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Pythagorean Fuzzy Full Implication Triple I Method and Its Application in Medical Diagnosis

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Abstract This paper is devoted to the research of full implication triple I method under Pythagorean fuzzy environment. We first propose the concepts of Pythagorean *t*-norm, Pythagorean *t*-conorm, residual Pythagorean fuzzy implication operator (RPFIO) and Pythagorean fuzzy biresiduum. The full implication triple I method for Pythagorean fuzzy modus ponens (PFMP) and Pythagorean fuzzy modus tollens (PFMT) are also established. In addition, the properties of full implication triple I method of PFMP and PFMT models including the robustness, continuity and reversibility are analyzed. Finally, a practical problem is discussed to demonstrate the effectiveness of the Pythagorean fuzzy full implication multiple I method in medical diagnosis. The advantages of the new method are also explained. Overall, compared with the existing methods, the proposed methods are based on logical reasoning rather than using aggregation operators, so they can more accurately and completely express decision information.

Keywords Full implication triple I method · Pythagorean fuzzy set · RPFIO · Robustness · Continuity

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1 Introduction

In order to solve various types of uncertainties and complex decision-making problems, the theory of fuzzy sets was proposed by Zadeh [30]. Later, Atanassov [1] extended the concept of fuzzy set and introduced the intuitionistic fuzzy set (IFS) theory. Because decision makers consider both membership degree and nonmembership degree in decision-making process, this theory is more accurately deals with the uncertainties and decision-making problems than fuzzy sets. However, the IFS needs to satisfy the restricted condition that the sum of the degree of membership and the degree of non-membership is less than or equal to 1. Under this restricted condition, the range of applications of IFSs is very narrow, and there are limitations in describing uncertainty and fuzziness problems. For example, when decision makers adopt 0.6 and 0.7 as membership degree and nonmembership degree to express their opinions, it is obviously beyond the range of applications of IFSs. Thus, Yager [28] proposed the Pythagorean fuzzy sets (PFSs) with a restricted condition that the square of the sum of the membership degree and nonmembership degree is less than or equal to 1. Obviously, the range of its applications of PFSs is more accurate and sufficient than that of IFSs. Since then, many scholars have studied the PFS and obtained a series of valuable research results [4–6, 8, 13, 15, 26, 29]. For example, Zhang and Xu [34] defined the related operations and properties of PFSs, and developed a TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) multi-attribute decision-making method. Zhang [35] initiated an interval-valued PFS and applied it to decision-making problems. In [13], the authors defined the concept of Pythagorean hesitant fuzzy sets through integrating hesitant fuzzy sets with PFSs. Ren [17] proposed a Pythagorean fuzzy multi-attribute group decision-making method based on TODIM (an acronym in Portuguese for Interactive Multi-criteria Decision Making). Ejegwa et al. [7] established some novel methods of computing correlation between PFSs via the three characteristic parameters of PFS by incorporating the ideas of Pythagorean fuzzy deviation, variance and covariance. In [3], they also introduced some new statistical techniques of computing correlation coefficient of PFSs by using Pythagorean fuzzy variance and covariance. In addition, through combining PFS and N-soft set, Zhang et al. [32] initiated the theory of Pythagorean fuzzy N-soft set and applied it to decision-making problem. Zhang and Ma [33] developed three-way decisions with decision-theoretic rough sets based on Pythagorean fuzzy covering and established four methods to address the expected loss expressed in the form of Pythagorean fuzzy numbers (PFNs).

Approximate reasoning is one of the most important topics for a theory dealing with uncertainty, and fuzzy reasoning is the main component of approximate reasoning. It is known that the most fundamental models of fuzzy reasoning are fuzzy modus pones (FMP) and fuzzy modus tollens (FMT), where FMP and FMT can, respectively, be expressed as follows:

FMP inference model: Suppose that: $\widetilde{X} \longrightarrow \widetilde{Y}$, and given: $\widetilde{X}^*$, calculate: $\widetilde{Y}^*$,
FMT inference model: Suppose that: $\widetilde{X} \longrightarrow \widetilde{Y}$, and given: $\widetilde{Y}^*$, calculate: $\widetilde{X}^*$,

where $\widetilde{X}, \widetilde{X}^*$ and $\widetilde{Y}, \widetilde{Y}^*$ are fuzzy sets over U and V, respectively. In 1973, Zadeh [31] introduced the composition rule of inference (CRI). This method was developed into the basic method of fuzzy reasoning. However, from the perspective of logical semantics, the CRI method lacks a strict logic foundation. Thus, Wang [19] provided a new method called the full implication reasoning method (triple I method) based on the standpoint of logical semantics in order to establish a strict logical foundation of fuzzy reasoning. Since then, the triple I method has received the widespread attention of many scholars [18, 21, 24, 27], and yielded a series of valuable research results. For example, the researchers investigated the approximation properties [27] of the triple I method, including the reversibility [16] and continuity [12, 27]. Dai and Pei [2] discussed the triple I method for the FMP and FMT inference models, and proved the robustness of the triple I method. Similarly, Lu [22] discussed the robustness of triple I method based on the fuzzy soft sets. By using the Hamming and uniform metrics, Liu and Wang [12] discussed the continuity of triple I method for fuzzy reasoning. Zheng and Shi [37] investigated triple I method based on the IFS, and proposed α-Triple I method of initionistic fuzzy reasoning. However, regardless of whether a fuzzy set or IFS is used, the range of applications is relatively narrow compared with that of PFS, and there are limitations associated with describing uncertainty and fuzziness problems. On the other hand, there have been few studies on the combination of PFSs with the fuzzy reasoning method. Therefore, in order to fill the research gaps in this field, we attempt to establish a triple I method of the PFMP and PFMT inference models. The innovations of this article are as follows. First, we propose the concepts of Pythagorean t-norm, Pythagorean t-conorm, RPFIO and Pythagorean fuzzy biresiduum. Furthermore, the degree of similarity between PFSs based on the Pythagorean fuzzy biresiduum is also established. Then, we construct the expressions for triple I method of the PFMP and PFMT inference models. Finally, some properties of triple I method based on PFMP and PFMT inference models including the robustness and continuity are explored.

The structure of this article is as follows. Section 2 reviews some basic definitions concerning t-norms, t-conorms, fuzzy implication operators and PFSs. In Sect. 3, the concepts of Pythagorean t-norm, Pythagorean t-conorm, and RPFIO are proposed. We construct a Pythagorean fuzzy biresiduum and define the degree of similarity between PFSs based on the Pythagorean fuzzy biresiduum in Sect. 4. Section 5 establishes the triple I method for the PFMP and PFMT inference models, constructs the expressions for the triple I method based on the PFMP and PFMT inference models and investigates its reversibility. In Sect. 6, the robustness and continuity of triple I method for PFMP and PFMT inference models based on degree of similarity are explored. In Sect. 7, a practical example is provided to illustrate the effectiveness and practicality of the triple I method based on the PFMP inference model. We conclude in Sect. 8.

2 Preliminaries

In this section, we shall provide several definitions that are necessary for our paper. This paper relates to two kinds of products: $\vee$-product and $\wedge$-product. That is, $\alpha \vee \beta = \max(\alpha, \beta)$, $\alpha \wedge \beta = \min(\alpha, \beta)$, $\sup\{\alpha_i | i \in I\} = \bigvee_{i \in I} \alpha_i$, and $\inf\{\alpha_i | i \in I\} = \bigwedge_{i \in I} \alpha_i$, where I is a nonempty index set.

Definition 2.1 ([11]) Consider a function $\triangle : [0,1]^2 \rightarrow [0,1]$ that satisfies the following three conditions: $\forall \alpha, \beta, \gamma \in [0,1]$:

(1) commutative: $\alpha \triangle \beta = \beta \triangle \alpha$;
(2) associative: $\alpha \triangle (\beta \triangle \gamma) = (\alpha \triangle \beta) \triangle \gamma$;
(3) monotonicity: $\alpha \leq \beta \Rightarrow \alpha \triangle \gamma \leq \beta \triangle \gamma$;
(4) $1 \triangle \alpha = \alpha$.

Then, $\triangle$ is said to be a t-norm. Similarly, if a function $\nabla : [0,1]^2 \to [0,1]$ is the associative, commutative, monotonic, and satisfies the condition $0\nabla\tilde{\alpha} = \tilde{\alpha}$, then ∇ is said to be a t-conorm. The functions $\triangle$ and ∇ are referred to as dual t-norm and t-conorm if they satisfy the formulas $\alpha\triangle\beta = 1-(1-\alpha)\nabla(1-\beta)$ and $\alpha\nabla\beta = 1-(1-\alpha)\triangle(1-\beta)$.

Definition 2.2 ([11]) Suppose that $\triangle$ is a t-norm, ∇ is a t-conorm and I is a nonempty index set. For all $\alpha, \beta_i \in [0,1]$, if $\alpha\triangle(\sup_{i\in I}\beta_i) = \sup_{i\in I}(\alpha\triangle\beta_i)$, then $\triangle$ is called a left-continuous t-norm. For all $\alpha_i, \beta \in [0,1]$, if $(\inf_{i\in I}\alpha_i)\nabla\beta = \inf_{i\in I}(\alpha_i\nabla\beta)$, then ∇ is called a right-continuous t-conorm.

Definition 2.3 ([11, 23]) Let $\triangle$ be a left-continuous t-norm. Define an operation $\to: [0,1]^2 \to [0,1]$ as $\alpha \to \beta = \sup\{\varepsilon \in [0,1] | \varepsilon\triangle\alpha \le \beta\}$ such that $(\triangle, \to)$ forms an adjoint pair, i.e., $\alpha\triangle\beta \le \gamma \Leftrightarrow \alpha \le \beta \to \gamma$; then, $\to$ is said to be a residual fuzzy implication operator.

Definition 2.4 ([36]) Let ∇ be a right-continuous t-conorm. Define a function $\ominus : [0,1]^2 \to [0,1]$ as $\alpha \to \beta = \inf\{\varepsilon \in [0,1] | \alpha \le \varepsilon\nabla\beta\}$ such that $(\nabla, \ominus)$ forms a co-adjoint pair, i.e., $\alpha \le \beta\nabla\gamma \Leftrightarrow \alpha \ominus \gamma \le \beta$, then $\ominus$ is said to be a residual fuzzy difference operator.

Definition 2.5 ([36]) Suppose that $(\triangle, \to)$ and $(\nabla, \ominus)$ are respectively the adjoint pair and co-adjoint pair, where $\triangle, \nabla$ are the dual t-norm and t-conorm. Then $\nabla, \to, \ominus$ are called the associated operators of $\triangle$.

Theorem 2.6 ([36]) *Suppose that $\nabla, \to, \ominus$ are the associated operators of $\triangle$. For all $\alpha, \beta \in [0,1]$,*

$$\alpha \ominus \beta = 1-(1-\beta) \to (1-\alpha).$$

Definition 2.7 ([11]) Let $\triangle$ be a t-norm and $\to$ be a residual fuzzy implication operator. $\forall\alpha, \beta \in [0,1]$, if $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$, then $\leftrightarrow$ is said to be a biresiduum associated with a residual fuzzy implication operator.

Theorem 2.8 ([10, 25]) *Let $\triangle$ be a t-norm and $\leftrightarrow$ be a biresiduum associated with a residual fuzzy implication operator; then, $\forall\alpha, \beta, \gamma, \lambda \in [0,1]$:*

(1) $\alpha \leftrightarrow 1 = \alpha$;
(2) $\alpha = \beta \Leftrightarrow \alpha \leftrightarrow \beta = 1$;
(3) $\alpha \leftrightarrow \beta = \beta \leftrightarrow \alpha$;
(4) $(\alpha \leftrightarrow \beta)\triangle\gamma \leftrightarrow \lambda) \le (\alpha\triangle\gamma) \leftrightarrow \beta\triangle\lambda)$;
(5) $(\alpha \leftrightarrow_P \beta)\triangle\gamma \leftrightarrow \lambda) \le (\alpha \to \gamma) \leftrightarrow \beta \to \lambda)$;
(6) $(\alpha \leftrightarrow \beta)\triangle\beta \leftrightarrow \gamma) \le \alpha \leftrightarrow \gamma$;
(7) $(\alpha \leftrightarrow \beta) \wedge (\gamma \leftrightarrow \lambda) \le (\alpha \vee \gamma) \leftrightarrow \beta \vee \lambda)$.

