Int. J. Fuzzy Syst. (2021) 23(7):2335–2347
https://doi.org/10.1007/s40815-021-01072-3

An Approach to Select the Investment Based on Bipolar Picture Fuzzy Sets

M. Sarwar Sindhu[1] · Tabasam Rashid[1] · Agha Kashif[1]

Received: 14 September 2020 / Revised: 9 November 2020 / Accepted: 12 February 2021 / Published online: 29 March 2021
© Taiwan Fuzzy Systems Association 2021

Abstract Motivated by bipolar fuzzy (B_pF) sets (B_pFSs) and picture fuzzy sets (P_cFSs), we introduced the aggregation operators for the novel extension of P_cFSs named as bipolar picture fuzzy sets (BP_cFSs). Firstly, various arithmetic rules, scores and accuracy functions of the BP_cFSs are presented. Secondly, some aggregation operators of BP_cFSs are constructed to accumulate the bipolar picture fuzzy (BP_cF) data. Thirdly, the features of these aggregation operators are penned and then based on these aggregation operators a multiple criteria decision-making (MCDM) approach is developed to resolve the vague data. In the end, an illustrated example of the investment of money is put forward to show the authenticity and efficacy of the suggested approach. Moreover, BP_cF-TOPSIS, BP_cF-VIKOR, and sensitivity analysis (SA) have been used to provide the strength and practicality of the proposed MCDM model.

Keywords Fuzzy sets · Picture fuzzy sets · Aggregation operators · Bipolar fuzzy sets · Linear programming model

1 Introduction

Coung [6] presented a concept of picture fuzzy sets (P_cFSs), a generalization of fuzzy sets (FSs) [35] in 2013. P_cFSs consists of three well-known degrees, belonging degree (BD), non-belonging degree (NBD) and neutral degree (ND) so that $0 \leq BD + NBD + ND \leq 1$. P_cFSs become a significant tool to handle the situations that have more answers like, yes, no, neutral and refusal. Later on, Cuong and Kreinovich [7] developed some operational rules for P_cFSs to deal with the picture fuzzy information accurately. Aggregation operators (AOs) are extensively used to accumulate the data under different extensions of FSs, for example, Xia et al. [31] introduced a chain of AOs for hesitant fuzzy (HF) data by using quasi arithmetic means, Wei [27] constructed variously prioritized AOs for aggregating HF data, Wei et al. [28] introduced HF Choquet integral AOs, HF Choquet ordered averaging (HFCOA) operator and HF Choquet ordered geometric (HFCOG) operator. Similarly, numerous experts work to develop AOs for P_cFSs [9, 13, 18, 29]. Recently, Zhang et al. [36] developed some Heronian mean (H_rM) AOs to accumulate the picture fuzzy numbers (P_cFNs) and formulated an MCDM approach for solving the multiple criteria problems. The Hamy mean (HM) operators are extended by Li et al. [15] for Pythagorean fuzzy sets (PFSs) and then they implemented these to get the solution of MCDM problems. Recently, HM operators are also applied by Wei et al. [30] under the framework of dual hesitant PFSs.

Nowadays, DMs are using B_pFSs [37, 38] as a significant tool to handle the vague and uncertain data in MCDM problems. A couple of elements, named as, the positive degree of membership (P_vDM) and the negative degree of membership (N_vDM), are used to represent an entity in B_pFSs and the range of these degrees always bounded in $[-1, 1]$. Many DMs have used the B_pFSs in their research articles [11, 14, 39–41]. Later on, Gul [10] developed accumulation operators for B_pF information. An idea of hesitant B_pFSs and its operational rules are presented by Wei et al. [32] in 2017. Lu et al. [16] introduced the idea of

✉ M. Sarwar Sindhu
sarwartajdin@gmail.com

[1] Department of Mathematics, University of Management and Technology, 54770 Lahore, Pakistan

bipolar 2-tuple linguistic fuzzy sets (B_p2TLFSs). Moreover, Xu and Wei [34] presented the concept of dual B_pFSs and developed various operational rules to handle it. Based on B_pFSs, Akram and Arshad [1] proposed the B_pF linguistic variables and B_pF numbers. Alghamdi et al. [3] established the MCDM technique with the help of B_pF framework. Akram et al. developed the B_pF-TOPSIS and B_pF-ELECTRE-I techniques for medical diagnosis [2]. Shumaiza et al. [20] proposed the Trapezoidal B_pF numbers to investigate the group decision-making problems.

Based on the literature review, it can be seen that both P_cFSs and B_pFSs have got much attention from the DMs. That is why we have created a novel extension called BP_cFSs by combining both P_cFSs and B_pFSs. BP_cFSs have additional information in the form of P_vBD and N_vBD which are not present in the P_cFSs due to which it helps the DMs more effectively than P_cFSs and B_pFSs in the decision-making process. In this article, we pay heed to three aspects: (1) presented some core operational laws for BP_cFSs and (2) established two aggregation operators, called bipolar picture fuzzy weighted average (BP_cFWA)and geometric (BP_cFWG) operators to accumulate BP_cF information. (3) developed score and accuracy functions to compare the outcomes of two BP_cF numbers.

Vanderbei [24] presented the idea of linear programming (LP) model that allows some objective function to be maximized or minimized according to the circumstances. LP model is capable the DMs to tackle the issues which they face in MCDM procedures. Many DMs implemented the LP model to handle the MCDM problems in various area of lives [4, 8, 12, 25]. Sometimes the decisions alter with the change of weights of criteria. It means that weights of criteria have a vital influence on decisions. Assigning the weights to criteria are difficult task for the DMs, however, Sindhu et al. [21, 22] have applied the LP technique to evaluate the weights. To avoid biases, we have used the TOPSIS to find out the objective function, and then LP technique has applied to determine the criteria's weights in this work.

The Rest of the article is planned as Sect. 2 encloses some preliminaries regarding bipolar picture fuzzy sets, LP model, and score function. The AOs like BP_cFWA and BP_cFWG operators are developed, and their properties are discussed in Sect. 3, respectively. Based on these operators an MCDM model is proposed in Sect. 4, and the developed model is then applied on a practical example about the selection of investment company in Sect. 5 to elaborate the validity and effectiveness. A comprehensive comparative analysis and a sensitivity analysis is performed to empower the proposed MCDM model in Sects. 6 and 7, respectively. A brief discussion and conclusions are penned in Sect. 8.

2 Preliminaries

This section comprises some notions like *FSs*, P_cFSs, B_pFSs and LP to support the BP_cFSs and MCDM model. Also, to compare the BP_cF numbers, novel score and accuracy functions are presented here.

Definition 1 [35] A fuzzy set (FS) F on a discourse set $X = \{x_1, x_2, \ldots, x_n\}$ is presented as:

$$F = \left\{ \left\langle x_i, m_F(x_i) \right\rangle \middle| x_i \in X \right\},$$

where, $m : X \to [0, 1]$.

Definition 2 [6] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, a picture fuzzy set P_c on X is defined as:

$$P_c = \left\{ \left\langle x_i, \alpha_{P_{c}(x_i)}, \gamma_{P_c}(x_i), \beta_{P_c}(x_i) \right\rangle \middle| x_i \in X, i = 1, 2, \ldots, n \right\},$$

where $\alpha_{P_c}(x_i)$, $\beta_{P_c}(x_i)$, $\gamma_{P_c}(x_i) \in [0, 1]$ are called the acceptance membership, neutral and rejection membership degrees of $x_i \in X$ to the set P_c, respectively, and $\alpha_{P_c}(x_i)$, $\gamma_{P_c}(x_i)$ and $\beta_{P_c}(x_i)$ fulfil the condition: $0 \leq \alpha_{P_c}(x_i) + \gamma_{P_c}(x_i) + \beta_{P_c}(x_i) \leq 1$, for all $x_i \in X$. Also $\zeta_{P_c}(x_i) = 1 - \alpha_{P_c}(x_i) - \gamma_{P_c}(x_i) - \beta_{P_c}(x_i)$, then $\zeta_{P_c}(x_i)$ is said to be a degree of refusal membership of $x_i \in X$ in P_c. For our convenience, we can write $P_k = (\alpha^k_{P_c}(x_i), \beta^k_{P_c}(x_i), \gamma^k_{P_c}(x_i))$ as the picture fuzzy numbers (P_cFNs) over a set P_c, where k is positive integer.

Definition 3 [29] Let $P = \left(\alpha_{P_c}(x_i), \gamma_{P_c}(x_i), \beta_{P_c}(x_i)\right)$, $P_1 = \left(\alpha^1_{P_c}(x_i), \gamma^1_{P_c}(x_i), \beta^1_{P_c}(x_i)\right)$ and $P_2 = \left(\alpha^2_{P_c}(x_i), \gamma^2_{P_c}(x_i), \beta^2_{P_c}(x_i)\right)$ be three P_cFNs, then arithmetic operations are listed as follows:

1. $P_1 \oplus P_2 = (\alpha^1_{P_c} + \alpha^2_{P_c} - \alpha^1_{P_c} \times \alpha^2_{P_c}, \gamma^1_{P_c} \times \gamma^2_{P_c}, \beta^1_{P_c} \times \beta^2_{P_c})$;
2. $P_1 \otimes P_2 = (\alpha^1_{P_c} \times \alpha^2_{P_c}, \gamma^1_{P_c} + \gamma^2_{P_c} - \gamma^1_{P_c} \times \gamma^2_{P_c}, \beta^1_{P_c} + \beta^2_{P_c} - \beta^1_{P_c} \times \beta^2_{P_c})$;
3. $\lambda P = (1 - (1 - \alpha_{P_c})^{\lambda}, \gamma^{\lambda}_{P_c}, \beta^{\lambda}_{P_c})$, where, $\lambda > 0$;
4. $P^{\lambda}_p = (\alpha^{\lambda}_{P_c}, 1 - (1 - \gamma_{P_c})^{\lambda}, 1 - (1 - \beta_{P_c})^{\lambda})$, where, $\lambda > 0$.

