Model

Given that the objective of the VS-HFLTS multi-criteria group decision-making portfolio optimization is to maximize the return within the acceptable maximum risk $D_{\max}$, the optimization model can be expressed as follows:

$$f(\eta) = \max E\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right)$$

$$s.t. \begin{cases} E\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right) = \frac{1}{T}\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right); \\ D\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left[\overline{E}(x_t) \cdot \eta_t - E\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right)\right]^2}; \\ D\left(\sum_{t=1}^{T} \overline{E}(x_t) \cdot \eta_t\right) \le D_{\max}; \\ \sum_{t=1}^{T} \eta_t = 1,\ t = 1, 2, \cdots, T; \\ -1 \le \eta_t \le 1,\ t = 1, 2, \cdots, T. \end{cases} \tag{28}$$

5.2 Maximum Risk Critical Values

The investors' personality and cultural background affect their risk attitude. Radical investors have a higher risk tolerance than neutral investors, and neutral than conservative investors. The portfolio optimization model can capture investors' different risk attitudes by setting different maximum critical values of variance D_{max}.

This paper employs the trisection approach [9] to determine D_{max} corresponding to different types of investors, as shown in Fig. 2. Assume that the fluctuation range of the maximum risk threshold D_{max} is [*minD*, *maxD*].

A. Suppose that the maximum risk threshold $D_{\max}$ corresponding to risk-seeking investors is D_1, $D_1 = maxD$;
B. *Suppose that the max*imum risk threshold $D_{\max}$ corresponding to risk-neutral investors is D_2, $D_2 = minD + 2/3(maxD - minD)$;

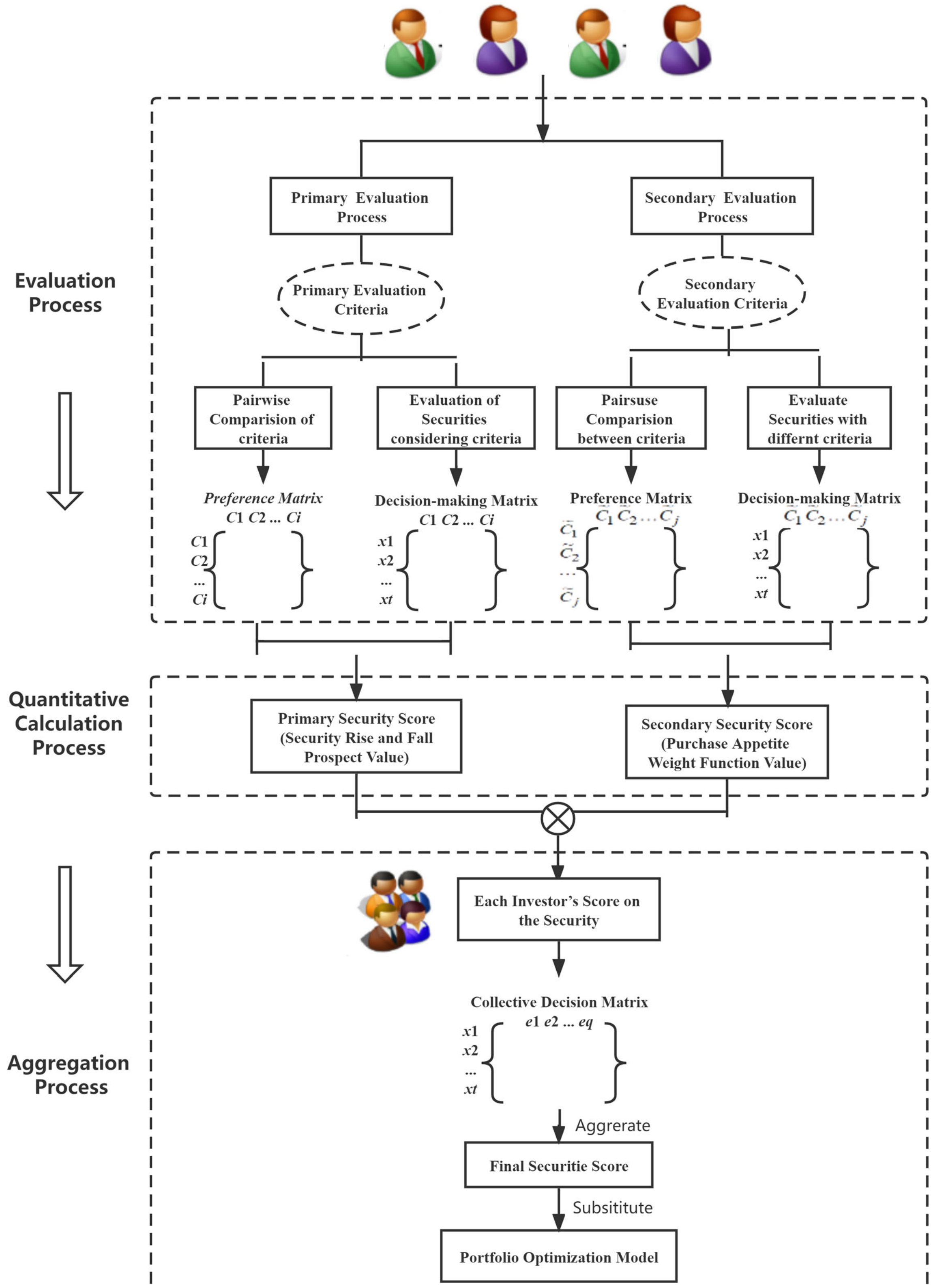


Fig. 1 VS-HFLTS multi-criteria group decision-making portfolio flowchart

C. Suppose that the maximum risk threshold $D_{\max}$ corresponding to risk-averse investors is D_3,

$$D_3 = minD + 1/3(maxD - minD).$$

It is worth noting that even the critical maximum risk threshold of risk-averse investors D_{max} cannot reach the minimum value of variance $minD$, whereas the maximum risk threshold of risk-seeking investors D_{max} can reach the maximum value of variance $maxD$.