Lemma 2.9 ([11]) *Let $X, X' : U \to [0,1]$ be two arbitrary functions; then,*

(1) $\bigwedge_{u\in U}(X(u) \leftrightarrow X'(u)) \le (\bigwedge_{u\in U} X(u)) \leftrightarrow (\bigwedge_{u\in U} X'(u))$;
(2) $\bigwedge_{u\in U}(X(u) \leftrightarrow X'(u)) \le (\bigvee_{u\in U} X(u)) \leftrightarrow (\bigvee_{u\in U} X'(u))$.

In [28], Yager et al. introduced the concept of PFSs.

Definition 2.10 ([28]) Let U be a universal set; then, a PFS P on U is expressed as follows:

$$P = \{(u, \mu_P(u), \eta_P(u)) | u \in U\},$$

where $\mu_P(u)$ and $\eta_P(u)$ denote the membership degree and nonmembership degree, respectively, with the condition that $0 \le (\mu_P(u))^2 + (\eta_P(u))^2 \le 1$.

We call $(\mu_P(u), \eta_P(u))$ a PFN, which can be written as $\tilde{p} = (\mu, \eta)$. Denote $P^* = \{(\mu_P(u), \eta_P(u)) \in [0,1]^2 | 0 \le (\mu_P(u))^2 + (\eta_P(u))^2 \le 1\}$. Suppose that $\tilde{p}_1 = (\mu', \eta')$ and $\tilde{p}_2 = (\mu'', \eta'')$ are two PFNs; then,

$$\tilde{p}_1 \vee \tilde{p}_2 = (\mu' \vee \mu'', \eta' \wedge \eta''), \tilde{p}_1 \wedge \tilde{p}_2 = (\mu' \wedge \mu'', \eta' \vee \eta'').$$

Definition 2.11 ([14]) Let $\tilde{p}_1 = (\mu', \eta')$ and $\tilde{p}_2 = (\mu'', \eta'')$ be two PFNs. The score function and the accuracy function of $\tilde{p}$ can be defined as $S(\tilde{p}) = \mu^2 - \eta^2$ and $H(\tilde{p}) = \mu^2 + \eta^2$, respectively. For $\tilde{p}_1 = (\mu', \eta')$ and $\tilde{p}_2 = (\mu'', \eta'')$,

(1) if $S(\tilde{p}_1) > S(\tilde{p}_2)$, then $\tilde{p}_1 > \tilde{p}_2$.
(2) if $S(\tilde{p}_1) = S(\tilde{p}_2)$, then

if $H(\tilde{p}_1) > H(\tilde{p}_2)$, then $\tilde{p}_1 > \tilde{p}_2$; if $H(\tilde{p}_1) = H(\tilde{p}_2)$, then $\tilde{p}_1 = \tilde{p}_2$.

According to Definition 2.11, we have $\mu' \ge \mu'', \eta' \le \eta''$. So $\tilde{p}_1 \ge \tilde{p}_2$.

3 Pythagorean Fuzzy Implication Operator

In this current section, we shall propose the concepts of Pythagorean t-norm, Pythagorean t-conorm, and RPFIO.

Let $\tilde{\alpha} = (\mu, \eta), \tilde{\beta} = (\mu', \eta') \in P^*$. Then, we define two binary operations $\triangle_{P^*}$ and ∇_{P^*} on P^* as follows:

$$\tilde{\alpha}\triangle_{P^*}\tilde{\beta} = (\sqrt{\mu^2\triangle\mu'^2}, \sqrt{\eta^2\nabla\eta'^2}),$$
$$\tilde{\alpha}\nabla_{P^*}\tilde{\beta} = (\sqrt{\mu^2\nabla\mu'^2}, \sqrt{\eta^2\triangle\eta'^2}).$$

Since $0 \le \mu^2 + \eta^2 \le 1, 0 \le \mu'^2 + \eta'^2 \le 1$, it follows from Definition 2.1 that

$$\begin{aligned}(\sqrt{\mu^2\triangle\mu'^2})^2 + (\sqrt{\eta^2\nabla\eta'^2})^2 &= \mu^2\triangle\mu'^2 + \eta^2\nabla\eta'^2 \\ &\le \mu^2\triangle\mu'^2 + (1-\mu^2)\nabla(1-\mu'^2) \\ &= \mu^2\triangle\mu'^2 + 1 - \mu^2\triangle\mu'^2 = 1,\end{aligned}$$

which implies that $\tilde{\alpha}\triangle_{P^*}\tilde{\beta} \in P^*$. By the same token, we can prove that $\tilde{\alpha}\nabla_{P^*}\tilde{\beta} \in P^*$.

Theorem 3.1 *Let $\triangle_{P^*}$ and ∇_{P^*} be two binary operations on P^*. Given that $\tilde{\alpha} = (\mu, \eta), \tilde{\beta} = (\mu', \eta'), \tilde{\gamma} = (\mu'', \eta'')$ are three PFNs, then $\triangle_{P^*}$ satisfies the following conditions*:

(1) $\tilde{\alpha}\triangle_{P^*}\tilde{\beta} = \tilde{\beta}\triangle_{P^*}\tilde{\alpha}$;
(2) $\tilde{\alpha}\triangle_{P^*}(\tilde{\beta}\triangle_{P^*}\tilde{\gamma}) = (\tilde{\alpha}\triangle_{P^*}\tilde{\beta})\triangle_{P^*}\tilde{\gamma}$;
(3) If $\tilde{\alpha} \le \tilde{\beta}$, then $\tilde{\alpha}\triangle_{P^*}\tilde{\gamma} \le \tilde{\beta}\triangle_{P^*}\tilde{\gamma}$;
(4) $(1,0)\triangle_{P^*}\tilde{\alpha} = \tilde{\alpha}$.

Similarly, ∇_{P^*} is associative, commutative and monotonic, and satisfies the condition $(0,1)\nabla_{P^*}\tilde{\alpha} = \tilde{\alpha}$.

Proof According to the associative law and the commutative law for $\triangle$ and ∇, both $\triangle_{P^*}$ and ∇_{P^*} are associative and commutative. On the one hand, when $\tilde{\alpha} \le \tilde{\beta}$, we have $\mu \le \mu', \eta \ge \eta'$. On the other hand, since $\triangle, \nabla$ is monotonic, we obtain

$$\begin{aligned}\tilde{\alpha}\triangle_{P^*}\tilde{\gamma} &= (\sqrt{\mu^2\triangle\mu''^2}, \sqrt{\eta^2\nabla\eta''^2}) \\ &\le (\sqrt{\mu'^2\triangle\mu''^2}, \sqrt{\eta'^2\nabla\eta''^2}) = \tilde{\beta}\triangle_{P^*}\tilde{\gamma}.\end{aligned}$$

Thus $\triangle_{P^*}$ is monotonic. Furthermore, $\triangle_{P^*}$ satisfies the boundary conditions. In fact,

$$\begin{aligned}(1,0)\triangle_{P^*}\tilde{\alpha} &= (1,0)\triangle_{P^*}(\mu,\eta) \\ &= (\sqrt{\mu^2\triangle 1}, \sqrt{\eta^2\nabla 0}) = (\mu,\eta) = \tilde{\alpha}.\end{aligned}$$

Similarly, we can prove that ∇_{P^*} is monotonic and satisfies the condition $(0,1)\nabla_{P^*}\tilde{\alpha} = \tilde{\alpha}$. □

Based on Theorem 3.1, we call $\triangle_{P^*}$ a Pythagorean *t*-norm induced from $\triangle$. And ∇_{P^*} is referred to as a Pythagorean *t*-conorm induced from ∇.

Theorem 3.2 *Let $\triangle$ be a left-continuous t-norm and ∇ be a right-continuous t-conorm. Then, $\triangle_{P^*}$ and ∇_{P^*}, respectively, satisfy the formulas*:

$$\tilde{\alpha}\triangle_{P^*}(\sup_{i\in I}\tilde{\beta}_i) = \sup_{i\in I}(\tilde{\alpha}\triangle_{P^*}\tilde{\beta}_i),$$
$$(\inf_{i\in I}\tilde{\alpha}_i)\nabla_{P^*}\tilde{\beta} = \inf_{i\in I}(\tilde{\alpha}_i\nabla_{P^*}\tilde{\beta}),$$

where I is a nonempty index set.

In that case, $\triangle_{P^*}$ is called a left-continuous Pythagorean t-norm on P^*, and ∇_{P^*} is called a right-continuous Pythagorean t-conorm on P^*.

Definition 3.3 Let $\triangle_{P^*}$ be a left-continuous Pythagorean *t*-norm. If a binary operation $\rightarrow_{P^*}$ on P^* satisfies $\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} = \sup\{\tilde{\varepsilon} \in P^* | \tilde{\varepsilon}\triangle_{P^*}\tilde{\alpha} \le \tilde{\beta}\}$, then $\rightarrow_{P^*}$ is said to be an RPFIO.

Theorem 3.4 *Let $\triangle_{P^*}$ be a Pythagorean t-norm and $\rightarrow_{P^*}$ be an RPFIO; then, the following holds*:

(1) $\tilde{\gamma}\triangle_{P^*}\tilde{\alpha} \le \tilde{\beta} \Leftrightarrow \tilde{\gamma} \le \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}$,
(2) $\tilde{\gamma}\triangle_{P^*}\tilde{\alpha} \ge \tilde{\beta} \Leftrightarrow \tilde{\gamma} \ge \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}$,
(3) $\tilde{\alpha} \ge \tilde{\gamma} \rightarrow_{P^*} \tilde{\beta} \Leftrightarrow \tilde{\gamma} \ge \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}$,
(4) $\tilde{\alpha} \rightarrow_{P^*} (\tilde{\beta} \rightarrow_{P^*} \tilde{\gamma}) = (\tilde{\alpha}\triangle_{P^*}\tilde{\beta}) \rightarrow_{P^*} \tilde{\gamma}$,
(5) $\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} = (1,0) \Leftrightarrow \tilde{\alpha} \le \tilde{\beta}$,
(6) $\tilde{\gamma} \le \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} \Leftrightarrow \tilde{\alpha} \le \tilde{\gamma} \rightarrow_{P^*} \tilde{\beta}$,
(7) $(1,0) \rightarrow_{P^*} \tilde{\alpha} = \tilde{\alpha}$,
(8) $\tilde{\gamma} \rightarrow_{P^*} (\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}) = \tilde{\alpha} \rightarrow_{P^*} (\tilde{\gamma} \rightarrow_{P^*} \tilde{\beta})$.