Definition 4 [37, 38] A B_pFS denoted by B_p on a universal set $X = \{x_1, x_2, \ldots, x_n\}$ is defined as follows:

$$B_p = \left\{ \left\langle x_i, (\alpha^+_{B_p(x_i)}, \beta^-_{B_p}(x_i)) \right\rangle \middle| x_i \in X, i = 1, 2, ..., n \right\},$$

where $\alpha^+_{B_p(x_i)} : X \to [0,1]$, $\beta^-_{B_p}(x_i) : X \to [-1,0]$ are named as P_vBD and N_vBD of $x_i \in X$ to B_p, respectively.

Definition 5 [10] Let B_p, B_p^1 and B_p^2 be any three B_pFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then several accumulation operators are listed as follows:

1. $B_p^1 \oplus B_p^2 = (\alpha_1^+ + \alpha_2^+ - \alpha_1^+ \times \alpha_2^+, -|\beta_1^-| \times |\beta_2^-|)$;
2. $B_p^1 \otimes B_p^2 = (|\alpha_1^-| \times |\alpha_2^-|, \beta_1^+ + \beta_2^+ - \beta_1^+ \times \beta_2^+)$;
3. $\kappa B_p = (1 - (1 - \alpha^+)^\kappa, -|\beta^-|)$, where, $\kappa > 0$;
4. $B_p^\kappa = (\alpha^+)^\kappa, -1 + |1 + \beta^-|^\kappa$, where, $\kappa > 0$;
5. $B_p^c = (1 - \alpha^+, |\beta^- - 1|$.

Inspired by B_pFSs and P_cFSs, we proposed the bipolar picture fuzzy sets (BP_cFSs) denoted by $\mathcal{P}$ is presented below,

Definition 6 [23] Suppose that $X = \{x_1, x_2, \ldots, x_n\}$ is a discourse, then the BP_cFSs $\mathcal{P}$ on X is presented as:

$$\mathcal{P} = \left\{ \left\langle x_i, (\tilde{P}_c^+(x_i), \tilde{P}_c^-(x_i)) \right\rangle \middle| x_i \in X, \quad i = 1, 2, \ldots, n \right\},$$

here $\tilde{P}_c^+(x_i) = (\alpha^+_{\mathcal{P}}(x_i), \gamma^+_{\mathcal{P}}(x_i), \beta^+_{\mathcal{P}}(x_i))$, $\tilde{P}_c^-(x_i) = (\alpha^-_{\mathcal{P}}(x_i), \gamma^-_{\mathcal{P}}(x_i), \beta^-_{\mathcal{P}}(x_i))$ satisfy the following condition: $0 \le (\alpha^+_{\mathcal{P}}(x_i) + \gamma^+_{\mathcal{P}}(x_i) + \beta^+_{\mathcal{P}}(x_i)) \le 1$ and $-1 \le (\alpha^-_{\mathcal{P}}(x_i) + \gamma^-_{\mathcal{P}}(x_i) + \beta^-_{\mathcal{P}}(x_i)) \le 0$ for all $x_i \in X$.

For the convenient, the duo, $\tilde{p_k}(x) = (\tilde{P}_c^{k+}(x), \tilde{P}_c^{k-}(x))$ is called the BP_cF number (BP_cFN) represented by $\tilde{p_k} = (\tilde{P}_c^{k+}, \tilde{P}_c^{k-})$ that fulfills the conditions: $\alpha^{k+}_{P_c}, \gamma^{k+}_{P_c}, \beta^{k+}_{P_c} \in [0,1]$, $\le \alpha^{k-}_{P_c}, \gamma^{k-}_{P_c}, \beta^{k-}_{P_c} \in [-1, 0]$, $0 \le \alpha^{k+}_{P_c} + \gamma^{k+}_{P_c} + \beta^{k+}_{P_c} \le 1$ and $-1 \le \alpha^{k-}_{P_c} + \gamma^{k-}_{P_c} + \beta^{k-}_{P_c} \le 0$.

Definition 7 [23] Let $\tilde{p} = (\alpha^+_{P_c}, \gamma^+_{P_c}, \beta^+_{P_c}, \alpha^-_{P_c}, \gamma^-_{P_c}, \beta^-_{P_c})$, $\tilde{p_1} = (\alpha^{1+}_{P_c}, \gamma^{1+}_{P_c}, \beta^{1+}_{P_c}, \alpha^{1-}_{P_c}, \gamma^{1-}_{P_c}, \beta^{1-}_{P_c})$ and $\tilde{p_2} = (\alpha^{2+}_{P_c}, \gamma^{2+}_{P_c}, \beta^{2+}_{P_c}, \alpha^{2-}_{P_c}, \gamma^{2-}_{P_c}, \beta^{2-}_{P_c})$ be three BP_cFNs, then the operational rules are penned as:

1. $\tilde{p_1} \oplus \tilde{p_2} = ((\alpha^{1+}_{P_c} + \alpha^{2+}_{P_c} - \alpha^{1+}_{P_c} \cdot \alpha^{2+}_{P_c}, \gamma^{1+}_{P_c} \cdot \gamma^{2+}_{P_c}, \beta^{1+}_{P_c} \cdot \beta^{2+}_{P_c}), -(\alpha^{1-}_{P_c} + \alpha^{2-}_{P_c} - \alpha^{1-}_{P_c} \cdot \alpha^{2-}_{P_c}), -|\gamma^{1-}_{P_c}| \cdot |\gamma^{2-}_{P_c}|, -|\beta^{1-}_{P_c}| \cdot |\beta^{2-}_{P_c}|)$;
2. $\tilde{p_1} \otimes \tilde{p_2} = (\alpha^{1+}_{P_c} \cdot \alpha^{2+}_{P_c}, \gamma^{1+}_{P_c} + \gamma^{2+}_{P_c} - \gamma^{1+}_{P_c} \cdot \gamma^{2+}_{P_c}, \beta^{1+}_{P_c} + \beta^{2+}_{P_c} - \beta^{1+}_{P_c} \cdot \beta^{2+}_{P_c}), -(|\alpha^{1-}_{P_c}| \cdot |\alpha^{2-}_{P_c}|, -(\gamma^{1-}_{P_c} + \gamma^{2-}_{P_c} - \gamma^{1-}_{P_c} \cdot \gamma^{2-}_{P_c}), -(\beta^{1-}_{P_c} + \beta^{2-}_{P_c} - \beta^{1-}_{P_c} \cdot \beta^{2-}_{P_c}))$;
3. $\lambda \tilde{p} = ((1 - (1 - \alpha^+_{P_c})^\lambda, (\gamma^+)^\lambda{}_{P_c}, (\beta^+)^\lambda_{P_c}), -(1 - |(1 - \alpha^-_{P_c})|^\lambda, -|(\gamma^-)^\lambda_{P_c}|, -|(\beta^-)^\lambda_{P_c}|))$, where, $\lambda > 0$;
4. $\tilde{p}^\lambda = ((\alpha^+_{P_c})^\lambda, 1 - (1 - \gamma^+_{P_c})^\lambda, 1 - (1 - \beta^+_{P_c})^\lambda), -|(\alpha^-_{P_c})^\lambda|, -(1 - |(1 - \gamma^-_{P_c})^\lambda|, -(1 - |(1 - \beta^-_{P_c})^\lambda|))$, where, $\lambda > 0$.

In order to compare the two BP_cFN, we presented the score and accuracy functions as follows:

Definition 8 Suppose that $\tilde{p_i} = (\tilde{P}^+_{ci}, \tilde{P}^-_{ci}), (i = 1, 2)$ are two BP_cFNs, then the score functions $\tilde{S^i_f}$ and accuracy function $\tilde{A^i_f}$ between two BP_cFNs are written as:

$$\tilde{S^i_f}(\tilde{p_i}) = \sum_{i=1}^{|\tilde{p_i}|} \frac{1 + \tilde{P}^+_{ci} + \tilde{P}^-_{ci}}{2},$$

and

$$\tilde{A^i_f}(\tilde{p_i}) = \sum_{i=1}^{|\tilde{p_i}|} \frac{\tilde{P}^+_{ci} - \tilde{P}^-_{ci}}{2},$$

then, we can compare two BP_cFNs on the basis of following characteristics:

1. if $\tilde{S^1_f}(\tilde{p_1}) > \tilde{S^2_f}(\tilde{p_2})$, then $\tilde{p_1}$ is superior to $\tilde{p_2}$ and written as, $\tilde{p_1} \succ \tilde{p_2}$;
2. if $\tilde{S^1_f}(\tilde{p_1}) < \tilde{S^2_f}(\tilde{p_2})$, then $\tilde{p_1}$ is inferior to $\tilde{p_2}$ and denoted as, $\tilde{p_1} \prec \tilde{p_2}$;

and if $\tilde{S^1_f}(\tilde{p_1}) = \tilde{S^2_f}(\tilde{p_2})$, then,

1. if $\tilde{A^1_f}(\tilde{p_1}) > \tilde{A^2_f}(\tilde{p_2})$, then $\tilde{p_1}$ is superior to $\tilde{p_2}$ and written as, $\tilde{p_1} \succ \tilde{p_2}$;
2. if $\tilde{A^1_f}(\tilde{p_1}) = \tilde{A^2_f}(\tilde{p_2})$, then $\tilde{p_1}$ is equivalent to $\tilde{p_2}$ and denoted as, $\tilde{p_1} \approx \tilde{p_2}$.