5.3 Settle the Maximum and Minimum Values of Variance

The extreme values of the VS-HFLTS multi-criteria group decision-making portfolio variance (i.e., $maxD$ and $minD$) can be obtained by solving the following optimization portfolio model.

$$d(\eta) = \max D\left(\sum_{t=1}^{T}\overline{E}(x_t)\cdot\eta_t\right) \text{or} \min D\left(\sum_{t=1}^{T}\overline{E}(x_t)\cdot\eta_t\right)$$
$$s.t.\begin{cases} E\left(\sum_{t=1}^{T}\overline{E}(x_t)\cdot\eta_t\right) = \frac{1}{T}\left(\sum_{t=1}^{T}\overline{E}(x_t)\cdot\eta_t\right); \\ D\left(\sum_{t=1}^{T}\overline{E}(x_t)\cdot\eta_t\right) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left[\overline{E}(x_t)\cdot\eta_t - E\left(\sum_{t=1}^{T}\overline{E}(x_t)\cdot\eta_t\right)\right]^2}; \\ \sum_{t=1}^{T}\eta_t = 1,\ t = 1, 2, \cdots, T; \\ -1 \le \eta_t \le 1,\ t = 1, 2, \cdots, T. \end{cases} \tag{29}$$

6 Application and Numerical Simulation

The rationality and effectiveness of the optimization model are verified by analyzing five selected securities from the perspectives of three institutional investors. The studied securities consist of five A-share stocks, namely Wuliangye (stock code 000858.SZ), Hengshun Vinegar (stock code 600305.SH), Jinyu Medical (stock code 603882.SH), Haier Zhijia (stock code 600690.SH), and Rongbai Technology (stock code 688005.SH). As shown in Table 2, data on these stocks were obtained from the research reports published by East Money.com on December 24, 2020. These collated A-share stocks' information is evaluated based on primary and secondary criteria. The primary criteria consider five dimensions: national policies, market environment, industry conditions, business operating conditions, and stock price trend in the past year, whereas the secondary criteria consider three dimensions: household income, market trend, and risk appetite.

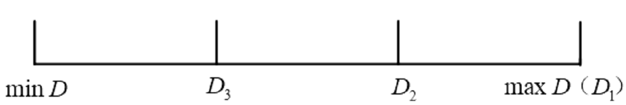


Fig. 2 Trisection diagram of $D_{\max}$ of investors with different risk appetites attitudes

6.1 VS-HFLTS Multi-criteria Group Decision-making Portfolio Valuation Process

The five stocks are denoted as $\{x_1, x_2, x_3, x_4, x_5\}$, and their corresponding investment ratios are $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$, given that $\sum_{t=1}^{5}\eta_t = 1$. The primary criteria are denoted as $\{C_1, C_2, C_3, C_4, C_5\}$, whereas the secondary criteria are $\{\tilde{C_1}, \tilde{C_2}, \tilde{C_3}\}$. Assuming that the investment goal is to maximize the return, the VS-HFLTS multi-criteria group decision-making portfolio optimization system can be used to obtain the HFLPM and the HFLDMM considering two-stage criteria. A risk-averse investor is taken as an example in the following detailed illustration of the portfolio evaluation process. The evaluation process by other types of investors is similar.

Step 1 Set up the HFLTS for the multi-criteria group decision-making portfolio evaluation process, as shown in Table 3.

Step 2 Taking the investor e_1 as an example, the investor e_1 performs pairwise comparisons of the five primary criteria, C_i (i = 1, 2, …, 5), and gives the natural language evaluations corresponding to the primary preference evaluation (Table 4). Adopt the context-free grammar G_H to convert the obtained natural language into linguistic expressions for primary preference evaluation (Table 5).

Step 3 Convert the linguistic expressions of primary preference evaluation into an IS-HFLE using the transformation function in Definition 3, and then obtain the investor e_1's primary IS-HFLPM $\mathbf{P}_S^{(1)}$ for the primary evaluation criteria C_i:

$$\mathbf{P}_S^{(1)} = \begin{bmatrix} \{S_0\} & \{S_1,S_2,S_3,S_4\} & \{S_2,S_3,S_4\} & \{S_1,S_2\} & \{S_3,S_4\} \\ \{S_{-4},S_{-3},S_{-2},S_{-1}\} & \{S_0\} & \{S_1,S_2\} & \{S_{-2},S_{-1}\} & \{S_2,S_3,S_4\} \\ \{S_{-4},S_{-3},S_{-2}\} & \{S_{-2},S_{-1}\} & \{S_0\} & \{S_2,S_3\} & \{S_1\} \\ \{S_{-2},S_{-1}\} & \{S_1,S_2\} & \{S_{-3},S_{-2}\} & \{S_0\} & \{S_2,S_3\} \\ \{S_{-4},S_{-3}\} & \{S_{-4},S_{-3},S_{-2}\} & \{S_{-1}\} & \{S_{-3},S_{-2}\} & \{S_0\} \end{bmatrix}$$

Step 4 The investor e_1 gives the natural language evaluations corresponding to the primary decision-making evaluation of the five securities, x_t (t = 1, 2, …, 5), with respect to the five primary criteria, C_i (i = 1, 2, …, 5) (Table 6). Adopt the context-free grammar G_H to convert the obtained natural language into linguistic expressions for the primary decision-making evaluation (Table 7).

Step 5 Convert the linguistic expressions of primary decision-making evaluation into a VS-HFLE using the transformation function in Definition 3, and then obtain the investor e_1's primary VS-HFLDMM $\mathbf{D}_S^{(1)}$ for the five

Table 2 Related information on investment target stocks (obtained from the research report of Eastmoney.com)

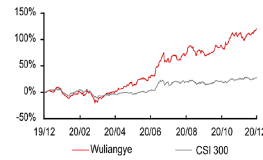
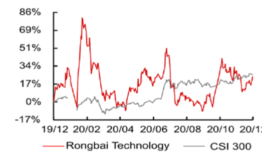
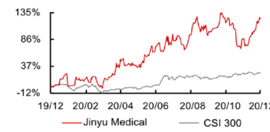
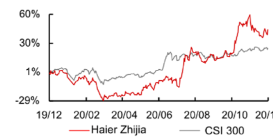
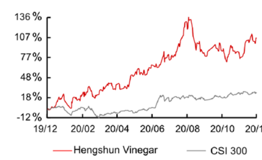
	Market environment	Industry conditions	Business operating conditions	The relative trend of stock prices in the past year
Wuliangye (Brewing industry)	Consumption has recovered less than expected due to the global pandemic	Increased industry competition; leading companies insist on making progress while maintaining stability	The target is likely to be met even in the pandemic situation; the price of wheat is steadily rising; the **overweight rating** is maintained	
Hengshun Vinegar (Food and drink)	Increased market competition	The only listed company in the industry; food safety issues cannot be ignored	After blindly diversifying, refocus on the main business; management efficiency needs to be improved; **neutral rating**	
Jinyu Medical (Medical industry)	Hospitals use Independent Clinical Laboratories (ICL) to control testing costs; the ICL penetration rate increases	The ICL industry has a significant scale effect, high concentration, and good growth prospects	The layout of the national ICL leader; comprehensive testing projects; laboratories turned losses into profits; the **overweight rating** is achieved	
Haier Zhijia (Appliance industry)	In the pandemic situation, domestic production capacity is exported overseas to expand overseas market share and create a wide range of resources	Disturbed by the pandemic; under short-term pressure	Successful privatization, retail transformation (high-end), cost reduction and efficiency improvement; profit release can be expected; the **long rating** is maintained	
Rongbai Technology (Material industry)	The high-nickel market volume has increased; ranking first in market share; raw material price risk is relatively high	The technology is advanced; the product pass rate has increased; the material system has scaled up; the energy density advantage is prominent	Major customers are locked in; the level of technology and the scale of production are gradually increasing; global leader in the high-nickel industry; the **long rating** is achieved	