Proof (1) From the definition of $\rightarrow_{P^*}$, we observe that if $\tilde{\gamma}\triangle_{P^*}\tilde{\alpha} \le \tilde{\beta}$, then $\tilde{\gamma} \le \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}$. When $\tilde{\gamma} \le \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}$, from the left-continuity and monotonicity of $\triangle_{P^*}$, we can obtain

$$\begin{aligned}\tilde{\gamma}\triangle_{P^*}\tilde{\alpha} &\le (\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta})\triangle_{P^*}\tilde{\alpha} \\ &= \sup\{\tilde{\varepsilon} \in P^* | \tilde{\varepsilon}\triangle_{P^*}\tilde{\alpha} \le \tilde{\beta}\}\triangle_{P^*}\tilde{\alpha} \\ &= \sup\{(\tilde{\varepsilon}\triangle_{P^*}\tilde{\alpha}) \in P^* | \tilde{\varepsilon}\triangle_{P^*}\tilde{\alpha} \le \tilde{\beta}\} = \tilde{\beta}.\end{aligned}$$

(2) When $\tilde{\gamma} \ge \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}$, from the left-continuity and monotonicity of $\triangle_{P^*}$, we can obtain

$$\begin{aligned}\tilde{\gamma}\triangle_{P^*}\tilde{\alpha} &\ge (\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta})\triangle_{P^*}\tilde{\alpha} \\ &= \sup\{\tilde{\varepsilon} \in P^* | \tilde{\varepsilon}\triangle_{P^*}\tilde{\alpha} \le \tilde{\beta}\}\triangle_{P^*}\tilde{\alpha} \\ &= \sup\{(\tilde{\varepsilon}\triangle_{P^*}\tilde{\alpha}) \in P^* | \tilde{\varepsilon}\triangle_{P^*}\tilde{\alpha} \le \tilde{\beta}\} = \tilde{\beta}.\end{aligned}$$

On the other hand, from the definition of $\rightarrow_{P^*}$, we have $\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} = \sup\{\tilde{\varepsilon} \in P^* | \tilde{\varepsilon}\triangle_{P^*}\tilde{\alpha} \le \tilde{\beta}\}$. When $\tilde{\gamma}\triangle_{P^*}\tilde{\alpha} \ge \tilde{\beta}$, we can obtain $\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} \le \tilde{\gamma}$.

(3) From the result (2), we can obtain $\tilde{\alpha} \ge \tilde{\gamma} \rightarrow_{P^*} \tilde{\beta} \Leftrightarrow \tilde{\alpha}\triangle_{P^*}\tilde{\gamma} \ge \tilde{\beta} \Leftrightarrow \tilde{\gamma}\triangle_{P^*}\tilde{\alpha} \ge \tilde{\beta} \Leftrightarrow \tilde{\gamma} \ge \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}$.

(4) Let $\tilde{x} = \tilde{\alpha} \rightarrow_{P^*} (\tilde{\beta} \rightarrow_{P^*} \tilde{\gamma}) \in P^*$. From the result (1), we have

$$\begin{aligned}\tilde{x} = \tilde{\alpha} \rightarrow_{P^*} (\tilde{\beta} \rightarrow_{P^*} \tilde{\gamma}) &\Longleftrightarrow \tilde{\alpha}\triangle_{P^*}\tilde{x} = \tilde{\beta} \rightarrow_{P^*} \tilde{\gamma} \\ &\Longleftrightarrow (\tilde{\alpha}\triangle_{P^*}\tilde{x})\triangle_{P^*}\tilde{\beta} = \tilde{\gamma} \Longleftrightarrow (\tilde{x}\triangle_{P^*}\tilde{\alpha})\triangle_{P^*}\tilde{\beta} = \tilde{\gamma} \\ &\Longleftrightarrow \tilde{x}\triangle_{P^*}(\tilde{\alpha}\triangle_{P^*}\tilde{\beta}) = \tilde{\gamma} \Longleftrightarrow \tilde{x} = (\tilde{\alpha}\triangle_{P^*}\tilde{\beta}) \rightarrow_{P^*} \tilde{\gamma}.\end{aligned}$$

(5)–(8) are straightforward. □

Theorem 3.5 *Suppose that $\rightarrow_{P^*}$ is an RPFIO, $\tilde{\alpha} = (\mu, \eta), \tilde{\beta} = (\mu', \eta') \in P^*$. Then*

$$\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} = (\sqrt{\mu^2 \rightarrow \mu'^2} \wedge \sqrt{1 - (\eta'^2 \ominus \eta^2)}, \sqrt{\eta'^2 \ominus \eta^2}).$$

Proof Let $\tilde{\varepsilon} = (\mu'', \eta'') = \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}$. For all $\tilde{\varepsilon}_i = (\mu''_i, \eta''_i) \in P^*$, it follows from Definition 3.3 that

$$\begin{aligned}\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} &= \sup\{\tilde{\varepsilon_i} \in P^* | \tilde{\varepsilon_i} \triangle_{P^*} \tilde{\alpha} \le \tilde{\beta}\} \\ &= \sup\{(\mu_i'', \eta_i'') \in P^* | \sqrt{\mu_i''^2 \triangle \mu^2} \le \mu', \sqrt{\eta_i''^2 \nabla \eta^2} \ge \eta'\} \\ &= \sup\{(\mu_i'', \eta_i'') \in P^* | \mu_i''^2 \triangle \mu^2 \le \mu'^2, \eta''^2_i \nabla \eta^2 \ge \eta'^2\}.\end{aligned}$$

By Theorem 3.4, we have

$$\begin{aligned}&\mu_i''^2 \triangle \mu^2 \le \mu'^2 \ \& \ \eta_i''^2 \nabla \eta^2 \ge \eta'^2, \\ &\quad \Rightarrow \mu^2 \rightarrow \mu'^2 \ge \mu_i''^2 \ \& \ \eta'^2 \ominus \eta^2 \le \eta_i''^2, \\ &\quad \Rightarrow \mu^2 \rightarrow \mu'^2 \ge \mu_i''^2 \ \& \ \eta'^2 \ominus \eta^2 \le \eta_i''^2, \\ &\quad 1 - (\eta'^2 \ominus \eta^2) \ge 1 - \eta_i''^2 \ge \mu_i''^2, \\ &\quad \Rightarrow \sqrt{\mu^2 \rightarrow \mu'^2} \ge \mu_i'' \ \& \ \sqrt{\eta'^2 \ominus \eta^2} \le \eta_i'', \\ &\quad \sqrt{1 - (\eta'^2 \ominus \eta^2)} \ge \sqrt{1 - \eta_i''^2} \ge \mu_i''.\end{aligned}$$

Therefore, if $\tilde{\varepsilon} = (\mu'', \eta'') = (\mu_i'', \eta_i'')$, we have

$$\begin{aligned}&\tilde{\varepsilon} = (\mu'', \eta'') = \tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} \le (\sqrt{\mu^2 \rightarrow \mu'^2} \\ &\quad \wedge \sqrt{1 - (\eta'^2 \ominus \eta^2)}, \sqrt{\eta'^2 \ominus \eta^2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\sqrt{\mu^2 \rightarrow \mu'^2} \wedge \sqrt{1 - (\eta'^2 \ominus \eta^2)}, \sqrt{\eta'^2 \ominus \eta^2}) \triangle_{P^*} \tilde{\alpha} \\ &\quad \le (\sqrt{\mu^2 \rightarrow \mu'^2}, \sqrt{\eta'^2 \ominus \eta^2}) \triangle_{P^*} (\mu, \eta) \\ &\quad \le (\sqrt{(\mu^2 \rightarrow \mu'^2) \triangle \mu^2}, \sqrt{(\eta'^2 \ominus \eta^2) \nabla \eta^2}) \\ &\quad = (\sqrt{\mu'^2}, \sqrt{\eta'^2}) = (\mu', \eta') = \tilde{\beta}.\end{aligned}$$

By applying the result (1) of Theorem 3.4, we obtain

$$\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta} = (\sqrt{\mu^2 \rightarrow \mu'^2} \wedge \sqrt{1 - (\eta'^2 \ominus \eta^2)}, \sqrt{\eta'^2 \ominus \eta^2}).$$

□

Example 3.6 The following four RPFIOs were induced by four residual fuzzy implication operators and residual fuzzy difference operators:

(1) When $\triangle$ is Gödel t-norm, $\alpha \triangle_G \beta = \alpha \wedge \beta$, and its associated operators are

$$\alpha \nabla_G \beta = \alpha \vee \beta, \quad \alpha \rightarrow_G \beta = \begin{cases} 1, & \alpha \le \beta, \\ \beta, & \alpha > \beta, \end{cases}$$

$$\beta \ominus_G \alpha = \begin{cases} 0, & \beta \le \alpha, \\ \beta, & \beta > \alpha. \end{cases}$$

Then

$$\tilde{\alpha} \rightarrow_{P_G^*} \tilde{\beta} = \begin{cases} (1, 0), & \mu \le \mu', \eta' \le \eta, \\ (\sqrt{1 - \eta'^2}, \eta'), & \mu \le \mu', \eta' > \eta, \\ (\mu', 0), & \mu > \mu', \eta' \le \eta, \\ (\mu', \eta'), & \mu > \mu', \eta' > \eta. \end{cases}$$

(2) When $\triangle$ is *Product* t-norm, $\alpha \triangle_\pi \beta = \alpha\beta$, and its associated operators are

$$\alpha \nabla_\pi \beta = \alpha + \beta - \alpha\beta, \quad \alpha \rightarrow_\pi \beta = \begin{cases} 1, & \alpha \le \beta, \\ \dfrac{\beta}{\alpha}, & \alpha > \beta. \end{cases}$$

$$\beta \ominus_\pi \alpha = \begin{cases} 0, & \beta \le \alpha, \\ \dfrac{\beta - \alpha}{1 - \alpha}, & \beta > \alpha. \end{cases}$$

Then

$$\tilde{\alpha} \rightarrow_{P_\pi^*} \tilde{\beta} = \begin{cases} (1, 0), & \mu \le \mu', \eta' \le \eta, \\ \left(\dfrac{\sqrt{1 - \eta'^2}}{\sqrt{1 - \eta^2}}, \dfrac{\sqrt{\eta'^2 - \eta^2}}{\sqrt{1 - \eta^2}}\right), & \mu \le \mu', \eta' > \eta, \\ \left(\dfrac{\mu'}{\mu}, 0\right), & \mu > \mu', \eta' \le \eta, \\ \left(\dfrac{\sqrt{1 - \eta'^2}}{\sqrt{1 - \eta^2}} \wedge \dfrac{\mu'}{\mu}, \dfrac{\sqrt{\eta'^2 - \eta^2}}{\sqrt{1 - \eta^2}}\right), & \mu > \mu', \eta' > \eta. \end{cases}$$

(3) When $\triangle$ is *Einstein* t-norm, $\alpha \triangle_\epsilon \beta = \frac{\alpha\beta}{1 + (1 - \alpha)(1 - \beta)}$, and its associated operators are

$$\alpha \nabla_\epsilon \beta = \frac{\alpha + \beta}{1 + \alpha\beta}, \quad \alpha \rightarrow_\epsilon \beta = \begin{cases} 1, & \alpha \le \beta, \\ \dfrac{2\beta - \alpha\beta}{\alpha + \beta - \alpha\beta}, & \alpha > \beta. \end{cases}$$

$$\beta \ominus_\epsilon \alpha = \begin{cases} 0, & \beta \le \alpha, \\ \dfrac{\alpha - \beta}{1 - \alpha\beta}, & \beta > \alpha. \end{cases}$$

Then

$$\tilde{\alpha} \rightarrow_{P_\epsilon^*} \tilde{\beta} = \begin{cases} (1, 0), & \mu \le \mu', \eta' \le \eta, \\ \left(\dfrac{\sqrt{1 - \eta^2 + \eta'^2 - \eta^2\eta'^2}}{\sqrt{1 - \eta^2\eta'^2}}, \dfrac{\sqrt{\eta^2 - \eta'^2}}{\sqrt{1 - \eta^2\eta'^2}}\right), & \mu \le \mu', \eta' > \eta, \\ \left(\dfrac{\sqrt{2\mu'^2 - \mu'^2\mu^2}}{\sqrt{\mu^2 + \mu'^2 - \mu^2\mu'^2}}, 0\right), & \mu > \mu', \eta' \le \eta, \\ \left(\dfrac{\sqrt{2\mu'^2 - \mu'^2\mu^2}}{\sqrt{\mu^2 + \mu'^2 - \mu^2\mu'^2}} \wedge \left(\dfrac{\sqrt{1 - \eta^2 + \eta'^2 - \eta^2\eta'^2}}{\sqrt{1 - \eta^2\eta'^2}}\right), \dfrac{\sqrt{\eta^2 - \eta'^2}}{\sqrt{1 - \eta^2\eta'^2}}\right), & \mu > \mu', \eta' > \eta. \end{cases}$$

(4) When $\triangle$ is R_0 t -norm, $\alpha \triangle_{R_0} \beta = \begin{cases} 0, & \alpha + \beta \le 1, \\ \alpha \wedge \beta, & \alpha + \beta > 1, \end{cases}$ and its associated operators are

$$\alpha \nabla_{R_0} \beta = \begin{cases} 1, & \alpha + \beta \ge 1, \\ \alpha \vee \beta, & \alpha + \beta < 1, \end{cases}$$

$$\alpha \rightarrow_{R_0} \beta = \begin{cases} 1, & \alpha \le \beta, \\ (1 - \alpha) \vee \beta, & \alpha > \beta. \end{cases}$$

$$\beta \ominus_{R_0} \alpha = \begin{cases} 0, & \beta \le \alpha, \\ \beta \wedge (1 - \alpha), & \beta > \alpha. \end{cases}$$

Then

$$\tilde{\alpha} \rightarrow_{P^*_{R_0}} \tilde{\beta} = \begin{cases} (1,0), & \mu \le \mu', \eta' \le \eta, \\ (\sqrt{1-\eta'^2} \vee \eta, \eta' \wedge \sqrt{1-\eta^2}), & \mu \le \mu', \eta' > \eta, \\ (\sqrt{1-\mu^2} \vee \mu', 0), & \mu > \mu', \eta' \le \eta, \\ ((\sqrt{1-\mu^2} \vee \mu') \wedge (\sqrt{1-\eta'^2} \vee \eta), \eta' \wedge \sqrt{1-\eta^2}), & \mu > \mu', \eta' > \eta. \end{cases}$$

4 Degree of Similarity Between PFSs Based on the Pythagorean Fuzzy Biresiduum

In this section, we propose the concept of the Pythagorean fuzzy biresiduum and define the degree of similarity between PFSs based on the Pythagorean fuzzy biresiduum.

