Int. J. Fuzzy Syst. (2022) 24(1):474–494
https://doi.org/10.1007/s40815-021-01150-6

Archimedean Compensatory Fuzzy Logic as a Pluralist Contextual Theory Useful for Knowledge Discovery

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Received: 19 January 2021 / Revised: 6 June 2021 / Accepted: 2 July 2021 / Published online: 22 August 2021

Abstract Compensatory Fuzzy Logic is a transdisciplinary axiomatic theory, different from the Classical Norm and Conorm approach to improving interpretability by natural language. Archimedean Compensatory Fuzzy Logic (ACFL), introduced recently, uses different properties and interpretations of involved truth values. Membership functions involved are not studied explicitly in fuzzy theories, even though it is essential in solving problems. The definition of parameterized families of membership functions is not rare in fuzzy literature. However, according to our review, each of those families has the same shape except the recently introduced Continuous Linguistic Variables. That has been a limitation in the expressiveness of linguistic values. Besides, except for Dombi's theory, these functions are often not related to logical operators. This paper aims to use ACFL to overcome each of these drawbacks. We generalize some fuzzy concepts, only using the ACFL generator function. A *Generalized Sigmoidal Function* and a *Generalized Linguistic Modifier* are s-shaped functions generated by it. Those elements define a parameterized family containing different shape functions like an increasing sigmoidal, decreasing sigmoidal and convex function; we call it a *Generalized Continuous Linguistic Variable*. This paper improves ACFL by unifying it into single theory elements like logic generator functions, linguistic modifiers, membership functions, and linguistic variables. The improved ACFL is not just a Pluralist Logic that makes compatible the classical approach of Norm and Conorm with CFL theory, but a contextual pluralist logic able to select a logic that better expresses specific contextual knowledge. This theory is valuable in Knowledge Discovery; because it creates new searching elements that allow selecting the 'best logic´ for a particular dataset. We develop knowledge discovery cases for different databases to illustrate it and show its data sensitivity.

Keywords Membership function · Sigmoidal · Archimedean compensatory fuzzy logic · Knowledge discovery

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1 Introduction

Membership functions are an essential part of fuzzy theory since the seminal paper on fuzzy sets by Lotfi Zadeh [1]. Dombi is an author who paid attention to this subject [2, 3]; he included parameterized membership functions such that each parameter is meaningful. He also incorporated logical operators as part of them.

However, the critical issue of explicitly defining what kind of membership function should be used in a fuzzy problem has been neglected in the literature. Sometimes this is reduced to the simplest ones, the trapezoidal or triangular. For practical purposes, s-shaped functions are more appropriate to define smooth transitions in the degree of importance or the strength of belief. The sigmoid function is an s-shaped function with particular interest [4–6]. The importance comes from the origin of its definition: the rate of belief of one person is defined as proportional to the product of their belief and the negation of belief.

Parameterized families of membership functions exist in the literature. However, every one of these families is a set of functions with the same shape, and as a consequence, it is difficult to express possible semantics. For instance, a family of parameterized sigmoidal functions can represent linguistic values [7] of the type 'high' or 'big' if they are increasing and 'low' or 'small' if they are decreasing, but never one of the types 'medium.'

To overcome semantics limitation, this paper presents a new parametrized family of membership functions generated through the Archimedean Compensatory Fuzzy Logic (ACFL), improved in this paper and previously defined in [8]. This logic consists of one Archimedean t-norm generated by one function. ACFL serves as a generator of one compensatory operator, one t-conorm, and other compensatory operator-defined from this t-conorm. Besides, it includes the most usual negation operator and a fuzzy order.

The original ACFL naturally extends Archimedean t-norm to Archimedean Logic, which adds to the t-norm. ACFL integrates its dual t-conorm and the negation operator. In [8], we demonstrate that from the same generator function of this Archimedean Logic, it is possible to generate a Compensatory parallel Logic, according to [9].

One predicate in one ACFL, containing the operators of conjunction, disjunction, and implication, can be considered the same in Archimedean Logic or Compensatory Logic. The most usual definition of universal quantifiers from t-norms has an equivalent definition in the compensatory operators' framework.

The main contributions of this paper are outlined as follows:

(1) We use an ACFL generator as a basis for defining a sigmoidal membership function that we call the *Generalized Sigmoidal Function*. From this Generalized Sigmoidal Function and one *Generalized Linguistic Modifier*, we also generalize the family of parameterized membership functions called *General Continue Linguistic Variable* (GLCV) [10] to any specific ACFL, now called Generalized Sigmoidal Functions.
(2) The GCLV can takes many shapes, depending on the parameters. Therefore, it can represent an infinite linguistic variable rather than a set of linguistic values. At least three basic shapes are included in the family, one increasing sigmoidal, one decreasing sigmoidal, and one convex. Some convex and not symmetric functions are part of the family, too. For the first time, one family of functions can represent such a variety of linguistic values.
(3) We enhance ACFL to transform it into a theory with the new property of involving in just one theoretical space, Fuzzy Multivalued Logics of two classes, generators functions, and essential semantic tools like modifiers, membership functions, and linguistic variables. This improved ACFL is a contextual pluralist logic able to expresses specific contextual knowledge in a selected logic.
(4) A family of membership functions generated with ACFL presents an advantage in the practical application of this GCLV. For example, in Knowledge Discovery (KD), one of these families can be fitted from a dataset to resolve an optimization problem over the space of continuous parameters.
(5) The results can be expressed in natural language, where the most possible linguistic value is included, or at least the essential ones. That is possible using the Principle of Representation of Linguistic variables and some algorithms developed to determine specific labels between the ones included in the correspondent predefined linguistic variable [10].

That Principle of Representation and the correspondent algorithms can be extended to ACFL, but it is not an objective of the present work. It will be developed in a new paper. This theoretical space produced by the generalizations and the study of their existence is a necessary step towards a Generalized Principle of Representation.

The interpretability is an essential concept and a desirable property in fuzzy logic; in this paper, we follow the perspective of interpretability that appears in [11]. Interpretability "is the property fulfilled by a logical theory, such that there is a two-sided relation between the results of the calculus upon the field of such theory over its objects using its operators, and on the other side, the meanings of them represented in the natural or professional language. The interchange between the logical theory calculus and the natural or professional language representation should be transparent, but not necessarily isomorphic, to represent the knowledge. This is a sort of generalization of

Knowledge Engineering. This is also one way to construct a transdisciplinary logical theory."

We aim to obtain interpretable fuzzy theories according to above definition. The interpretability of the Compensatory Fuzzy Logic (CFL) is a starting point for this goal. The ACFL is a theoretical combination of two different logics; it combines two logics according to the order and diverse meanings. It is a way to link the classical definitions of t-norm and t-conorm with CFL.

Logic Pluralism was a precedent of Scientific Pluralism, which began with particular works of the Vienna Circle [12] and, lastly, have reemerged in a new way and following those pioneer developments [2]. Bivalued Logics like Boolean Logic, Intuitionistic Logic, Dual Logic, and other approaches are different logic, satisfying different logical values. Those logical approaches have a meaningful presence in Fuzzy Logic. The understanding of the group or individual logic as the specific way of thinking and reasoning of different groups and persons is another outlook of Pluralism in Logics. Another important manifestation of Pluralism in Logics is Contextual Pluralism; it considers logic should depend on the content of propositions and predicates with which logic works.

CFL emerged with a transdisciplinary axiomatic approach, different from the Classical Norm and Conorm approach to improving interpretability by natural language [13]. Lastly, ACFL was introduced, harmonizing both connectives sets by properties and different interpretations of the truth values [8].

The unified treatment of modifiers, membership functions, and Continue Linguistic Variables treated in this paper as generalizations and useful correspondent propositions enhance that theory, joining the associated to each ACFL logic function and Archimedean and Compensatory connectives with these other relevant elements of Fuzzy Logic. That is a new element that does not present till now in Fuzzy Logic. That property is joined to the already mentioned one, which explains the apparent contradiction of CFL with the classical approach of Norm and Conorm of Fuzzy Logic.

A very general approach for KD is obtained where predicates of fuzzy logic are discovered, including parameters concerning the membership functions and the specific Compensatory Logic and Archimedean Logics. This process is a practical manifestation of the Contextual Pluralism of ACFL, allowing the selection of the particular ACFL with the truer universal proposition of a predicate [14, 15].

These two properties allow us to call ACFL the first Pluralist Fuzzy Logic theory, a significant name in the context of Knowledge Discovery, to concrete the Tolerance Principle of Carnap, the initiator of the Logic Pluralism and the principal creator of Logical Positivism.

This paper is organized as follows: Sect. 2 summarizes the main concepts further used. Section 3 presents the main results. Section 4 introduces an ACFL based on the Exponential Logarithmic function (ACFL-ELF) and illustrates the application of the theory in Knowledge Discovery using it with different datasets. Finally, in Sect. 5, we state the conclusions.

2 Preliminaries

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ the extended real line of the extended real numbers [16].

Definitions A t-norm is a function T: $[0,1]\times[0,1] \to [0,1]$ having the following properties [17]:

(i). **Commutativity** $T(x,y) = T(y,x)$.
(ii). **Monotonicity (increasing)** $T(x,y) \leq T(u,v)$, if $x \leq u$ and $y \leq v$.
(iii). **Associativity** $T(x,T(y,z)) = T(T(x,y),z)$.
(iv). **One as a neutral element** $T(x,1) = x$.

A well-known property of t-norms is $T(x,y) \leq \min(x,y)$.

Let $g_T : [0,1]\times\mathbb{N} \to [0,1]$ an auxiliary recursive operator associated with the t-norm T, defined by.

$$g_T(x,n) = \begin{cases} x & \text{for } n = 1 \\ T(x, g_T(x,n-1)), & \text{for } n > 1. \end{cases}$$

For simplicity, we will symbolize $g_T(x,n) = T\underbrace{(x,x,x,\ldots,x)}_{n-\text{times}}$, for $n > 1$.

Definitions A t-norm $T : [0,1]^2 \to [0,1]$ is said to be Archimedean if and only if it satisfies for every $(x,y) \in (0,1)^2$ there is a natural number n such that $T\underbrace{(x,x,\ldots,x)}_{n\,\text{times}} < y$.

It can be observed that:

(1) If $T : [0,1]^2 \to [0,1]$ t-norm Archimedean, then for every $x \in (0,1), T(x,x) < x$.
(2) Let be $T : [0,1]^2 \to [0,1]$ t-norm. $T : [0,1]^2 \to [0,1]$ t-norm Archimedean if and only if it satisfies for every
$$x \in (0,1),\ \lim_{n\to\infty} T\left(\underbrace{x,x,x,\ldots,x}_{n-\text{times}}\right) = 0.$$

A t-norm $T : [0,1]^2 \to [0,1]$ is continuous if and only if for every pair of convergent sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in $[0,1]$, $\lim_{n\to\infty} T(x_n,y_n) = T\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right)$.

For every continuous Archimedean t-norm there exists a continuous decreasing function $f : [0,1] \to [0,+\infty]$ satisfying $f(1) = 0$, such that [17]:

$$T(x_1,\ldots,x_n) = f^{(-1)}\left(\sum_{i=1}^{n} f(x_i)\right), \quad (1)$$

where $f^{(-1)}$ is the pseudo inverse of f, and $L = \lim_{x\to 0^+} f(x)$,

$$f^{(-1)}(z) = \begin{cases} f^{-1}(z), & \text{if } z \in [f(1), L) \\ 0 & \text{if } z \in [L, +\infty] \end{cases}. \quad (2)$$

There are operators of the form [18]:

$$M_f(x_1,\ldots,x_n) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right). \quad (3)$$

In (3), if f is a strictly monotone continuous function in the real extended line, it corresponds to the family of the quasi-arithmetic means, where f is the additive *generator* (f is not unique) [18, 19]. This family has been studied in detail by Kolmogorov [20] and Aczél and Alsina [21].

