Int. J. Fuzzy Syst. (2021) 23(1):107–127
https://doi.org/10.1007/s40815-020-01002-9

An Advanced Optimization Technique for Smart Production Using α-Cut Based Quadrilateral Fuzzy Number

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Received: 1 April 2020 / Revised: 23 September 2020 / Accepted: 3 November 2020 / Published online: 5 January 2021

Abstract In the design phase of a new smart product, production costs are unpredictable due to location, transport, and engineering design. In these situations, consequently, cost optimization becomes ambiguous. This paper presents a methodology to obtain optimization through a fuzzy linear programming problem (FLPP) in which fuzzy numbers signify the right-side parameters. The comparative investigation of modeling and optimizing creation cost through a new α-cut based quadrilateral fuzzy number is proposed to solve the fuzzy linear programming and the necessary operations on the proposed number. Due to the probabilistic increase and decrease in the accessibility of the various constraints, the actual expected total cost fluctuates. In this respect, a unique situation of instability is incorporated, and reasonable models to reduce the cost of eradication in the creation process are presented. The main endeavor is made to look at the credibility of optimized cost utilizing the α-cut based quadrilateral FLPP models, and the outcome is contrasted with its augmentation. The data of the production cost of RCF Kapurthala is taken, and the creation expenses of various mentors from the year 2010–2011 are considered as input parameters. The aggregate cost is focused on the objective function. The least low, lower, upper, and most upper bounds are computed for each situation, and then systems of optimized fuzzy LPP are constructed. The credibility of quadrilateral fuzzy LPP concerning all situations is obtained and using this membership grade, the minimum, and highest minimum costs are illustrated.

Keywords Production planning · Cost-effective analysis · Uncertainty · Fuzzy number · Fuzzy linear programming problem

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1 Introduction

The construction of a new product has been significant research interest in the field of production management for some time, and the optimization of it has attracted considerable interest over the last three decades [1–5]. Consequently, production cost optimization has become an essential component for every productive and sustainable production company. Cost minimization is an appropriate criterion for competitive efficiency, monopoly, revenue-free objectives, and various environments. Also, it is steady with the maximization of the expected income of the business, but the rate of production is unpredictable [6]. Conceptual expansion is commonly used and is essential for measuring welfare, performance, technological exchange, input replacement, size, the scale of economies, and many other critical monetary concepts [7].

Moreover, when production is unpredictable, the validity and essence of cost minimization are much less evident. It culminated in a multitude of methods and algorithms designed to address particular problems in the literature. The fuzzy set theory provides an optimal method for modeling productivity by incorporating production project dynamism and addressing the subjective and probabilistic complexity of productivity factors. A mixed-integer non-linear mathematical model was developed by [8] in which the fuzzy set theory represented the uncertainties. The introduction of fuzzy sets was an invention by Zadeh [9] that provided the graduation of the membership relationship. The applications of fuzzy sets, including decision making [10], probability [11], control theory [12], medical studies [13], and the characterization of complex systems [14] are currently used in the majority of scientific disciplines.

Fuzzy number, a crucial component of the fuzzy set theory, is very prominent when it appears to describe the unknown phenomena in real issues. Two unique perspectives can be seen as fuzzy numbers, the first one is their membership function, and the second being their alpha-cut. The two considerations are equivalent, and one may be superior to the other, depending on the details that we want to consider.

The interpretation of fuzzy numbers from an alpha-cut perspective is an interval approach. In contrast, with the assistance of various arithmetic operations with the prerequisites, the different characteristics of the fuzzy number could be found. Different features of the fuzzy number could also be identified [15] using the various arithmetic operations with the necessary criteria. To study some properties of fuzzy arithmetic operations, [16] and Guerra have analyzed the decomposition of fuzzy numbers and have compared the proposed approximation to standard fuzzy arithmetic. Taleshian and Rezvani [17] presented two trapezoidal fuzzy numbers with methods for solving the multiplication operation. Banerjee [18] mentioned the four basic arithmetic operations of generalized trapezoidal fuzzy numbers. While using the definition of distribution and complementary distribution functions, Garg and Ansha [19] studied the basic arithmetic operations for two generalized positive parabolic fuzzy numbers. As a significant piece of mathematical programming, linear programming is one of the regularly applied operation research systems. Due to the vulnerability of objective objects and the fluctuation of human muses, there are many situations in which target values, technological coefficients, and assets cannot be precisely incorporated into the linear programming model. Fuzzy numbers of techniques have been proposed to deal with fuzzy linear programming problems. Numerous researchers have extensively studied a fuzzy linear program, a triangular and trapezoidal fuzzy linear programming model. In this paper, we proposed a newly constructed α-cut based quadrilateral fuzzy number in the right-hand side of the fuzzy linear programming problem, which gives a clear picture of the optimization in the uncertain conditions general framework of FLP [20] we assess the optimal values of different situations.

The rest of the paper is arranged accordingly: Sect. 2 provides the literature review of certain related studies. Section 3 briefly summarizes the essential preliminaries related to the theory of fuzzy set and fuzzy number in optimization. The proposed membership function of α-cut based quadrilateral number and its arithmetic operations are introduced in Sects. 4 and 5 that presents the model of FLPP using the proposed fuzzy number. Section 6 discusses the case study and identifies data to illustrate the idea presented in this paper, the numerical representation of the α-cut based quadrilateral fuzzy number. Eventually, Sect. 7 provides some concrete conclusions and recommendations for future research.

2 Literature Review

Many manufacturers are experiencing tight market competition leading to massive economic growth. Production time and quality are therefore increasingly important in terms of competitiveness, and consumer products can be found at lower prices, improved quality, and faster shipments. Several methods have been used to optimize the cost of production items. In [21], two criteria for an optimum production policy were described:

(a) the maximum production rates and
(b) the minimum cost.