Definition 4.1 Given that $\tilde{\alpha}$ and $\tilde{\beta}$ are two PFNs, if $\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta} = (\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}) \wedge (\tilde{\beta} \rightarrow_{P^*} \tilde{\alpha})$, then $\leftrightarrow_{P^*}$ is said to be a Pythagorean fuzzy biresiduum associated with the RPFIO.

Theorem 4.2 *Let $\triangle_{P^*}$ be a Pythagorean t-norm, $\rightarrow_{P^*}$ be a RPFIO and $\leftrightarrow_{P^*}$ be a Pythagorean fuzzy biresiduum; then, the following holds:*

(1) $\tilde{\alpha} \leftrightarrow_{P^*} (1,0) = \tilde{\alpha}$;

(2) $\tilde{\alpha} = \tilde{\beta} \Leftrightarrow \tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta} = (1,0)$;

(3) $\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta} = \tilde{\beta} \leftrightarrow_{P^*} \tilde{\alpha}$;

(4) $(\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta}) \triangle_{P^*} (\tilde{\gamma} \leftrightarrow_{P^*} \tilde{\lambda}) \le (\tilde{\alpha} \triangle_{P^*} \tilde{\gamma}) \leftrightarrow_{P^*} (\tilde{\beta} \triangle_{P^*} \tilde{\lambda})$;

(5) $(\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta}) \triangle_{P^*} (\tilde{\gamma} \leftrightarrow_{P^*} \tilde{\lambda}) \le (\tilde{\alpha} \rightarrow_{P^*} \tilde{\gamma}) \leftrightarrow_{P^*} (\tilde{\beta} \rightarrow_{P^*} \tilde{\lambda})$;

(6) $(\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta}) \triangle_{P^*} (\tilde{\beta} \leftrightarrow_{P^*} \tilde{\gamma}) \le \tilde{\alpha} \leftrightarrow_{P^*} \tilde{\gamma}$;

(7) $(\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta}) \wedge (\tilde{\gamma} \leftrightarrow_{P^*} \tilde{\lambda}) \le (\tilde{\alpha} \vee \tilde{\gamma}) \leftrightarrow_{P^*} (\tilde{\beta} \vee \tilde{\lambda})$.

Proof Suppose that $\tilde{\alpha} = (\mu, \eta), \tilde{\beta} = (\mu', \eta'), \tilde{\gamma} = (\mu^*, \eta^*)$ and $\tilde{\lambda} = (\mu^{*'}, \eta^{*'})$. From the definition of $\leftrightarrow_{P^*}$ and Theorem 3.4, the results from (1) to (3) are straightforward.

(4) Suppose that $\bar{\eta}^2 = 1 - \eta^2, \bar{\eta}'^2 = 1 - \eta'^2, \bar{\eta}^{*2} = 1 - \eta^{*2}$ and $\bar{\eta}^{*'2} = 1 - \eta^{*'2}$. From Definition 4.1 and the concepts of the Pythagorean t-norm, we can obtain

$$\begin{aligned}
&(\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta}) \triangle_{P^*} (\tilde{\gamma} \leftrightarrow_{P^*} \tilde{\lambda}) \\
&\quad = ((\tilde{\alpha} \rightarrow_{P^*} \tilde{\beta}) \wedge (\tilde{\beta} \rightarrow_{P^*} \tilde{\alpha})) \triangle_{P^*} ((\tilde{\gamma} \rightarrow_{P^*} \tilde{\lambda}) \wedge (\tilde{\lambda} \rightarrow_{P^*} \tilde{\gamma})) \\
&\quad = (\sqrt{\mu^2 \rightarrow \mu'^2} \wedge \sqrt{\bar{\eta}^2 \rightarrow \bar{\eta}'^2}, \sqrt{1 - \bar{\eta}^2 \rightarrow \bar{\eta}'^2}) \\
&\quad \wedge (\sqrt{\mu'^2 \rightarrow \mu^2} \wedge \sqrt{\bar{\eta}'^2 \rightarrow \bar{\eta}^2}, \sqrt{1 - \bar{\eta}'^2 \rightarrow \bar{\eta}^2}) \triangle_{P^*} \\
&(\sqrt{\mu^{*2} \rightarrow \mu^{*'2}} \wedge \sqrt{\bar{\eta}^{*2} \rightarrow \bar{\eta}^{*'2}}, \sqrt{1 - \bar{\eta}^{*2} \rightarrow \bar{\eta}^{*'2}}) \\
&\quad \wedge (\sqrt{\mu^{*'2} \rightarrow \mu^{*2}} \wedge \sqrt{\bar{\eta}^{*'2} \rightarrow \bar{\eta}^{*2}}, \sqrt{1 - \bar{\eta}^{*'2} \rightarrow \bar{\eta}^{*2}}) \\
&\quad = (\sqrt{((\mu^2 \leftrightarrow \mu'^2) \wedge (\bar{\eta}^2 \leftrightarrow \bar{\eta}'^2)) \triangle ((\mu^{*2} \leftrightarrow \mu^{*'2}) \wedge (\bar{\eta}^{*2} \leftrightarrow \bar{\eta}^{*'2}))}, \\
&\quad \sqrt{(1 - \bar{\eta}'^2 \leftrightarrow \bar{\eta}^2) \nabla (1 - \bar{\eta}^{*'2} \leftrightarrow \bar{\eta}^{*2})}) \\
&\quad \le (\sqrt{(\mu^2 \leftrightarrow \mu'^2) \triangle (\mu^{*2} \leftrightarrow \mu^{*'2})} \wedge \sqrt{(\bar{\eta}^2 \leftrightarrow \bar{\eta}'^2) \triangle (\bar{\eta}^{*2} \leftrightarrow \bar{\eta}^{*'2})}, \\
&\sqrt{1 - ((\bar{\eta}'^2 \leftrightarrow \bar{\eta}^2) \triangle (\bar{\eta}^{*'2} \leftrightarrow \bar{\eta}^{*2}))}) \\
&\quad \le (\sqrt{(\mu^2 \triangle \mu^{*2}) \leftrightarrow (\mu'^2 \triangle \mu^{*'2})} \wedge \sqrt{(\bar{\eta}^2 \triangle \bar{\eta}^{*2}) \leftrightarrow (\bar{\eta}'^2 \triangle \bar{\eta}^{*'2})}, \\
&\sqrt{1 - ((\bar{\eta}'^2 \triangle \bar{\eta}^{*'2}) \leftrightarrow (\bar{\eta}^2 \triangle \bar{\eta}^{*2}))}) \\
&\quad = (\sqrt{\mu^2 \triangle \mu^{*2}}, \sqrt{1 - \bar{\eta}^2 \triangle \bar{\eta}^{*2}}) \leftrightarrow_{P^*} \sqrt{\mu'^2 \triangle \mu^{*'2}}, \sqrt{1 - \bar{\eta}'^2 \triangle \bar{\eta}^{*'2}}) \\
&\quad = (\sqrt{\mu^2 \triangle \mu^{*2}}, \sqrt{\eta^2 \nabla \eta^{*2}}) \leftrightarrow_{P^*} (\sqrt{\mu'^2 \triangle \mu^{*'2}}, \sqrt{\eta'^2 \nabla \eta^{*'2}}) \\
&\quad = (\mu, \eta) \triangle_{P^*} (\mu^*, \eta^*) \leftrightarrow_{P^*} (\mu', \eta') \triangle_{P^*} (\mu^{*'}, \eta^{*'}) \\
&\quad = (\tilde{\alpha} \triangle_{P^*} \tilde{\gamma}) \leftrightarrow_{P^*} (\tilde{\beta} \triangle_{P^*} \tilde{\lambda}).
\end{aligned}$$

(5) It is similar to the proof of (4).

(6) According to the definitions of $\leftrightarrow_{P^*}$ and $\triangle_{P^*}$, we have

$$\begin{aligned}
&(\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta}) \triangle_{P^*} (\tilde{\beta} \leftrightarrow_{P^*} \tilde{\gamma}) \\
&\quad \le (\sqrt{(\mu^2 \leftrightarrow \mu'^2) \triangle (\mu'^2 \leftrightarrow \mu^{*2})} \wedge \sqrt{(\bar{\eta}^2 \leftrightarrow \bar{\eta}'^2) \triangle (\bar{\eta}'^2 \leftrightarrow \bar{\eta}^{*2})}, \\
&\sqrt{1 - ((\bar{\eta}'^2 \leftrightarrow \bar{\eta}^2) \triangle (\bar{\eta}^{*2} \leftrightarrow \bar{\eta}'^2))}) \\
&\quad \le (\sqrt{\mu^2 \leftrightarrow \mu^{*2}} \wedge \sqrt{\bar{\eta}^2 \leftrightarrow \bar{\eta}^{*2}}, \\
&\sqrt{1 - (\bar{\eta}^2 \leftrightarrow \bar{\eta}^{*2})}) \\
&\quad = (\sqrt{\mu^2 \rightarrow \mu^{*2}} \wedge \sqrt{\bar{\eta}^2 \rightarrow \bar{\eta}^{*2}} \wedge \sqrt{\mu^{*2} \rightarrow \mu^2} \\
&\quad \wedge \sqrt{\bar{\eta}^{*2} \rightarrow \bar{\eta}^2}, \sqrt{1 - (\bar{\eta}^2 \rightarrow \bar{\eta}^{*2})} \vee \sqrt{1 - (\bar{\eta}^{*2} \rightarrow \bar{\eta}^2)}) \\
&\quad = ((\mu, \eta) \rightarrow_{P^*} (\mu^*, \eta^*)) \wedge ((\mu^*, \eta^*) \rightarrow_{P^*} (\mu, \eta)) \\
&\quad = \tilde{\alpha} \leftrightarrow_{P^*} \tilde{\gamma}.
\end{aligned}$$