In (1), if f is strictly decreasing and continuous function,

$$T(x_1,\ldots,x_n) = f^{(-1)}\left(n.f\left(M_f(x_1,\ldots,x_n)\right)\right)$$

and

$$M_f(x_1,\ldots,x_n) = f^{-1}\left(\frac{1}{n} f(T(x_1,\ldots,x_n))\right).$$

Considering $f(x) = -\ln(x)$, $f^{(-1)}(z) = \begin{cases} e^{-z}, & \text{if } z \in [0,+\infty) \\ 0, & \text{if } z = +\infty \end{cases}$, $T(x_1,\ldots,x_n) = \prod_{i=1}^{n} x_i$ and $M_f(x_1,\ldots,x_n) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$ as the geometric mean. And further $f(x) = \frac{1}{x} - 1$, $f^{(-1)}(z) = \begin{cases} \dfrac{1}{z+1}, & \text{if } z \in [0,+\infty) \\ 0, & \text{if } z = +\infty \end{cases}$, $T(x_1,\ldots,x_n) = \frac{1}{\sum_{i=1}^{n} \frac{1}{x_i} - n + 1}$ and $M_f(x_1,\ldots,x_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$ as the harmonic mean.

We say $L = (c_c, c_T, d_c, d_T, o, n)$ is an ACFL if we count on the following fuzzy operators [8]:

(1) $c_T : [0,1]^2 \to [0,1]$ is an Archimedean t-norm generated by some f which fulfills Eq. 1.

(2) $d_T : [0,1]^2 \to [0,1]$ satisfies $d_T(x_1,x_2) = 1 - c_T(1-x_1, 1-x_2)$ for every $x = (x_1,x_2) \in [0,1]^2$.

(3) $c_c : [0,1]^n \to [0,1]$ satisfies Eq. 3 for f used in point 1.

4) $d_c : [0,1]^n \to [0,1]$, where for a vector $x = (x_1,x_2,\ldots,x_n) \in [0,1]^n$, $d_c(x_1,x_2,\ldots,x_n) = 1 - c_c(1-x_1, 1-x_2, \ldots, 1-x_n)$ and $c_c(x_1,x_2,\ldots,x_n) = 1 - d_c(1-x_1, 1-x_2, \ldots, 1-x_n)$.

(5) $o : [0,1]^2 \to [0,1]$ is a fuzzy order, $o(x,y) = 0.5(c_c(x) - c_c(y)) + 0.5$.

(6) $n : [0,1] \to [0,1]$ is the negation operator with equation $n(x) = 1 - x$.

Let be L an ACFL. The universal quantifier is defined as the conjunction c_T. That is to say, for $X = \{x_1,\ldots,x_n\} \subset [0,1]$, $\forall_T x_i \in X = c_T(x_1,\ldots,x_n)$ and equivalently, $\forall_c x_i \in X = c_c(x_1,x_2,\ldots,x_n)$.

The compensatory operators satisfy the following axioms [9]:

(i). **Compensation** $\min(x_1,x_2,\ldots,x_n) \le c_c(x_1,x_2,\ldots,x_n) \le \max(x_1,x_2,\ldots,x_n)$.

(ii). **Symmetry or Commutativity** $c_c(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n) = c_c(x_1,x_2,\ldots,x_j,\ldots,x_i,\ldots,x_n)$.

(iii). **Strict Growth** If $x_1 = y_1$, $x_2 = y_2$, …, $x_{i-1} = y_{i-1}$, $x_{i+1} = y_{i+1}$,…, $x_n = y_n$ are different to zero and $x_i > y_i$ then $c_c(x_1,x_2,\ldots,x_n) > c_c(y_1,y_2,\ldots,y_n)$.

(iv). **Veto**: If $x_i = 0$ for any i then $c_c(x) = 0$.

(v). **Fuzzy Reciprocity** $o(x,y) = n[o(y,x)]$.

(vi). **Fuzzy Transitivity** If $o(x,y) \ge 0.5$ and $o(y,z) \ge 0.5$, then $o(x,z) \ge \max(o(x,y), o(y,z))$.

(vii). **De Morgan's Laws**

$$n(c_c(x_1,x_2,\ldots,x_n)) = d_c(n(x_1), n(x_2), \ldots, n(x_n)),$$

$$n(d_c(x_1,x_2,\ldots,x_n)) = c_c(n(x_1), n(x_2), \ldots, n(x_n)).$$

Other properties satisfied by an ACFL are the following:

1. **Compensation** $\min(x_1,x_2,\ldots,x_n) \le d_c(x_1,x_2,\ldots,x_n) \le \max(x_1,x_2,\ldots,x_n)$.
2. **Symmetry or Commutativity** $d_c(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n) = d_c(x_1,x_2,\ldots,x_j,\ldots,x_i,\ldots,x_n)$.
3. **Strict Growth** If $x_1 = y_1$, $x_2 = y_2$, …, $x_{i-1} = y_{i-1}$, $x_{i+1} = y_{i+1}$,…, $x_n = y_n$ are different to one and $x_i > y_i$ then $d_c(x_1,x_2,\ldots,x_n) > d_c(y_1,y_2,\ldots,y_n)$.
4. **Veto** If $x_i = 1$ for any i then $d_c(x) = 1$.

Definition Let ***f(x)*** a **strictly decreasing** real-valued function, $X = \{x_1,\ldots,x_n\} \subset [0,1]$. *f*(*x*) generates an ACFL L if *f*(*x*) generates the correspondent conjunctions c_T and d_T.

Proposition 1 *Let **f(x)** a **strictly decreasing** real-valued function, which generates an ACFL L.* $X = \{x_i \in [0,1], \text{for} i = 1,2,\ldots,n\}$.

If ***p*** and ***q*** are **predicates** that can be evaluated in vectors with components in X, then it is verified, $\forall_{\mathrm{T}} \boldsymbol{x}_{\mathrm{k}} p(\boldsymbol{x}_{\mathrm{k}}) \leq \forall_{\mathrm{T}} \boldsymbol{x}_{\mathrm{k}} q(\boldsymbol{x}_{\mathrm{k}})$ if and only if $\forall_{\mathbf{c}} \boldsymbol{x}_{\mathbf{k}} \boldsymbol{p}(\boldsymbol{x}_{\mathbf{k}}) \leq \forall_{\mathbf{c}} \boldsymbol{x}_{\mathbf{k}} \boldsymbol{q}(\boldsymbol{x}_{\mathbf{k}})$, where k is the index for the set of a vector with components in X.

3 Generalizations in the Framework of ACFL

This section contains concepts from ACFL's theory.

3.1 Generalized Linguistic Modifier

Definition 1 Given an ACFL L and its **function generator** f, we shall define a ***Generalized Linguistic Modifier*** $m(x, L)$ by the following equation: $x_L^a = f^{(-1)}(af(x))$, where $a \in \mathbb{R}^+$ and x_L^a **denotes** $m(x, L)$.

Let $f(x) = -g(\ln(x))$, for $x \in (0, 1]$, in the expression of *Generalized Linguistic Modifier. We* will call g ***Secondary Generator Function*** **of** L. In which case, $g(u) = -f(e^u)$ for $u \in (-\infty, 0]$, $g^{-1}(z) = \ln(f^{-1}(-z))$. If $a > 0$ and $x = 1$, then $f(x) = 0$, $x_L^a = f^{(-1)}(0) = 1$ and $g^{-1}(0) = 0$.

Because $f(x)$ is bounded and taking into account $\lim\limits_{a \to 0^+} f^{(-1)}(af(x)) = 1$, we define $x_L^0 = 1$.

Taking into account (2) for $L = +\infty$ and extending the definition of the pseudo inverse of g such that an odd function results in the extended real line, we have:

$$g^{(-1)}(z) = \begin{cases} \ln(f^{-1}(-z)), & \text{if } z \in (-\infty, 0] \\ -\ln(f^{-1}(z)), & \text{if } z \in (0, +\infty) \\ -\infty, & \text{if } z = -\infty \\ +\infty, & \text{if } z = +\infty. \end{cases} \tag{4}$$

Remark 1 Let us observe that x_L^a generalizes de equation: $c_T\underbrace{(x, x, \ldots, x)}_{n\,\text{times}} = f^{(-1)}\underbrace{(f(x) + f(x) + \cdots + f(x))}_{n\,\text{times}} = f^{(-1)}(nf(x))$, where $n \in \mathbb{N}$.

Supposing $x, a \in \mathbb{R}^+$, $0 < x \leq 1$, and odd n if $g(x) = x^n$ and $g^{-1}(x) = \sqrt[n]{x}$ then $f(x) = -(\ln(x))^n$, $f^{-1}(x) = e^{-\sqrt[n]{x}}$ and $x_L^a = f^{-1}(af(x)) = e^{a^{\frac{1}{n}} \cdot \ln(x)}$. In Fig. 1 it is observed $x_L^a = f^{-1}(af(x)) \to 1$, when $a \to 0^+$. While $x_L^a = f^{-1}(af(x)) \to x$, when $a \to 1^-$.

Another case is for $x, a, b \in \mathbb{R}^+$, $0 < x \leq 1$ and $b > 1$ and odd natural n. If $g(x) = \frac{x^n}{(\ln(b))^n}$, $g^{-1}(x) = \ln(b).\sqrt[n]{x}$ then $f(x) = -(\log_b(x))^n$, $f^{-1}(x) = e^{-\ln(b)\sqrt[n]{x}} = b^{-\sqrt[n]{x}}$ and $x_L^a = f^{-1}(af(x)) = x^{\sqrt[n]{a}}$ does not depend on b. It is also true $x_L^a = f^{-1}(af(x)) \to 1$, when $a \to 0^+$. While $x_L^a = f^{-1}(af(x)) \to x$, when $a \to 1^-$.

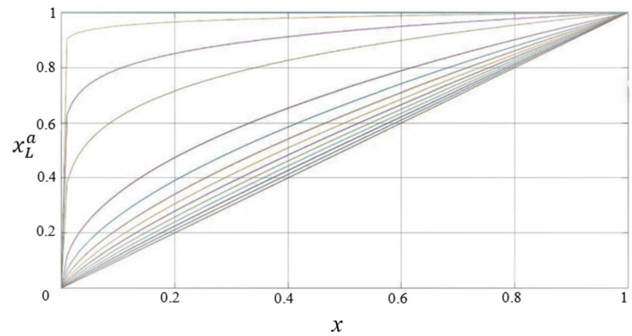

Fig. 1 Graphs $x_L^a = e^{a^{\frac{1}{n}} \cdot \ln(x)}$, for $n = 3$, and different values of positive a and $a \to 0^+$

3.2 Generalized Sigmoidal Function

Definition 2 Let the ACFL L and g **its Secondary Generator Function**. We say that $S_g(x) = \frac{1}{1 + e^{-g^{(-1)}(x)}}$ is a *Generalized Sigmoidal Function.*

It is also well-known the theory of sigmoidal membership functions that we shall use further [4–6].

Definition 3 Let the ACFL L and g its Secondary Generator Function. We say $S_g(x; \alpha, \gamma) = \frac{1}{1 + e^{-g^{(-1)}(\alpha(x - \gamma))}}$ is a ***Parameterized Generalized Sigmoidal Function***.

3.3 Generalized Continuous Linguistic Variables

Definition 4 The ***Generalized Continuous Linguistic Variable***, with parameters $\alpha, \gamma \in \mathbb{R}$, $\alpha > 0$, and $m \in [0, 1]$, generated by the secondary generator function g of an ACFL L, is defined as follows:

$$GCLV_L(x; \alpha, \gamma, m) = \frac{C_T\left(S_g(x; \alpha, \gamma)_L^m, \left(1 - S_g(x; \alpha, \gamma)\right)_L^{1-m}\right)}{M} \tag{5}$$

where c_T is a continuous Archimedean t-norm in L, and M is the maximum of $c_T\left(S_g(x; \alpha, \gamma)_L^m, \left(1 - S_g(x; \alpha, \gamma)\right)_L^{1-m}\right)$ in $\mathbb{R}$. $S_g(x; \alpha, \gamma)_L^m$ and $\left(1 - S_g(x; \alpha, \gamma)\right)_L^{1-m}$ are the generalized linguistic modifiers over $S_g(x; \alpha, \gamma)$ and $1 - S_g(x; \alpha, \gamma)$, respectively.