Table 3 The HFLTS of multi-criteria group decision-making portfolio

Criteria	HFLTS
Primary preference evaluation	S_p = {S_{-4} = **totally unimportant**, S_{-3} = **very unimportant**, S_{-2} = **moderately unimportant**, S_{-1} = **slightly unimportant**, S_0 = **no judgment**, S_1 = **slightly important**, S_2 = **moderately important**, S_3 = **very important**, S_4 = **totally important**}
Primary decision-making evaluation	S_d = {$S_{v(-4)}$ = **completely negate**, $S_{v(-3)}$ = **strongly negate**, $S_{v(-2)}$ = **moderately negate**, $S_{v(-1)}$ = **slightly negate**, $S_{v(0)}$ = **no judgment**, $S_{v(1)}$ = **slightly confirm**, $S_{v(2)}$ = **moderately confirm**, $S_{v(3)}$ = **strongly confirm**, $S_{v(4)}$ = **completely confirm**}
Secondary preference evaluation	$\tilde{S}_p$ = {$\tilde{S}_{-4}$ = **totally unimportant**, $\tilde{S}_{-3}$ = **very unimportant**, $\tilde{S}_{-2}$ = **moderately unimportant**, $\tilde{S}_{-1}$ = **slightly unimportant**, $\tilde{S}_0$ = **no judgment**, $\tilde{S}_1$ = **slightly important**, $\tilde{S}_2$ = **moderately important**, $\tilde{S}_3$ = **very important**, $\tilde{S}_4$ = **totally important**}
Secondary decision-making evaluation	$\tilde{S}_d$ = {$\tilde{S}_{v(-4)}$ = **short selling all**, $\tilde{S}_{v(-3)}$ = **short selling a lot**, $\tilde{S}_{v(-2)}$ = **short selling half**,$\tilde{S}_{v(-1)}$ = **short selling a little**, $\tilde{S}_{v(0)}$ = **hold**, $\tilde{S}_{v(1)}$ = **take long a little**, $\tilde{S}_{v(2)}$ = **take long half**, $\tilde{S}_{v(3)}$ = **take long a lot**, $\tilde{S}_{v(4)}$ = **take long all**}

Table 4 Natural language for primary preference evaluation

	C1	C2	C3	C4	C5)
C_1	No idea	Important	More important	Important, but not very important	At least particularly important
C_2	Not important, but not at all unimportant	No idea	A little important, but not particularly important	A little unimportant, but not particularly unimportant	At least moderately important
C_3	Less important	A little unimportant, but not particularly unimportant	No idea	At least moderately important, but not completely important	Some important
C_4	Not important, but not very unimportant	Some important, but not much important	At least moderately unimportant, but not at all unimportant	No idea	At least moderately important, but not completely important
C_5	At least not particularly important	At least moderately unimportant	Some unimportant	At least moderately unimportant, but not at all unimportant	No idea

Table 5 Linguistic expressions for primary preference evaluation

	C_1	C_2	C_3	C_4	C_5
C_1	**No judgment**	Between **slightly important** and **totally important**	At least **moderately important**	Between **slightly important** and **moderately important**	Between **very important** and **totally important**
C_2	Between **totally unimportant** and **slightly unimportant**	**No judgment**	Between **slightly important** and **moderately important**	Between **moderately unimportant** and **slightly unimportant**	Between **moderately important** and **totally important**
C_3	At least **moderately unimportant**	Between **moderately unimportant** and **slightly unimportant**	**No judgment**	Between **moderately important** and **very important**	**Slightly important**
C_4	Between **moderately unimportant** and **slightly unimportant**	Between **slightly important** and **moderately important**	Between **very unimportant** and **moderately unimportant**	**No judgment**	Between **moderately important** and **very important**
C_5	Between **totally unimportant** and **very unimportant**	Between **totally unimportant** and **moderately unimportant**	**Slightly unimportant**	Between **very unimportant** and **moderately unimportant**	**No judgment**

securities, x_t (t = 1, 2, …, 5), with respect to the five primary criteria, C_i (i = 1, 2, …, 5):

Step 6 Repeat Steps 2 to 3 to obtain the investor e_1's secondary IS-HFLPM $\tilde{\mathbf{P}}_S^{(1)}$ for the secondary evaluation criteria $\tilde{C}_j$:

$$\tilde{\mathbf{P}}_S^{(1)} = \begin{bmatrix} \{\tilde{S}_0\} & \{\tilde{S}_{-4}, \tilde{S}_{-3}\} & \{\tilde{S}_{-2}, \tilde{S}_{-1}\} \\ \{\tilde{S}_3, \tilde{S}_4\} & \{\tilde{S}_0\} & \{\tilde{S}_2, \tilde{S}_3\} \\ \{\tilde{S}_1, \tilde{S}_2\} & \{\tilde{S}_{-3}, \tilde{S}_{-2}\} & \{\tilde{S}_0\} \end{bmatrix}$$

Step 7 Repeat Steps 4 to 5 to obtain the investor e_1's e$_1$ secondary VS-HFLDMM $\tilde{\mathbf{D}}_S^{(1)}$ for the five securities, x_t (t = 1, 2, …, 5), with respect to the three secondary criteria, $\tilde{C}_j (j$ = 1, 2, 3):

$$\tilde{\mathbf{D}}_S^{(1)} = \begin{bmatrix} \{\tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}, \tilde{S}_{v(3)}\} \\ \{\tilde{S}_{v(-3)}, \tilde{S}_{v(-2)}, \tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(-2)}, \tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} \\ \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(0)}, \tilde{S}_{v(1)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} \\ \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}\} & \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} \\ \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} & \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} & \{\tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} \end{bmatrix}$$

Step 8 Use the above steps to obtain the investor e_2's and the investor e_3's primary evaluation preference matrices $\mathbf{P}_S^{(2)}$ and $\mathbf{P}_S^{(3)}$, primary decision-making matrices $\mathbf{D}_S^{(2)}$ and $\mathbf{D}_S^{(3)}$, secondary evaluation preference matrices $\tilde{\mathbf{P}}_S^{(2)}$ and $\tilde{\mathbf{P}}_S^{(3)}$, and secondary decision-making matrices $\tilde{\mathbf{D}}_S^{(2)}$ and $\tilde{\mathbf{D}}_S^{(3)}$:

$$\mathbf{P}_S^{(2)} = \begin{bmatrix} \{S_0\} & \{S_1, S_2, S_3\} & \{S_2, S_3, S_4\} & \{S_1, S_2\} & \{S_3, S_4\} \\ \{S_{-3}, S_{-2}, S_{-1}\} & \{S_0\} & \{S_1, S_2\} & \{S_{-2}, S_{-1}, S_0\} & \{S_3, S_4\} \\ \{S_{-4}, S_{-3}, S_{-2}\} & \{S_{-2}, S_{-1}\} & \{S_0\} & \{S_2, S_3\} & \{S_1, S_2\} \\ \{S_{-2}, S_{-1}\} & \{S_0, S_1, S_2\} & \{S_{-3}, S_{-2}\} & \{S_0\} & \{S_2, S_3\} \\ \{S_{-4}, S_{-3}\} & \{S_{-4}, S_{-3}\} & \{S_{-2}, S_{-1}\} & \{S_{-3}, S_{-2}\} & \{S_0\} \end{bmatrix}$$