(7) It follows from the definition of $\leftrightarrow_{P^*}$ that

$$
\begin{aligned}
&(\tilde{\alpha} \leftrightarrow_{P^*} \tilde{\beta}) \wedge (\tilde{\gamma} \leftrightarrow_{P^*} \tilde{\lambda})\\
&\quad = (\sqrt{((\mu^2 \leftrightarrow \mu'^2) \wedge (\bar{\eta}^2 \leftrightarrow \bar{\eta}'^2)) \wedge ((\mu^{*2} \leftrightarrow \mu^{*'2}) \wedge (\bar{\eta}^{*2} \leftrightarrow \bar{\eta}^{*'2}))},\\
&\sqrt{(1 - \bar{\eta}'^2 \leftrightarrow \bar{\eta}^2) \vee (1 - \bar{\eta}^{*'2} \leftrightarrow \bar{\eta}^{*2})})\\
&\quad \le (\sqrt{(\mu^2 \vee \mu^{*2}) \leftrightarrow (\mu'^2 \vee \mu^{*'2})}\\
&\quad \wedge \sqrt{(\bar{\eta}^2 \vee \bar{\eta}^{*2}) \leftrightarrow (\bar{\eta}'^2 \vee \bar{\eta}^{*'2})},\\
&\sqrt{1 - ((\bar{\eta}'^2 \vee \bar{\eta}^{*'2}) \leftrightarrow (\bar{\eta}^2 \vee \bar{\eta}^{*2}))})\\
&\quad = ((\mu \vee \mu^*, \eta \wedge \eta^*) \to_{P^*} (\mu' \vee \mu^{*'}, \eta' \wedge \eta^{*'}))\\
&\quad \wedge ((\mu' \vee \mu^{*'}, \eta' \wedge \eta^{*'}) \to_{P^*} (\mu \vee \mu^*, \eta \wedge \eta^*))\\
&\quad = (\mu \vee \mu^*, \eta \wedge \eta^*) \leftrightarrow_{P^*} (\mu' \vee \mu^{*'}, \eta' \wedge \eta^{*'})\\
&\quad = (\tilde{\alpha} \vee \tilde{\gamma}) \leftrightarrow_{P^*} (\tilde{\beta} \vee \tilde{\lambda}).
\end{aligned}
$$

□

Lemma 4.3 *Let $\widetilde{X}, \widetilde{X}'$ be two PFSs over U; then,*

(1) $\bigwedge_{u\in U}(\widetilde{X}(u) \leftrightarrow_{P^*} \widetilde{X}'(u)) \le (\bigwedge_{u\in U} \widetilde{X}(u)) \leftrightarrow_{P^*} (\bigwedge_{u\in U} \widetilde{X}'(u))$;

(2) $\bigwedge_{u\in U}(\widetilde{X}(u) \leftrightarrow_{P^*} \widetilde{X}'(u)) \le (\bigvee_{u\in U} \widetilde{X}(u)) \leftrightarrow_{P^*} (\bigvee_{u\in U} \widetilde{X}'(u))$.

Proof It is similar to the proof of the result (4) of Theorem 4.2. □

Definition 4.4 Suppose that $\to_{P^*}$ is an RPFIO and $\leftrightarrow_{P^*}$ is a Pythagorean fuzzy biresiduum associated with RPFIO. Given that X and X' are two PFSs over U, and $\delta \in P^*$, define

$$
S_P(X, X') = \bigwedge_{u\in U}\{\widetilde{X}(u) \leftrightarrow_{P^*} \widetilde{X}'(u)\} = \bigwedge_{u\in U}\{(\widetilde{X}(u) \to_{P^*} \widetilde{X}'(u)) \wedge (\widetilde{X}'(u) \to_{P^*} \widetilde{X}(u))\};
$$

then, $S_P(X, X')$ is referred to as the degree of similarity of X and X' induced from the RPFIO. If $S_P(X, X') \ge \delta$, then we say that X and X' are δ-equal, and denote $X = (\delta)X'$.

5 The Triple I Method of PFMP Model and PFMT Model

In this section, we shall extend the full implication triple I method of the FMP model and FMT model proposed by Wang to the PFMP and PFMT environments, and establish the Pythagorean fuzzy full implication triple I method.

Now, PFMP inference model is given as follows:

$$
\begin{array}{llll}
\text{Suppose that} & \widetilde{X}(u) & \longrightarrow_{P^*} & \widetilde{Y}(v) \\
\text{And given} & \widetilde{X}^*(u) & & \\
\hline
\text{Calculate} & & & \widetilde{Y}^*(v)
\end{array}
\tag{1}
$$

The principle of triple I method for PFMP inference model is to seek the optimal $\widetilde{Y}^*$ over V such that the expression

$$
(\widetilde{X}(u) \to_{P^*} \widetilde{Y}(v)) \to_{P^*} (\widetilde{X}^*(u) \to_{P^*} \widetilde{Y}^*(v)), \tag{2}
$$

takes the largest possible value for any $v \in V$. That is to say, the conclusion $\widetilde{Y}^*$ of Eq. (1) is the smallest PFS over V satisfying

$$
(\widetilde{X}(u) \to_{P^*} \widetilde{Y}(v)) \to_{P^*} (\widetilde{X}^*(u) \to_{P^*} \widetilde{Y}^*(v)) = (1, 0), \tag{3}
$$

where $\to_{P^*}$ is a RPFIO, $\widetilde{X}^*(u) = (\mu_p^*(u), \eta_p^*(u))$, $\widetilde{X}(u) = (\mu_p(u), \eta_p(u))$, $\widetilde{Y}^*(v) = (\mu_p^{*'}(v), \eta_p^{*'}(v))$, and $\widetilde{Y}(v) = (\mu_p'(v), \eta_p'(v))$.

Theorem 5.1 *Let $\widetilde{X}^*, \widetilde{X}$ be PFSs over U, and $\widetilde{Y}^*, \widetilde{Y}$ be PFSs over V, respectively. Given that $\triangle_{P^*}$ is a Pythagorean t-norm and $\to_{P^*}$ is a RPFIO, then the triple I solution of Eq. (1) can be expressed as follows:*

$$
\widetilde{Y}^*(v) = \bigvee_{u\in U}\{\widetilde{X}^*(u) \triangle_{P^*} (\widetilde{X}(u) \to_{P^*} \widetilde{Y}(v))\}. \tag{4}
$$

Proof Firstly, we prove that $\widetilde{Y}^*(v)$ determined by Eq. (4) satisfies Eq. (3). It follows from Eq. (4) that

$$
\widetilde{Y}^*(v) \ge \widetilde{X}^*(u) \triangle_{P^*} (\widetilde{X}(u) \to_{P^*} \widetilde{Y}(v)).
$$

By Theorem 3.4, we obtain

$$
\widetilde{X}^*(u) \to_{P^*} \widetilde{Y}^*(v) \ge \widetilde{X}(u) \to_{P^*} \widetilde{Y}(v),
$$

thus

$$
(\widetilde{X}(u) \to_{P^*} \widetilde{Y}(v)) \to_{P^*} (\widetilde{X}^*(u) \to_{P^*} \widetilde{Y}^*(v)) = (1, 0).
$$

Secondly, we prove that $\widetilde{Y}^*$ is the smallest PFSs over V satisfying Eq. (3). Supposing that $\widetilde{Z}^*$ is a PFS over V satisfying Eq. (3), we have

$$
(\widetilde{X}(u) \to_{P^*} \widetilde{Y}(v)) \to_{P^*} (\widetilde{X}^*(u) \to_{P^*} \widetilde{Z}^*(v)) = (1, 0).
$$

By Theorem 3.4, we obtain

$$
\widetilde{X}(u) \to_{P^*} \widetilde{Y}(v) \le \widetilde{X}^*(u) \to_{P^*} \widetilde{Z}^*(v).
$$

Thus

$$
\widetilde{Z}^*(v) \ge \widetilde{X}^*(u) \triangle_{P^*} (\widetilde{X}(u) \to_{P^*} \widetilde{Y}(v)).
$$

In light of that, it follows from Eq. (4) that $\widetilde{Y}^*(v) \le \widetilde{Z}^*(v)$. □

Corollary 5.2 *If $\triangle_{P^*}$ is a Pythagorean t-norm and $\to_{P^*}$ is a RPFIO, then the triple I solution of PFMP inference model is $\widetilde{Y}^*(v) = (\mu_p^{*'}(v), \eta_p^{*'}(v)) \in P^*$, where*

$$\mu_p^{*'}(v) = \bigvee_{u\in U}\{\sqrt{\mu_p^*(u)^2 \triangle((\mu_p(u)^2 \rightarrow \mu_p'(v)^2)} \wedge(1-(\eta_p'(v)^2 \ominus \eta_p(u)^2)))\},$$
$$\eta_p^{*'}(v) = \bigvee_{u\in U}\{\sqrt{\eta_p^*(u)^2 \nabla(\eta_p'(v)^2 \ominus \eta_p(u)^2)}\}.$$

Example 5.3 Let $U=\{u_1,u_2,u_3\}$ and $V=\{v_1,v_2,v_3\}$. From PFMP model we consider the following form:

If $\rightarrow_{P^*}$ is the RPFIO induced from Gödel t-norm $\triangle_G$, then according to the Pythagorean fuzzy triple I method, the solution Y^* can be obtained as follows:

Suppose that	$\widetilde{X}(u)=\{(u_1,0.8,0.2),(u_2,0.7,0.5),(u_3,0.9,0.3)\}$
	$\longrightarrow_P \widetilde{Y}(v)=\{(v_1,0.7,0.3),(v_2,0.9,0.3),(v_3,0.2,0.8)\}$
And given	$\widetilde{X}^*(u)=\{(u_1,0.5,0.6),(u_2,0.9,0.2),(u_3,0.8,0.2)\}$
Calculate	$\widetilde{Y}^*$

$$\widetilde{Y}^*(v_1)=\bigvee_{u\in U}\{\widetilde{X}^*(u)\triangle_{P^*}(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v_1))\}=(0.9,0.2),$$
$$\widetilde{Y}^*(v_2)=\bigvee_{u\in U}\{\widetilde{X}^*(u)\triangle_{P^*}(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v_2))\}=(0.9,0.2),$$
$$\widetilde{Y}^*(v_3)=\bigvee_{u\in U}\{\widetilde{X}^*(u)\triangle_{P^*}(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v_3))\}=(0.2,0.8).$$

Therefore, we obtain the following solution of Pythagorean fuzzy triple I method based on PFMP inference model:

$$\widetilde{Y}^*(v)=\{(v_1,0.9,0.2),(v_2,0.9,0.2),(v_3,0.2,0.8)\}.$$

The following theorem reveals that the triple I method for PFMP inference model is reversible.

Theorem 5.4 *Let $\widetilde{X}^*,\widetilde{X}$ be PFSs over U, and $\widetilde{Y}^*,\widetilde{Y}$ be PFSs over V, respectively. Given that $\triangle_{P^*}$ is a Pythagorean t-norm, and $\rightarrow_{P^*}$ is a RPFIO. If $\widetilde{X}^*=\widetilde{X}$, then $\widetilde{Y}^*=\widetilde{Y}$.*

Proof For all $u\in U$, if $\widetilde{X}^*(u)=\widetilde{X}(u)$, then it follows from Eq. (4) that

$$\widetilde{Y}^*(v)=\bigvee_{u\in U}\{\widetilde{X}(u)\triangle_{P^*}(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\}.$$

Noting that $\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v)\le\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v)$, then we have

$$\widetilde{X}(u)\triangle_{P^*}(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\le\widetilde{Y}(v),$$

which implies that $\widetilde{Y}^*(v)\le\widetilde{Y}(v)$.

On the other hand, suppose that$\widetilde{X}(u)=\widetilde{X}^*(u)=(1,0)$ for $u\in U$. Form Eq. (4) and Theorem 3.4, we obtain

$$\widetilde{Y}^*(v)\ge\widetilde{X}^*(u)\triangle_{P^*}(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v)) = (1,0)\triangle_{P^*}((1,0)\rightarrow_{P^*}\widetilde{Y}(v))=\widetilde{Y}(v).$$

Therefore, $\widetilde{Y}^*(v)=\widetilde{Y}(v)$ for all $v\in V$. □

In what follows, PFMT inference model is given as follows:

$$\begin{array}{llll} \text{Suppose that} & \widetilde{X}(u) & \longrightarrow_{P^*} & \widetilde{Y}(v) \\ \text{And given} & & & \widetilde{Y}^*(v) \\ \hline \text{Calculate} & \widetilde{X}^*(u) & & \end{array} \tag{5}$$

The principle of triple I method for PFMT inference model is to seek the optimal $\widetilde{X}^*$ over U such that the expression Eq. (2) takes the largest possible value for any $u\in U$. That is to say, the conclusion $\widetilde{X}^*$ of Eq. (5) is the largest PFS over U satisfying Eq. (3) for all $u\in U$.