Taking into account $f(x) = -g(\ln(x))$ is the generator of L, we shall define the following definition.

Definition 5 Let L an ACFL and g its secondary generator function, then the t-norm generated by g is the equation:

$$\mathrm{c_T}(x_1, x_2) = e^{-g^{(-1)}(-g(\ln(x_1)) - g(\ln(x_2)))}. \tag{6}$$

From Definition 5 and recalling the definition of L we state the following:

$$d_T(x_1, x_2) = 1 - e^{-g^{(-1)}(-g(\ln(1-x_1)) - g(\ln(1-x_2)))}, \tag{7}$$

$$c_c(x_1, x_2, \ldots, x_n) = e^{-g^{(-1)}\left(\frac{-\sum_{i=1}^{n} g(\ln(x_i))}{n}\right)}, \tag{8}$$

and

$$d_c(x_1, x_2, \ldots, x_n) = 1 - e^{-g^{(-1)}\left(\frac{-\sum_{i=1}^{n} g(\ln(1-x_i))}{n}\right)}. \tag{9}$$

When it makes sense, and $g^{(-1)}$ can be differentiated with a continuous derivate, it is easy to check that the ***Parameterized Generalized Sigmoidal Function*** **satisfies** the following ordinary differential equation:

$$\frac{dS_g(x;\alpha,\gamma)}{dx} = \alpha.\frac{dg^{(-1)}}{dx}(\alpha(x-\gamma)).S_g(x;\alpha,\gamma).(1 - S_g(x;\alpha,\gamma)). \tag{10}$$

Although we defined the secondary generator function g and many other terms based on this function without a demonstration, we will prove that these definitions make sense in the propositions below.

Proposition 2 *Given $g^{-1}: \mathbb{R} \to \mathbb{R}$ an odd, unbounded, strictly increasing function, continuous and concave in $\mathbb{R}^+$, such that $g: [0,1] \to \mathbb{R}$ exists. They are sufficient and necessary conditions to the following two properties:*

(1) $S_g(x;\alpha,\gamma) = \frac{1}{1+e^{-g^{-1}(\alpha(x-\gamma))}}$ *is a sigmoidal function, i.e.,* $\lim_{x\to+\infty} S_g(x;\alpha,\gamma) = 1$, $\lim_{x\to-\infty} S_g(x;\alpha,\gamma) = 0$, *it is increasing in $\mathbb{R}$; $S_g(x) = S_g(x;1,0)$ concave in $\mathbb{R}^+$, convex in $\mathbb{R}^-$ and symmetric to (0,0.5).*

(2) $f(x) = -g(\ln(x))$ *is the inverse of* $f^{-1}(x) = e^{-g^{-1}(x)}$ *and generator of a continuous Archimedean t-norm.*

Proof Firstly, we shall prove the proposition for $S_g(x) = \frac{1}{1+e^{-g^{-1}(x)}}$.

Considering that g^{-1} is an odd function and, therefore, also g, $f(1) = -g(\ln(1)) = -g(0) = 0$.

The strictly increasing, unbounded, and continuity of g^{-1} implies g is strictly increasing and continuity in $\mathbb{R}$, particularly in [0, 1]. g is strictly increasing in [0, 1] and $f(1) = 0$, then $f([0,1]) \subseteq [0,+\infty]$. Hence, f is a generator of a t-norm.

Besides, from the oddness of g^{-1}, $e^{-g^{-1}(x)}e^{-g^{-1}(-x)} = e^{-(g^{-1}(x)+g^{-1}(-x))} = 1$. This is equivalent to the condition $\frac{1}{1+e^{-g^{-1}(x)}} + \frac{1}{1+e^{-g^{-1}(-x)}} = 1$, which means that $S_g(x)$ is symmetric respect to (0,0.5).

Let us recall that g^{-1} is not bounded and continuous, then, $\lim_{x\to+\infty} S_g(x) = \lim_{x\to+\infty} \frac{1}{1+e^{-g^{-1}(x)}} = \frac{1}{1+e^{-\lim_{x\to+\infty} g^{-1}(x)}} = 1$. Similarly, we can demonstrate that $\lim_{x\to-\infty} S_g(x) = 0$.

It is easy to prove that this result can be generalized to the parameterized sigmoidal function $S_g(x;\alpha,\gamma) = \frac{1}{1+e^{-g^{-1}(\alpha(x-\gamma))}}$.

Let us note that g^{-1} is convex in $\mathbb{R}^-$ because it is concave in $\mathbb{R}^+$ and odd.

The convexity of g^{-1} in $\mathbb{R}^-$, considering that the function $y = \frac{1}{1+e^{-x}}$ is also convex in $\mathbb{R}^-$ and that they are increasing, as a consequence, that the composition $S_g(x)$ is also convex in $\mathbb{R}^-$.

The concavity of $S_g(x)$ in $\mathbb{R}^+$ is a consequence of the oddness of g^{-1} and its convexity in $\mathbb{R}^-$.

Now, let us suppose the necessary conditions are satisfied. $f(x) = -g(\ln(x))$ and $f(x)$ is a generator of a continuous Archimedean t-norm if $f(x)$ is decreasing and $f(1) = 0$. Therefore, g has to be increasing and $g(0) = 0$. Hence, $g^{-1}(0) = 0$ and $g^{-1}(x)$ are increasing. Moreover, $g^{-1}(x)$ is strictly increasing and continuous because of the definition of t-norm generators.

$S_g(x)$ is symmetric to the point (0,0.5) if $g^{-1}(x)$ is odd. Note that $\frac{1}{1+e^{-g^{-1}(x)}} + \frac{1}{1+e^{-g^{-1}(-x)}} = 1$ is the condition of symmetry to the point (0,0.5), and it is equivalent to $e^{-(g^{-1}(x)+g^{-1}(-x))} = 1$ for every $x \in \mathbb{R}$ or $g^{-1}(x) + g^{-1}(-x) = 0$, which means that g^{-1} is odd.

The existence of g is a consequence of property 2.

$\lim_{x\to+\infty} S_g(x) = 1$ and $\lim_{x\to-\infty} S_g(x) = 0$ imply $g^{-1}(x)$ is unbounded.

Finally, R+'s concavity has to be satisfied in an S-shaped function, taking into account g^{-1} is odd and hence $S_g(x)$ is convex in $\mathbb{R}^-$.

Now, we shall restrict our attention to $g(x)$ satisfying the conditions in Proposition 2.

From the above, it follows that the graph $S_g(x;\alpha,\gamma)$ is concave downward in $(\gamma,+\infty)$, concave up inwards $(-\infty,\gamma)$ and symmetric concerning point $(\gamma, 0.5)$.

Let us remark that some consequences of Definition 4 are: $GCLV_L(x;\alpha,\gamma,1) = S_g(x;\alpha,\gamma)$ and $GCLV_L(x;\alpha,\gamma,0) = 1 - S_g(x;\alpha,\gamma)$.

In the ACFL framework, the generalization of the linguistic modifier's definitions, the sigmoidal function, and the Generalized Continuous Linguistic Variable increase interpretability.

Let us remark that Definition 3 under the conditions of Proposition 2 is also a generalization of the sigmoidal functions as a function of a distance, see [20]. $u(x) = \frac{1}{1+d(x,x_0)}$ is another point of view to understand this membership function, where $d(x,x_0)$ is the distance of the points to certain point of reference x_0, $d(x, x_0) = 0$ if $x = x_0$ and $d(x, x_0) = +\infty$ if $u(x) = 0$. The expression $d(x, x_0)$ can be thought of as the result of the composition $h(x - x_0)$, where $h(x) = |x|$ non-negative function. Considering $x_0 = \gamma$, in classical sigmoid function $h(x) = e^{-\alpha x}$ and in the generalization $h(x) = e^{-g^{-1}(\alpha x)}$.

Another example is due to Dombi, see [3], where $d(x) = \left(\frac{\nu}{1-\nu}\right)^{1-\lambda}\left(\frac{b-x}{x-a}\right)^{\lambda}$ (distance between a given object and a standard (ideal)), $u(x) = \frac{(1-\nu)^{1-\lambda}(x-a)^{\lambda}}{(1-\nu)^{1-\lambda}(x-a)^{\lambda} + \nu^{\lambda-1}(b-x)^{\lambda}}, \lambda > 1$, (a,b) is the interval on which $u(x) > 0$, λ is the sharpness parameter, and ν is the expectation level.

In the following, we illustrate the proposed theory with some examples.

3.4 Examples

Example 1 If $f(x) = -\ln(x)$ ifs given, then $f^{-1}(x) = e^{-x}$ and $g(x) = g^{-1}(x) = id(x)$, which generates the ACFL L, such that $c_T(x_1, x_2) = x_1 x_2$ is the product t-norm and $d_T(x_1, x_2) = x_1 + x_2 - x_1 x_2$ is its dual t-conorm. c_c is the geometric mean and $d_c(x_1, x_2, \ldots, x_n) = 1 - c_c(1 - x_1, 1 - x_2, \ldots, 1 - x_n)$.

If α = 10 and γ = 5 then $S_g(x; 10, 5) = \frac{1}{1+e^{-10(x-5)}}$ and

$$GCLV_L(x; 10, 5, 0.2) = \frac{S_g^{0.2}(x; 10, 5)\left(1 - S_g(x; 10, 5)\right)^{0.8}}{\max_{x\in\mathbb{R}}\left[S_g^{0.2}(x; 10, 5)\left(1 - S_g(x; 10, 5)\right)^{0.8}\right]}$$

$$GCLV_L\left(x; 10, 5, \frac{1}{2}\right) = \frac{\sqrt{S_g(x; 10, 5)\left(1 - S_g(x; 10, 5)\right)}}{\max_{x\in\mathbb{R}}\left[\sqrt{S_g(x; 10, 5)\left(1 - S_g(x; 10, 5)\right)}\right]}$$

$$GCLV_L(x; 10, 5, 0.8) = \frac{S_g^{0.8}(x; 10, 5)\left(1 - S_g(x; 10, 5)\right)^{0.2}}{\max_{x\in\mathbb{R}}\left[S_g^{0.8}(x; 10, 5)\left(1 - S_g(x; 10, 5)\right)^{0.2}\right]}$$

Let us note $g(x)$ satisfies the conditions of Proposition 2, and the generalized linguistic modifier coincides with the classical power function (Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11).

See the two figures below:

Example 2 For $g^{-1}(x) = \sqrt[3]{x}$, $g(x) = x^3$, $f(x) = -\ln^3(x)$, $f^{-1}(x) = e^{-\sqrt[3]{x}}$.