$$\mathbf{D}_S^{(2)} = \begin{bmatrix} \{S_{v(-1)}, S_{v(0)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}\} & \{S_{v(1)}, S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(1)}, S_{v(2)}\} & \{S_{v(0)}, S_{v(1)}\} \\ \{S_{v(0)}\} & \{S_{v(-2)}, S_{v(-1)}\} & \{S_{v(0)}, S_{v(1)}, S_{v(2)}\} & \{S_{v(-2)}, S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}\} \\ \{S_{v(0)}, S_{v(1)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(0)}, S_{v(1)}, S_{v(2)}, S_{v(3)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}, S_{v(3)}\} \\ \{S_{v(-1)}, S_{v(0)}\} & \{S_{v(1)}, S_{v(2)}\} & \{S_{v(1)}, S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(3)}, S_{v(4)}\} \\ \{S_{v(1)}, S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(1)}, S_{v(2)}, S_{v(3)}\} & \{S_{v(3)}\} & \{S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(3)}, S_{v(4)}\} \end{bmatrix} \tilde{\mathbf{P}}_S^{(2)} = \begin{bmatrix} \{\tilde{S}_0\} & \{\tilde{S}_{-4}, \tilde{S}_{-3}\} & \{\tilde{S}_{-2}, \tilde{S}_{-1}, \tilde{S}_0\} \\ \{\tilde{S}_3, \tilde{S}_4\} & \{\tilde{S}_0\} & \{\tilde{S}_2, \tilde{S}_3\} \\ \{\tilde{S}_0, \tilde{S}_1, \tilde{S}_2\} & \{\tilde{S}_{-3}, \tilde{S}_{-2}\} & \{\tilde{S}_0\} \end{bmatrix}$$

$$\tilde{\mathbf{D}}_S^{(2)} = \begin{bmatrix} \{\tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}, \tilde{S}_{v(3)}\} \\ \{\tilde{S}_{v(-3)}, \tilde{S}_{v(-2)}, \tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(-2)}, \tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} \\ \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(1)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} \\ \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}, \tilde{S}_{v(3)}\} & \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} \\ \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} & \{\tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} & \{\tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} \end{bmatrix}$$

$$\mathbf{P}_S^{(3)} = \begin{bmatrix} \{S_0\} & \{S_0, S_1, S_2, S_3, S_4\} & \{S_3, S_4\} & \{S_1, S_2\} & \{S_4\} \\ \{S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0\} & \{S_0\} & \{S_0, S_1, S_2\} & \{S_{-2}, S_{-1}\} & \{S_2, S_3, S_4\} \\ \{S_{-4}, S_{-3}\} & \{S_{-2}, S_{-1}, S_0\} & \{S_0\} & \{S_2, S_3\} & \{S_0, S_1\} \\ \{S_{-2}, S_{-1}\} & \{S_1, S_2\} & \{S_{-3}, S_{-2}\} & \{S_0\} & \{S_2, S_3\} \\ \{S_{-4}\} & \{S_{-4}, S_{-3}, S_{-2}\} & \{S_{-1}, S_0\} & \{S_{-3}, S_{-2}\} & \{S_0\} \end{bmatrix}$$

$$\mathbf{D}_S^{(3)} = \begin{bmatrix} \{S_{v(-1)}, S_{v(0)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}\} & \{S_{v(1)}, S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(1)}, S_{v(2)}\} & \{S_{v(0)}, S_{v(1)}\} \\ \{S_{v(0)}\} & \{S_{v(-2)}, S_{v(-1)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}\} & \{S_{v(-2)}, S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}\} \\ \{S_{v(0)}, S_{v(1)}\} & \{S_{v(1)}, S_{v(2)}\} & \{S_{v(1)}, S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(0)}, S_{v(1)}, S_{v(2)}, S_{v(3)}\} & \{S_{v(-1)}, S_{v(0)}, S_{v(1)}, S_{v(2)}, S_{v(3)}\} \\ \{S_{v(-1)}, S_{v(0)}\} & \{S_{v(1)}, S_{v(2)}, S_{v(3)}\} & \{S_{v(1)}, S_{v(2)}, S_{v(3)}\} & \{S_{v(2)}, S_{v(3)}\} & \{S_{v(2)}, S_{v(3)}, S_{v(4)}\} \\ \{S_{v(1)}, S_{v(2)}, S_{v(3)}\} & \{S_{v(1)}, S_{v(2)},\} & \{S_{v(2)}, S_{v(3)}\} & \{S_{v(2)}, S_{v(3)}, S_{v(4)}\} & \{S_{v(3)}, S_{v(4)}\} \end{bmatrix}$$

$$\tilde{\mathbf{P}}_S^{(3)} = \begin{bmatrix} \{\tilde{S}_0\} & \{\tilde{S}_{-4}, \tilde{S}_{-3}\} & \{\tilde{S}_{-2}\} \\ \{\tilde{S}_3, \tilde{S}_4\} & \{\tilde{S}_0\} & \{\tilde{S}_1, \tilde{S}_2, \tilde{S}_3\} \\ \{\tilde{S}_2\} & \{\tilde{S}_{-3}, \tilde{S}_{-2}, \tilde{S}_{-1}\} & \{\tilde{S}_0\} \end{bmatrix}$$

$$\tilde{\mathbf{D}}_S^{(3)} = \begin{bmatrix} \{\tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}, \tilde{S}_{v(3)}\} \\ \{\tilde{S}_{v(-3)}, \tilde{S}_{v(-2)}, \tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}\} & \{\tilde{S}_{v(-2)}, \tilde{S}_{v(-1)}, \tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} \\ \{\tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(0)}, \tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}, \tilde{S}_{v(3)}\} \\ \{\tilde{S}_{v(1)}, \tilde{S}_{v(2)}\} & \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}\} & \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} \\ \{\tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} & \{\tilde{S}_{v(2)}, \tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} & \{\tilde{S}_{v(3)}, \tilde{S}_{v(4)}\} \end{bmatrix}$$

6.2 VS-HFLTS Multi-criteria Group Decision-Making Portfolio Quantitative Calculation

The actual evaluation process requires the investors to be as objective and fair as possible when performing the preference evaluation of different criteria. The evaluation itself has included information such as investor's personal knowledge, subjective cognition, and value preferences. Therefore, the preference evaluation terms adopt uniformly distributed semantics, and the linguistic scale function is used when scoring the preference evaluation matrix. As the degree of semantics increases, the value of the linguistic scale function increases uniformly in Section. 6.1. The linguistic scale function can be expressed as follows:

$$f(S_\alpha) = \frac{\alpha}{\tau} . \quad (30)$$

However, an investor's psychology psychological state may undergo subtle changes when trading in securities, especially when assessing stock's potential price trend and their subsequent desire to purchase the security. Regardless of whether the investor is risk-averse or risk-seeking, as the degree of evaluation deepens (i.e., the linguistic term