2.1 Optimization Models for Production Cost Analysis

The optimal policy, analyzed for processes, indicates a clear trend in a variable dimension generated by a known distribution process. For a cleaner production–transportation preparation issue in manufacturing plants, a multi-objective optimization model has been suggested by [22]. In [23], the reset and non-compliance costs were combined to ensure an economic model equalizing resetting and non-resetting. Wang et al. [24] used a novel multi-objective linear programming model to evaluate capacity planning's cost-effectiveness and relationship with suppliers. Kazaz and Sloan [25] address the combination of critical ratios to determine the optimum policy and tradeoffs involving production profits, maintenance costs, and process-related effects. Still, they did not consider reduced quality costs. Chen and Huang [26] developed optimization models to assess the optimal process to reduce production costs and quality losses due to different quality features. To solve integrated production decision-making problems in a fuzzy environment with multi-component, multi-supplier, and multi-source machines under reparable remanufacturing systems, a fuzzy multi-object linear programming model was created by [27]. Kapur et al. [28] and Wen and Mergen [29] addressed design issues but exceeded short-term requirements during the manufacturing process. A modified cost minimization model of Wen and Mergen [29] was proposed by [30] with a linear, asymmetrical, and quadratic measuring of loss in quality of products within specification to determine the optimal mean of the process. A time estimate algorithm for the different manufacturing systems has been proposed by [31] based on the fuzzy linear regression analysis using quadratic programming. McNally et al. [32] improve the understanding of how project protocols influence the aspects of product innovation and concluded that the project protocols have an indirect impact on the financial performance of the company by supporting products and discontinuing promotions. A novel optimization approach was used in [33] to optimize the market share of the new product by taking advantage of the partial value expectations of potential customers for the production and development process. This approach converges globally and can use established heuristic techniques to speed up convergence. Jafari et al. [34] studied the uncertainty that was taken into account the preferences for nurses and the number of surplus nurses. Four different types of fuzzy solutions were put to overcome the uncertainties in the research problem. The fuzzy models were then built based on the fuzzy solutions proposed to provide a more flexible solution for policymakers. Zhang and Tseng [35] suggested a structural approach based on product and process modeling, which discussed the complexities and created a bridge between diversity and costs in two phases. Phase I developed models and studied the product and process variations. Phase II explored the relationship between various goods and costs and identifies additional key contributors, namely cost drivers for variety in the product family. Zengin and Ada [36] analyzed the objective cost function to control the cost of the product while promoting quality requirements as required by the customer. The article was also designed to create a cost control system that would promote the achievement of target costs, particularly in small- and medium-sized enterprises.

Optimization models for process optimization indicate that production costs have been minimized, and the quality loss has been suggested with different quality characteristics [26]. The goal of the models is to achieve the optimum mean process based on the manufacturer and the consumer's minimum estimated total costs. A non-linear integer programming model was introduced in [37], which was developed to address the cost-optimization issue, considering the impact of the overall float loss. An integer linear programming decision support model was introduced by [38] to enable the optimal management and distribution of water resources. The model is intended to minimize the total cost of water, including the economic costs of treatment and distribution and the related costs to the environment. Hong et al. [39] focused on distribution allocation in a two-stage supply chain with fixed costs, conceived as a model of integer programming. A heuristic Ant Colony Optimization (ACO) was created to solve the model.

Interval estimates are straightforward methods of estimating a project by splitting the project into project packages and using statistical distributions to approximate each package. The approach to the range estimates applies to companies and analysts and is more similar to what the experts themselves articulate, making an approach technically more comfortable to use.

2.2 Fuzzy Linear Programming Problems Through Fuzzy Numbers

Wang and Peng [40] transformed the FNLPP problem into the linear programming interval number and applied the designs of FNLPP's the r-best optimal solution and r-worst optimum solution to the problem. Shaheen et al. [41] provided an alternative approach to range estimation based on the fuzzy set theory. It has provided a technique for the extraction and processing of fuzzy numbers by experts within the fuzzy framework of the study. A Fuzzy State Estimation (FSE) model is used by [42] to model the uncertainty in estimating the state of the power system, based on the optimization of restricted linear programming. Uncertain measurements are expressed as fuzzy numbers with a triangular and trapezoidal membership function with medium and propagated values. In the article Jagadeeswari and Nayagam [43], efforts were made for using the distance function in terms of the α-cuts to address the problem of triangular approximations of the fuzzy parabolic numbers. A new nearest trapezoidal approach operator with expected interval survival was prescribed in [44]. Chen and Cheng [45] presented the subjective perspectives of decision-makers with trapezoidal fuzzy numbers in linguistic terms. An FLFP solution procedure where objective function, capital, and technical coefficients are fuzzy triangle numbers has been proposed in [46]. Ebrahimnejad and Tavana [47] proposed an approach to address FLP problems in which symmetric fuzzy trapezoidal numbers are interpreted as objective function coefficients and right-side values, while real numbers are the components of the matrix coefficient. An approach has been suggested by [48] to solve the FFLP problem, with a symmetric trapezoidal fuzzy number representing the parameters without any conversion of crisp equivalent problems. Three cases of linear programming problems, such as real numbers, type-1 fuzzy numbers, and type-2 fuzzy sets, were discussed in [49]. A complete linear defuzzification function defined in a trapezoidal fuzzy number subsection of a fuzzy number vector space is the best way to solve a linear programming problem with real objects in the type-1 fuzzy linear programming. The theorem α-level representation was the approach for obtaining the optimal type-2 solution. Dong and Wan [50] developed a new approach for the linear fuzzy system in which trapezoidal fuzzy numbers (TrFNs) were used to represent all objective coefficients, technology coefficients, and tools. Also, the proposed model of the paper is not only mathematically extensive as well as the degree of recognition of the fuzzy limitations was violated adequately. Karimi et al. [51] provides the best–worst method for addressing multi-attribute decision-making (MADM) issues in the fuzzy situation. Then the weight of the criterion is entirely determined by a fuzzy linear mathematical model. In addition to that, all the weights are quantified by fuzzy triangular numbers. Bolos et al. [52] discussed a new hybrid model using linear schedules and fuzzy numbers to achieve tangible assets in the business. This hybrid model is suggested as the basis of decision variables, objective function coefficients, and a matrix of constraints for the resolution in the form of triangular fuzzy numbers. de Andrés–Sánchez [53] improves the applicability of the formula's fuzzy version by proposing and analyzing three triangular approximations where triangular variability numbers are the underlying asset price, volatility, and free interest rate. As far as various decision-making issues with intuitive trapezoidal interval information were concerned, some operating laws were established concerning intuitive trapezoidal fuzzy interval numbers, score function, and the precise functioning of trapezoidal fuzzy intuitive interval numbers. A model optimization based on the attribute weight maximization deviation method was developed by Wei [54]. And, in order to address the limitation of Wei's method Kaur et al. [55] a modified linear programming model was proposed to define the optimal weight vector of the attributes.