Definition 9 [24]. The modified LP model is presented as:

$$\begin{aligned}
\textit{Maximize}: \quad & Z = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n \\
\textit{Subjectto}: \quad & a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \cdots + a_{1n}y_n \le b_1 \\
& a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \cdots + a_{2n}y_n \le b_2 \\
& \vdots \\
& a_{m1}y_1 + a_{m2}y_2 + a_{m3}y_3 + \cdots + a_{mn}y_n \le b_m \\
& y_1, y_2, \ldots, y_n \ge 0,
\end{aligned}$$

where m represents the number of constraints, and n denotes the cardinality of decision variables $(y_1, y_2, \ldots, y_n)$, respectively. The solution $(y_1, y_2, \ldots, y_n)$ is called a feasible solution if it fulfills all the given limitations.

3 BP_cF Aggregation Operators

In this section, bipolar picture fuzzy weighted average (BP_cFWA)and geometric (BP_cFWG) operators are established to accumulate the BP_cF data.

Definition 10 Let $\tilde{p_k} = (\tilde{P}^+_c, \tilde{P}^-_c)$, where, $k = 1, 2, 3, \ldots, n$ be a set of BP_cFNs, a BP_cFWA operator is defined as:

$$BP_cFWA(\tilde{p_1},\tilde{p_2},\ldots,\tilde{p_n}) = \oplus_{k=1}^{n} w_k \cdot \tilde{p_k}.$$
$$BP_cFWA(\tilde{p_1},\tilde{p_2},\ldots,\tilde{p_n}) = ((1-\Pi_{k=1}^{n}(1-\alpha_{P_c}^{+})^{w_k},$$
$$\Pi_{k=1}^{n}(\gamma_{P_c}^{+})^{w_k}, \Pi_{k=1}^{n}(\beta_{P_c}^{+})^{w_k}),$$
$$-(1-|\Pi_{k=1}^{n}(1-\alpha_{P_c}^{-})^{w_k}|, -\Pi_{k=1}^{n}|(\gamma_{P_c}^{-})^{w_k}|,$$
$$-\Pi_{k=1}^{n}|(\beta_{P_c}^{-})^{w_k}|)),$$

where $w_k = (w_1, w_2, \ldots, w_n)^T$ is the weight's vector that connected $\tilde{p_k}$ and sustaining the limitations: $w_k > 0$ and $\sum_{k=1}^{n} w_k = 1$.

Definition 11 Let $\tilde{p_k} = (\tilde{P_c^+}, \tilde{P_c^-})$, where, $k = 1,2,3,\ldots,n$ be a set of BP_cFNs, a BP_cFWG operator is presented as:

$$BP_cFWG(\tilde{p_1},\tilde{p_2},\ldots,\tilde{p_n}) = \otimes_{k=1}^{n}(\tilde{p_k})^{w_k},$$
$$= \Pi_{k=1}^{n}(\alpha_{P_c}^{+})^{w_k}, 1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{+})^{w_k},$$
$$1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{+})^{w_k}),$$
$$-\Pi_{k=1}^{n}|(\alpha_{P_c}^{-})^{w_k}|, -(1-\Pi_{k=1}^{n}|(1-\gamma_{P_c}^{-})^{w_k}|),$$
$$-(1-\Pi_{k=1}^{n}|1-(\beta_{P_c}^{-})^{w_k}|)),$$

where $w_k = (w_1, w_2, \ldots, w_n)^T$ is the weight's vector that connected $\tilde{p_k}$ and sustaining the limitations: $w_k > 0$ and $\sum_{k=1}^{n} w_k = 1$.

Theorem 1 *The BP_cFWG operator returns a BP_cFN with* $BP_cFWG(\tilde{p_1},\tilde{p_2},\ldots,\tilde{p_n}) = \otimes_{k=1}^{n}(\tilde{p_k})^{w_k}$.

Proof We can prove the Theorem 1 by using mathematical induction on n as follows: (1) for $n = 2$, we get

$$(\tilde{p_1})^{w_1} = ((\alpha_{P_c}^{1+})^{w_1}, 1-(1-\gamma_{P_c}^{1+})^{w_1}, 1-(1-(\beta_{P_c}^{1+})^{w_1},$$
$$-|(\alpha_{P_c}^{1-})^{w_1}|, -(1-|(1-\gamma_{P_c}^{1-})^{w_1}|), -(1-|1-(\beta_{P_c}^{1-})^{w_1}|)),$$
$$(\tilde{p_2})^{w_2} = ((\alpha_{P_c}^{2+})^{w_2}, 1-(1-\gamma_{P_c}^{2+})^{w_2}, 1-(1-(\beta_{P_c}^{2+})^{w_2}),$$
$$-(|(\alpha_{P_c}^{2-})^{w_2}|, -(1-|(1-\gamma_{P_c}^{2-})^{w_2}|), -(1-|1-(\beta_{P_c}^{2-})^{w_2}|)).$$
$$(\tilde{p_1})^{w_1}(\tilde{p_2})^{w_2} = (\alpha_{P_c}^{1+})^{w_1}(\alpha_{P_c}^{2+})^{w_2}, 1-(1-\gamma_{P_c}^{1+})^{w_1}$$
$$(1-\gamma_{P_c}^{2+})^{w_2}, 1-(1-(\beta_{P_c}^{1+})^{w_1})(1-(\beta_{P_c}^{2+})^{w_2},$$
$$-(|(\alpha_{P_c}^{1-})^{w_1}|(|(\alpha_{P_c}^{2-})^{w_2}|, -(1-|(1-\gamma_{P_c}^{1-})^{w_1}|)(1-|\gamma_{P_c}^{2-})^{w_2}|),$$
$$-(1-|1-(\beta_{P_c}^{1-})^{w_1}|)(|1-(\beta_{P_c}^{2-})^{w_2}|).$$

Thus Theorem 1 holds for $n = 2$. Suppose that it holds for $n = i$, where $i < k$, that is

$$BP_cFWG(\tilde{p_1},\tilde{p_2},\ldots,\tilde{p_i}) = \Pi_{k=1}^{n}(\alpha_{P_c}^{i+})^{w_i}, 1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{i+})^{w_i},$$
$$1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{i+})^{w_i}), -\Pi_{k=1}^{n}(\alpha_{P_c}^{i-})^{w_i},$$
$$-(1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{i-})^{w_i}), -(1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{i-})^{w_i}))),$$

then, when $n = i+1$, by the operational rules in Theorem 1, we get

$$\Pi_{k=1}^{i+1}(\tilde{p_k})^{w_k} = \Pi_{k=1}^{i}(\tilde{p_k})^{w_k} \cdot (\tilde{p_{i+1}})^{w_{i+1}},$$
$$= (\Pi_{k=1}^{i}(\alpha_{P_c}^{i+})^{w_i}, 1-\Pi_{k=1}^{i}(1-\gamma_{P_c}^{i+})^{w_i},$$
$$1-\Pi_{k=1}^{i}(1-(\beta_{P_c}^{i+})^{w_i}), -\Pi_{k=1}^{i}(\alpha_{P_c}^{i-})^{w_i},$$
$$-(1-\Pi_{k=1}^{i}(1-\gamma_{P_c}^{i-})^{w_i}),$$
$$-(1-\Pi_{k=1}^{i}(1-(\beta_{P_c}^{i-})^{w_i})) \cdot ((\alpha_{P_c}^{(i+1)+})^{w_{i+1}},$$
$$1-(1-\gamma_{P_c}^{(i+1)+})^{w_{i+1}}, 1-(1-(\beta_{P_c}^{(i+1)+})^{w_{i+1}}),$$
$$-(\alpha_{P_c}^{(i+1)-})^{w_{i+1}}, -(1-(1-\gamma_{P_c}^{(i+1)-})^{w_{i+1}}),$$
$$-(1-(1-(\beta_{P_c}^{(i+1)-})^{w_{i+1}})),$$
$$= \Pi_{k=1}^{i+1}(\alpha_{P_c}^{k+})^{w_k},$$
$$1-\Pi_{k=1}^{i+1}(1-\gamma_{P_c}^{i+})^{w_i}, 1-\Pi_{k=1}^{i+1}(1-\beta_{P_c}^{k+})^{w_k}),$$
$$-(\Pi_{k=1}^{i+1}(\alpha_{P_c}^{k-})^{w_k}, -(1-\Pi_{k=1}^{i+1}(1-\gamma_{P_c}^{k-})^{w_k}),$$
$$-(1-\Pi_{k=1}^{i+1}(1-(\beta_{P_c}^{k-})^{w_k})),$$

which reveals that Theorem 1 holds for $n = i+1$. Hence, we can say that Theorem 1 satisfies for all n. Then clearly,

$$(\Pi_{k=1}^{n}(\alpha_{P_c}^{k+})^{w_k}, 1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{k+})^{w_k},$$
$$1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{k+})^{w_k}) \in [0,1],$$

and

$$\Pi_{k=1}^{n}(\alpha_{P_c}^{k+})^{w_k} + 1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{k+})^{w_k}$$
$$+1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{k+})^{w_k}) \le 1,$$

also,

$$(-\Pi_{k=1}^{n}(\alpha_{P_c}^{k-})^{w_k}), -(1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{k-})^{w_k}),$$
$$-(1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{k-})^{w_k}) \in [-1,0],$$

and

$$(-\Pi_{k=1}^{n}(\alpha_{P_c}^{k-})^{w_k}) - (1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{k-})^{w_k})$$
$$-(1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{k-})^{w_k}) \le -1$$

Hence, $BP_cFWG(\tilde{p_1},\tilde{p_2},\ldots,\tilde{p_i})$ form a BP_cFN.