Theorem 5.5 *Let $\widetilde{X}^*,\widetilde{X}$ be PFSs over U, and $\widetilde{Y}^*,\widetilde{Y}$ be PFSs over V, respectively. Given that $\triangle_{P^*}$ is a Pythagorean t-norm and $\rightarrow_{P^*}$ is a RPFIO, then the triple I solution of Eq. (5) can be expressed as follows:*

$$\widetilde{X}^*(u)=\bigwedge_{v\in V}\{(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\rightarrow_{P^*}\widetilde{Y}^*(v)\}. \tag{6}$$

Proof Firstly, we prove that $\widetilde{X}^*(u)$ determined by Eq. (6) satisfies Eq. (3). It follows from Eq. (6) that

$$\widetilde{X}^*(u)\le(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\rightarrow_{P^*}\widetilde{Y}^*(v).$$

By Theorem 3.4, we obtain

$$(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\rightarrow_{P^*}(\widetilde{X}^*(u)\rightarrow_{P^*}\widetilde{Y}^*(v))=(1,0).$$

Secondly, we prove that $\widetilde{X}^*$ is the largest PFSs over U satisfying Eq. (3). Supposing that $\widetilde{H}^*$ is a PFS over U satisfying Eq. (3), we have

$$(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\rightarrow_{P^*}(\widetilde{H}^*(u)\rightarrow_{P^*}\widetilde{Y}^*(v))=(1,0).$$

By Theorem 3.4, we obtain

$$\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v)\le\widetilde{H}^*(u)\rightarrow_{P^*}\widetilde{Y}^*(v).$$

Thus

$$\widetilde{H}^*(u)\le(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\rightarrow_{P^*}\widetilde{Y}^*(v).$$

In light of that, it follows from Eq. (6) that $\widetilde{H}^*(u)\le\widetilde{X}^*(u)$. □

Corollary 5.6 *If $\triangle_{P^*}$ is a Pythagorean t-norm and $\rightarrow_{P^*}$ is a RPFIO, then the triple I solution of PFMT inference model is $\widetilde{X}^*(u)=(\mu_p^*(u),\eta_p^*(u))\in P^*$, where*

$$\mu_p^*(u) = \bigwedge_{v\in V}\{\sqrt{(((\mu_p(u)^2 \rightarrow_{P^*} \mu_p'(u)^2) \wedge (1-(\eta_p'(v)^2 \ominus \eta_p(u)^2))) \rightarrow_{P^*} \mu_p^{*\prime}(v)^2)}$$
$$\wedge \sqrt{(1-(\eta_p^{*\prime}(v)^2 \ominus (\eta_p'(v)^2 \ominus \eta_p(u)^2)))},$$
$$\eta_p^*(u) = \bigwedge_{v\in V}\{\sqrt{\eta_p^{*\prime}(v)^2 \ominus (\eta_p'(v)^2 \ominus \eta_p(u)^2)}\}.$$

Example 5.7 Let $U=\{u_1,u_2,u_3\}$ and $V=\{v_1,v_2,v_3\}$. From PFMT model we consider the following form:

$$\begin{array}{ll} \textit{Suppose that} & \widetilde{X}(u)=\{(u_1,0.5,0.5),(u_2,0.9,0.4),(u_3,0.8,0.2)\} \\ & \quad \longrightarrow_P \widetilde{Y}(v)=\{(v_1,0.7,0.3),(v_2,0.9,0.3),(v_3,0.7,0.2)\} \\ \textit{And given} & \widetilde{Y}^*(v)=\{(v_1,0.8,0.3),(v_2,0.8,0.4),(v_3,0.6,0.3)\} \\ \hline \textit{Calculate} & \qquad\qquad\qquad\qquad \widetilde{X}^* \end{array}$$

If $\rightarrow_{P^*}$ is the RPFIO induced from Gödel t-norm $\triangle_G$, then according to the Pythagorean fuzzy triple I method, the solution X^* can be obtained as follows:

$$\widetilde{X}^*(u_1) = \bigwedge_{v\in V}\{(\widetilde{X}(u_1) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}^*(v)\} = (0.6, 0.4),$$
$$\widetilde{X}^*(u_2) = \bigwedge_{v\in V}\{(\widetilde{X}(u_2) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}^*(v)\} = (0.6, 0.4),$$
$$\widetilde{X}^*(u_3) = \bigwedge_{v\in V}\{(\widetilde{X}(u_3) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}^*(v)\} = (0.6, 0).$$

Therefore, we obtain the following solution of Pythagorean fuzzy triple I method based on PFMT inference model:

$$\widetilde{X}^*(u) = \{(u_1, 0.6, 0.4), (u_2, 0.6, 0.4), (u_3, 0.6, 0)\}.$$

The following theorem reveals that the triple I method for PFMT inference model is reversible.

Theorem 5.8 *Let $\widetilde{X}^*, \widetilde{X}$ be PFSs over U, and $\widetilde{Y}^*, \widetilde{Y}$ be PFSs over V, respectively. Given that $\triangle_{P^*}$ is a Pythagorean t-norm and $\rightarrow_{P^*}$ is a RPFIO. If $\widetilde{Y}^* = \widetilde{Y}$, then $\widetilde{X}^* = \widetilde{X}$.*

Proof For all $v \in V$, if $\widetilde{Y}^*(v) = \widetilde{Y}(v)$, then it follows from Eq. (4) that

$$\widetilde{X}^*(u) = \bigwedge_{v\in V}\{(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}(v)\}.$$

Noting that $\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v) \leq \widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)$, then we have

$$\widetilde{X}(u) \leq (\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}(v),$$

which implies that $\widetilde{X}^*(u) \geq \widetilde{X}(u)$.

On the other hand, noting that $\widetilde{X}^*(u) \rightarrow_{P^*} \widetilde{Y}(v) \leq \widetilde{X}^*(u) \rightarrow_{P^*} \widetilde{Y}(v)$, according to the result (3) of Theorem 3.4, we have

$$\widetilde{X}^*(u) \leq (\widetilde{X}^*(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}(v).$$

From Eq. (6) and Theorem 3.4, we obtain

$$\widetilde{X}^*(u) \leq (\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}(v)$$
$$\Rightarrow \widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v) \leq \widetilde{X}^*(u) \rightarrow_{P^*} \widetilde{Y}(v)$$
$$\Rightarrow \widetilde{X}(u) \geq (\widetilde{X}^*(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}(v) \geq \widetilde{X}^*(u).$$

Therefore, $\widetilde{X}^*(u) = \widetilde{X}(u)$ for all $u \in U$. □

6 The Robustness and Continuity Properties of Triple I Method Based on PFMP and PFMT Inference Models

In what follows we prove the robustness of the triple I method based on PFMP and PFMT inference models. Let $\widetilde{X}^*, \widetilde{X}, \widetilde{X}'$ and $\widetilde{X}^{*'}$ be PFSs over U, and $\widetilde{Y}^*, \widetilde{Y}, \widetilde{Y}'$ and $\widetilde{Y}^{*'}$ be PFSs over V, respectively. Given that $\triangle_{P^*}$ is a Pythagorean t-norm, $\rightarrow_{P^*}$ is a RPFIO and $\leftrightarrow_{P^*}$ is a Pythagorean fuzzy biresiduum associated with RPFIO, δ_1, δ_2 and δ_3 are three PFNs.

Theorem 6.1 *Assume that $S_P(\widetilde{X}, \widetilde{X}') \geq \delta_1, S_P(\widetilde{Y}, \widetilde{Y}') \geq \delta_2$ and $S_P(\widetilde{X}^*, \widetilde{X}^{*'}) \geq \delta_3$. Given that $\widetilde{Y}^*$ and $\widetilde{Y}^{*'}$ are triple I solutions of PFMP $(\widetilde{X}, \widetilde{Y}, \widetilde{X}^*)$ and PFMP $(\widetilde{X}', \widetilde{Y}', \widetilde{X}^{*'})$ given by the model (1), respectively, then $S_P(\widetilde{Y}^*, \widetilde{Y}^{*'}) \geq \delta_1 \triangle_{P^*} \delta_2 \triangle_{P^*} \delta_3$.*

Proof According to the results (4) and (5) of Theorem 4.2 and Eq. (4), we have

$$S_P(\widetilde{Y}^*, \widetilde{Y}^{*'})$$
$$= \bigwedge_{v\in V}\{\widetilde{Y}^*(v) \leftrightarrow_{P^*} \widetilde{Y}^{*'}(v)\}$$
$$= \bigwedge_{v\in V}\{\bigvee_{u\in U}\{\widetilde{X}^*(u)\triangle_{P^*}(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v))\}$$
$$\leftrightarrow_{P^*} \bigvee_{u\in U}\{\widetilde{X}^{*'}(u)\triangle_{P^*}(\widetilde{X}'(u) \rightarrow_{P^*} \widetilde{Y}'(v))\}\}$$
$$\geq \bigwedge_{v\in V}\bigwedge_{u\in U}\{(\widetilde{X}^*(u)\triangle_{P^*}(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)))$$
$$\leftrightarrow_{P^*} (\widetilde{X}^{*'}(u)\triangle_{P^*}(\widetilde{X}'(u) \rightarrow_{P^*} \widetilde{Y}'(v)))\}$$
$$\geq \bigwedge_{v\in V}\bigwedge_{u\in U}\{(\widetilde{X}^*(u) \leftrightarrow_{P^*} \widetilde{X}^{*'}(u))\triangle_{P^*}((\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v))$$
$$\leftrightarrow_{P^*} (\widetilde{X}'(u) \rightarrow_{P^*} \widetilde{Y}'(v)))\}$$
$$\geq \bigwedge_{v\in V}\bigwedge_{u\in U}\{(\widetilde{X}^*(u) \leftrightarrow_{P^*} \widetilde{X}^{*'}(u))\triangle_{P^*}(\widetilde{X}(u)$$
$$\leftrightarrow_{P^*} \widetilde{X}'(u))\triangle_{P^*}(\widetilde{Y}(v) \leftrightarrow_{P^*} \widetilde{Y}'(v))\}$$
$$\geq \delta_1 \triangle_{P^*} \delta_2 \triangle_{P^*} \delta_3.$$

□

Theorem 6.2 *Assume that* $S_P(\widetilde{X},\widetilde{X}')\geq\delta_1, S_P(\widetilde{Y},\widetilde{Y}')\geq\delta_2$ *and* $S_P(\widetilde{Y}^*,\widetilde{Y}^{*'})\geq\delta_3$. *Given that* $\widetilde{X}^*$ *and* $\widetilde{X}^{*'}$ *are triple I solutions of PFMT* $(\widetilde{X},\widetilde{Y},\widetilde{Y}^*)$ *and PFMT* $(\widetilde{X}',\widetilde{Y}',\widetilde{Y}^{*'})$ *given by the model* (5), *respectively, then* $S_P(\widetilde{X}^*,\widetilde{X}^{*'})\geq\delta_1\triangle_{P^*}\delta_2\triangle_{P^*}\delta_3$.