According to the literature, it seemed that most researchers found optimizations in uncertainty due to fuzzy linear programming problems with different fuzzy triangular and trapezoidal numbers. The α-cut based quadrilateral fuzzy number proposed in these articles is the generalization of the triangular and trapezoidal fuzzy numbers. Also, a new fuzzy linear programming model is built with the proposed fuzzy number to demonstrate the practical, real-world situations. The following section explains the preliminary research.

3 Preliminaries and Basic Definitions

In this section, we analyze some of the fuzzy set theory's fundamental principles and terminology used for the other sections of this paper.

Definition 3.1 [56] A fuzzy set $\tilde{B}$ is defined on universe set Y defined as follows:

$$\tilde{A} = \left\{(y, \mu_{\tilde{B}}(y) : y \in Y\right\}, \tag{1}$$

where $\mu_{\tilde{B}}(y)$ represents the membership function of the fuzzy set whose value lies in the interval [0,1].

Definition 3.2 A fuzzy set $\tilde{B}$ is called normal if there exists at least one element $y \in Y$ with

$$\mu_{\tilde{B}}(y) = 1 \tag{2}$$

Definition 3.3 Let $\tilde{B}$ be a fuzzy set in Y and $\alpha \in [0, 1]$. The *α-cut* of a fuzzy set $\tilde{B}$ in Y is the crisp set $\tilde{B}^{\alpha}$ given by

$$\tilde{B}^{\alpha} = \{y|\, \mu_{\tilde{B}}(Y) \geq \alpha\} \tag{3}$$

Definition 3.4 The *support* of the fuzzy set $\tilde{B}$ of Y is defined as follows:

$$\mathrm{supp}\left(\tilde{B}\right) = \{y|\, \mu_{\tilde{B}}(Y) > 0\} \tag{4}$$

Definition 3.5 A convex fuzzy set $\tilde{B}$ of Y is defined as follows:

$$\mu_{\tilde{B}}\{\lambda y_1 + (1-\lambda)y_2 = \min\{\mu_{\tilde{B}}(y_1), \mu_{\tilde{B}}(y_2)\}, \text{ where } 0 = \lambda = 1 \tag{5}$$

It is said to be a non-convex fuzzy set if the above inequality does not hold.

Definition 3.6 A fuzzy set $\tilde{B}$ in $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions:

(i) $\tilde{B}$ is normal,
(ii) It is convex fuzzy set,
(iii) It is closed in $[0, 1]$ and

(iv) The support of $\tilde{B}$ is bounded.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by $F(\mathbb{R})$: It is well known that if $\tilde{B} \in F(\mathbb{R})$, then the α-cut of $\tilde{B}$ is a closed interval for every $\alpha \in [0, 1]$, i.e., closed, bounded, and convex subset of $\mathbb{R}$. Therefore, the closed interval is denoted by $\tilde{B}_{\alpha} = \left[\widetilde{B_{\alpha}}^{L}, \widetilde{B_{\alpha}}^{U}\right]$. If $\widetilde{B_{\alpha}}^{L} \geq 0 \forall \alpha \in [0, 1]$, then $\tilde{B}$ is called a non-negative fuzzy number.

Definition 3.7 A fuzzy number $\tilde{B} = \left(\beta_i - \varepsilon_i, \beta_i, \beta_i + \varepsilon_i^*\right)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{B}}(Y) = \begin{cases} \dfrac{y-(\beta_i-\varepsilon_i)}{\varepsilon_i} & \beta_i-\varepsilon_i \le y \le \beta_i \\ 1 & y=\beta_i \\ \dfrac{y-(\beta_i-\varepsilon_i^*)}{\varepsilon_i^*} & \beta_i \le y \le \beta_i+\varepsilon_i^* \end{cases} \tag{6}$$

If $\varepsilon_i = \varepsilon_i^*$, it is called a symmetric triangular fuzzy number, otherwise non-symmetric. A triangular fuzzy number $\left(\beta_i-\varepsilon_i, \beta_i, \beta_i+\varepsilon_i^*\right)$ is said to be non-negative fuzzy number iff $\beta_i-\varepsilon_i \ge 0$.

Definition 3.8 A fuzzy number $\tilde{B} = \left(\beta_i-\varepsilon_i, \beta_i, \beta_i^*, \beta_i^*+\varepsilon_i^*\right)$ is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{B}}(Y) = \begin{cases} \dfrac{y-(\beta_i-\varepsilon_i)}{\varepsilon_i} & \beta_i-\varepsilon_i \le y \le \beta_i \\ 1 & \beta_i \le y \le \beta_i^* \\ \dfrac{y-(\beta_i^*-\varepsilon_i^*)}{\varepsilon_i^*} & \beta_i^* \le y \le \beta_i^*+\varepsilon_i^* \end{cases} \tag{7}$$

If $\varepsilon_i = \varepsilon_i^*$, it is called the symmetric trapezoidal fuzzy number, otherwise non-symmetric. A trapezoidal fuzzy number $\left(\beta_i-\varepsilon_i, \beta_i, \beta_i^*, \beta_i^*+\varepsilon_i^*\right)$ is said to be non-negative fuzzy number iff $\beta_i-\varepsilon_i \ge 0$.