Theorem 2 *Let* $\tilde{p_k} = (\alpha^{k+}, \gamma^{k+}, \beta^{k+}, \alpha^{k-}, \gamma^{k-}, \beta^{k-})$ *be a collection BP_cFNs, then the BP_cFWG operator hold the following properties*:

1. *Idempotent*,
2. *Monotonic*,
3. *Bounded*,
4. *Commutative.*

Proof

1. Let $\tilde{p_1} = \tilde{p_2} =, \ldots, = \tilde{p_n} = \tilde{p} = (\alpha^+, \gamma^+, \beta^+, \alpha^-, \gamma^-, \beta^-)$, then,

$$BP_cFWG(\tilde{p_1},\tilde{p_2},\ldots,\tilde{p_n}) = (\Pi_{k=1}^{n}(\alpha_{P_c}^{k+})^{w_k},$$
$$1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{k+})^{w_k}, 1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{k+})^{w_k},$$
$$-\Pi_{k=1}^{n}(\alpha_{P_c}^{k-})^{w_k}), -(1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{k-})^{w_k}),$$
$$-(1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{k-})^{w_k}),$$
$$=(\Pi_{k=1}^{n}(\alpha_{P_c}^{+})^{w_k}, 1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{+})^{w_k},$$
$$1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{+})^{w_k}, -\Pi_{k=1}^{n}(\alpha_{P_c}^{-})^{w_k}),$$
$$-(1-\Pi_{k=1}^{n}(1-\gamma_{P_c}^{-})^{w_k}), -(1-\Pi_{k=1}^{n}(1-(\beta_{P_c}^{-})^{w_k}),$$
$$=((\alpha_{P_c}^{+})^{\sum_{k=1}^{n}w_n}, 1-(1-\gamma_{P_c}^{+})^{\sum_{k=1}^{n}w_n},$$
$$1-(1-(\beta_{P_c}^{+})^{\sum_{k=1}^{n}w_n}, -(\alpha_{P_c}^{-})^{\sum_{k=1}^{n}w_n}),$$
$$-(1-(1-\gamma_{P_c}^{-})^{\sum_{k=1}^{n}w_n}), -(1-(1-(\beta_{P_c}^{-})^{\sum_{k=1}^{n}w_n}),$$
$$=((\alpha_{P_c}^{+}), 1-(1-\gamma_{P_c}^{+}), 1-(1-(\beta_{P_c}^{+}),$$
$$-(\alpha_{P_c}^{-})), -(1-(1-\gamma_{P_c}^{-})), -(1-(1-(\beta_{P_c}^{-})).$$

Since $\sum_{k=1}^{n} w_n = 1$, then we get, $BP_cFWG(\tilde{p_1}, \tilde{p_2},\ldots,\tilde{p_n}) = (\alpha_{P_c}^{+}, \gamma_{P_c}^{+}, \beta_{P_c}^{+}, -\alpha_{P_c}^{-}, -\gamma_{P_c}^{-}, -\beta_{P_c}^{-}) = \tilde{p}$, which is required.

2. Let $\alpha_{i_j}^{+} \geq \alpha_{\theta_j}^{+}, \gamma_{i_j}^{+} \geq \gamma_{\theta_j}^{+}, \beta_{i_j}^{+} \leq \beta_{\theta_j}^{+}, \alpha_{i_j}^{-} = \alpha_{\theta_j}^{-}, \gamma_{i_j}^{-} = \gamma_{\theta_j}^{-}$ and $\beta_{i_j}^{-} = \beta_{\theta_j}^{-}$. Consider

$$\Rightarrow \alpha_{i_j}^{+} \geq \alpha_{\theta_j}^{+}$$
$$\Rightarrow (\alpha_{i_j}^{+})^{w_k} \geq (\alpha_{\theta_j}^{+})^{w_k}$$
$$\Rightarrow \Pi_{k=1}^{n}(\alpha_{i_j}^{+})^{w_k} \geq \Pi_{k=1}^{n}(\alpha_{\theta_j}^{+})^{w_k}$$

also,

$$\gamma_{i_j}^{+} \geq \gamma_{\theta_j}^{+}$$
$$\Rightarrow 1-\gamma_{i_j}^{+} \leq 1-\gamma_{\theta_j}^{+}$$
$$\Rightarrow \Pi_{k=1}^{n}(1-\gamma_{i_j}^{+})^{w^k} \leq \Pi_{k=1}^{n}(1-\gamma_{\theta_j}^{+})^{w_k}$$
$$\Rightarrow 1-\Pi_{k=1}^{n}(1-\gamma_{i_j}^{+})^{w_k} \geq \Pi_{k=1}^{n}1-(1-\gamma_{\theta_j}^{+})^{w_k}$$

now take,

$$\beta_{i_j}^{+} \leq \beta_{\theta_j}^{+}$$
$$\Rightarrow \Pi_{k=1}^{n}(\beta_{i_j}^{+})^{w_k} \leq \Pi_{k=1}^{n}(\beta_{\theta_j}^{+})^{w_k}$$
$$\Rightarrow 1-\Pi_{k=1}^{n}(\beta_{i_j}^{+})^{w_k} \geq 1-\Pi_{k=1}^{n}(\beta_{\theta_j}^{+})^{w_k}$$
$$\Rightarrow 1-(1-\Pi_{k=1}^{n}(\beta_{i_j}^{+})^{w_k}) \leq 1-(1-\Pi_{k=1}^{n}(\beta_{\theta_j}^{+})^{w_k}).$$

3. Let $\tilde{b_i} = (\alpha_i^{+}, \gamma_i^{+}, \beta_i^{+}, \alpha_i^{-}, \gamma_i^{-}, \beta_i^{-})$ with $i = 1,2,\ldots,k$, $\tilde{b_p} = (\alpha_{\max i}^{+}, \gamma_{\max i}^{+}, \beta_{\max i}^{+}, \alpha_{\max i}^{-}, \gamma_{\max i_j}^{-}, \beta_{\max i}^{-})$ and $\tilde{b_n} = (\alpha_{\min i}^{+}, \gamma_{\min i}^{+}, \beta_{\min i}^{+}, \alpha_{\min i}^{-}, \gamma_{\min i}^{-}, \beta_{\min i}^{-})$ be a set of BP_cFNs, then we have to prove that BP_cFWG is bounded, that is, $\tilde{b_n} < BP_cFWG(\tilde{b_1},\tilde{b_2},\ldots,\tilde{b_k}) < \tilde{b_p}$. It follows from properties (1) and (2) as, $BP_cFWG(\tilde{b_1}, \tilde{b_2},\ldots,\tilde{b_k}) \geq BP_cFWG(\tilde{b_n},\tilde{b_n},\ldots,\tilde{b_n}) = \tilde{b_n}$, $BP_cFWG(\tilde{b_1},\tilde{b_2},\ldots,\tilde{b_k}) \leq BP_cFWG(\tilde{b_p}, \tilde{b_p},\ldots,\tilde{b_p}) = \tilde{b_p}$, $\Rightarrow \tilde{b_n} < BP_cFWG(\tilde{b_1},\tilde{b_2},\ldots,\tilde{b_k}) < \tilde{b_p}$.
4. Let, $\tilde{a}_i = (\alpha_i^{+}, \gamma_i^{+}, \beta_i^{+}, \alpha_i^{-}, \gamma_i^{-}, \beta_i^{-})$and- for all $i, \theta = 1,2,...,k$ be two sets of BP_cFNs, then have to prove that BP_cFWG is commutative, that is, $BP_cFWG(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_k) = BP_cFWG(\tilde{b_1},\tilde{b_2},\ldots,\tilde{b_k})$. Since $\tilde{b_\theta}$ is any permutation of $\tilde{a_i}$, then $\otimes_{i=1}^{k}(\tilde{a}_i)^{w_i} = \otimes_{\theta=1}^{k}(\tilde{b_\theta})^{w_\theta}$. Hence, $BP_cFWG(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_k) = BP_cFWG(\tilde{b_1},\tilde{b_2},\ldots,\tilde{b_k})$.

Theorem 3 *The BP_cFWA operator returns a BP_cFN with* $BP_cFWA(\tilde{p_1},\tilde{p_2},\ldots,\tilde{p_n}) = \oplus_{k=1}^{n} w_k \cdot \tilde{p_k}$.

Proof The proof of this Theorem is obvious.

Theorem 4 *Let $\tilde{p_k} = (\alpha^{k+}, \gamma^{k+}, \beta^{k+}, \alpha^{k-}, \gamma^{k-}, \beta^{k-})$ be a collection BP_cFNs, then the BP_cFWA operator satisfy the following properties*:

1. *Idempotent*,
2. *Monotonic*,
3. *Bounded*,
4. *Commutative.*

Proof We can prove it by adopting the same steps as Theorem 2.

4 MCDM Approach for BP_cF Environment

By considering BP_cF aggregation operators established in Sect. 3, an MCDM model is presented to handle the BP_cF information. Suppose that,

$Q = \{Q_1, Q_2,\ldots,Q_n\}$ and $V = \{V_1, V_2,\ldots,V_m\}$ are the discrete collection of alternatives and criteria, respectively. If the DMs gave the various esteems to the alternative $Q_i (i = 1,2,\ldots,n)$ under the criteria $V_j (j = 1,2,\ldots,m)$. A bipolar picture fuzzy decision matrix $B_{pc} = [b_{ij}]_{n\times m}$ is constructed on the basis of BP_cF information. Since the weights of the criteria have an excessive impact, thereby a weighing vector of criteria is provided as $w = (w_1, w_2, w_3,\ldots,w_j)^T$, where $\sum_{j=1}^{m} w_j = 1, j = 1,2,\ldots,m$ and $w_j > 0$ can be evaluated by implementing the LP model described in Definition 9. The MCDM model based on proposed BP_cF aggregation operators has the following steps.

Step 1. Based on the BP_cF information provided by DM form a bipolar picture fuzzy decision matrix denoted by $B_{pc} = [b_{ij}]_{n\times m}$.