$c_T(x_1, x_2) = e^{\sqrt[3]{\ln^3(x_1) + \ln^3(x_2)}}$, $d_T(x_1, x_2) = 1 - e^{\sqrt[3]{\ln^3(1-x_1) + \ln^3(1-x_2)}}$, $c_c(x_1, x_2, \ldots, x_n) = e^{\sqrt[3]{\frac{\sum_{i=1}^{n} \ln^3(x_i)}{n}}}$ and $d_c(x_1, x_2, \ldots, x_n) = 1 - e^{\sqrt[3]{\frac{\sum_{i=1}^{n} \ln^3(1-x_i)}{n}}}$.

$$S_g(x; 10, 5) = \frac{1}{1 + e^{-\sqrt[3]{10(x-5)}}}.$$

$x_L^m = f^{-1}(mf(x)) = e^{\sqrt[3]{m \ln^3(x)}} = x^{\sqrt[3]{m}}$, $x_L^{1-m} = x^{\sqrt[3]{1-m}}$ and hence:

$$GCLV_L(x; 10, 5, 0.2) = \frac{e^{\sqrt[3]{\ln^3\left(S_g(x;10,5)^{\sqrt[3]{0.2}}\right) + \ln^3\left(\left(1 - S_g(x;10,5)\right)^{\sqrt[3]{0.8}}\right)}}}{\max_{x\in\mathbb{R}}\left[e^{\sqrt[3]{\ln^3\left(S_g(x;10,5)^{\sqrt[3]{0.2}}\right) + \ln^3\left(\left(1 - S_g(x;10,5)\right)^{\sqrt[3]{0.8}}\right)}}\right]}$$

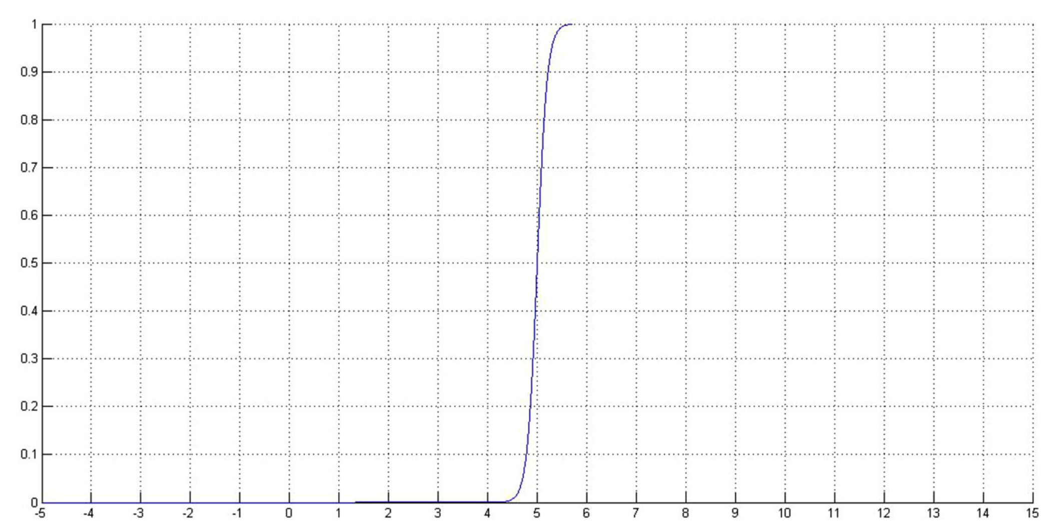

Fig. 2 Graph of a Parameterized Generalized Sigmoidal Function based on $g(x) = \mathrm{id}(x)$ or $GCLV_L(x; 10, 5, 1)$

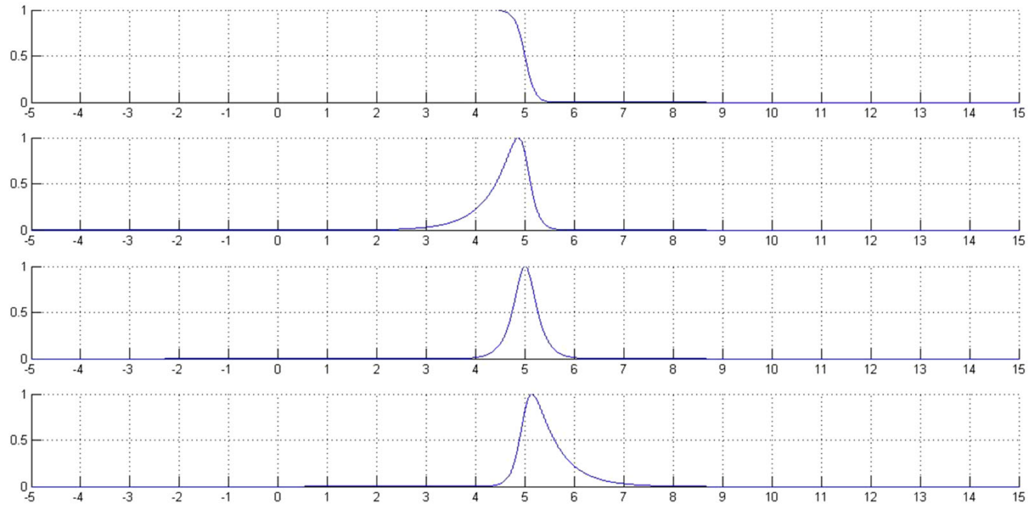

Fig. 3 From top to bottom, graphs of $GCLV_L(x;10,5,0)$, $GCLV_{\mathrm{L}}(x;10,5,0.2)$, $GCLV_L\left(x;10,5,\frac{1}{2}\right)$ and $GCLV_L(x;10,5,0.8)$, based on $g(x) = \mathrm{id}(x)$

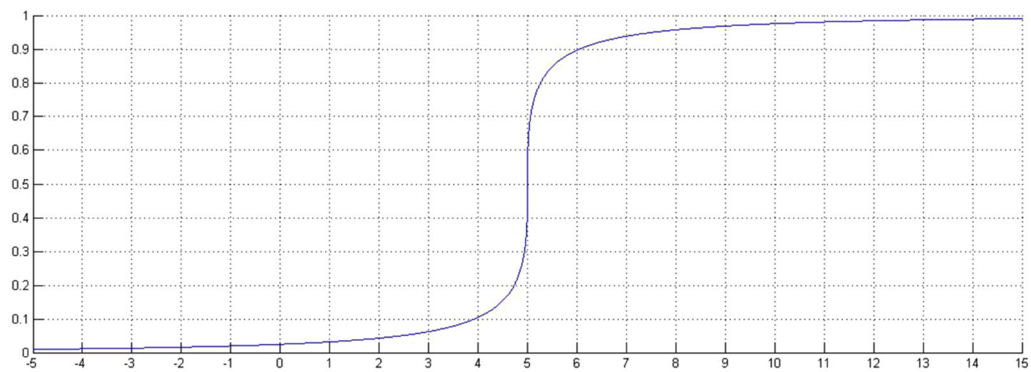

Fig. 4 Graph of a Parameterized Generalized Sigmoidal Function based on $g(x) = x^3$ or $GCLV_L(x;10,5,1)$

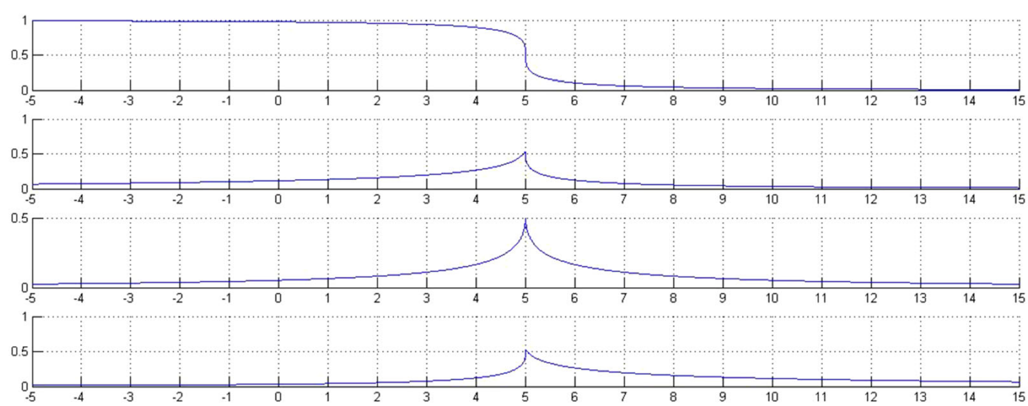

Fig. 5 From top to bottom, graphs of $GCLV_L(x;10,5,0)$, $GCLV_{\mathrm{L}}(x;10,5,0.2)$, $GCLV_L\left(x;10,5,\frac{1}{2}\right)$ and $GCLV_L(x;10,5,0.8)$, based on $g(x) = x^3$

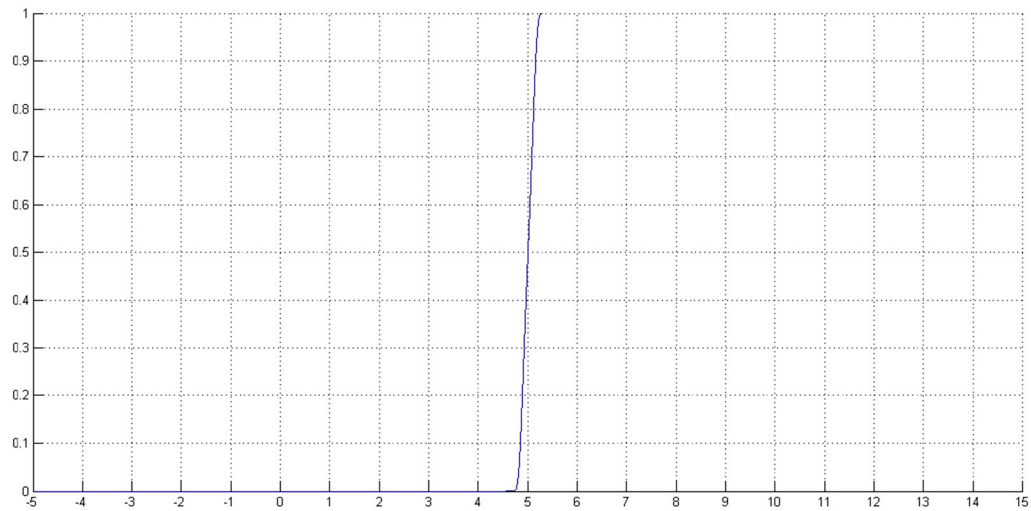

Fig. 6 Graph of a Parameterized Generalized Sigmoidal Function based on $g(x) = \operatorname{arcsinh}(x)$ or $GCLV_L(x; 10, 5, 1)$

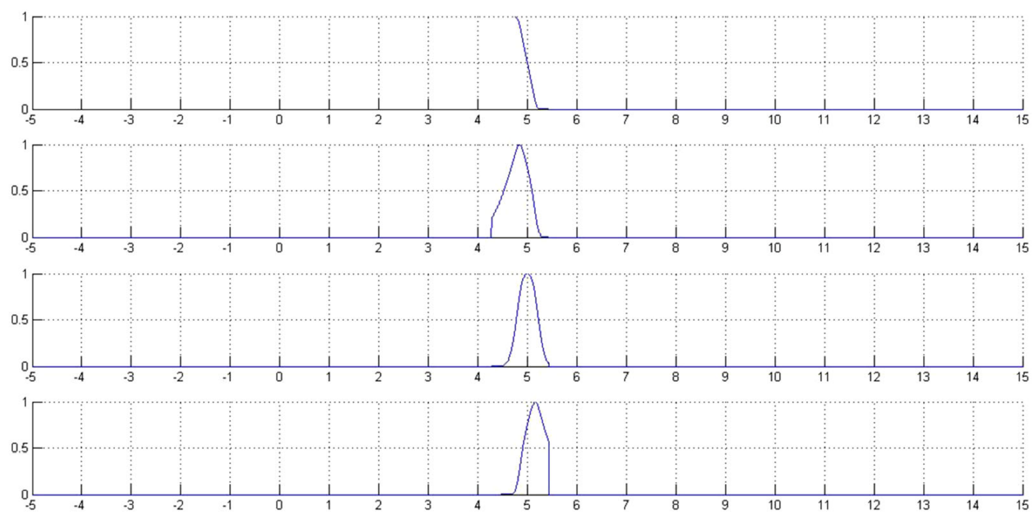

Fig. 7 From top to bottom, graphs of $GCLV_L(x; 10, 5, 0)$, $GCLV_L(x; 10, 5, 0.2)$, $GCLV_L\left(x; 10, 5, \frac{1}{2}\right)$ and $GCLV_L(x; 10, 5, 0.8)$, based on $g(x) = \operatorname{arcsinh}(x)$