3.1 Fuzzy Linear Programming

The standard form fuzzy linear programming is represented by Zimmermann [20]:

$$\begin{aligned} &\text{Max } Z = \sum_{j=1}^{n} c_j y_j \\ &\text{Subject to} \sum_{j=1}^{n} a_{ij} y_j \le \tilde{\beta}_i \\ &\text{Where, } y \ge 0, i,j \in \mathbb{N}, \end{aligned} \tag{8}$$

where $\widetilde{\beta}_i$ is the right triangle fuzzy number, the membership grade of which is defined as follows:

$$\tilde{\beta}_i = \begin{cases} 1 & \text{When } y \le \beta_i \\ \dfrac{\beta_{ii}+\varepsilon_i-y}{\varepsilon_i} & \text{When } \beta_i \le y \le \beta_i+\varepsilon_i \\ 0 & \text{when } y \ge \beta_i+\varepsilon_i \end{cases} \tag{9}$$

Therefore, the membership grade of the right triangle fuzzy number is shown in Fig. 1

The right-hand coefficient is the membership function, i.e., constraint available. The lower and upper constraints of the optimum values must be determined to optimize such a problem. The lower bound value ($\boldsymbol{Z_l}$) is

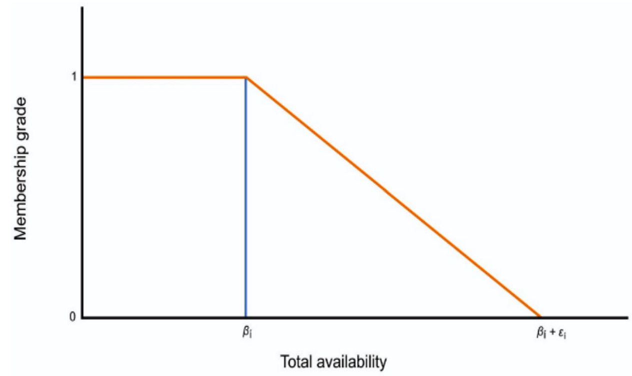

Fig. 1 Representation of the membership function for $\widetilde{\beta}_i$

$$\begin{aligned} &\text{Max } \boldsymbol{Z_l} = \sum_{j=1}^{n} c_j y_j \\ &\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \le \beta_i \\ &\text{Where, } y_j \ge 0, i,j \in \mathbb{N} \end{aligned} \tag{10}$$

The optimal values upper bound ($\boldsymbol{Z_u}$) is as follows:

$$\begin{aligned} &\text{Max } \boldsymbol{Z_u} = \sum_{j=1}^{n} c_j y_j \\ &\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \le \beta_i + \varepsilon_i \\ &\text{Where, } y_j \ge 0, i,j \in \mathbb{N}, \end{aligned} \tag{11}$$

where ε_i is an increase in the probabilistic availability of restrictions. In this case, the total probabilistic increase of access to restrictions is determined by the right coefficient.

The Simplex method can now be used to find a solution for the lower and upper bounds of the LPPs. Using these lower and upper bounds, the optimized fuzzy LPP is obtained as follows:

$$\begin{aligned} &\text{Max } Z = \gamma \\ &\text{Subject to } \gamma(Z_u - Z_l) - \sum_{j=1}^{n} c_j y_j \le -Z_l \\ &\gamma\varepsilon_i + \sum_{j=1}^{n} a_{ij} y_j \le \beta_i + \varepsilon_i, \end{aligned} \tag{12}$$

where $y_j \ge 0, i,j \in \mathbb{N}$, and $\gamma \in (0,1)$ is membership grade.

3.2 Objective of the Proposed Fuzzy Number

The grade of satisfaction at β_i is 1 in the triangular fuzzy number *i.e.*, $\left(\beta_i-\varepsilon_i, \beta_i, \beta_i+\varepsilon_i^*\right)$, but the grade of satisfaction from $\beta_i-\varepsilon_i$ to β_i is determined by the angle of elevation, the scale from 0 to 1, and from β_i to $\beta_i+\varepsilon_i^*$ is defined by the angle of the depression, which is from 1 to 0.

Similarly, the level of satisfaction for β_i to β_i^* is 1 for a trapezoidal number $\left(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i^*\right)$. Nevertheless, the degree of satisfaction from $\beta_i - \varepsilon_i$ to β_i is shown by an elevated angle from 0 to 1 and from β_i^* to $\beta_i^* + \varepsilon_i^*$ by a slump angle of 1 to 0. In some cases, however, these fuzzy numbers do not reflect the actual description of realistic situations. In this article, a newly developed fuzzy number is suggested, "α−" cut based on a fuzzy quadrilateral number $\left(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i^*\right)$ in which the degree of satisfaction from $\beta_i - \varepsilon_i$ to β_i is expressed by an angle of elevation, the range of which is from 0 to β and between β_i to β_i^* is ß to 1, where $\beta \in [0, 1]$. The depression angle with a range of 1 to 0 is also represented from β_i^* to $\beta_i^* + \varepsilon_i^*$.

4 Proposed Membership Function for the α-cut Based Quadrilateral Fuzzy Number

A fuzzy number $\tilde{\boldsymbol{B}} = \langle(\beta_i - \varepsilon_i, \beta_i, \ \beta_i^*, \beta_i^* + \varepsilon_i^*) | \beta_i, \beta_i^* \in \mathbb{R}, \text{ and } \varepsilon_i, \varepsilon_i^* \in \mathbb{R}^+\rangle$, is said to be a **α-cut based quadrilateral fuzzy number** if its membership function $B_i(y) : \mathbb{R} \to [0, 1]$ satisfies the following properties:

(i) It is compact in $\mathbb{R}$.
(ii) It is continuous over $\mathbb{R}$.
(iii) It is monotonic increasing on $[\beta_i - \varepsilon_i, \beta_i^*]$ and monotonic decreasing on $[\beta_i^*, \beta_i^* + \varepsilon_i^*]$.
(iv) It is zero for all $y \in (-\infty, \beta_i - \varepsilon_i) \cup \left(\beta_i^* + \varepsilon_i^* \infty\right)$.
(v) It is normal.
(vi) It is a triangular fuzzy number when $\beta = 0$ and $\beta = \frac{\varepsilon_i}{\beta_i^* - (\beta_i - \varepsilon_i)} \forall \alpha \in [0, 1]$.
(vii) It is a trapezoidal fuzzy number when $\beta = 1$, $\forall \alpha \in [0, 1]$.
(viii) It is convex according to (vi), (vii) and $\theta_1 \geq \theta_2 \forall \alpha \leq \beta, \alpha, \beta \neq 0$.
(ix) It is non-convex when $\theta_1 < \theta_2 \forall \alpha \leq \beta, \alpha, \ \beta \neq 0$. where $\theta_1 = \tan^{-1}\left(\frac{\beta}{\epsilon_i}\right)$ and $\theta_2 = \tan^{-1}\left(\frac{\bar{\beta}}{\beta_i^* - \beta_i}\right)$.