Step 2. In order to obtain the objective function, the following steps of the TOPSIS technique are adopted:

Step 2 (a) Determine the BP_cF positive ideal solution (B_pFPIS) denoted by B^+_{pc} and BP_cF negative ideal solution (B_pFNIS)represented by B^-_{pc}, respectively.

$$B^+_{pc} = \left(\max_j(\alpha^+_{ij}), \max_j(\gamma^+_{ij}), \max_j(\beta^+_{ij})), \min_j(\alpha^-_{ij}), \min_j(\gamma^-_{ij}), \min_j(\beta^-_{ij})) \right)$$
$$B^-_{pc} = \left((\min_j(\alpha^+_{ij}), \min_j(\gamma^+_{ij}), \min_j(\beta^+_{ij})), \max_j(\alpha^-_{ij}), \max_j(\gamma^-_{ij}), \max_j(\beta^-_{ij})) \right).$$

Step 2 (b) For any two BP_cFSs L and Q, the weighted distance measure presented by Sindhu et al. [23] is given below, compute the similarity measure $\tilde{S}_{pc}(L, Q)$ as: $\tilde{S}_{pc}(L, Q) = 1 - D^w_{pc}(L, Q)$, where

$$D^w_{pc}(L, Q) = \sum_{i=1}^{n} w_j \left(\begin{array}{l} \left[\begin{array}{l} |\alpha^{l+}_{B_{Pc}}(x_i) - \alpha^{q+}_{B_{Pc}}(x_i)| + |\gamma^{l+}_{B_{Pc}}(x_i) - \gamma^{q+}_{B_{Pc}}(x_i)| + |\beta^{l+}_{B_{Pc}}(x_i) - \beta^{q+}_{B_{Pc}}(x_i)| \\ +|\alpha^{l-}_{B_{Pc}}(x_i) - \alpha^{q-}_{B_{Pc}}(x_i)| + |\gamma^{l-}_{B_{Pc}}(x_i) - \gamma^{q-}_{B_{Pc}}(x_i)| + |\beta^{l-}_{B_{Pc}}(x_i) - \beta^{q-}_{B_{Pc}}(x_i)| \end{array} \right] + \\ \max \left[\begin{array}{l} |\alpha^{l+}_{B_{Pc}}(x_i) - \alpha^{q+}_{B_{Pc}}(x_i)|, |\gamma^{l+}_{B_{Pc}}(x_i) - \gamma^{q+}_{B_{Pc}}(x_i)|, |\beta^{l+}_{B_{Pc}}(x_i) - \beta^{q+}_{B_{Pc}}(x_i)| \\ , |\alpha^{l-}_{B_{Pc}}(x_i) - \alpha^{q-}_{B_{Pc}}(x_i)|, |\gamma^{l-}_{B_{Pc}}(x_i) - \gamma^{q-}_{B_{Pc}}(x_i)|, |\beta^{l-}_{B_{Pc}}(x_i) - \beta^{q-}_{B_{Pc}}(x_i)| \end{array} \right] \end{array} \right).$$

Step 2 (c) Evaluate the degree of similarity as: $\tilde{S}^+_{Pci}(B_i, \Delta^+) = 1 - D^w_{Pc}(B_i, B^+_{pc})$, $\tilde{S}^-_{Pci}(B_i, \Delta^-) = 1 - D^w_{Pc}(B_i, B^-_{pc})$, where, $1 \leq i \leq n$.

Step 2 (d) Compute the objective function Z as: $Z = \sum_{i=1}^{n} (\tilde{S}^+_{Pci}(B_i, B^+_{pc}) - \tilde{S}^-_{Pci}(B_i, B^-_{pc}))$.

Step 3. With the help of LP model as described in Definition 9, evaluate the weights of criteria by maximizing the objective function Z under the given constraints.

Step 4. Applying the BP_cFWA and BP_cFWG operators to process the information in matrix $B_{pc} = [b_{ij}]_{n \times m}$ to accumulate the information of the alternatives $Q_i (i = 1, 2, \ldots, n)$.

Step 5. Based on Definition 8, evaluate the score values $\tilde{S^i_f}$ of $Q_i (i = 1, 2, \ldots, n)$.

Step 6. Arrange all the alternatives $Q_i (i = 1, 2, \ldots, n)$ from highest to lowest values of $\tilde{S^i_f}$ obtained in Step 5 and then rank them to choose the best one. The highest and lowest values of $\tilde{S^i_f}$ indicate the best and worst alternatives, respectively.

5 Practical Example

From the beginning, it has been a big problem for the investors to invest their money in something that can make maximum profit. To do this, investors have to resort to different companies (alternatives). Suppose that an investment company wants to invest a sum of money in a favourable field to make the maximum possible profit. The investment company has formed a committee that will select one of the best of the proposed four companies (alternatives):(1) a car company (B_1); (2) a food company (B_2); (3) a computer company (B_3); and (4) an arms company (B_4). The committee must decide by considering the following three beneficial criteria: (1) the risk; (2) the growth; and (3) the customer satisfaction. The weights of the criteria are completely unknown and can find out by utilizing objective function computed with the help of TOPSIS under some constraints as given in Step 3. The four possible companies (alternatives) are to be evaluated under the above three criteria in the form of BP_cFNs, as

$$B_{pc} = \begin{pmatrix} \langle 0.5,0.4,0.1,-0.3,-0.4,-0.2\rangle & \langle 0.6,0.3,0.1,-0.5,-0.3,-0.2\rangle & \langle 0.4,0.4,0.2,-0.2,-0.4,-0.4\rangle \\ \langle 0.4,0.4,0.2,-0.4,-0.3,-0.2\rangle & \langle 0.4,0.1,0.3,-0.4,-0.3,-0.1\rangle & \langle 0.5,0.5,0.0,-0.1,-0.2,-0.5\rangle \\ \langle 0.6,0.2,0.1,-0.3,-0.2,-0.1\rangle & \langle 0.7,0.2,0.1,-0.4,-0.3,-0.2\rangle & \langle 0.3,0.4,0.2,-0.2,-0.3,-0.4\rangle \\ \langle 0.6,0.3,0.1,-0.4,-0.2,-0.1\rangle & \langle 0.4,0.1,0.4,-0.3,-0.2,-0.3\rangle & \langle 0.4,0.5,0.1,-0.1,-0.4,-0.4\rangle \end{pmatrix}$$

shown in the following simplified bipolar picture fuzzy decision matrix (BP_cFDM) denoted by $B_{pc} = [b_{ij}]_{4\times3}$.

Step 1. Formulate the information given by the DM as a BP_cFDM,

$B_{pc} = [b_{ij}]_{4\times3}$.

Step 2. Based on TOPSIS, by using positive and negative ideal solutions given below: $B^{+}_{pc} = \{\langle 0.6,0.4,0.2,-0.3,-0.2,-0.1\rangle\langle 0.7,0.3,0.4,-0.3,-0.2,-0.1\rangle\langle 0.5,0.5,0.2,-0.1,-0.2,-0.4\rangle\}$ $B^{-}_{pc} = \{\langle 0.4,0.2,0.1,-0.4,-0.4,-0.2\rangle\langle 0.4,0.1,0.1,-0.5,-0.3,-0.3\rangle\langle 0.4,0.4,0,-0.2,-0.4,-0.5\rangle\}$, we get the objective function as: $-0.4w_1 + 0.8w_2$.

Step 3. By applying the LP model, the criteria's weights under some constraints are computed as:

$$\begin{aligned} \textit{Maximize}: \quad & Z = -0.4w_1 + 0.8w_2 \\ \textit{Subjectto}: \quad & 10w_1 + 8w_2 + 12w_3 \geq 10, \\ & 10w_1 + 8w_2 + 12w_3 \leq 10.5, \\ & 8w_1 + 11w_2 + 7w_3 \geq 8, \\ & 8w_1 + 11w_2 + 7w_3 \leq 8.5, \\ & 12w_1 + 15w_2 + 12w_3 \geq 12, \\ & 12w_1 + 15w_2 + 12w_3 \leq 12.5, \\ & w_1 + w_2 + w_3 = 1, \\ & w_1, w_2, w_3 \geq 0, \end{aligned}$$

$w_1 = 0.4000$, $w_2 = 0.3960$ and $w_3 = 0.2000$.

Step 4. Applying the BP_cFWG operators to accumulate the information given in $B_{pc} = [b_{ij}]_{4\times3}$, we get,

$$R = \begin{pmatrix} \langle 0.5154, 0.0024, 0.1159, -0.3403, -0.0024, -0.2312\rangle \\ \langle 0.4198, 0.0010, 0, -0.3043, -0.0008, -0.1837\rangle \\ \langle 0.5563, 0.0007, 0.1159, -0.3115, -0.0008, -0.1752\rangle \\ \langle 0.5563, 0.0007, 0.1159, -0.2715, -0.0007, -0.2058\rangle \end{pmatrix}$$

Step 5. Based on Definition 8, the score values $\tilde{S^i_f}$ of R are obtained as: $\tilde{S^1_f} = 0.5299$, $\tilde{S^2_f} = 0.4660$, $\tilde{S^3_f} = 0.0927$, $\tilde{S^4_f} = 0.0975$.

Step 6. According to the score values of $\tilde{S^i_f}$ obtained in Step 5, the arrangement of the alternatives is, $B_1 \succ B_2 \succ B_4 \succ B_3$, hence, B_1 (alternative) is the best one.

5.1 Computation for BP_cFWA Operator

In the current subsection, Steps 4 to 6 are repeated for BP_cFWA operator:

Step $\acute{4}$. Applying the BP_cFWA operators to accumulate the information given in $B_{pc} = [b_{ij}]_{4\times3}$, we have,

$$\acute{R} = \begin{pmatrix} \langle 0.5240, 0.3582, 0.1159, -0.3698, -0.3582, -0.2312\rangle \\ \langle 0.4203, 0.2424, 0, -0.3480, -0.2780, -0.1837\rangle \\ \langle 0.5993, 0.2312, 0.1159, -0.3227, -0.2563, -0.1752\rangle \\ \langle 0.4888, 0.2161, 0.1747, -0.3069, -0.2312, -0.2058\rangle \end{pmatrix}$$

Step $\acute{5}$. Based on Definition 8, the score values $\tilde{S^i_f}$ of R are obtained as: $\tilde{S^1_f} = 0.5194$, $\tilde{S^2_f} = 0.4265$, $\tilde{S^3_f} = 0.0961$, $\tilde{S^4_f} = 0.0679$.