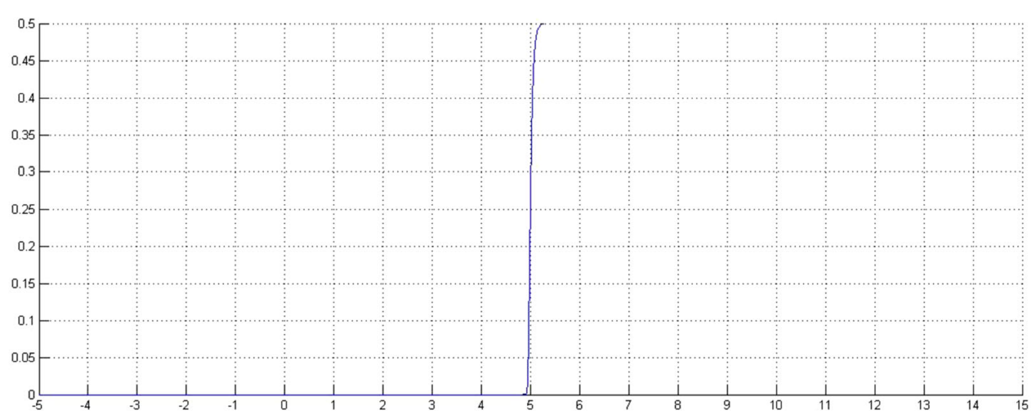

Fig. 8 Graph of a Parameterized Generalized Sigmoidal Function based on $g(x) = \frac{x}{\ln(10)}$ or $GCLV_L(x; 10, 5, 1)$

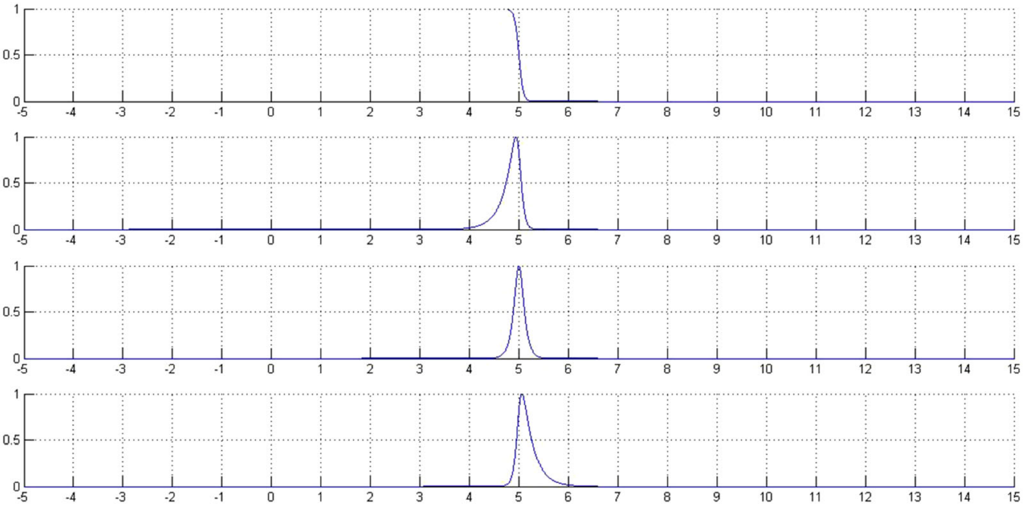

Fig. 9 From top to bottom, graphs of $GCLV_L(x;10,5,0)$, $GCLV_L(x;10,5,0.2)$, $GCLV_L\left(x;10,5,\frac{1}{2}\right)$ and $GCLV_L(x;10,5,0.8)$, based on $g(x)=\frac{x}{\ln(10)}$

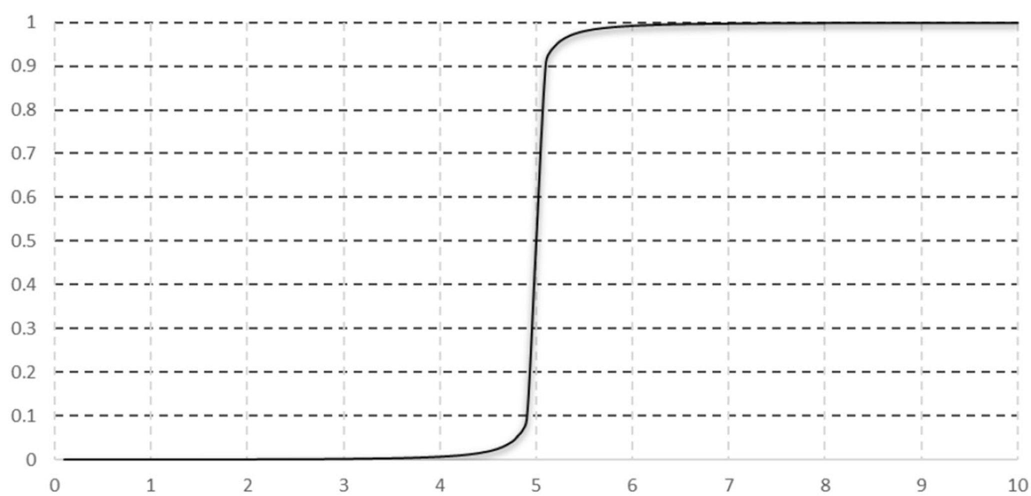

Fig. 10 Graph of a Parameterized Generalized Sigmoidal Function based on $g(x)=\frac{x^n}{\ln^n(b)}$, where $b=10$ and $n=3$ or $GCLV(x;10,5,1)$

$$GCLV_L\left(x;10,5,\frac{1}{2}\right) = \frac{e^{\sqrt[3]{\ln^3\left(S_g(x;10,5)^{\sqrt[3]{1/2}}\right)+\ln^3\left(\left(1-S_g(x;10,5)\right)^{\sqrt[3]{1/2}}\right)}}}{\max_{x\in\mathbb{R}}\left[e^{\sqrt[3]{\ln^3\left(S_g(x;10,5)^{\sqrt[3]{1/2}}\right)+\ln^3\left(\left(1-S_g(x;10,5)\right)^{\sqrt[3]{1/2}}\right)}}\right]}$$

$$GCVL_L(x;10,5,0.8) = \frac{e^{\sqrt[3]{\ln^3\left(S_g(x;10,5)^{\sqrt[3]{0.8}}\right)+\ln^3\left(\left(1-S_g(x;10,5)\right)^{\sqrt[3]{0.2}}\right)}}}{\max_{x\in\mathbb{R}}\left[e^{\sqrt[3]{\ln^3\left(S_g(x;10,5)^{\sqrt[3]{0.8}}\right)+\ln^3\left(\left(1-S_g(x;10,5)\right)^{\sqrt[3]{0.2}}\right)}}\right]}$$

In general, secondary generator functions with formula $g^{-1}(x)=x^{1/n}$ are valid, where n is an odd natural number.

Example 3 For $g^{-1}(x)=\sinh(x)$, $g(x)=\operatorname{arcsinh}(x)$, $f(x)=-\operatorname{arcsinh}(\ln(x))$, $f^{-1}(x)=e^{-\sinh(x)}$.

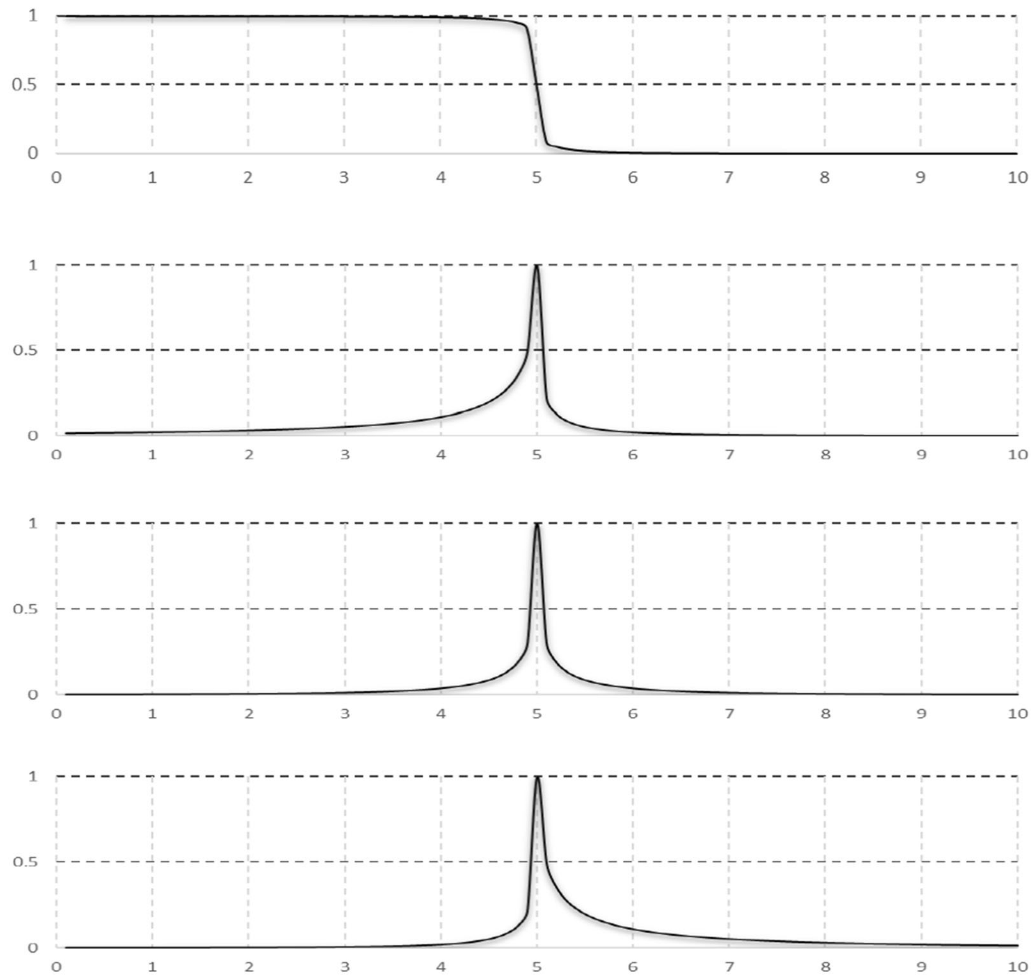

Fig. 11 From top to bottom, graphs of $GCLV_L(x;10,5,0)$, $GCLV_L(x;10,5,0.2)$, $GCLV_L\left(x;10,5,\frac{1}{2}\right)$ and $GCLV_L(x;10,5,0.8)$, based on $g(x)=\frac{x^n}{\ln^n(b)}$

$c_T(x_1,x_2)=e^{\sinh(\operatorname{arcsinh}(\ln(x_1))+\operatorname{arcsinh}(\ln(x_2)))}$, $d_T(x_1,x_2)=1-e^{\sinh(\operatorname{arcsinh}(\ln(1-x_1))+\operatorname{arcsinh}(\ln(1-x_2)))}$, $c_c(x_1,x_2,\cdots,x_n)=e^{\sinh\left(\frac{\sum_{i=1}^n \operatorname{arcsinh}(\ln(x_i))}{n}\right)}$ and $d_c(x_1,x_2,\cdots,x_n)=1-e^{\sinh\left(\frac{\sum_{i=1}^n \operatorname{arcsinh}(\ln(1-x_i))}{n}\right)}$.