Table 6 Natural language for primary decision-making evaluation

	C_1	C_2	C_3	C_4	C_5
x_1	At most, a small drop, not much influence	At most a small drop, but no sharp gains	Increase	Rise, but not so much	Move up a bit
x_2	No idea	At least a small decline, but not a big decline	At most drop a little, but not climb much	The rise and fall will not be great	At most fall a little, but not skyrocket
x_3	Won't fall, but won't rise much	Go up a bit, but not go up significantly	At least not slide	Not fall, but no sharp gains	Fluctuates slightly, but the overall trend is upward
x_4	At most marginally softer, not much influence	At least marginally higher, but not absolutely soar	At most a slight decline, still maintaining an upward trend	At least surge	Sharp gains
x_5	At least go up a little	Go up, but no sharp gains	At least a moderate increase, but not 100% sure	At least a moderate increase	Rise a lot

Table 7 Linguistic expressions for primary decision-making evaluation

	C_1	C_2	C_3	C_4	C_5
x_1	Between **slightly negate** and **no judgment**	Between **slightly negate** and **moderately confirm**	At least **slightly confirm**	Between **slightly confirm** and **moderately confirm**	Between **no judgment** and **slightly confirm**
x_2	**No judgment**	Between **moderately negate** and **slightly confirm**	Between **slightly negate** and **moderately confirm**	Between **moderately negate** and **moderately confirm**	Between **slightly negate** and **moderately confirm**
x_3	Between **no judgment** and **slightly confirm**	Between **slightly confirm** and **moderately confirm**	At least **no judgment**	Between **no judgment** and **strongly confirm**	Between **slightly negate** and **strongly confirm**
x_4	Between **slightly negate** and **no judgment**	Between **slightly confirm** and **strongly confirm**	At most **slightly negate**	At least **moderately confirm**	At least **strongly confirm**
x_5	At least **slightly confirm**	Between **slightly confirm** and **moderately confirm**	Between **moderately confirm** and **strongly confirm**	At least **moderately confirm**	At least **strongly confirm**

subscript increases), the linguistic value contained in the evaluation increases gradually, but this increasing trend slows down (i.e., the value interval of adjacent language evaluation gradually decreases). In other words, as the variable subscript increases, the extent of the rise and fall of securities will certainly increase, but the increasing degree will gradually decrease. For example, an investor's evaluation of a stock "definitely a sharp growth" and "take long all" is more rigorous than "maybe marginally higher" and "take long a little." Taking into account the changes in investors' trading psychology, the decision-making evaluation employs VS-HFLTS, and the corresponding decision-making matrix quantitative calculation uses linguistic value functions in Section. 6.1. Suppose that the appetite is to take long when the stock is bullish, and the corresponding HFLVF is positive, the appetite is to short selling when the stock is bearish, and the HFLVF is negative. The HFLVF can be expressed as follows:

$$g\left(S_{v(l)}\right) = [v(l)]^{C}. \tag{31}$$

After substituting the Eq. (27) into the Eq. (2), the variable subscript hesitant fuzzy linguistic value function (VS-HFLVF) can be derived as

$$g\left(S_{v(l)}\right) = \begin{cases} l^{C\cdot\alpha}, & l = 0,\ 1,\ \cdots,\ \tau, \\ -(\gamma)^{C}\cdot(-l)^{C\cdot\beta}, & l = -\tau,\ \cdots,\ 0. \end{cases}$$

where α and β indicate the investor's sensitivity to gains/take long securities and losses/short-selling securities, respectively; γ indicates the investor's aversion to losses; and C is a variable subscript parameter. The VS-HFLVF must have the following properties:

A. The primary HFLVF g passes through the origin: $g(S_0) = 0$;
B. The primary HFLVF g passes through the first and third quadrants: If $v(l) > 0$, $g\ (S_{v(l)}) > 0$; if $v(l) < 0$, $g\ (S_{v(l)}) < 0$;
C. The primary HFLVF is convex: $0 < C{\cdot}\alpha < 1$ and $0 < C{\cdot}\beta < 1$;

As the primary hesitant fuzzy linguistic subscript increases, the differences between adjacent primary hesitant fuzzy linguistic term subscripts gradually decrease. Therefore, C is determined as 1/3 in the subsequent numerical simulation.

The primary preference matrices and decision matrices of the three institutional investors are calculated using the process introduced in Section. 4.1, and the results are as follows:

$$\mathbf{D}_S^{(1)} = \begin{bmatrix} -0.6552 & 0.2288 & 1.2768 & 1.1127 & 0.5000 \\ 0 & -1.4580 & 0.2288 & -0.1381 & 0.2288 \\ 0.5000 & 1.1127 & 1.0215 & 0.9014 & 0.4591 \\ -0.6552 & 1.2019 & 0.6328 & 1.3691 & 1.4409 \\ 1.2768 & 1.1127 & 1.3028 & 1.2768 & 1.4409 \end{bmatrix}$$

$$\mathbf{P}_S^{(1)} = \begin{bmatrix} 0 & 0.6250 & 0.7500 & 0.3750 & 0.8750 \\ -0.6250 & 0 & 0.3750 & -0.3750 & 0.7500 \\ -0.7500 & -0.3750 & 0 & 0.6250 & 0.2500 \\ -0.3750 & 0.3750 & -0.6250 & 0 & 0.6250 \\ -0.8750 & -0.7500 & -0.2500 & -0.6250 & 0 \end{bmatrix}$$

$$\mathbf{D}_S^{(3)} = \begin{bmatrix} -0.6522 & 0.2288 & 1.2768 & 1.1127 & 0.5000 \\ 0 & -1.4580 & 0.2288 & -0.1381 & 0.2288 \\ 0.5000 & 1.1127 & 1.2768 & 0.9014 & 0.4591 \\ -0.6552 & 1.2019 & 1.2019 & 1.3028 & 1.3691 \\ 1.2019 & 1.1127 & 1.3028 & 1.3691 & 1.4409 \end{bmatrix}$$

$$\mathbf{D}_S^{(2)} = \begin{bmatrix} -0.6522 & 0.2288 & 1.2768 & 1.1127 & 0.5000 \\ 0 & -1.4580 & 0.7418 & -0.1381 & 0.2288 \\ 0.5000 & 0.2288 & 0.6328 & 0.9014 & 0.4591 \\ -0.6522 & 1.1127 & 1.2768 & 1.3691 & 1.4409 \\ 1.2768 & 1.2019 & 1.3802 & 1.3691 & 1.4409 \end{bmatrix}$$