Proof According to the result (5) of Theorem 4.2 and Eq. (6), we have

$$\begin{aligned}
&S_P(\widetilde{X}^*,\widetilde{X}^{*'})\\
&=\bigwedge_{u\in U}\{\widetilde{X}^*(u)\leftrightarrow_{P^*}\widetilde{X}^{*'}(u)\}\\
&\quad=\bigwedge_{u\in U}\{\bigwedge_{v\in V}\{(\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\rightarrow_{P^*}\widetilde{Y}^*(v)\}\\
&\leftrightarrow_{P^*}\bigwedge_{v\in V}\{(\widetilde{X}'(u)\rightarrow_{P^*}\widetilde{Y}'(v))\rightarrow_{P^*}\widetilde{Y}^{*'}(v)\}\}\\
&\quad\geq\bigwedge_{u\in U}\bigwedge_{v\in V}\{((\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\rightarrow_{P^*}\widetilde{Y}^*(v))\\
&\leftrightarrow_{P^*}((\widetilde{X}'(u)\rightarrow_{P^*}\widetilde{Y}'(v))\rightarrow_{P^*}\widetilde{Y}^{*'}(v))\}\\
&\quad\geq\bigwedge_{u\in U}\bigwedge_{v\in V}\{((\widetilde{X}(u)\rightarrow_{P^*}\widetilde{Y}(v))\\
&\leftrightarrow_{P^*}(\widetilde{X}'(u)\rightarrow_{P^*}\widetilde{Y}'(v)))\triangle_{P^*}(\widetilde{Y}^*(v)\leftrightarrow_{P^*}\widetilde{Y}^{*'}(v))\}\\
&\quad\geq\bigwedge_{u\in U}\bigwedge_{v\in V}\{(\widetilde{X}(u)\leftrightarrow_{P^*}\widetilde{X}'(u))\triangle_{P^*}(\widetilde{Y}(v)\\
&\leftrightarrow_{P^*}\widetilde{Y}'(v))\triangle_{P^*}(\widetilde{Y}^*(v)\leftrightarrow_{P^*}\widetilde{Y}^{*'}(v))\}\\
&\quad\geq\delta_1\triangle_{P^*}\delta_2\triangle_{P^*}\delta_3.
\end{aligned}$$

□

Remark 6.3 According to Theorems 6.1 and 6.2, we know that the triple I methods based on PFMP and PFMT models have the same robustness.

Example 6.4 Assume that $S_P(\widetilde{X},\widetilde{X}')\geq\delta_1, S_P(\widetilde{Y},\widetilde{Y}')\geq\delta_2$, $S_P(\widetilde{X}^*,\widetilde{X}^{*'})\geq\delta_3$, and $\delta_i=(\delta_{i\mu},\delta_{i\eta}),(i=1,2,3)$. Given that $\widetilde{Y}^*$ and $\widetilde{Y}^{*'}$ are triple I solutions of PFMP $(\widetilde{X},\widetilde{Y},\widetilde{X}^*)$ and PFMP $(\widetilde{X}',\widetilde{Y}',\widetilde{X}^{*'})$ given by the model (1), respectively.

(1) If $\rightarrow_{P^*}$ is a RPFIO induced by Gödel t-norm, then

$$S_P(\widetilde{Y}^*,\widetilde{Y}^{*'})\geq(\delta_{1\mu}\wedge\delta_{2\mu}\wedge\delta_{3\mu},\delta_{1\eta}\vee\delta_{2\eta}\vee\delta_{3\eta}).$$

(2) If $\rightarrow_{P^*}$ is a RPFIO induced by Product t-norm, then

$$\begin{aligned}&S_P(\widetilde{Y}^*,\widetilde{Y}^{*'})\\&\geq(\delta_{1\mu}\delta_{2\mu}\delta_{3\mu},\sqrt{\delta_{1\eta}^2+\delta_{2\eta}^2+\delta_{3\eta}^2-\delta_{1\eta}^2\delta_{2\eta}^2-\delta_{1\eta}^2\delta_{3\eta}^2-\delta_{2\eta}^2\delta_{3\eta}^2+\delta_{1\eta}^2\delta_{2\eta}^2\delta_{3\eta}^2}).\end{aligned}$$

(3) If $\rightarrow_{P^*}$ is a RPFIO induced by Einstein t-norm, then

$$\begin{aligned}&S_P(\widetilde{Y}^*,\widetilde{Y}^{*'})\\&\geq\left(\frac{\delta_{1\mu}\delta_{2\mu}\delta_{3\mu}}{\sqrt{4-2\delta_{1\mu}^2-2\delta_{2\mu}^2-2\delta_{3\mu}^2+\delta_{1\mu}^2\delta_{2\mu}^2+\delta_{1\mu}^2\delta_{3\mu}^2+\delta_{2\mu}^2\delta_{3\mu}^2}},\right.\\&\left.\sqrt{\frac{\delta_{1\eta}^2+\delta_{2\eta}^2+\delta_{3\eta}^2+\delta_{1\eta}^2\delta_{2\eta}^2\delta_{3\eta}^2}{1+\delta_{1\eta}^2\delta_{2\eta}^2+\delta_{1\eta}^2\delta_{3\eta}^2+\delta_{2\eta}^2\delta_{3\eta}^2}}\right).\end{aligned}$$

(4) If $\rightarrow_{P^*}$ is a RPFIO induced by R_0 t-norm, then

$$S_P\left(\widetilde{Y}^*,\widetilde{Y}^{*'}\right)\geq(\sqrt{\delta_{1\mu}^2\triangle\delta_{2\mu}^2\triangle\delta_{3\mu}^2},\sqrt{\delta_{1\eta}^2\nabla\delta_{2\eta}^2\nabla\delta_{3\eta}^2}\Big).$$

Remark 6.5 In Example 6.4, as per Theorem 6.2 we can also obtain the same robustness of the triple I method for PFMT based on four RPFIOs induced by Gödel t-norm, Product t-norm, Einstein t-norm and R_0 t-norm.

In what follows we prove the continuity of the triple I method based on PFMP and PFMT inference models.

The triple I method for the PFMP inference model is a mapping $h: P^*(U)\rightarrow P^*(V)$, i.e., for any input $\widetilde{X}^*$ over U, there exists a corresponding output $\widetilde{Y}^*=h(\widetilde{X}^*)$ over V, where $P^*(U)$ and $P^*(V)$ denote the set of all PFSs on the universes U and V, respectively.

Definition 6.6 Let $\widetilde{X}_1^*,\widetilde{X}_2^*$ be PFSs over U. If for all $\widetilde{\varphi}\in P^*$, there exists $\widetilde{\omega}\in P^*$ such that $s_P(h(\widetilde{X}_1^*),h(\widetilde{X}_2^*))>\widetilde{\varphi}$ whenever $s_P(\widetilde{X}_1^*,\widetilde{X}_2^*)>\widetilde{\omega}$, then the method h is said to be uniformly continuous. If $s_P(h(\widetilde{X}^*),h(\widetilde{X}))>\widetilde{\varphi}$ whenever $s_P(\widetilde{X}^*,\widetilde{X})>\widetilde{\omega}$ for any $\widetilde{X}^*$ over U, then the method h is said to be continuous at $\widetilde{X}$.

Remark 6.7 If the method h is uniformly continuous, then the method h is continuous for any PFSs over U.

Theorem 6.8 *Let* $\widetilde{X}_1^*,\widetilde{X}_2^*$ *and* $\widetilde{X}$ *be PFSs over* U, *and* $\widetilde{Y}_1^*,\widetilde{Y}_2^*$ *and* $\widetilde{Y}$ *be PFSs over* V, *respectively. Given that* $\triangle_{P^*}$ *is a Pythagorean t-norm. Then the triple I method for the PFMP inference model is uniformly continuous.*

Proof For all $\widetilde{\varphi}\in P^*$, let $\widetilde{\omega}=\widetilde{\varphi}$. When $s_P(\widetilde{X}_1^*,\widetilde{X}_2^*)>\widetilde{\omega}$, according to Definition 4.4 and Theorem 4.2 we can obtain

$$
\begin{aligned}
&s_P(\widetilde{Y}_1^*, \widetilde{Y}_2^*)\\
&\quad = \bigwedge_{v\in V}\{\widetilde{Y}_1^*(v) \leftrightarrow_{P^*} \widetilde{Y}_2^*(v)\}\\
&\quad = \bigwedge_{v\in V}\Bigg\{\bigvee_{u\in U}\{\widetilde{X}_1^*(u)\triangle_{P^*}(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v))\}\\
&\leftrightarrow_{P^*} \bigvee_{u\in U}\{\widetilde{X}_2^*(u)\triangle_{P^*}(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v))\}\Bigg\}\\
&\quad \geq \bigwedge_{v\in V}\bigwedge_{u\in U}\{(\widetilde{X}_1^*(u)\triangle_{P^*}(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)))\\
&\leftrightarrow_{P^*} (\widetilde{X}_2^*(u)\triangle_{P^*}(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)))\}\\
&\quad \geq \bigwedge_{v\in V}\bigwedge_{u\in U}\{(\widetilde{X}_1^*(u)\\
&\leftrightarrow_{P^*} \widetilde{X}_2^*(u))\triangle_{P^*}((\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)) \leftrightarrow (\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)))\}\\
&\quad \geq \bigwedge_{v\in V}\bigwedge_{u\in U}\{(\widetilde{X}_1^*(u) \leftrightarrow_{P^*} \widetilde{X}_2^*(u))\}\\
&\quad \geq \bigwedge_{v\in V}\widetilde{\omega} = \widetilde{\varphi}.
\end{aligned}
$$

Therefore, the triple I method for the PFMP inference model is uniformly continuous. □

The triple I method based on the PFMT inference model is a mapping $z : P^*(V) \rightarrow P^*(U)$, i.e., for any input $\widetilde{Y}^*$ over V, there exists a corresponding output $\widetilde{X}^* = z(\widetilde{Y}^*)$ over U.

Definition 6.9 Let $\widetilde{Y}_1^*, \widetilde{Y}_2^*$ be PFSs over V. If for all $\widetilde{\varphi} \in P^*$, there exists $\widetilde{\omega} \in P^*$ such that $s_P(z(\widetilde{Y}_1^*), z(\widetilde{Y}_2^*)) > \widetilde{\varphi}$ whenever $s_P(\widetilde{Y}_1^*, \widetilde{Y}_2^*) > \widetilde{\omega}$, then the method z is said to be uniformly continuous. If $s_P(z(\widetilde{Y}^*), z(\widetilde{Y})) > \widetilde{\varphi}$ whenever $s_P(\widetilde{Y}^*, \widetilde{Y}) > \widetilde{\omega}$ for any $\widetilde{Y}^*$ over V, then the method z is said to be continuous at $\widetilde{Y}$.

Theorem 6.10 *Let $\widetilde{X}_1^*, \widetilde{X}_2^*$ and $\widetilde{X}$ be PFSs over U, and $\widetilde{Y}_1^*, \widetilde{Y}_2^*$ and $\widetilde{Y}$ be PFSs over V, respectively. Given that $\triangle_{P^*}$ is a Pythagorean t-norm. Then the triple I method for the PFMT inference model is uniformly continuous.*

Proof For all $\widetilde{\varphi} \in P^*$, let $\widetilde{\omega} = \widetilde{\varphi}$. When $s_P(\widetilde{Y}_1^*, \widetilde{Y}_2^*) > \widetilde{\omega}$, according to Definition 4.4 and Theorem 4.2 we can obtain

$$
\begin{aligned}
&s_P(\widetilde{X}_1^*, \widetilde{X}_2^*)\\
&\quad = \bigwedge_{u\in U}\{\widetilde{X}_1^*(u)\\
&\leftrightarrow_{P^*} \widetilde{X}_2^*(u)\}\\
&\quad = \bigwedge_{u\in U}\Bigg\{\bigwedge_{v\in V}\{(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}_1^*(v)\}\\
&\leftrightarrow_{P^*} \bigwedge_{v\in V}\{(\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}_2^*(v)\}\Bigg\}\\
&\quad \geq \bigwedge_{u\in U}\bigwedge_{v\in V}\{((\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}_1^*(v))\\
&\leftrightarrow_{P^*} ((\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)) \rightarrow_{P^*} \widetilde{Y}_2^*(v))\}\\
&\quad \geq \bigwedge_{u\in U}\bigwedge_{v\in V}\{((\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v))\\
&\leftrightarrow_{P^*} (\widetilde{X}(u) \rightarrow_{P^*} \widetilde{Y}(v)))\triangle_{P^*}(\widetilde{Y}_1^*(v) \leftrightarrow_{P^*} \widetilde{Y}_2^*(v))\}\\
&\quad \geq \bigwedge_{u\in U}\bigwedge_{v\in V}\{(\widetilde{Y}_1^*(v) \leftrightarrow_{P^*} \widetilde{Y}_2^*(v))\}\\
&\quad \geq \bigwedge_{u\in U}\widetilde{\omega} = \widetilde{\varphi}.
\end{aligned}
$$

Therefore, the triple I method for the PFMT inference model is uniformly continuous. □

7 Application Example

In this section, we apply the PFMP inference model to solve medical diagnosis problem.