Step $\acute{6}$. According to the score values of $\tilde{S^i_f}$ obtained in Step 5, the arrangement of the alternatives is, $B_1 \succ B_2 \succ B_3 \succ B_4$, that is, the best alternative is B_1.

6 Comparative Analysis

6.1 Comparative Analysis with BP_cF-TOPSIS

A comparative study of the proposed BP_cF MCDM approach with other MCDM techniques like the BP_cF-TOPSIS presented by Sindhu et al. [23], and VIKOR [17] technique is penned in this section. The MCDM problem provided in Sect. 5 is solved by BP_cF-TOPSIS to compare the outcomes obtained from these methods. When solving the problem given in Sect. 5 by using BP_cF-TOPSIS, the steps for evaluating the weights of criteria are the same as that presented in Step 2 of the proposed MCDM model. So, we can repeat the first three steps of the proposed MCDM approach and then move to the next step to achieve the best alternative as:

Step 4. Evaluate the degree of similarity $\tilde{S}^{+}_{Pci}$ and $\tilde{S}^{-}_{Pci}$ by using the formulae as described in Step 2(*c*) between each alternative and the elements of B^{+}_{pc} and B^{-}_{pc}, respectively, we get,

$\tilde{S}^{+}_{Pc1} = 0.8608$; $\tilde{S}^{+}_{Pc2} = 0.7962$; $\tilde{S}^{+}_{Pc3} = 0.7859$ $\tilde{S}^{+}_{Pc4} = 0.7961$,

and $\tilde{S}^{-}_{Pc1} = 0.7810$; $\tilde{S}^{-}_{Pc1} = 0.8306$; $\tilde{S}^{-}_{Pc1} = 0.8310$; $\tilde{S}^{-}_{Pc1} = 0.8258$.

Step 5. The relative closeness R_{Ci} of alternative B_i with respect to the B_PFPIS B^{+}_{pc} is obtained on the basis of following formula as:

$$R_{Ci} = \frac{\tilde{S}^{+}_{Pci}}{\tilde{S}^{+}_{Pci} + \tilde{S}^{-}_{Pci}}.$$

$R_{C1} = 0.5243$; $R_{C2} = 0.4894$; $R_{C3} = 0.4861$; $R_{C4} = 0.4908$. The ranking order is obtained as: $B_1 \succ B_4 \succ B_2 \succ B_3 \succ$, that is B_1 is the best alternative which matches with the proposed MCDM model perfectly.

6.2 Comparative Analysis with BP_cF VIKOR

In the present subsection, based on VIKOR technique, an MCDM investigation approach named BP_cF- VIKOR is developed and implemented to solve the MCDM problems. The MCDM problem provided in Sect. 5 resolved with the help of BP_cF- VIKOR approach by considering the following steps:

Step 1. Repeat first three Steps of the proposed MCDM model.

Step 4. For any two BP_cFSs L and Q, the distance measure presented by Sindhu et al. [23] is given below,

$$D_{pc}(L,Q) = \frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} |\alpha^{l+}_{B_{Pc}}(x_i) - \alpha^{q+}_{B_{Pc}}(x_i)| + |\gamma^{l+}_{B_{Pc}}(x_i) - \gamma^{q+}_{B_{Pc}}(x_i)| + |\beta^{l+}_{B_{Pc}}(x_i) - \beta^{q+}_{B_{Pc}}(x_i)| \\ +|\alpha^{l-}_{B_{Pc}}(x_i) - \alpha^{q-}_{B_{Pc}}(x_i)| + |\gamma^{l-}_{B_{Pc}}(x_i) - \gamma^{q-}_{B_{Pc}}(x_i)| + |\beta^{l-}_{B_{Pc}}(x_i) - \beta^{q-}_{B_{Pc}}(x_i)| \end{bmatrix} + \max\begin{bmatrix} |\alpha^{l+}_{B_{Pc}}(x_i) - \alpha^{q+}_{B_{Pc}}(x_i)|, |\gamma^{l+}_{B_{Pc}}(x_i) - \gamma^{q+}_{B_{Pc}}(x_i)|, |\beta^{l+}_{B_{Pc}}(x_i) - \beta^{q+}_{B_{Pc}}(x_i)| \\ , |\alpha^{l-}_{B_{Pc}}(x_i) - \alpha^{q-}_{B_{Pc}}(x_i)|, |\gamma^{l-}_{B_{Pc}}(x_i) - \gamma^{q-}_{B_{Pc}}(x_i)|, |\beta^{l-}_{B_{Pc}}(x_i) - \beta^{q-}_{B_{Pc}}(x_i)| \end{bmatrix}\right).$$

Based on the above distance measure, evaluate the values of α_i, β_i and η_i as:

$$\alpha_i = \sum_{j=1}^{n} w_j \frac{D_{Pc}(B^{+}_{pc}, b_{ij})}{D_{Pc}(B^{+}_{pc}, B^{-}_{pc})},$$

$$\beta_i = \max_j [w_j \frac{D_{Pc}(B^{+}_{pc}, b_{ij})}{D_{Pc}(B^{+}_{pc}, B^{-}_{pc})}],$$

$$\eta_i = \frac{v(\alpha_i - \alpha^{\star})}{(\alpha^{-} - \alpha^{\star})} + \frac{(1-v)(\beta_i - \beta^{\star})}{(\beta^{-} - \beta^{\star})},$$

where, $\alpha^{\star} = \min_i \alpha_i, \alpha^{-} = \max_i \alpha_i$ and $\beta^{\star} = \min_i \beta_i, \beta^{-} = \max_i \beta_i$, and v is the weight of the strategy of the majority of the criteria, and its esteems always lie in the interval [0, 1]., and generally, the value of v cab be assumed as $v = 0.5$. The values of α_i, β_i, and η_i, where $i = 1, 2, 3, 4$ obtained by the above formulae are penned in Table 1.

Arrange the alternatives according to the values of α_i, β_i, and η_i from lower to higher-order. From these values of α_i, β_i, and η_i, we get three ranking arrangements that are further used to suggest the compromise solution of the options. The term η_i is called the measure of separation of B_i from the superior option, which represents that the minimum value of η_i gives the superior option. The compromise solution of the alternative B_1 is computed when it has a minimum value of η_i and satisfied the following two conditions:

Condition 1. An acceptable advantage for decision-making: $\eta(B^2) - \eta(B^1) \geq \frac{1}{m}$, where m is the number of alternatives, B^1 and B^2 are the first two alternatives in η_i.

Condition 2. Acceptable stability for decision-making: This condition describes that if the alternative B^1 has superior ranking according to the values of η_i then it must also be the superior according to the values obtained by α_i and/or β_i: If any one of the two conditions is not fulfilled, a collection of compromise solutions are gotten as follows:

- B^1 and B^2 if only the **Condition 2.** is not satisfied, or
- If the **Condition 1.** is not satisfied, then this compromise solution contains the alternatives $B^1, B^2, \ldots, B^m$ is determined as: $\eta(B^K) - \eta(B^1) \geq \frac{1}{m}$ for largest K.

Table 1 Results obtained α_i, β_i, and η_i

Alternatives	α_i	β_i	η_i
B_1	1.1419	1.0604	0
B_2	1.1654	1.0839	0.3614
B_3	1.2368	1.1545	0.8893
B_4	1.2486	1.1662	1.0000
Ranking	$B_1 \succ B_2 \succ B_3 \succ B_4$	$B_1 \succ B_2 \succ B_3 \succ B_4$	$B_1 \succ B_2 \succ B_3 \succ B_4$

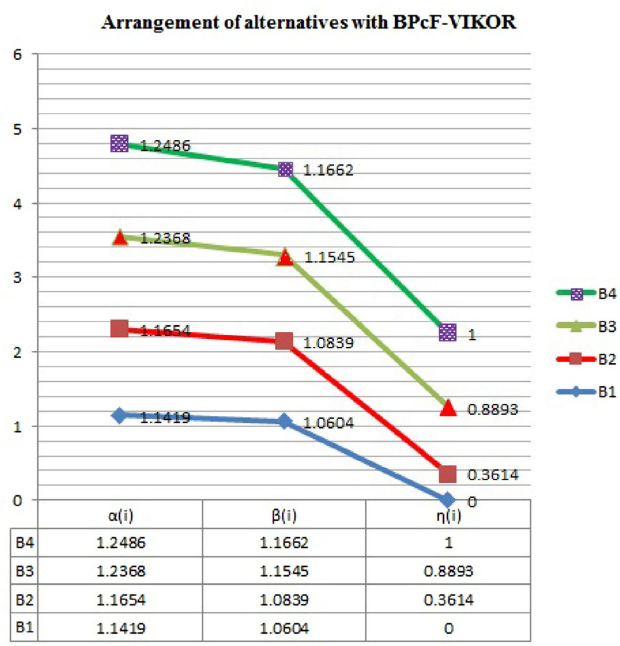

Fig. 1 Ranking order of alternatives

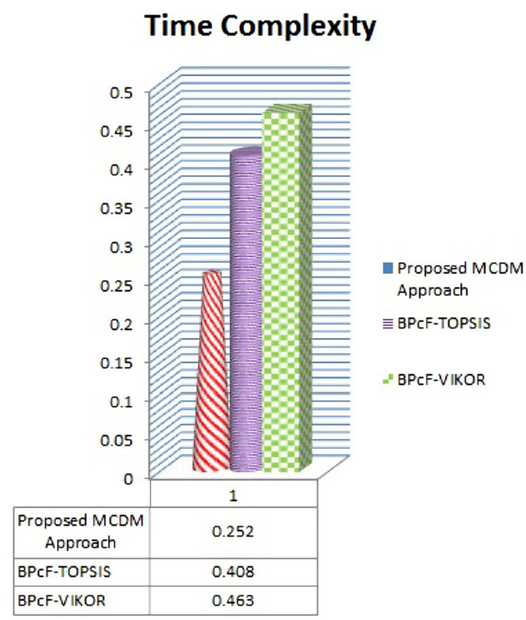

Fig. 2 Comparison of time complexity

Step 5. Since the **Condition 1.** and **Condition 2.** are satisfied, and hence the alternative B^1 is the superior one which is illustrated graphically in Fig. 1.