$$\mathbf{P}_S^{(2)} = \begin{bmatrix} 0 & 0.5000 & 0.7500 & 0.3750 & 0.8750 \\ -0.5000 & 0 & 0.3750 & -0.2500 & 0.8750 \\ -0.7500 & -0.3750 & 0 & 0.6250 & 0.3750 \\ -0.3750 & 0.2500 & -0.6250 & 0 & 0.6250 \\ -0.8750 & -0.8750 & -0.3750 & -0.6250 & 0 \end{bmatrix}$$

$$\mathbf{P}_S^{(3)} = \begin{bmatrix} 0 & 0.5000 & 0.8750 & 0.3750 & 1.0000 \\ -0.5000 & 0 & 0.2500 & -0.3750 & 0.7500 \\ -0.8750 & -0.2500 & 0 & 0.6250 & 0.3750 \\ -0.3750 & 0.3750 & -0.6250 & 0 & 0.6250 \\ -1.0000 & -0.7500 & -0.6250 & -0.6250 & 0 \end{bmatrix}.$$

The secondary preference matrices and decision matrices of the three institutional investors are calculated using the process introduced in Section. 4.2 and the results are as follows:

$$\tilde{\mathbf{P}}_S^{(1)} = \begin{bmatrix} 0 & -0.8750 & -0.3750 \\ 0.8750 & 0 & 0.6250 \\ 0.3750 & -0.6250 & 0 \end{bmatrix} \tilde{\mathbf{D}}_S^{(1)} = \begin{bmatrix} 0.7418 & 0.2288 & 0.9014 \\ -0.4165 & -0.1381 & 1.1127 \\ 1.1127 & 0.5000 & 1.2768 \\ 1.1127 & 1.3028 & 1.3691 \\ 1.3691 & 1.3691 & 1.4409 \end{bmatrix}$$

$$\tilde{\mathbf{D}}_S^{(1)} = \begin{bmatrix} 0.7418 & 0.2288 & 0.9014 \\ -0.4165 & -0.1381 & 1.1127 \\ 1.1127 & 0.5000 & 1.2768 \\ 1.1127 & 1.3028 & 1.3691 \\ 1.3691 & 1.3691 & 1.4409 \end{bmatrix}$$

$$\tilde{\mathbf{P}}_S^{(2)} = \begin{bmatrix} 0 & -0.8750 & -0.2500 \\ 0.8750 & 0 & 0.6250 \\ 0.2500 & -0.6250 & 0 \end{bmatrix} \tilde{\mathbf{D}}_S^{(2)} = \begin{bmatrix} 0.7418 & 0.2288 & 1.2019 \\ -0.4165 & -0.4790 & 1.1127 \\ 1.1127 & 1.0000 & 1.2768 \\ 1.1127 & 1.2019 & 1.3691 \\ 1.3691 & 1.4409 & 1.4409 \end{bmatrix}$$

$$\tilde{\mathbf{P}}_S^{(3)} = \begin{bmatrix} 0 & -0.8750 & -0.5000 \\ 0.8750 & 0 & 0.5000 \\ 0.5000 & -0.5000 & 0 \end{bmatrix} \tilde{\mathbf{D}}_S^{(3)} = \begin{bmatrix} 0.7418 & -0.1034 & 1.2019 \\ -0.7449 & -0.1381 & 1.1127 \\ 0.7418 & 0.7418 & 1.2019 \\ 1.1127 & 1.3028 & 1.3691 \\ 1.4409 & 1.3691 & 1.4409 \end{bmatrix}$$

The final score matrix of the three institutional investors for the five selected stocks is calculated using the process introduced in Section 4.3 and the results are as follows:

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 0.0161 & -0.0027 & 0.0637 & 0.0955 & 0.1618 \\ 0.0204 & -0.0009 & 0.0499 & 0.0876 & 0.1557 \\ 0.1158 & -0.0406 & 0.1946 & 0.2054 & 0.4770 \end{bmatrix} \end{array}$$

The final scores of the five selected stocks obtained after aggregating the decision-making opinions of all three investors in a group are 0.0763, -0.0249, 0.1401, 0.1549, and 0.2136. After substituting the final scores of the five stocks into the portfolio optimization model expressed as the Eq. (24) in Section. 5.1, the following non-linear programming calculation can be obtained:

$$f(\eta) = \max E$$
$$s.t.\begin{cases} E = \frac{1}{5}(0.0763\eta_1 - 0.0249\eta_2 + 0.1401\eta_3 + 0.1594\eta_4 + 0.2136\eta_5)\,; \\ D = \sqrt{\frac{1}{5}\left[(0.0763\eta_1 - E)^2 + (-0.0249\eta_2 - E)^2 + (0.1401\eta_3 - E)^2 + (0.1594\eta_4 - E)^2 + (0.2136\eta_5 - E)^2\right]}\,; \\ D \le 0.1075\,; \\ \sum_{i=1}^{5}\eta_i = 1,\ i = 1, 2, \cdots, 5\,; \\ -1 \le \eta_i \le 1,\ i = 1, 2, \cdots, 5\,. \end{cases} \tag{32}$$

where the maximum and minimum values of the hesitant fuzzy linguistic portfolio variance are solved by Matlab Software using the model (25), and the results are $maxD = 0.3018$, $minD = 0.0105$. Therefore, the maximum risk threshold D_{max} corresponding to risk-averse investor is $D_{max} = minD + 1/3(maxD - minD) = 0.1075$.

The solution of the Eq. (28) shows that the optimal investment ratios are -0.3501, $-$ 1.0000, 0.8572, 0.8102, and 0.6827. This leads to the portfolio to have the largest return, the maximum value is $\max E = 0.0786$, and the corresponding risk is at the extreme value of 0.1076 that can be tolerated.

7 Discussion of Simulation Results

7.1 Comparison Between VS-HFLTS and IS-HFLTS

This section takes traditional conservative investor (parameter values $\alpha = \beta = 0.88$, $\gamma = 2.25$, $\zeta = \xi = 0.5$) as an example to compare the VS-HFLTS and the traditional IS-HFLTS. The corresponding numerical simulations demonstrate the comparison results of the two HFLTS, which indicate the effectiveness and superiority of the VS-HFLTS portfolio decision-making proposed in this paper.

The comparison results between VS-HFLTS and IS-HFLTS are shown in Table 8. It can be seen that when the subscript value of the linguistic term is positive (i.e., holding a bullish or take long attitude toward the stock), the VS-HFLTS has smaller subscript values than that of IS-HFLTS. This is because the VS-HFLTS introduces investors' psychological risk factors into investment decision evaluation. Considering that conservative investors are more sensitive to the psychological perception of returns than other kinds of investors, and their sensitivity will become stronger with the strengthening of evaluation terms. Therefore, not only the subscript of VS-HFLTS is smaller than that of IS-HFLTS, but also the absolute value of the deviation between subscripts of VS-HFLTS shows a decreasing trend. When the subscript value of the linguistic term is negative (i.e., holding a bearish or short-selling attitude toward the stock), the VS-HFLTS has larger subscript values than that of IS-HFLTS. The reason is that the VS-HFLTS not only considers the degree of investors' psychological perception when facing risk (parameters α, β) but also considers the degree of investors' aversion to risk (parameterγ). Therefore, although the VS-HFLTS subscript is smaller than IS-HFLTS (more negative), the absolute value of the deviation between the subscripts of its language terms still maintains a decreasing trend.