$$S_g(x;10,5)=\frac{1}{1+e^{-\sinh(10(x-5))}}.$$

$x_L^m=e^{\sinh(m\operatorname{arcsinh}(\ln(x)))}$, $x_L^{1-m}=e^{\sinh((1-m)\operatorname{arcsinh}(\ln(x)))}$ and therefore,

$$GCLV_L(x;10,5,0.2)=\frac{e^{\sinh\left(\operatorname{arcsinh}\left(\ln\left(S_g(x;10,5)_L^{0.2}\right)\right)+\operatorname{arcsinh}\left(\ln\left(1-S_g(x;10,5)\right)_L^{0.8}\right)\right)}}{\max_{x\in\mathbb{R}}\left[e^{\sinh\left(\operatorname{arcsinh}\left(\ln\left(S_g(x;10,5)_L^{0.2}\right)\right)+\operatorname{arcsinh}\left(\ln\left(1-S_g(x;10,5)\right)_L^{0.8}\right)\right)}\right]}.$$

$$GMMF_L\left(x;10,5,\frac{1}{2}\right)=\frac{e^{\sinh\left(\operatorname{arcsinh}\left(\ln\left(S_g(x;10,5)_L^{0.5}\right)\right)+\operatorname{arcsinh}\left(\ln\left(1-S_g(x;10,5)\right)_L^{0.5}\right)\right)}}{\max_{x\in\mathbb{R}}\left[e^{\sinh\left(\operatorname{arcsinh}\left(\ln\left(S_g(x;10,5)_L^{0.5}\right)\right)+\operatorname{arcsinh}\left(\ln\left(1-S_g(x;10,5)\right)_L^{0.5}\right)\right)}\right]}.$$

$$GMMF_L(x;10,5,0.8)=\frac{e^{\sinh\left(\operatorname{arcsinh}\left(\ln\left(S_g(x;10,5)_L^{0.8}\right)\right)+\operatorname{arcsinh}\left(\ln\left(1-S_g(x;10,5)\right)_L^{0.2}\right)\right)}}{\max_{x\in\mathbb{R}}\left[e^{\sinh\left(\operatorname{arcsinh}\left(\ln\left(S_g(x;10,5)_L^{0.8}\right)\right)+\operatorname{arcsinh}\left(\ln\left(1-S_g(x;10,5)\right)_L^{0.2}\right)\right)}\right]}.$$

Example 4 For $g^{-1}(x)=\ln(10)x$, $g(x)=\frac{x}{\ln(10)}$, $f(x)=-\log_{10}x$ and $f^{-1}(x)=e^{-\ln(10)x}=10^{-x}$.

$c_T(x_1,x_2)=10^{\log_{10}x_1+\log_{10}x_2}=x_1x_2$, $d_T(x_1,x_2)=x_1+x_2-x_1x_2$. Like in Example 1, c_c is the geometric mean $d_c(x_1,x_2,\ldots,x_n)=1-c_c(1-x_1,1-x_2,\ldots,1-x_n)$.

$$S_g(x;10,5) = \frac{1}{1+10^{-10(x-5)}}.$$

$x_L^m = x^m$, $x_L^{1-m} = x^{1-m}$ and therefore,

$$GCLV_L(x;10,5,0.2) = \frac{S_g^{0.2}(x;10,5)\left(1-S_g(x;10,5)\right)^{0.8}}{\max_{x\in\mathbb{R}}\left[S_g^{0.2}(x;10,5)\left(1-S_g(x;10,5)\right)^{0.8}\right]}$$

$$GCLV_L\left(x;10,5,\frac{1}{2}\right) = \frac{\sqrt{S_g(x;10,5)\left(1-S_g(x;10,5)\right)}}{\max_{x\in\mathbb{R}}\left[\sqrt{S_g(x;10,5)\left(1-S_g(x;10,5)\right)}\right]}$$

$$GCLV_L(x;10,5,0.8) = \frac{S_g^{0.8}(x;10,5)\left(1-S_g(x;10,5)\right)^{0.2}}{\max_{x\in\mathbb{R}}\left[S_g^{0.8}(x;10,5)\left(1-S_g(x;10,5)\right)^{0.2}\right]}$$

The finite linear combination of a set of real functions g satisfying conditions in Proposition 2 is itself a Secondary Generator Function. The finite linear combination's scalar values should be non-negative, and at least one of them should not be null.

The composition of a finite set of real functions g satisfying conditions in Proposition 2 is also a Secondary Generator Function.

$g^{-1}(x) = \ln(10)x$ can be generalized to $g^{-1}(x) = \ln(b)x$, where b is a real number, $b > 1$.

Besides, we can interpret the factor $[g^{-1}]'$ from the differential equation $\frac{dS_g(x;\alpha,\gamma)}{dx} = \alpha[g^{-1}]'(\alpha(x-\gamma))S_g(x;\alpha,\gamma)\left(1-S_g(x;\alpha,\gamma)\right)$, as the speed of growth of the sigmoidal function.

If $S_g(x;\alpha,\gamma)$ is a utility function, $\alpha[g^{-1}]'(x)$ is the rate of risk aversion for $x<\gamma$ and of risk propensity for $x>\gamma$.

In Example 1 $[g^{-1}]'(x) = 1$, in Example 2 $[g^{-1}]'(x) = \frac{1}{3}x^{-2/3}$, in Example 3 $[g^{-1}]'(x) = \cosh(x)$ and in Example 4 $[g^{-1}]'(x) = \ln(10)$.

Given W, a linguistic variable over a continuous variable set X. Membership functions can represent W's primary terms and its linguistic modifiers in $GCLV_L(x;\alpha,\gamma,m)$, where L is an ACFL.

The next section contains a solution to a Knowledge Discovery problem to demonstrate the proposed theory's applicability developed so far.

4 Application to Knowledge Discovery

We apply ACFL to knowledge discovery. The relation of each ACFL with modifiers and membership functions makes the discovery and their linguistic interpretation richer.

A generating function is constructed through the composition of functions related to Example 1, Example 2, and Example 4. In this way, a generating function is obtained, which is a composition of exponential and logarithmic functions, generalizing those observed in the examples.

4.1 Compensatory Archimedean Fuzzy Logic Based on an Exponential-Logarithmic Function

The result of this generalization is called an ACFL based on an exponential-logarithmic function (ACFL-ELF). Which hereafter will be denoted L for simplicity.

Be $x,a,b\in\mathbb{R}$, $0<x\le 1$, $b>1$, and n odd natural number.

If $g_1^{-1}(x) = \sqrt[n]{x}, g_1(x) = x^n, f_1(x) = -(\ln(x))^n, f_1^{-1}(x) = e^{-\sqrt[n]{x}}$

and $g_2^{-1}(x) = \ln(b), g_2(x) = \frac{x}{\ln(b)}, f_2(x) = -\log_b(x)$, $f_2^{-1}(x) = e^{-\ln_b x} = b^{-x}$

then a new function g can be defined, as follows: $g(x) = g_1 o g_2(x) = g_1(g_2(x)) = \frac{x^n}{\ln^n(b)}$. Thus, $g^{-1}(x) = \ln(b)\sqrt[n]{x}, f(x) = -(\log_b(x))^n$ and $f^{-1}(x) = b^{-\sqrt[n]{x}}$.

$c_T(x_1,x_2) = b^{\sqrt[n]{\log_b^n(x_1)+\log_b^n(x_2)}}$,

$d_T(x_1,x_2) = 1 - b^{\sqrt[n]{\log_b^n(1-x_1)+\log_b^n(1-x_2)}}$, $c_c(x_1,x_2,\ldots,x_n) = b^{\sqrt[n]{\frac{\sum_{i=1}^n \log_b^n(x_i)}{n}}}$ and $d_c(x_1,x_2,\ldots,x_n) = 1 - b^{\sqrt[n]{\frac{\sum_{i=1}^n \log_b^n(1-x_i)}{n}}}$.

$$S_g(x;10,5) = \frac{1}{1+b^{-\sqrt[n]{(10(x-5)}}}$$

and therefore,

$$GCLV_L(x;10,5,0.2) = \frac{b^{\sqrt[n]{\log_b^n\left(s_g(x;10,5)_L^{0.2}\right)+\log_b^n\left(1-s_g(x;10,5)_L^{0.8}\right)}}}{\max_{x\in\mathbb{R}}\left[b^{\sqrt[n]{\log_b^n\left(s_g(x;10,5)_L^{0.2}\right)+\log_b^n\left(1-s_g(x;10,5)_L^{0.8}\right)}}\right]}.$$

$$GCLV_L\left(x;10,5,\frac{1}{2}\right) = \frac{b^{\sqrt[n]{\log_b^n\left(s_g(x;10,5)_L^{0.5}\right)+\log_b^n\left(1-s_g(x;10,5)_L^{0.5}\right)}}}{\max_{x\in\mathbb{R}}\left[b^{\sqrt[n]{\log_b^n\left(s_g(x;10,5)_L^{0.5}\right)+\log_b^n\left(1-s_g(x;10,5)_L^{0.5}\right)}}\right]}.$$

$$GCLV_L(x;10,5,0.8) = \frac{b^{\sqrt[n]{\log_b^n\left(s_g(x;10,5)_L^{0.8}\right)+\log_b^n\left(1-s_g(x;10,5)_L^{0.2}\right)}}}{\max_{x\in\mathbb{R}}\left[b^{\sqrt[n]{\log_b^n\left(s_g(x;10,5)_L^{0.8}\right)+\log_b^n\left(1-s_g(x;10,5)_L^{0.2}\right)}}\right]}.$$

4.2 Algorithms for ACFL-ELF Logic

We modify the genetic algorithm based on genetic programming (GA-GP) [21], which comprises two genetic algorithms that work together in the predicate discovery process and the predicate optimization. The second

α_1	γ_1	m_1	α_2	γ_2	m_2	α_3	γ_3	m_3	α_4	γ_4	m_4	f	b	e	TV

Fig. 12 Representation structure of an individual from the population with four attributes

algorithm OGCLV optimizes a GCLV membership function defined in that work.

In this case, no predicates are discovered because, for a given dataset, a predicate of type $p \leftrightarrow q$ is built, whose construction structure is described in Sect. 4.3. Furthermore, for each atom of this predicate, the parameters of its corresponding GCLV are optimized. This process is carried out using the modified OGCLV algorithm, called optimization of a GCLV defined through an exponential-logarithmic function (OGCLV-ELF).

Algorithm 1 corresponds to OGCLV-ELF, which estimate the predicate parameters using the optimization process of GA-GP) [21], but changing the individuals' structure used for the evolution and the logic operators used to evaluate a predicate in Lines 12 and 4. The settings used in [21] for the genetic operators are kept in this OGCLV-ELF algorithm.

Algorithm 1 : OGCLV-ELF

```
Input: generations, max_population, predicate.
Output: individual0, which has the predicate highest truth value.
1:  population = random_population( ); // initialize parameters for the predicate
2:  evaluate(population);
3:  for i = 1 to generations // repeat while number of generation is not reached
4:      sort(population); // sort from highest to lowest truth value
5:      for j = 1 to (max_population * 0.95) //cross 95% of the population
6:          if j ≤ (max_population * 0.1) //10% of the best individuals are crossed
7:              kids = cross_deck(individual_j, individual_j+1);
8:          else //crosses 85% of the population randomly
9:              kids = kids ∪ cross_deck(individual_rand, individual_rand);
10:     for j = 1 to (max_population * 0.05) //mutates 5% of the population
11:         kids = kids ∪ mutation(individual_rand);
12:     evaluate(kids);
13:     replace(population, kids) //replace the worst individuals with kids
14: return individual0
```

Algorithm 1: Evolutionary knowledge discovery algorithm.

The representation structure of an individual in OGCLV-ELF is different from the structure in OGCLV. Having a number n of attributes, plus the attribute class, that is part of the predicate. Each attribute i requires three parameters $(\alpha_i, \gamma_i, m_i)$ to define its GCLV. The algorithm can navigate between the different generating functions f. However, for this case, it is fixed in the ACFL-ELF logic, which requires two segments to store the base of a logarithmic function b and an exponential value e. The last segment is used to contain the truth value (TV) of the discovered individual.

In Fig. 12, the construction of an individual with four attributes is represented. The parameters subscript represents the corresponding attribute; besides, each individual of a population is constructed by randomly selecting the values of $\alpha_{\min}^{\max}$, $\gamma_{\min}^{\max}$, and m_0^1.