The outcomes obtained by BP_cF- TOPSIS and BP_cF-VIKOR are coincides completely with the proposed MCDM approach which shows the authentication and reliability of our model.

6.3 Time Complexity

In this subsection, a brief and comprehensive comparative have been performed by using time complexity (TC) analysis. TC is measured by the number of fundamental operations which is executed for given information as a function of input size k (k is the number of alternatives). Generally, AHP presented by Saaty [19] is used to find out the weights of criteria in MCDM processes. However, it has an issue in time complexity, when the number of criteria increases in MCDM problems then AHP takes much time to execute. To overcome this problem, we introduced the LP technique to evaluate the weights of criteria which takes less time to execute the big data calculations. By using MATLAB, we have $\frac{1}{4}$seconds, $\frac{2}{5}$seconds and $\frac{23}{50}$seconds TC values (run time) obtained by proposed MCDM approach, BP_cF- TOPSIS and BP_cF- VIKOR, respectively. Figure 2 reveals the comparison based on TC between the proposed, BP_cF- TOPSIS and BP_cF- VIKOR approaches.

The analysis indicates that our proposed MCDM approach take minimum TC value that is our approach is better than BP_cF- TOPSIS and BP_cF- VIKOR approaches.

7 Sensitivity Analysis

Generally, the data for MCDM problems are unclear and imprecise so, there is a need for an instrument that provides us with an effective decision that's why we use the SA in this regard. SA can be applied to investigate the variation of outcomes by changing the weights of criteria. In this section, weighted SA is used to examine the effect on the outcomes achieved by the proposed model after changing the weights of criteria. A formula introduced by Alireza [5] is implemented to get the new weight vector for the criteria

Table 2 Results obtained for altering the weights of criteria

Alternatives	Original (BP_cFWG)	Increment in w_1	Increment in w_2	Increment in w_3
B_1	0.5299	0.5333	0.5133	0.5267
B_2	0.4660	0.4208	0.4483	0.4862
B_3	0.0927	0.1160	0.0878	0.0576
B_4	0.0975	0.1009	0.0902	0.0776

Table 3 Results obtained for altering the weights of criteria

Alternatives	Original (BP_cFWA)	Increment in w_1	Increment in w_2	Increment in w_3
B_1	0.5194	0.5242	0.5033	0.5161
B_2	0.4265	0.4140	0.3692	0.4733
B_3	0.0961	0.1152	0.0680	0.0808
B_4	0.0679	0.1017	0.0407	0.0591

and then investigate the behavior of the outcomes attained by the proposed model. We altered the weights of criteria individually by adding 0.2 and determine the effect on the outcomes. The results obtained by altering the values of weights of criteria are illustrated in Tables 2 and 3.

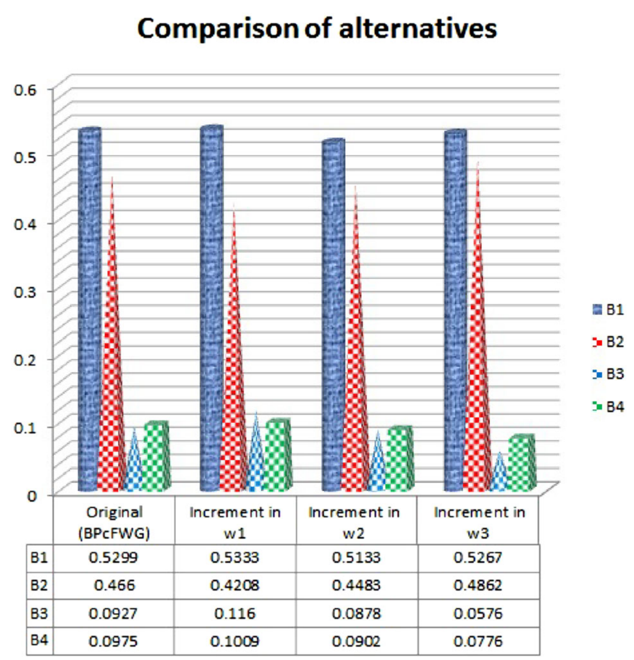

Fig. 3 Ranking comparison with distinct weights of criteria

Figures 3 and 4 show the slight fluctuation among the alternatives after altering the weights of criteria by adding 0.2 in each weight, it reveals that a slight change is occurred in the numeric values of score function, however, the order of ranking remains same which empower our proposed model.

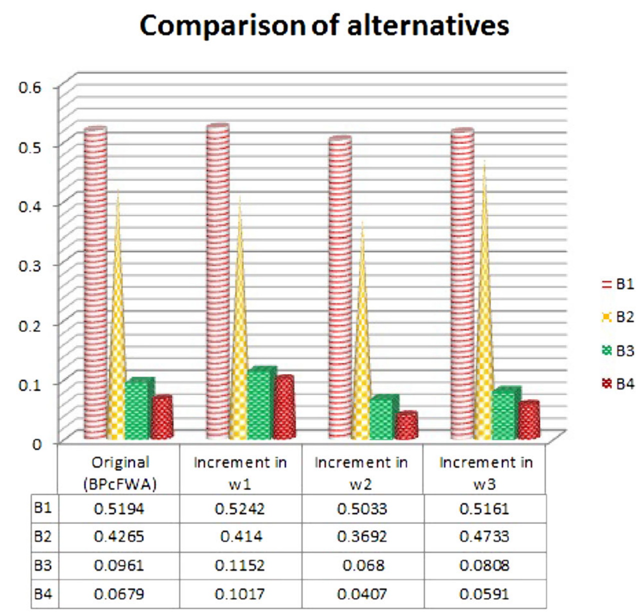

Fig. 4 Ranking comparison with distinct weights of criteria

8 Discussion and Conclusions

The main purpose of this work is to make it easy for DMs to make decisions. AOs are presented for BP_cFSs which accumulate more information in a better way. The features and characteristics of AOs like BP_cFWA and BP_cFWG operators are discussed comprehensively. An MCDM approach is proposed to deal with the uncertain, vague and incomplete information under the framework of BP_cFSs. However, existing techniques [26, 33] deal with the intuitionistic fuzzy and picture fuzzy environment rather than BP_cF environment. Consequently, the suggested MCDM approach based on BP_cFWG and BP_cFWA operators

Table 4 Comparison with proposed MCDM model, BP_cF-TOPSIS and BP_cF-VIKOR

Alternatives	BP_cFWG	BP_cFWA	BP_cF-TOPSIS	BP_cF-VIKOR
B_1	0.5299	0.5194	0.5243	0
B_2	0.4660	0.4265	0.4894	0.3614
B_3	0.0927	0.0961	0.4861	0.8893
B_4	0.0975	0.0679	0.4908	1.0000

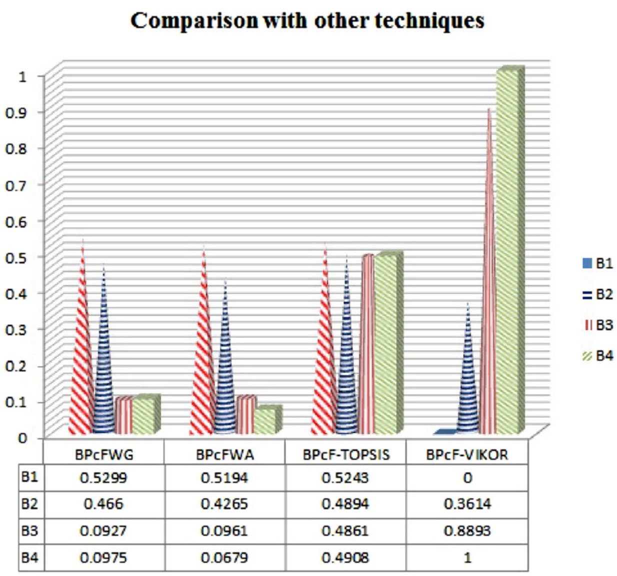

Fig. 5 Ranking order of alternatives

reveals the trustworthiness in the field of rational manipulation. The results obtained by P_cFWG, BP_cFWA, BP_cF-TOPSIS, and BP_cF-VIKOR are penned in Table 4.

Figure 5 reveals a comparative analysis of proposed MCDM approach with other techniques graphically.

AOs play a vital role, to sum up, the information in the decision-making procedure, and therefore, the current article presented a couple of novel AOs for BP_cFSs, named as BP_cFWA and BP_cFWG operators. Various features of the endorsed operators are presented and then, we have used these operators to remedy MCDM problems. There has always been a problem for DMs to allocate the weights to criteria. To overcome this problem, we have used the LP model to find out the weights of criteria so that favouritism can be eliminated. Based on BP_cFWA and BP_cFWG operators, an MCDM model is presented to resolve a money investment problem for validity and effectiveness. These basic AOs can help us to develop generalized weighted AOs, Bonferroni, and Hamy mean for BP_cF environment in future.

Acknowledgements The authors would like to thank all reviewers and the editor in chief for their insightful comments that have improved the quality of the final manuscript.

Compliance with Ethical Standards

Conflicts of interest All authors has no conflict of interest.