A fuzzy number $\tilde{\boldsymbol{B}} = \left\langle\left(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i^*\right) | \beta_i, \beta_i^* \in \mathbb{R}, \text{ and } \varepsilon_i, \varepsilon_i^* \in \mathbb{R}^+\right\rangle$ is called a quadrilateral fuzzy number, and its membership function is defined by

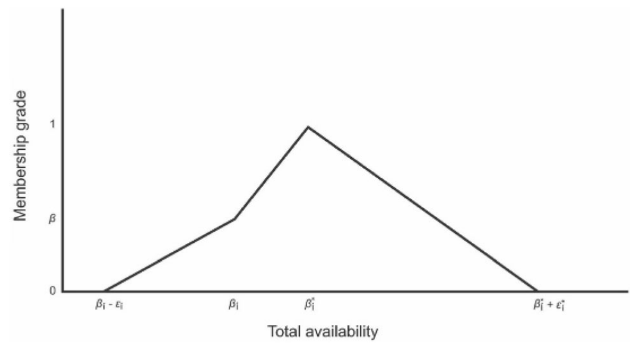

Fig. 2 Membership grade for the fuzzy quadrilateral number

$$B_i(y) = \begin{cases} \beta + \left(\dfrac{y - \beta_i}{\beta_i^* - \beta_i}\right)\bar{\beta} & \text{When } \beta_i \leq y \leq \beta_i^* \\ \left(\dfrac{y - \beta_i + \varepsilon_i}{\varepsilon_i}\right)\beta & \text{When } \beta_i - \varepsilon_i \leq y \leq \beta_i \\ \dfrac{\beta_i^* + \varepsilon_i^* - y}{\varepsilon_i^*} & \text{When } \beta_i^* \leq y \leq \beta_i^* + \varepsilon_i^* \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where β represents the membership grade and $\bar{\beta}$ represents the complement of β.

Meanwhile, the quadrilateral membership function shown in Fig. 2 is most frequently used to represent the α-cut of the fuzzy quadrilateral number $\tilde{\boldsymbol{B}}$.

The α-cut of the fuzzy quadrilateral number $\tilde{\boldsymbol{B}}$ in closed interval is defined as follows:

$$B_\alpha = \begin{cases} \left[\beta_i + \alpha\left(\beta_i^* - \beta_i\right), \left(\beta_i^* + \varepsilon_i^*\right) - \alpha\varepsilon_i^*\right] & \beta = 0, \alpha \in [0, 1] \\ \left[(\beta_i - \varepsilon_i) + \alpha\varepsilon_i, \left(\beta_i^* + \varepsilon_i^*\right) - \alpha\varepsilon_i^*\right] & \beta = 1, \alpha \in [0, 1] \\ \left[(\beta_i - \varepsilon_i) + \alpha\left(\beta_i^* - \beta_i + \varepsilon_i\right), \left(\beta_i^* + \varepsilon_i^*\right) - \alpha\varepsilon_i^*\right] & \beta = \dfrac{\varepsilon_i}{\beta_i^* - (\beta_i - \varepsilon_i)}, \alpha \in [0, 1] \\ I_1 \cup I_2 \cup I_3 & \alpha \leq \beta, \alpha, \beta \neq 0 \end{cases} \tag{14}$$

Where, $I_1 = \left[\beta_i - \varepsilon_i, (\beta_i - \varepsilon_i) + \dfrac{\alpha\varepsilon_i}{\beta_1}\right), I_2 = \left[\beta_i, \beta_i + \dfrac{\bar{\alpha} - \beta}{\bar{\beta}}\left(\beta_i^* - \beta_i\right)\right)$ and

$I_3 = \left[\beta_i^*, \left(\beta_i^* + \varepsilon_i^*\right) - \alpha\varepsilon_i^*\right)$

4.1 Proposed Arithmetic Operations Between α-Cut Based Quadrilateral Fuzzy Numbers

In this section, the improved arithmetic operations have been proposed between α-cut based quadrilateral fuzzy numbers by using α-cut.

Let $\tilde{B}_i^p = (\beta_i^p - \varepsilon_i^p, \beta_i^p, \beta_i^{p*}, \beta_i^{p*} + \varepsilon_i^{p*})$ and $\tilde{B}_i^q = (\beta_i^q - \varepsilon_i^q, \beta_i^q, \beta_i^{q*}, \beta_i^{q*} + \varepsilon_i^{q*})$ be two α-cut based quadrilateral fuzzy numbers with membership functions B_i^p and B_i^q respectively, which can be written as

$$B_i^p(y) = \begin{cases} \beta^p + \left(\dfrac{y-\beta_i^p}{\beta_i^{p*}-\beta_i^p}\right) \times \overline{\beta^p} & \text{When } \beta_i^p \le y \le \beta_i^{p*} \\ \left(\dfrac{y-\beta_i^p+\varepsilon_i^p}{\varepsilon_i^p}\right) \times \beta^p & \text{When } \beta_i^p - \varepsilon_i^p \le y \le \beta_i^p \\ \dfrac{\beta_i^{p*}-\varepsilon_i^{p*}-y}{\varepsilon_i^{p*}} & \text{When } \beta_i^{p*} \le y \le \beta_i^{p*}+\varepsilon_i^{p*} \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where β^p represents the membership grade and $\overline{\beta^p}$ represents the complement of β^p.