The calculation of the true value of a predicate requires the logic operators described in Sect. 3.2 and an implication operator, particularly the s-implication $I_s(x,y) = d(n(x),y)$, where d and n are the disjunction and negation operators, respectively. The equivalence operator is also defined as $e(x,y) = c(i(x,y), i(y,x))$, which is valid for any implication and conjunction operator.

Furthermore, the universality quantifier, which allows us to evaluate the truth value of a predicate p of a logic L over the entire data set, is expressed by Eq. 11 and described in Algorithm 2.

$$\forall = f^{-1}\left(\frac{\sum_{i=1}^{n} f(p_i)}{n}\right). \tag{11}$$

In Algorithm 1, for each individual obtained, containing a set of parameters, the evaluate function in Line 12 and implicit in Line 4 invocates Algorithm 2 to calculate the true value of the predicate of type $p \leftrightarrow q$. Algorithm 1 uses these parameters and the dataset on which Algorithm 1 seeks to adjust the parameters.

Algorithm 2 : *For-All-Value*

Input: The dataset with n attributes, a class attribute, and r rows; each attribute includes its value x and parameters (α, γ, m) required to calculate the GCLV true value defined in Eq 5. The parameters e and b of the ```++++çç.

Output: *for_all*, which is the truth value of the evaluated predicate of type $p \leftrightarrow q$.

for $i = 1$ **to** r *// evaluates the predicate for each record in the dataset*

$antecedent = c_T\left(GCLV(attrib\_1_i), \ldots, GCLV(attrib\_n_i)\right)$ *//the antecedent conjunction operation*

$$= b^{\sqrt[e]{\log_b^e(GCLV(attrib\_1_i)) + \cdots + \log_b^e(GCLV(attrib\_n_i))}}$$

$consequent = GCLV(attrib\_c_i)$ *//the consequent is obtained for the attribute class*

$imp1 = 1 - b^{\sqrt[e]{\log_b^e(antecedent) + \log_b^e(consequent)}}$ *// the first predicate implication*

$imp2 = 1 - b^{\sqrt[e]{\log_b^e(consequent) + \log_b^e(antecedent)}}$ *// the second predicate implication*

$equivalence = b^{\sqrt[e]{\log_b^e(imp1) + \log_b^e(imp2)}}$ *// the equivalence operation*

$sum\_for\_all \mathrel{+}= \log_b^e(equivalence)$ *// the truth value is accumulated using conjunction*

$for\_all = b^{\sqrt[e]{\frac{sum\_for\_all}{n}}}$ // the inverse function obtained from the sum for all dataset truth values

return *for_all*

Algorithm 2: Calculation of the degree of membership of a predicate.

4.3 Knowledge Discovery and Interpretation of Results Using an ACFL-ELF

An ACFL-ELF is used to discover knowledge expressed in the form of logical predicates. From the results obtained, a predicate is taken to express it in terms of simple natural language.

Table 1 shows the classification datasets that were used in the KD process by this logic. A dataset that uses l different qualitative values is treated to transform it into

Table 1 Datasets used in the knowledge discovery process and their descriptions

Dataset	Class attribute	Number of records	Number of attributes	Location
Car	Name: class Class type: string	1728	7	https://archive.ics.uci.edu/ml/datasets/Car+Evaluation
Dermatology	Name: class class type: string	366	35	https://archive.ics.uci.edu/ml/datasets/Dermatology
Logistic	Name: D-mest type: real	60	29	https://www.dropbox.com/sh/w9e3glio3ngepcp/AAD5i0rbgWwZLGkwOGbK1hMda?dl=0
Glass	Name: class class type: string	214	10	https://archive.ics.uci.edu/ml/datasets/Glass+Identification
Bupa	Name: Drink class type: real	345	7	https://archive.ics.uci.edu/ml/datasets/liver+disorders
Indian	Name: class class type: integer	768	9	http://archive.ics.uci.edu/ml/machine-learning-databases/pima-indians-diabetes/
Iris	Name: class class type: string	150	5	https://archive.ics.uci.edu/ml/datasets/Iris

Table 2 Values of the parameters obtained after the optimization process

Dataset car

Buying			Maint			Doors			Persons			Lug Boot			Safety			Class
α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	Singleton
0.93	50	0.62	1.52	60	0.41	1.31	48	0.45	0.58	47	0.26	1.43	64	0.54	1.32	74	0.61	Accessible car (1)
1.11	51	0.66	1.61	48	0.65	1.00	48	0.37	1.06	45	0.54	0.25	59	0.36	1.79	64	0.39	Good car(2)
2.14	46	0.49	0.33	51	0.32	1.86	42	0.55	1.51	60	0.32	1.43	64	0.47	0.79	59	0.52	Unaccessible car (3)
0.45	62	0.47	2.40	55	0.47	0.08	62	0.48	0.65	61	0.36	1.05	62	0.68	2.33	63	0.59	Very good car (4)

Dataset dermatology

Erythema			Scaling			Definite borders			Itching			Koebner Phenomenon			Polygonal Papules			Class
α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	Singleton
1.18	48	0.71	0.32	46	0.84	1.39	78	0.55	1.33	77	0.11	1.96	40	0.00	1.72	43	0.03	Chronic dermatitis (1)
2.91	53	0.79	1.90	53	0.91	1.80	40	0.95	0.08	58	0.16	2.77	57	0.99	1.15	45	0.51	Lichen (2)

Dataset logistic

PIB1			PIB2			INF1			INF2			CPD1			CPD2			D Mest		
α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m
1.27	77	0.80	0.19	73	0.97	1.00	55	0.92	1.13	49	0.82	0.80	68	0.00	2.17	57	0.24	0.81	59	0.00

Dataset glass

Refractive			Na			Mg			Al			Si			K			Class
α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	Singleton
2.19	75	0.13	0.64	66	0.22	0.71	50	0.68	2.56	70	0.28	0.50	64	0.65	0.41	41	0.18	Wind float (1)
0.54	74	0.56	0.74	70	0.64	1.14	67	0.17	2.95	75	0.46	2.86	79	0.50	2.24	60	0.67	Wind not float (2)
2.23	43	0.47	1.60	42	0.00	2.80	64	0.00	2.33	45	1.00	1.40	76	0.00	1.05	65	0.11	Container (3)
1.62	50	0.57	1.17	74	0.44	0.28	44	0.58	0.54	52	0.53	1.53	61	0.47	1.27	41	0.23	Headlamps (4)
1.99	64	0.46	1.72	57	0.99	3.02	57	0.00	0.75	51	0.11	1.04	65	0.48	0.78	71	0.52	Table ware (5)
0.73	77	0.68	0.78	77	0.79	0.81	47	0.59	0.30	69	0.24	2.74	51	0.37	1.10	59	0.15	Very wind float (6)

Dataset Bupa

Mcv			Alkphos			Sgpt			Sgot			Gammagt			Drinks		
α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m
2.23	76	1.00	1.49	48	0.33	1.44	41	0.31	0.92	40	0.00	0.23	48	0.21	3.03	74	0.68

Dataset Iris

Sepal length			Sepal width			Petal lenght			Petal width			Class
α	γ	m	α	γ	m	α	γ	m	α	γ	m	Singleton
0.56	68	0.41	0.25	74	0.30	0.13	70	0.46	0.35	69	0.36	Setosa (1)
1.06	57	0.43	1.33	52	0.62	1.95	69	0.35	1.52	65	0.44	Virginica (2)
0.35	56	0.41	0.41	44	0.60	0.41	65	0.54	2.25	47	0.55	Versicolor (3)

Dataset Indian Pima

Times pregnant			Glucose concentration			Blood pressure			Skin thickness			Insulin			BMI			Diabetes
α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	α	γ	m	Singleton
2.54	72	0.59	0.59	62	0.66	0.94	61	0.39	0.45	69	0.20	2.22	62	0.25	1.42	79	0.44	Diabetes (1)

integer values in the range of 1 to *l*. The data are normalized in a range of values of [0, 100].

For classification, a predicate is defined with the form "P if and only if Q". The general predicate structure we use in this searching for each dataset is the following:

$$\forall_{i \in \text{dataset}}(attrib_1(i) \wedge attrib_2(i) \wedge attrib_{\ldots}(i) \wedge attrib_n(i)) \leftrightarrow class(i).$$

When we refer to $attrib_i$, it means there is a linguistic variable representing this attribute for this predicate. The work through linguistic variables is one of this theory's advantages. For simplicity, when we refer to discovered predicates, it means the set of parameters discovered

The next classification problem seeks to solve an optimization problem whose problem consists of a pair (x, y), where x is a set of attributes, and y is a set of results associated with each vector. Moreover, in this case, for each element x, the parameters that best fit the optimization of each element's parameters in **y** are optimized. Besides, the objective is to show that for each dataset, a function is better adapted to the analyzed data, which allows us to observe this system's sensitivity.

For a dataset with integer-type classes, the classes were evaluated separately using a singleton membership function for each class, assigning the true value of 1.0 to the current class and 0.0 to the other classes. For example, Iris dataset with three classes, three independent optimizations were performed to discover three predicates' parameters, one per class. In case a dataset has classes of real type, only the parameters of one predicate were found using GCLV.

For a GCLV, the values of α, γ, m, and the parameters of the generator function ACFL-ELF are optimized; the objective is to maximize the predicate's true value

The parameters to be optimized are delimited employing the following intervals $\alpha \in [.05 - 3]$ where the sharpness of the function is determined, $\gamma \in [30 - 60] = 0.5$ fuzzy value, and $m \in [0, 1]$. For the ACFL-ELF, which uses values of the logarithmic base b and exponent n, b is taken in the interval [1, 7], and belongs to the set of real numbers, while n is taken in the interval [1, 15] and must be an odd number.

The objective function also evaluates the proposed predicate based on the data found in the data set for each record. This is done using the quantifier $\forall$, calculated through a compensatory conjunction operator (c). The algorithm performs a search for the best parameters based on the proposed predicate, giving the best parameter configuration.

In resume, the steps that are carried out through the execution of an optimization algorithm to discover the best parameters for each dataset are the following:

Step 1. Normalization of the dataset within the interval [0,100].

Step 2. Solving of a nonlinear optimization problem with the class of real type:

$$P(\alpha, \gamma, m) = \max_{\alpha,\gamma,m}\{\forall_{x \in \text{dataset}}(GCLV_L(x_{\text{attrib}_1}; \alpha_{\text{attrib}_1}, \gamma_{\text{attrib}_1}, m_{\text{attrib}_1}) \wedge GCLV_L(x_{\text{attrib}_2}; \alpha_{\text{attrib}_2}, \gamma_{\text{attrib}_2}, m_{\text{attrib}_2}) \wedge \ldots \wedge GCLV_L(x_{\text{attrib}_n}; \alpha_{\text{attrib}_n}, \gamma_{\text{attrib}_n}, m_{\text{attrib}_n})) \leftrightarrow GCLV_L(x_{\text{class}}; \alpha_{\text{class}}, \gamma_{\text{class}}, m_{\text{class}})\}$$

Table 3 Truth values obtained through the logarithmic-exponential function ACFL-ELF with optimized parameters of the generator function f (logarithmic base b and exponent n)

Dataset	Classes	Parameter	Value	True value
Car	Car accessible	log	2.01	0.9304522
		exp	15	
	Car good	log	2.09	0.91626229
		exp	15	
	Car unaccessible	log	2.04	0.92654275
		exp	15	
	Car very good	log	2.03	0.93132237
		exp	15	
Dermatology	Cronica	log	2.02	0.55443044
		exp	9	
	Lichen	log	2.04	0.55790857
		exp	9	
Empresa	d mest	log	3.23	0.67023823
		exp	3	
Glass	Wind float	log	15	0.84416932
		exp	3	
	Wind not float	log	2.03	0.83581373
		exp	15	
	Container	log	2.01	0.65317164
		exp	3	
	Headlamps	log	2.02	0.8562308
		exp	15	
	Tableware	log	2.44	0.66706136
		exp	3	
	v wind float	log	2.05	0.79473997
		exp	11	
Bupa	Drinks	log	3.71	0.99516633
		exp	1	
Iris	Setosa	log	2	0.85640946
		exp	11	
	Versicolor	log	2.02	0.87864615
		exp	15	
	Virginica	log	2.12	0.86550224
		exp	15	
Indian Pima	Diabetes	log	2.06	0.84735278
		exp	15	