Ethical Approval This article does not contain any studies with animals performed by any of the authors.

Informed Consent Informed consent was obtained from all individual participants included in the study.

References

1. Akram, M., Arshad, M.: A novel trapezoidal bipolar fuzzy TOPSIS method for group decision-making. Group Decis. Negot. **28**, 565–584 (2019)
2. Akram, M., Shumaiza, Arshad, M.: Bipolar fuzzy TOPSIS and bipolar fuzzy ELECTRE-I methods to diagnosis. Comput. Appl. Math. **39**(1), 7 (2020)
3. Alghamdi, M.A., Alshehri, N.O., Akram, M.: Multi-criteria decision-making methods in bipolar fuzzy environment. Int. J. Fuzzy Syst. **20**, 2057–2064 (2018)
4. Aliyev, R.R.: Interval linear programming based decision making on market allocations. Procedia Comput. Sci. **120**, 47–52 (2017)
5. Alireza, A., Abbas, A.: Sensitivity analysis of TOPSIS technique: he results of change in the weight of one attribute on the final ranking of alternatives. J. Optim. Ind. Eng. **7**, 23–28 (2011)
6. Cuong, B. C.: Picture fuzzy sets, a new concept for computational intelligence problems. Published in proceedings of the third world congress on information and communication technologies. Hanoi, Vietnam, pp. 1–6 (2013)
7. Cuong, B.C., Kreinovich, V.: Picture fuzzy sets. J. Comput. Sci. Cybernet. **30**(4), 409–420 (2014)
8. Chen, S.M., Han, W.H.: A new multiattribute decision making method based on multiplication operations of interval-valued intuitionistic fuzzy values and linear programming methodology. Inf. Sci. **429**, 421–432 (2018)
9. Garg, H.: Some picture fuzzy aggregation operators and their applications to multicriteria decision-making. Arab. J. Sci. Eng. **42**(12), 5275–5290 (2017)
10. Gul, Z.: Some bipolar fuzzy aggregations operators and their applications in multicriteria group decision making (Doctoral dissertation, M. Phil Thesis). (2015)
11. Han, Y., Shi, P., Chen, S.: Bipolar-valued rough fuzzy set and its applications to the decision information system. IEEE Trans. Fuzzy Syst. **23**(6), 2358–2370 (2015)
12. He, P., Ng, T.S., Su, B.: Energy-economic recovery resilience with Input-Output linear programming models. Energy Econ. **68**, 177–191 (2017)

13. Jana, C., Senapati, T., Pal, M., Yager, R.R.: Picture fuzzy Dombi aggregation operators: application to MADM process. Appl. Softw. Comput. **74**, 99–109 (2018)
14. Lu, M., Busemeyer, J.R.: Do traditional Chinese theories of Yi Jing (Yin-Yang) and Chinese medicine go beyond western concepts of mind and matter. Mind Matter. **12**(1), 37–59 (2014)
15. Li, Z., Wei, G., Lu, M.: Pythagorean fuzzy Hamy mean operators in multiple attribute group decision making and their application to supplier selection. Symmetry. **10**(10), 505 (2018)
16. Lu, M., Wei, G., Alsaadi, F.E., Hayat, T., Alsaedi, A.: Bipolar 2-tuple linguistic aggregation operators in multiple attribute decision making. J. Intell. Fuzzy Syst. **33**(2), 1197–1207 (2017)
17. Opricovic, S., Tzeng, G.H.: Compromise solution by MCDM methods: a comparative analysis of VIKOR and TOPSIS. Eur. J. Oper. Res. **156**, 445–455 (2004)
18. Peng, X., Dai, J.: Algorithm for picture fuzzy multiple attribute decision making based on new distance measure. Int. J. Uncertain Quantif. **7**, 177–187 (2017)
19. Saaty, R.W.: The analytic hierarchy process-what it is and how it is used. Math. Model. **9**(3–5), 161–176 (1987)
20. Shumaiza, Akram, M., Al-Kenani, A. N., Alcantud, J. C. R.: , Group Decision-Making Based on the VIKOR Method with Trapezoidal Bipolar Fuzzy Information. Symmetry 11: 1313 (2019)
21. Sindhu, M. S., Rashid, T., Kashif, A., Guirao, J. L.: Multiple criteria decision making based on probabilistic interval-valued hesitant fuzzy sets by using LP methodology. Discrete Dyn Nat Soc. 2019 12 pages (2019)
22. Sindhu, M.S., Rashid, T., Kashif, A.: Modeling of linear programming and extended TOPSIS in decision making problem under the framework of picture fuzzy sets. PloS ONE (2019). https://doi.org/10.1371/journal.pone.0220957
23. Sindhu, M.S., Ahsan, M., Rafiq, A., Khan, I.A.: Multiple criteria decision making based on bipolar picture fuzzy sets and extended TOPSIS. J. Math. Comput. Sci. **23**(1), 49–57 (2021)
24. Vanderbei, R.J.: Linear Programming: Foundations and Extensions. Springer, Berlin, Heidelberg (2014)
25. Wang, C.Y., Chen, S.M.: An improved multiattribute decision making method based on new score function of interval-valued intuitionistic fuzzy values and linear programming methodology. Inf. Sci. **411**, 176–184 (2017)
26. Wang, C., Zhou, X., Tu, H., Tao, S.: Some geometric aggregation operators based on picture fuzzy sets and their application in MADM. Ital. J. Pure Appl. Math. **37**, 477–492 (2017)
27. Wei, G.: Hesitant Fuzzy prioritized operators and their application to multiple attribute group decision making. Knowl Based Syst. **31**, 176–182 (2012)
28. Wei, G., Zhao, X., Wang, H., Lin, R.: Pythagorean hesitant fuzzy choquet integral aggregation operators and their applications to multiple attribute decision making. Int. Interdiscip. J. **15**(2), 441–448 (2012)
29. Wei, G.W.: Picture fuzzy aggregation operators and their application to multiple attribute decision making. J. Intell. Fuzzy Syst. **33**, 713–724 (2017)
30. Wei, G., Wang, J., Wei, C., Wei, Y., Zhang, Y.: Dual hesitant pythagorean fuzzy Hamy mean operators in multiple attribute decision making. IEEE Access **7**, 86697–86716 (2019)
31. Xia, M., Xu, Z., Chen, N.: Some hesitant fuzzy aggregation operators with their application in group decision making. Group Decis. Negot. **22**(2), 259–279 (2013)
32. Wei, G., Alsaadi, F.E., Hayat, T., Alsaedi, A.: Hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. J. Intell. Fuzzy Syst. **33**(2), 1119–1128 (2017)
33. Xu, Z.S., Yager, R.R.: Some geometric aggregation operators based on intuitionistic fuzzy sets. Int. J. Gen. Syst. **35**, 417–433 (2006)
34. Xu, X.R., Wei, G.W.: Dual hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. Int. J. Knowl.-Based Intell. Eng. Syst. **21**(3), 155–164 (2017)
35. Zadeh, L.A.: Fuzzy sets. Inf. Control. **8**(3), 338–353 (1965)
36. Zhang, H., Zhang, R., Huang, H., Wang, J.: Some picture fuzzy Dombi Heronian mean operators with their application to multi-attribute decision-making. Symmetry **10**(11), 593 (2018)
37. Zhang, W.R.: Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In NAFIPS/IFIS/NASA'94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige. IEEE, pp. 305–309 (1994)
38. Zhang, W.R.: (Yin)(Yang) Bipolar fuzzy sets. In: 1998 IEEE International Conference on Fuzzy Systems Proceedings. IEEE World Congress on Computational Intelligence (Cat. No. 98CH36228) IEEE, 01 835–840 (1998)
39. Zhang, W.R., Zhang, L.: Bipolar logic and bipolar fuzzy logic. Inf. Sci. **165**(3–4), 265–287 (2004)
40. Zhang, W.R., Zhang, J.H., Shi, Y., Chen, S.: Bipolar linear algebra and Yin Yang-N-element cellular networks for equilibrium-based bio system simulation and regulation. J. Biol. Syst. **17**(4), 547–576 (2009)
41. Zhang, W.R.: Equilibrium energy and stability measures for bipolar decision and global regulation. Int. J. Fuzzy Syst. **5**(2), 114–122 (2003)

M. Sarwar Sindhu M. SARWAR SINDHU received the M. Phill degree from department of mathematics, National University of Computer and Emerging Sciences (FAST-NUCES), Lahore, Pakistan, 2013. He is currently a PhD scholar at department of mathematics, University of Management and Technology, Lahore, Pakistan. His main area of interest is MCDM based on linear programming optimization for some extensions fuzzy sets. He has authored 10 journal articles in well-known journals.

Tabasam Rashid Tabasam Rashid received the Ph.D. degree in Mathematics from National University of Computer and Emerging Sciences, Pakistan, in 2015. He is teaching Mathematical courses from January 2010. Now he is working as Associate Professor at University of Management and Technology, Lahore, Pakistan. He has authored more than 80 journal articles to professional journals. He is currently the Reviewer of American Mathematical Society and peer reviewer of several international journals. He has been selected as "Best Researcher (2015 and 2017)", "Won the research award on highest number of HEC recognized publications" and "Won the research award for highest impact factor publication" at University of Management and Technology, Lahore, Pakistan. He has supervised three PhD and fifteen MS students. His current research interests include fuzzy sets, fuzzy decision making, clustering algorithms, computing with words, similarity measures, aggregation operators, preference relations, etc.

Agha Kashif Agha Kashif received his M.Sc. degree and M.S. degree in Mathematics from G.C. University, Lahore. He received his Doctorate in Mathematics from FAST NUCES, Lahore, Pakistan. He is currently working as an Associate Professor at the Department of Mathematics, UMT, Lahore, Pakistan. His research interests include graph theory, algebra, combinatorics, BCK-algebra and fuzzy mathematics.