Further, the HFLVF is utilized to quantify the score of the elements of the VS-HFLTS, and the HFLSF is adopted to that of the traditional IS-HFLTS. The scoring comparison results are shown in Table 9. It can be seen that with the increase of the subscript of the linguistic term, the HFLVF and the HFLSF both show an increasing trend, which is in line with the quantitative evaluation of natural semantics. The difference is that the value of the HFLSF is symmetrically distributed and increases uniformly with the increase of the subscript; the value of the HFLVF gradually decreases with the increase of the subscript of the linguistic term, and the interval between adjacent linguistic evaluation functions gradually decreases. Because the investor gives a more rigorous evaluation, the more cautious its evaluation attitude is. However, the VS-HFLVF at the positive linguistic term subscript is larger than the integer subscript hesitant fuzzy linguistic scale function (IS-HFLSF). The reason is that the VS-HFLVF not only considers investors' cautious evaluation attitude when investing in the actual financial market, but also considers investors' sensitivity to profit and loss (conservative investors are more sensitive to gains and losses). At the value of the negative linguistic term subscript, the HFLVF based on the variable subscript not only considers investors' sensitivity to risk (parameters α, β), but also considers investors' aversion degree to risk (parameterγ). Hence, the value of VS-HFLVF is smaller than that of IS-HFLSF.

Table 8 Subscripts comparison between VS-HFLTS and IS-HFLTS

Subscripts	$-$ 4	$-$ 3	$-$ 2	$-$ 1	0	1	2	3	4
IS-HFLTS	$-$ 4	$-$ 3	$-$ 2	$-$ 1	0	1	2	3	4
VS-HFLTS	$-$ 7.62	$-$ 5.92	$-$ 4.10	$-$ 2.25	0	1	1.84	2.63	3.39

Table 9 Comparison between VS-HFLVF and IS-HFLSF

	− 4	− 3	− 2	− 1	0	1	2	3	4
IS-HFLSF value $f(S_\alpha)$	− 1.0000	− 0.7500	− 0.5000	− 0.2500	0	0.2500	0.5000	0.7500	1.0000
VS-HFLVF value $g(S_{v(l)})$	− 1.9677	− 1.8085	− 1.6057	− 1.3103	0	1	1.2254	1.3802	1.5017

Table 10 The investment ratio between VS-HFLTS and IS-HFLTS

	x_1	x_2	x_3	x_4	x_5	Value of max$E$	Corresponding value of D
Integer subscript	− 0.1357	− 1	0.4985	0.5439	0.0933	0.0148	0.0423
Variable subscript	− 0.3561	− 1	0.8572	0.8102	0.6827	0.0786	0.1076

According to the process of portfolio quantitative method proposed in this paper, the variable subscript hesitant fuzzy linguistic value function and the IS-HFLSF are used to score each stock, respectively, and comprehensively. The VS-HFLTS portfolio optimization model and the IS-HFLTS portfolio optimization model are solved separately by Eq. (28), and the optimal investment ratio of the corresponding conservative group investment decision-makers can be obtained. The results are shown in Table 10. It can be seen that both models obtain the maximum return value at the extreme value of risk, and the variable subscript hesitant fuzzy linguistic portfolio optimization model obtains the maximum return value of the portfolio at the risk critical value of 0.1076, and the maximum return is 0.0786. While the integer subscript hesitant fuzzy linguistic portfolio optimization model obtains the maximum return value of the portfolio at the risk critical value of 0.0423, the maximum return is 0.0148. Obviously, the maximum return and the corresponding risk value both satisfy the VS-HFLTS > IS-HFLTS. The reason is that the VS-HFLTS takes into account the psychological perception of conservative investors, and hence the VS- HFLTS portfolio quantitative calculation system scored five stocks higher than the IS-HFLTS's.

7.2 Comparison Between Different Types of Investors

In the previous sections, investors who participated in group decision-making were assumed to be conservative risk appetite types. The portfolio optimization for different types of investors can be analyzed by adjusting the parameters to capture the investors' risk attitude and the degree of investors' loss aversion. For risk-neutral investors, let $\alpha = \beta = 1$, $\gamma = 2.14$; for risk-seeking investors, let $\alpha = \beta = 1.12$, $\gamma = 2.09$.

Section 6.3 compares and analyzes the VS-HFLTS multi-criteria group decision portfolio evaluation results of three different types of investors. The final scores of the five selected securities based on the evaluations given by the three types of investors are obtained using the VS-HFLTS multi-criteria group decision-making portfolio calculation system proposed in this paper. The results are shown in Fig. 3. The final scores given by the three types of investors all satisfy $x_5 > x_4 > x_3 > x_1 > x_2$, which is in line with the research report information shown in Table 2.

Matlab Software is used to solve the VS-HFLTS portfolio optimization model and obtain the optimal investment ratios. As can be seen from the results shown in Fig. 4, to achieve an optimal allocation of funds, all investor groups

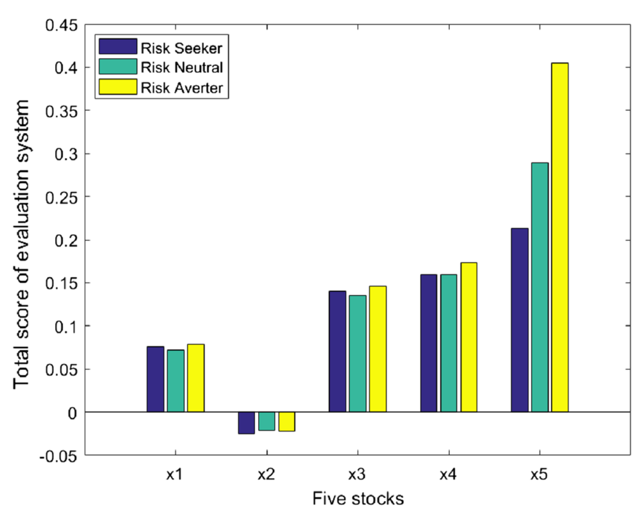


Fig. 3 Final scores of five stocks given by the three investor types of investors