$$B_i^q(y) = \begin{cases} \beta^q + \left(\dfrac{y-\beta_i^q}{\beta_i^{q*}-\beta_i^q}\right) \times \overline{\beta^q} & \text{When } \beta_i^q \le y \le \beta_i^{q*} \\ \left(\dfrac{y-\beta_i^q+\varepsilon_i^q}{\varepsilon_i^q}\right) \times \beta^q & \text{When } \beta_i^q - \varepsilon_i^q \le y \le \beta_i^q \\ \dfrac{\beta_i^{q*}-\varepsilon_i^{q*}-y}{\varepsilon_i^{q*}} & \text{When } \beta_i^{q*} \le y \le \beta_i^{q*}+\varepsilon_i^{q*} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where β^q represents the membership grade and $\overline{\beta^q}$ represents the complement of β^q. where β_i^p, β_i^{p*}, β_i^q and β_i^{q*} are real numbers, $\varepsilon_i^p, \varepsilon_i^{p*}, \varepsilon_i^q$ and ε_i^{q*} are the positive real numbers, such that $\beta^p \le \beta^q$. Take $\beta^s \in [\beta^p, \beta^q]$, then make a β^s-cut of fuzzy number $\tilde{B_i^q}$ such that $\tilde{B_i^q}$ is transformed into a new α-cut based quadrilateral fuzzy number as $\tilde{B_i^{q+}} = (\beta_i^q - \varepsilon_i^q,\ \beta_i^{q+}, \beta_i^{q*},\ \beta_i^{q*} + \varepsilon_i^{q*})$, where $\beta_i^{q+} = \beta_i^q - \varepsilon_i^q + \frac{\beta^r}{\beta^q} \times \varepsilon_i^q$ for membership function. Clearly if $\beta^p = \beta^q$ then $\beta^r = \beta^p = \beta^q$, $\beta_i^{q+} = \beta_i^q$ and hence the new α-cut based quadrilateral fuzzy number B_i^{q+} is same as that of α-cut based quadrilateral fuzzy number B_i^q.

Now the a β^S-cut of fuzzy number $\tilde{B_i^p}$ and $\tilde{B_i^q}$ become the α^p-cut of the α-cut based quadrilateral fuzzy number $\widetilde{\boldsymbol{B^p}}$ in closed interval is defined as follows:

$$B_\alpha^p = \begin{cases} [\beta_i^p + \alpha^p(\beta_i^{p*} - \beta_i^p), (\beta_i^{p*} + \varepsilon_i^{p*}) - \alpha^p \varepsilon_i^{p*}] & \beta^p = 0, \alpha^p \in [0,1] \\ [(\beta_i^p - \varepsilon_i^p) + \alpha^p \varepsilon_i^p, (\beta_i^{p*} + \varepsilon_i^{p*}) - \alpha^p \varepsilon_i^{p*}] & \beta^p = 1, \alpha^p \in [0,1] \\ [(\beta_i^p - \varepsilon_i^p) + \alpha^p(\beta_i^{p*} - \beta_i^p + \varepsilon_i^p), (\beta_i^{p*} + \varepsilon_i^{p*}) - \alpha^p \varepsilon_i^{p*}] & \beta^p = \dfrac{\varepsilon_i^p}{\beta_i^{p*} - (\beta_i^p - \varepsilon_i^p)}, \alpha^p \in [0,1] \\ I_1^p \cup I_2^p \cup I_3^p & \alpha^p \le \beta^p, \alpha^p, \beta^p \ne 0 \end{cases} \tag{17}$$

Where, $I_1^p = \left[\beta_i^p - \varepsilon_i^p, (\beta_i^p - \varepsilon_i^p) + \dfrac{\alpha^p \varepsilon_i^p}{\beta^p}\right)$, $I_2^p = \left[\beta_i^p, \beta_i^p + \dfrac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}} (\beta_i^{p*} - \beta_i^p)\right)$ and

$I_3^p = [\beta_i^{p*}, (\beta_i^{p*} + \varepsilon_i^{p*}) - \alpha^p \varepsilon_i^{p*})$

The α^q-cut based quadrilateral fuzzy number $\widetilde{\boldsymbol{B^q}}$ in closed interval is defined as follows:

$$B_\alpha^q = \begin{cases} [\beta_i^q + \alpha^q(\beta_i^{q*} - \beta_i^q), (\beta_i^{q*} + \varepsilon_i^{q*}) - \alpha^q \varepsilon_i^{q*}] & \beta^q = 0, \alpha^q \in [0,1] \\ [(\beta_i^q - \varepsilon_i^q) + \alpha^q \varepsilon_i^q, (\beta_i^{q*} + \varepsilon_i^{q*}) - \alpha^q \varepsilon_i^{q*}] & \beta^q = 1, \alpha^q \in [0,1] \\ [(\beta_i^q - \varepsilon_i^q) + \alpha^q(\beta_i^{q*} - \beta_i^q + \varepsilon_i^q), (\beta_i^{q*} + \varepsilon_i^{q*}) - \alpha^q \varepsilon_i^{q*}] & \beta^q = \dfrac{\varepsilon_i^q}{\beta_i^{q*} - (\beta_i^q - \varepsilon_i^q)}, \alpha^q \in [0,1] \\ I_1^q \cup I_2^q \cup I_3^q & \alpha^q \le \beta^q, \alpha^q, \beta^q \ne 0 \end{cases} \tag{18}$$

Where, $I_1^q = \left[\beta_i^q - \varepsilon_i^q, (\beta_i^q - \varepsilon_i^q) + \frac{\alpha^q \varepsilon_i^q}{\beta^q}\right)$, $I_2^q = \left[\beta_i^q, \beta_i^q + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}} (\beta_i^{q*} - \beta_i^q)\right)$ and $I_3^q = [\beta_i^{q*}, (\beta_i^{q*} + \varepsilon_i^{q*}) - \alpha^q \varepsilon_i^{q*})$

Thus, we quantify improved arithmetical operations based on these β^s-cuts: addition, subtraction, scalar propagation, division, etc., between the two quadrilateral fuzzy numbers.

Theorem 4.1 *Addition of two α-cut based quadrilateral fuzzy numbers* $\tilde{B_i^p} = (\beta_i^p - \varepsilon_i^p,\ \beta_i^p, \beta_i^{p*},\ \beta_i^{p*} + \varepsilon_i^{p*})$ *and* $\tilde{B_i^q} = (\beta_i^q - \varepsilon_i^q,\ \beta_i^q, \beta_i^{q*},\ \beta_i^{q*} + \varepsilon_i^{q*})$ *with two different confidence levels generates a α-cut based quadrilateral fuzzy number* $\tilde{B_i^s} = \tilde{B_i^p} + \tilde{B_i^q} = (\ \beta_i^s - \varepsilon_i^s,\ \beta_i^s, \beta_i^{s*},\ \beta_i^{s*} + \varepsilon_i^{s*})$ *where*

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p) + (\beta_i^q - \varepsilon_i^q) \tag{19}$$

$$\beta_i^s = \beta_i^p + \beta_i^q - \frac{\beta^s}{\beta^p} \varepsilon_i^q \tag{20}$$

$$\beta_i^{s*} = \beta_i^{p*} + \beta_i^{q*} \tag{21}$$

$$\beta_i^{s*} + \varepsilon_i^{s*} = (\beta_i^{p*} + \varepsilon_i^{p*}) + (\beta_i^{q*} + \varepsilon_i^{q*}) \tag{22}$$

Proof See Appendix 1.