At the beginning of 2020, a new type of Corona Virus Disease-2019 (COVID-19) caused the massive pandemic situation all over the world, and many people died from respiratory failure or other related complications. This virus is more contagious than other viruses and can cause greater damage to the body. Therefore, quickly identifying infected persons and conducting isolation treatment is the key to controlling the epidemic.

Suppose that U is a set of the symptoms with COVID-19 including fever (x_1), cough (x_2), fatigue (x_3), trouble breathing (x_4) and sore throat (x_5). Suppose that the data of a close contact with the related symptoms is represented by the PFS $\widetilde{X}$ over U, described as

$$
\begin{aligned}
\widetilde{X} = \Big\{&(x_1, (0.89, 0.18)), (x_2, (0.91, 0.17)),\\
&(x_3, (0.92, 0.10)), (x_4, (0.97, 0.05)), (x_5, (0.89, 0.19))\Big\}.
\end{aligned}
$$

According to the score function of PFNs, we know that the score value of PFN (0.86, 0.20) is approximately 0.70. Further, assume that 70% of patients with the basic symptoms represented by $\widetilde{X}$ is suffering from COVID-19, which can be described by the PFN (0.86, 0.20). Therefore,

Table 1 Data on the related symptoms of v_1, v_2, v_3 and v_4

X^*	x_1	x_2	x_3	x_4	x_5
v_1	(0.53, 0.70)	(0.63, 0.45)	(0.45, 0.56)	(0.42, 0.67)	(0.55, 0.61)
v_2	(0.92, 0.15)	(0.86, 0.23)	(0.75, 0.26)	(0.77, 0.13)	(0.93, 0.20)
v_3	(0.67, 0.47)	(0.89, 0.13)	(0.68, 0.30)	(0.81, 0.23)	(0.88, 0.29)
v_4	(0.66, 0.45)	(0.75, 0.23)	(0.56, 0.87)	(0.57, 0.62)	(0.61, 0.43)

we can obtain a Pythagorean fuzzy inference rule $\widetilde{X} \rightarrow_{p^*} \widetilde{Y}$, where $\widetilde{Y}$ is a PFN, denoted as $\widetilde{Y} = (\text{yes}, (0.86, 0.20))$.

Now, suppose that $V = \{v_1, v_2, v_3, v_4\}$ is a family of four different patients and their medical data of COVID-19 can be expressed as PFS $\widetilde{X}^*$ over U, which is shown in Table 1.

In the following, we adopt two methods to solve the medical diagnosis problem.

In Method 1, we use $\rightarrow_{P^*_\pi}$ as the Pythagorean fuzzy implication operator. Therefore, as per Eq. (4) we can obtain an inference result based on triple I method for PFMP, which is expressed as the following PFS $\widetilde{Y}^*$ over V,

$$\begin{aligned}\widetilde{Y}^*(v_1) = &\bigvee_{x_j \in U} \{\widetilde{X}^*(x_j) \triangle_{P^*} (\widetilde{X}(x_j) \rightarrow_{P^*} \widetilde{Y}(\text{yes}))\}\\
=&\Big((0.53, 0.70) \triangle_{P^*_\pi} ((0.89, 0.18) \rightarrow_{P^*_\pi} (0.86, 0.20))\Big)\\
&\vee \Big((0.63, 0.45) \triangle_{P^*_\pi} ((0.91, 0.17) \rightarrow_{P^*_\pi} (0.86, 0.20))\Big) \vee \Big((0.45, 0.56) \triangle_{P^*_\pi} ((0.92, 0.10)\\
&\rightarrow_{P^*_\pi} (0.86, 0.20))\Big) \vee \Big((0.42, 0.67) \triangle_{P^*_\pi} ((0.97, 0.05) \rightarrow_{P^*_\pi} (0.86, 0.20))\Big)\\
&\vee \Big((0.55, 0.61) \triangle_{P^*_\pi} ((0.87, 0.19) \rightarrow_{P^*_\pi} (0.86, 0.20))\Big)\\
=&(0.5121, 0.7266) \vee (0.5954, 0.5088)\\
&\vee (0.4207, 0.6366) \vee (0.3724, 0.7340) \vee (0.5437, 0.6231)\\
=&(0.5954, 0.5088).\end{aligned}$$

Similarly, we have

$$\widetilde{Y}^*(v_2) = \bigvee_{x_j \in U} \{\widetilde{X}^*(x_j) \triangle_{P^*} (\widetilde{X}(x_j) \rightarrow_{P^*} \widetilde{Y}(\text{yes}))\} = (0.9193, 0.2253),$$
$$\widetilde{Y}^*(v_3) = \bigvee_{x_j \in U} \{\widetilde{X}^*(x_j) \triangle_{P^*} (\widetilde{X}(x_j) \rightarrow_{P^*} \widetilde{Y}(\text{yes}))\} = (0.8699, 0.2230),$$
$$\widetilde{Y}^*(v_4) = \bigvee_{x_j \in U} \{\widetilde{X}^*(x_j) \triangle_{P^*} (\widetilde{X}(x_j) \rightarrow_{P^*} \widetilde{Y}(\text{yes}))\} = (0.7088, 0.3123).$$

Calculate the score values for all v_i by the score function, $S(v_1) = 0.0956, S(v_2) = 0.7944, S(v_3) = 0.7070, S(v_4) = 0.4049$.

So we know that the possibilities of v_1, v_2, v_3 and v_4 infected with COVID-19 are 9%, 79%, 70% and 40%, respectively.

In Method 2, by using $\rightarrow_{P^*_\epsilon}$ as the Pythagorean fuzzy implication operator, we obtain the triple I solution Y^* as follows:

$$\begin{aligned}\widetilde{Y}^* =&\{(v_1, (0.5898, 0.4540)), (v_2, (0.9208, 0.1348)),\\
&(v_3, (0.8709, 0.1314)), (v_4, (0.7060, 0.2324))\}.\end{aligned}$$

Calculate the score values for all v_i by the score function, $S(v_1) = 0.1417, S(v_2) = 0.8297, S(v_3) = 0.7411, S(v_4) = 0.4445$.

So we know that the possibilities of v_1, v_2, v_3 and v_4 infected with COVID-19 are 14%, 83%, 74% and 44%, respectively.

7.1 Comparative Analysis with the Other Methods

To expand on the advantages of the developed methods, we compare them with the existing methods by solving the same example, such as the Pythagorean fuzzy weighted geometric (PFWG) [9] operator, the Pythagorean fuzzy weighted averaging (PFWA) [9] operator, the q-rung orthopair fuzzy weighted geometric (q-ROFWG) [14] operator, the q-rung orthopair fuzzy weighted averaging (q-ROFWA) [14] operator, the q-rung orthopair fuzzy Muirhead means (q-ROFMM) [20] and the q-rung orthopair fuzzy dual Muirhead means (q-ROFDMM) [20]. By applying the above mentioned methods, we obtain the comparison results shown in Table 2.

From Table 2, we observe that the optimal ranking results by the different methods are essentially the same, even though the score functions are different in different methods. Therefore, the decision-making methods based on the Pythagorean fuzzy triple I method proposed by us are reasonable and valid. On the other hand, although the ranking results based on the Pythagorean fuzzy triple I method proposed by us are the same as those obtained by using the PFWG, PFWA, q-ROFWG, q-ROFWA, q-ROFMM and q-ROFDMM methods, these methods using different aggregation operators to aggregate the medical data of each patients may result in a lack of logical reasoning in these methods. For example, in Table 1 we can observe that since v_2 has relatively high medical indicators for the common symptoms of COVID-19, it is most likely to be a COVID-19 patient. The medical indicators of v_1 are relatively normal, therefore, the possibilities of v_1 being a COVID-19 patient is relatively small. However, the

Table 2 The comparison analysis with the different methods

Methods	The score function	Ranking
PFWA [9]	$S(v_1) = 0.7961, S(v_2) = 0.9990, S(v_3) = 0.9952, S(v_4) = 0.9276$	$v_2 > v_3 > v_4 > v_1$
PFWG [9]	$S(v_1) = -0.9022, S(v_2) = -0.048, S(v_3) = -0.3119, S(v_4) = -0.8986$	$v_2 > v_3 > v_4 > v_1$
q-ROFWA [14] (q = 3)	$S(v_1) = 0.6679, S(v_2) = 0.9967, S(v_3) = 0.9859, S(v_4) = 0.8512$	$v_2 > v_3 > v_4 > v_1$
q-ROFWG [14] (q = 3)	$S(v_1) = -0.8124, S(v_2) = 0.0585, S(v_3) = -0.2130, S(v_4) = -0.8417$	$v_2 > v_3 > v_1 > v_4$
q-ROFMM [20] (q = 3)	$S(v_1) = -0.0689, S(v_2) = 0.7183, S(v_3) = 0.5929, S(v_4) = 0.1880$	$v_2 > v_3 > v_4 > v_1$
q-ROFDMM [20] (q = 3)	$S(v_1) = -0.1175, S(v_2) = 0.6679, S(v_3) = 0.5025, S(v_4) = -0.0200$	$v_2 > v_3 > v_4 > v_1$
Method 1 (in this paper)	$S(v_1) = 0.0956, S(v_2) = 0.7944, S(v_3) = 0.7070, S(v_4) = 0.4049$	$v_2 > v_3 > v_4 > v_1$
Method 2 (in this paper)	$S(v_1) = 0.1417, S(v_2) = 0.8297, S(v_3) = 0.7411, S(v_4) = 0.4445$	$v_2 > v_3 > v_4 > v_1$

obtained results by using the PFWG, PFWA, q-ROFWG, q-ROFWA, q-ROFMM and q-ROFDMM methods are not so satisfying. For instance, the PFWG method finds that the possibilities of v_1 being a COVID-19 patient is 80%, and the possibilities of v_2 being a COVID-19 patient is 99%. Obviously, this result is unreasonable and inconsistent with the real case. Compared with these methods, the methods we propose are based on the Pythagorean fuzzy triple I method. In other words, the methods we propose focus on logical reasoning, so the novel methods are more logical and more consistent with the real case. To clarify, from Table 2 we can know that the results obtained by the novel methods in this paper have obvious differences. From the result by using Method 1, we observe that the possibilities of v_1 being a COVID-19 patient is 9%, and the possibilities of v_2 being a COVID-19 patient is 80%, which is consistent with the real case. However, as previously stated, the existing methods including the PFWG, PFWA, q-ROFWG, q-ROFWA, q-ROFMM and q-ROFDMM methods fail in this regard. In view of the above analysis, the advantage of our proposed methods is that it can more easily distinguish high-risk individuals and low-risk individuals and improve their recognition, which is very helpful for medical professionals to make the best choice.

8 Conclusion

In this study, we attempt to establish the triple I method for PFMP and PFMT inference models. We first propose the concepts of Pythagorean t-norm, Pythagorean t-conorm, RPFIO and Pythagorean fuzzy biresiduum. Furthermore, some of interesting properties of triple I method of PFMP and PFMT inference models are analyzed, including the robustness, continuity and reversibility. Finally, the triple I method of PFMP is applied to practical problems. By comparing the triple I method with the existing methods, the novel method in this paper is easier to classify and rank high-risk individuals and low-risk individuals, so it is more flexible and reasonable. In the future, the research on the fusion of PFSs and other reasoning methods is expected to become an interesting topic, and its application is also a topic worthy of in-depth study.

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