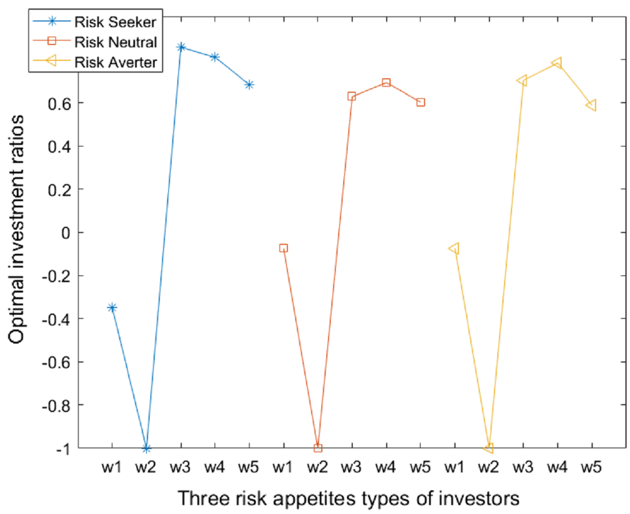


Fig. 4 Optimal portfolios for the three investor types investors

should make a "short" decision for securities x_1 and x_2, and a "long" decision for securities x_3, x_4, and x_5. Moreover, the distribution trend of the investment ratio with respect to the different types of investors is roughly the same. The investment ratios of the five securities are in line with the previous final scores of securities; that is, lower values can be observed in the case of securities x_1 and x_2, whereas x_3, x_4, and x_5 exhibit higher values.

As shown in Fig. 5 that the three types of group decision investors have achieved the corresponding maximum return at the extreme risk value, and the maximum return of the portfolio obtained by investors of different preference types and the corresponding risk value all meet: risk-seeker > risk-neutral > risk-averse. A comparison of the three portfolios presented in Fig. 5 reveals that the portfolio of the risk-seeking investor group achieves the highest maximum return but also the highest maximum risk level, whereas the portfolio of the risk-averse investor group exhibits the lowest maximum return but has the lowest maximum risk threshold. The results of numerical simulations reflect the investment phenomenon of high returns being accompanied by high risks and low risks reducing returns, thus demonstrating the reasonable effectiveness of the proposed model. Furthermore, the proposed quantitative calculation methods can find the effective frontiers of the multi-criteria group decision-making portfolios using the VS-HFLTS depicted in Fig. 6. The red, green, and yellow curves represent the effective frontiers of the portfolio for risk-seeking, risk-neutral, and risk-averse investor groups, respectively. Whereas, the three colored areas represent the investment opportunities of the three different investor types. The obtained results are in line with the theory that risk-seeking investors are willing to take on higher risks when pursuing higher returns and overall have more investment opportunities.

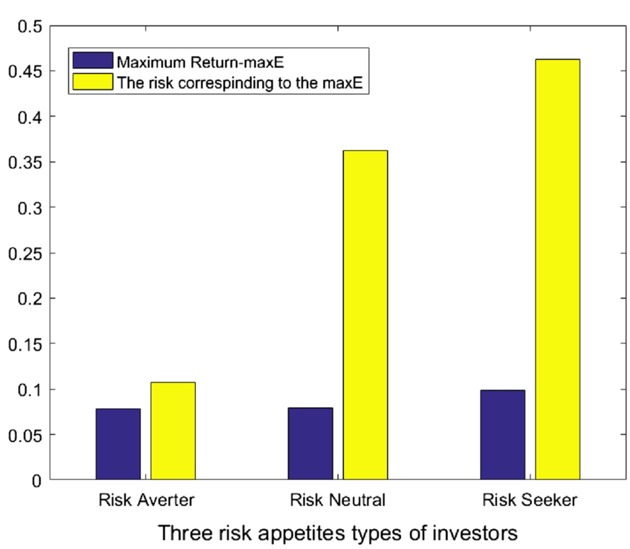


Fig. 5 The maximum returns and corresponding risk values of different investor

8 Conclusion

This paper builds a portfolio optimization model using the VS-HFLTS that captures a multi-criteria group decision-making process of evaluating securities. The equivalent non-linear model is solved to analyze the optimal portfolio strategies, and numerical simulation is performed to verify the effectiveness of the proposed calculation methods. The specific contributions are as follows:

1. The VS-HFLTS proposed in this paper is based on the definition of the value function in cumulative prospect theory. Therefore, the VS-HFLTS reflects the uncertainty regarding the financial product transaction, as it considers the investor's risk preference and sensitivity to gains and losses. Compared with the traditional IS-HFLTS, the VS-HFLTS portfolio selection can lead to a

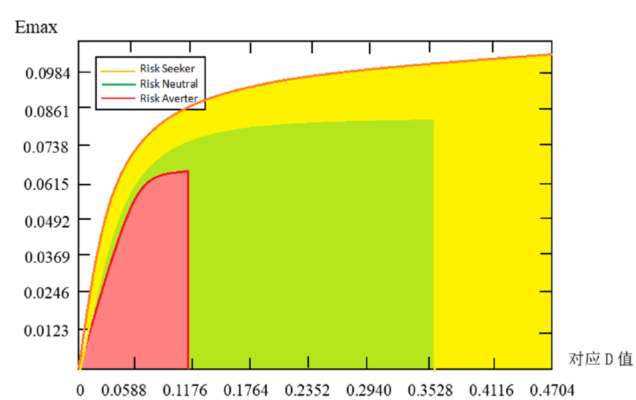


Fig. 6 The effective frontiers of portfolios for different investor types

better portfolio strategy, that is, the portfolio has greater returns or less risk. The application of behavioral finance to multi-criteria group decision-making portfolio selection provides a new research idea for portfolio optimization, which has certain theoretical significance.

2. The traditional IS-HFLSF does not take into account the psychological perception of investors facing risks and benefits, while the VS-HFLVF can reflect the psychological changes in the evaluations given by different types of investors. Besides, the VS-HFLTS portfolio quantitative calculation system is exploited to quantitatively score different natural linguistic evaluation opinions, which is not only in line with investors who are not only completely rational but also more conform to actual investment scenarios, which has strong application value in portfolio decision-making in the actual securities market.
3. The proposal of the HFLVF simplifies the calculation of the approximation and distance between different semantics, which also opens up a new research perspective for the problem that HFLTS does not have a unified algorithm for distance and approximation. In addition, since the HFLVF itself reflects the degree of hesitation, and the degree of hesitation can further affect the weight of each investor. Therefore, the HFLVF subtly avoids the difficulty of calculating preferences between different criteria and has certain mathematical significance.

In future research, we can consider group decision-making portfolios with non-cooperative behaviors. In addition, we can also pay attention to the group decision-making portfolios when the group consensus cannot be reached. At the same time, how to more effectively combine the actual data of securities' returns and risks in the financial market with the fuzzy portfolio decision-making method is also one of the difficulties to be solved in the future.

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Authors Contributions XH wrote the manuscript, provided the methodology, solved the portfolio model, and carried out numerical simulation. XZ gave methodological guidance and reviewed the manuscript.

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Data Availability The data and material used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Code Availability Some or all code generated or used during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare no competing interests.

Ethical Approval Not applicable.

Consent to Participate Not applicable.

Consent for Publication Not applicable.

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