Theorem 4.2 *If* $\tilde{B} = \langle (\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i^*) | \beta_i, \beta_i^* \in \mathbb{R}, \text{and}\ \varepsilon_i, \varepsilon_i^* \in \mathbb{R}^+ \rangle$ *be a α-cut based quadrilateral fuzzy number then* $k\tilde{B}$ *is again a α-cut based quadrilateral fuzzy number given by*

$$k\tilde{B} = \begin{cases} (k(\beta_i - \varepsilon_i), k\beta_i, k\beta_i^*, k(\beta_i^* + \varepsilon_i^*)) & ; k > 0 \\ (k(\beta_i^* + \varepsilon_i^*), k\beta_i^*, k\beta_i, (k(\beta_i - \varepsilon_i)) & ; k < 0 \end{cases} \tag{23}$$

For $k > 0$ the membership function is given by

$$B_{ki}(y) = \begin{cases} \beta + \left(\dfrac{y - k\beta_i}{\beta_i^* - \beta_i}\right) \times \bar{\beta} & \text{When } k\beta_i \le y \le k\beta_i^* \\ \left(\dfrac{y - k(\beta_i - \varepsilon_i)}{\varepsilon_i}\right) \times \beta & \text{When } k(\beta_i - \varepsilon_i) \le y \le k\beta_i \\ \dfrac{k(\beta_i^* + \varepsilon_i^*) - y}{\varepsilon_i^*} & \text{When } k\beta_i^* \le y \le k(\beta_i^* + \varepsilon_i^*) \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

For $k < 0$ the membership function is given by

$$B_{ki}(y) = \begin{cases} \beta + \left(\dfrac{y - k\beta_i^*}{\beta_i - \beta_i^*}\right) \times \bar{\beta} & \text{When } k\beta_i^* \le y \le k\beta_i \\ \left(\dfrac{y - k(\beta_i^* + \varepsilon_i^*)}{\varepsilon_i^*}\right) \times \beta & \text{When } k(\beta_i^* + \varepsilon_i^*) \le y \le k\beta_i^* \\ \dfrac{k(\beta_i - \varepsilon_i) - y}{\varepsilon_i} & \text{When } k\beta_i \le y \le k(\beta_i - \varepsilon_i) \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Proof See Appendix 1.

Theorem 4.3 *Subtraction of two α-cut based quadrilateral fuzzy numbers* $\tilde{B}_i^p = (\beta_i^p - \varepsilon_i^p,\ \beta_i^p, \beta_i^{p*},\ \beta_i^{p*} + \varepsilon_i^{p*})$ *and* $\tilde{B}_i^q = (\ \beta_i^q - \varepsilon_i^q,\ \beta_i^q, \beta_i^{q*},\ \beta_i^{q*} + \varepsilon_i^{q*})$ *with two different confidence levels generates a* $\alpha -$ *cut based quadrilateral fuzzy number* $\tilde{B}_i^s = \tilde{B}_i^p - \tilde{B}_i^q = (\ \beta_i^s - \varepsilon_i^s,\ \beta_i^s, \beta_i^{s*},\ \beta_i^{s*} + \varepsilon_i^{s*})$ *where*

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p) - (\beta_i^{q*} + \varepsilon_i^{q*}) \tag{26}$$

$$\beta_i^s = \beta_i^p - (\beta_i^{q*} + \varepsilon_i^{q*}) - \frac{\beta^s}{\beta^p}\varepsilon_i^{q*} \tag{27}$$

$$\beta_i^{s*} = \beta_i^{p*} - \beta_i^{q*} \tag{28}$$

$$\beta_i^{s*} + \varepsilon_i^{s*} = (\beta_i^{p*} + \varepsilon_i^{p*}) - (\beta_i^q - \varepsilon_i^q) \tag{29}$$

Proof Followed from Theorems 4.1 and 4.2, so we omit here.

Theorem 4.4 *Multiplication of two α-cut based quadrilateral fuzzy numbers* $\tilde{B}_i^p = (\ \beta_i^p - \varepsilon_i^p,\ \beta_i^p, \beta_i^{p*},\ \beta_i^{p*} + \varepsilon_i^{p*})$ *and* $\tilde{B}_i^q = (\ \beta_i^q - \varepsilon_i^q,\ \beta_i^q, \beta_i^{q*},\ \beta_i^{q*} + \varepsilon_i^{q*})$ *with two different confidence levels generates a quadrilateral fuzzy number* $\tilde{B}_i^s = \tilde{B}_i^p \times \tilde{B}_i^q = (\ \beta_i^s - \varepsilon_i^s,\ \beta_i^s, \beta_i^{s*},\ \beta_i^{s*} + \varepsilon_i^{s*})$ *where*

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p)(\beta_i^q - \varepsilon_i^q) \tag{30}$$

$$\beta_i^s = \beta_i^p(\beta_i^q - \varepsilon_i^q) + \frac{\beta^s}{\beta^q}\beta_i^p\varepsilon_i^q \tag{31}$$

$$\beta_i^{s*} = \beta_i^{p*}\beta_i^{q*} \tag{32}$$

$$\beta_i^{s*} + \varepsilon_i^{s*} = (\beta_i^{p*} + \varepsilon_i^{p*})(\beta_i^{q*} + \varepsilon_i^{q*}) \tag{33}$$

$$B_i^s(y) = \begin{cases} \beta^s + \dfrac{-K_1 + \sqrt{K_1^2 + 4M_1(y - N_1)}}{2M_1} & \text{When } \beta_i^s \le y \le \beta_i^{s*} \\ \dfrac{-K_2 + \sqrt{K_2^2 + 4M_2(y - N_2)}}{2M_2} & \text{When } \beta_i^s - \varepsilon_i^s \le y \le \beta_i^s \\ \dfrac{K_3 + \sqrt{K_3^2 + 4M_3(y - N_3)}}{2M_3} & \text{When } \beta_i^{s*} \le y \le \beta_i^{s*} + \varepsilon_i^{s*} \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

where

$K_1 = \frac{\beta_i^p\left(\beta_i^{q*} - \beta_i^q\right)}{\overline{\beta^q}} + \frac{\beta_i^q\left(\beta_i^{p*} - \beta_i^p\right)}{\overline{\beta^s}},$