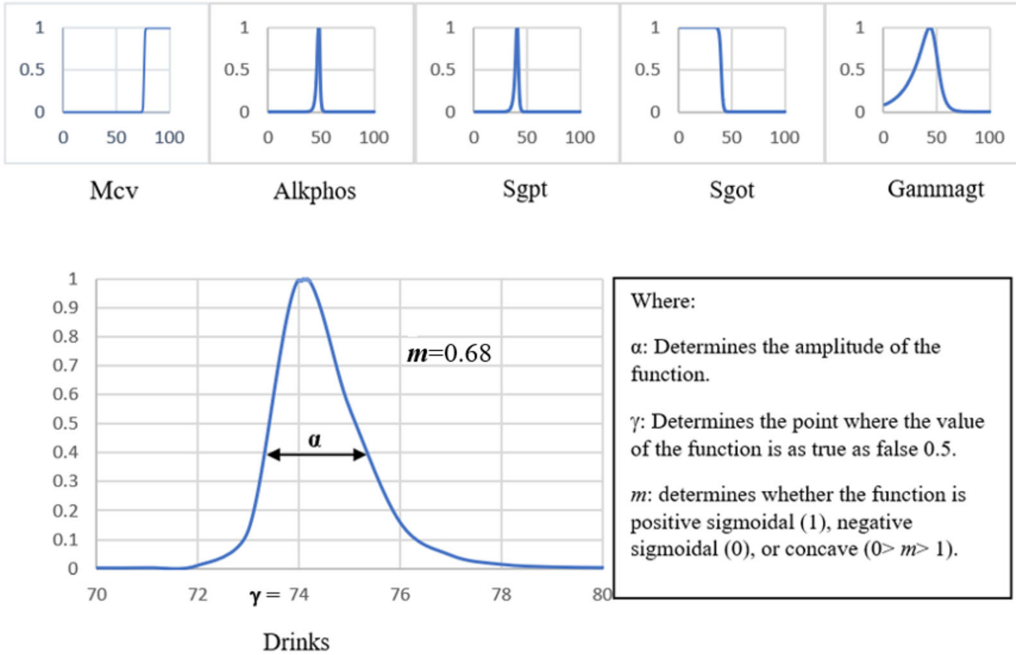

Fig. 13 GCLVs resulting from the optimization of the BUPA instance predicate parameters

a. For an integer class, the GCLV is a Singleton membership function.
b. In this representation, for each attribute to be optimized and according to the proposed predicate, the GCLV is associated with an ACFL-ELF. A value of α, γ, and m is calculated for each attribute optimized by a GCLV. For the generator function f, the b and n parameters are optimized.
c. For each element of the dataset, the operator $\wedge$ determines the t-norm associated with f, the attributes' conjunction.
d. The equivalence operator $\leftrightarrow$ is also defined by f; besides, it can be seen that the GCLV is defined through f.

Step 3. Converting the dataset to its initial value.

The OGCLV-ELF algorithm presented in Sect. 4.2 executed the optimization process, with 30 executions per set of parameters to discover. As an advantage, the top predicate was selected because it is a generalization of the if-then rule. The configuration of this algorithm is a crossing percentage of 80% and a 20% of mutation rate.

Tables 2 and 3 shows the results obtained through the optimization algorithm for each dataset, where for each proposed predicate, the best parameter optimization is obtained that adapts more accurately to the defined class. Table 2 shows α, γ, and m, while Table 3 shows b and n.

Using the optimized parameters and a universality quantifier, the truth value, obtained through evaluating the discovered predicate for each optimized class in the entire dataset, is calculated; the results are presented in Table 3.

Table 3 shows that the generating function of ACFL-ELF sensitively adapts the parameters to the data. Both the logarithmic and exponential values are different in each generated family through a GCLV that models the data's behavior. These results also show that this sensitivity allows us to obtain truth values calculated through the universality quantifier corresponding to high acceptance values.

Once the optimization process results have been obtained, the acquired knowledge can be expressed approximately in terms of a simple natural language. As an example, we selected the predicate generated for the bupa.txt instance, where the predicate to optimize is the following: $mcv \wedge Alkphos \wedge Sgpt \wedge Sgot \wedge Gammagt \leftrightarrow Drinks$. This predicate's objective is to discover how combinations of the results of a series of blood tests it is possible to find out how many alcoholic beverages the test subject has consumed.

By discovering the best parameters of GCLV and ACFL-ELF for all the dataset records, we can express the equivalence predicate using the parameters m and γ. For the Bupa data set, the natural lenguaje expression is below. This expression corresponds with Fig. 13, which shows the way in which the GCLV adapts the parameters to each of the attributes of the dataset.

For every individual that presents a mean corpuscular volume (mcv) that exceeds (m = 1) 94 units (γ = 76) and an alkaline phosphatase (alkpohos) that tends (m = 0.33) to

Table 4 Equivalence predicates expressed in simple natural language through the obtained parameters

Dataset	Attributes parameters													
Bupa	γ	m	γ	m	γ	m	γ	m	γ	m	γ	m		
	76.00	1.00	48.00	0.33	41.00	0.31	40.00	0.00	48.00	0.21	74.00	0.68		
	Interpretation													
	For every element that has an mcv that exceeds 76 units and that its Alkphos tends to 48 units and that its sgpt tends to 41 units and that its sgot is less than 40 units and its gammagt tends to 48 units, it is equivalent to having consumed around 74 units of alcohol													
Car	γ	m	γ	m	γ	m	γ	m	γ	m	γ	m		
	50.00	0.62	60.00	0.41	48.00	0.45	47.00	0.26	64.00	0.54	74.00	0.61		
	Interpretation													
	For every element that is buying that tends to 50 units and a maint that tends to 60 units and doors that tends to 48 units and persons that tends to 47 units and a lug boot that tends to 64 units and safety that tends to 74 units equals one accessible car													
Dermatology	γ	m	γ	m	γ	m	γ	m	γ	m	γ	m		
	48.00	0.71	46.00	0.84	78.00	0.55	77.00	0.11	40.00	0.00	43.00	0.03		
	Interpretation													
	For every element that has erythema that tends to 48 units and a scaling that tends to 46 units and a definite borders that tends to 78 units and itching that tends to 77 units and a koebner phenomenon less than 40 units and polygonal papules smaller than 43 units is equivalent to having chronic dermatitis													
Logistic	γ	m	γ	m	γ	m	γ	m	γ	m	γ	m	γ	m
	77.00	0.80	73.00	0.97	55.00	0.92	49.00	0.82	68.00	0.00	57.00	0.24	59.00	0.00
	Interpretation													
	Every element that GDP1 tends to 77 units and GDP2 exceeds 73 units and that INF1 exceeds 55 units and that INF2 tends to 49 units and that CPD1 is less than 68 units, and that CPD2 tends to 57 units is equivalent to having values below 59 units													
Glass	γ	m	γ	m	γ	m	γ	m	γ	m	γ	m		
	75.00	0.13	66.00	0.22	67.00	0.17	75.00	0.46	64.00	0.65	41.00	0.18		
	Interpretation													
	For every element that has a refractive that tends to 75 units and a Na that tends to 66 units and an mg that tends to 50 units and an ai that tends to 70 units, and a si that tends to 64 units, and a k that tends to 41 units is equivalent to being a wind float glass													
Iris	γ	m	γ	m	γ	m	γ	m						
	68.00	0.41	74.00	0.30	70.00	0.46	69.00	0.36						
	Interpretation													
	Every element with a sepal length tends to 68 units and a sepal width that tends to 74 units and a petal length that tends to 70 units, and a petal width that tends to 69 units is equivalent to an iris setosa													
Indian pima	γ	m	γ	m	γ	m	γ	m	γ	m	γ	m		
	72.00	0.59	62.00	0.66	61.00	0.39	69.00	0.20	62.00	0.25	79.00	0.44		
	Interpretation													
	For every item that has times pregnant that tends to 72 units and a glucose concentration that tends to 62 units and a blood pressure that tends to 61 units and a skin thickness that tends to 69 units and insulin that tends to 62 units, and a BMI that tends to 79 units is equivalent to having diabetes													

78 units ($\gamma = 48$) and an alanine aminotransferase (Sgpt) that tends ($m = 0.31$) towards 41 units ($\gamma = 41$) and an aspartate aminotransferase (Sgot) that is less ($m = 0$) than 36 units ($\gamma = 40$) and a gamma-glutamyl transpeptidase (Gammagt) that tends ($m = 0.21$) towards 147 units ($\gamma = 48$) equals an amount that tends ($m = 0.68$) to 15 alcoholic drinks ($\gamma = 74$).

Table 4 provides an interpretation of the results in the seven instances used as an example in this section. We obtained these interpretations using the discovered parameters and the dataset, as explained with the Bupa dataset.

Compensatory fuzzy logic (LDC) has been used to build models of supervised learning. In [22], the order picking optimization problem was addressed, transformed into a classification problem for its solution. In [23], the research purpose was to build a classification model for tissue discrimination in Magnetic Resonance brain images. In [10], the authors applied an LDC extension to built classifiers models for the red wine dataset and the Gas Furnace dataset. The previous two research compared their work with various literature algorithms and outperformed them. The works reviewed uses classification accuracy measures and others indicators. Their results show the LDC potential in classification, and consequently, of the proposed Archimedean—compensatory fuzzy logic. In this work, no classification is made since it goes outside the work scope, leaving this important task as future work.

5 Concluding Remarks

We have established by this paper a relationship between an Archimedean Compensatory Fuzzy Logic (ACFL) and important concepts of Fuzzy Logic. Using the generator function of an ACFL, we defined two concepts, the Generalized Linguistic Modifier, and the Generalized Sigmoidal Function. We demonstrated necessary and sufficient conditions for the Generalized Sigmoidal Function to be s-shaped. These two concepts were utilized to extend the parameterized family of membership functions, called *General Continuous Linguistic Variable* (GCLV), to any ACFL.

This association can be applied to Knowledge Discovery. We illustrated that applicability in real-life examples. Besides, we showed that specific ACFL Logics and GCLV functions for each variable are better adapted to different data sets. The experimental results showed sensitivity to them.

As a result of this work, we made a theoretical improvement to a previous ACFL to join in just one theory the next elements: Fuzzy Multivalued Logics of two different classes, generators functions, and relevant semantic tools like modifiers, membership functions, and linguistic variables.

The improved ACFL is a contextual Pluralist Logic. First, because it makes compatible the classical approach of Norm and Conorm with CFL theory. Second, because it can compare logics for expressing as better as possible specific contextual knowledge. These properties make ACFL the first Pluralist Fuzzy Logic theory relevant in the context of Knowledge Discovery.

We illustrated how the proposed theory could be used to obtain interpretable and accurate equivalence rules, adjusting simultaneously different concepts of Fuzzy Set and Fuzzy Logic, as the membership function and theirs shapes, and the ACFL. We are working on obtaining better natural language expressions for the obtained predicates.

Based on this work, it is intended to carry out tests concerning other generating functions and explore the results through different heuristics. Considering our previous work, we could carry out experiments that allow us to make data inferences through the predicates learned using ACFL.

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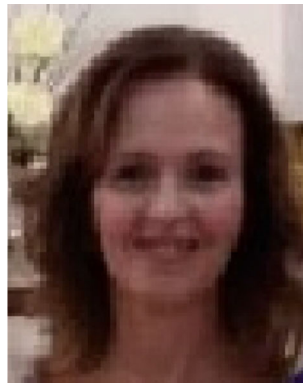

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