$K_2 = \frac{(\beta_i^p - \varepsilon_i^p)\left(\varepsilon_i^{q+} - \varepsilon_i^q\right)}{\overline{\beta^q}} + \frac{(\beta_i^q - \varepsilon_i^q)\varepsilon_i^p}{\overline{\beta^s}}$

$K_3 = (\beta_i^{p*} + \varepsilon_i^{p*})(\varepsilon_i^{q*}) + (\beta_i^{q*} + \varepsilon_i^{q*})\varepsilon_i^{p*}$

$M_1 = \frac{\left(\beta_i^{p*} - \beta_i^p\right)\left(\beta_i^{q*} - \beta_i^q\right)}{\overline{\beta^s\beta^q}},\ M_2 = \frac{\varepsilon_i^p\left(\varepsilon_i^{q+} - \varepsilon_i^q\right)}{\overline{\beta^s\beta^q}},\ M_3 = \varepsilon_i^{p*}\varepsilon_i^{q*}$

$N_1 = \beta_i^p\beta_i^q,\ N_2 = (\beta_i^p - \varepsilon_i^p)(\beta_i^q - \varepsilon_i^q),\ N_3 = (\beta_i^{p*} + \varepsilon_i^{p*})(\beta_i^{q*} + \varepsilon_i^{q*})$

Proof See Appendix 2.

Theorem 4.5 *If* $\tilde{\boldsymbol{B}} = \left(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i^*\right)$ *represents a α-cut based quadrilateral fuzzy number then the inverse of* $\tilde{\boldsymbol{B}}$ *, i.e.,* $\tilde{\boldsymbol{B}}^{-1} = \left(\left(\beta_i^* + \varepsilon_i^*\right)^{-1}, \left(\beta_i^*\right)^{-1}, (\beta_i)^{-1}, (\beta_i - \varepsilon_i)^{-1}\right)$ *is also a α-cut based quadrilateral fuzzy number whose membership function is given by*

$$B_i^{-1}(y) = \begin{cases} \beta^* + \left(\dfrac{y - \left(\beta_i^*\right)^{-1}}{(\beta_i)^{-1} - \left(\beta_i^*\right)^{-1}}\right)\overline{\beta^*} & \text{When}\left(\beta_i^*\right)^{-1} \le y \le (\beta_i)^{-1} \\ \left(\dfrac{y - \left(\beta_i^* + \varepsilon_i^*\right)^{-1}}{\left(\beta_i^*\right)^{-1} - \left(\beta_i^* + \varepsilon_i^*\right)^{-1}}\right)\beta^* & \text{When}\left(\beta_i^* + \varepsilon_i^*\right)^{-1} \le y \le \left(\beta_i^*\right)^{-1} \\ \dfrac{(\beta_i - \varepsilon_i)^{-1} - y}{(\beta_i - \varepsilon_i)^{-1} - (\beta_i)^{-1}} & \text{When}(\beta_i)^{-1} \le y \le (\beta_i - \varepsilon_i)^{-1} \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

where β^* *represents the membership grade and* $\overline{\beta^*}$ *represents the complement of* β^*.

Theorem 4.6. *Division of two α-cut based quadrilateral fuzzy numbers* $\tilde{B}_i^p. = (\ \beta_i^p - \varepsilon_i^p,\ \beta_i^p, \beta_i^{p*},\ \beta_i^{p*} + \varepsilon_i^{p*})$ *and* $\tilde{B}_i^q = (\ \beta_i^q - \varepsilon_i^q,\ \beta_i^q, \beta_i^{q*},\ \beta_i^{q*} + \varepsilon_i^{q*})$ *with two different confidence levels generates a α-cut based quadrilateral fuzzy number* $\tilde{B}_i^s = \frac{\tilde{B}_i^p}{\tilde{B}_i^q} = \tilde{B}_i^p \times \widetilde{(B_i^q)}^{-1} = (\beta_i^s - \varepsilon_i^s,\ \beta_i^s, \beta_i^{s*},\ \beta_i^{s*} + \varepsilon_i^{s*})$

where

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p) \times (\beta_i^{q*} + \varepsilon_i^{q*})^{-1} \tag{36}$$

$$\beta_i^s = \beta_i^p \times (\beta_i^{q*})^{-1} \tag{37}$$

$$\beta_i^{s*} = \beta_i^{p*} \times \left(\frac{1}{(\beta_i^q - \varepsilon_i^q) + \frac{\beta^s}{\beta^q}\varepsilon_i^q}\right) \tag{38}$$

$$\beta_i^{s*} + \varepsilon_i^{s*} = (\beta_i^{p*} + \varepsilon_i^{p*}) \times (\beta_i^q - \varepsilon_i^q)^{-1} \tag{39}$$

Proof As $\tilde{B}_i^q = (\beta_i^q - \varepsilon_i^q,\ \beta_i^q, \beta_i^{q*},\ \beta_i^{q*} + \varepsilon_i^{q*})$.Thus $\frac{1}{\tilde{B}_i^q} = \left(\frac{1}{\beta_i^{q*} + \varepsilon_i^{q*}}, \frac{1}{\beta_i^{q*}}, \frac{1}{\tilde{B}_i^q}, \frac{1}{\beta_i^q - \varepsilon_i^q}\right)$ and $\tilde{B}_i^p \times \frac{1}{\tilde{B}_i^q}$. Therefore, the proof of this theorem is similar to Theorem 4.4.

5 Fuzzy Linear Programming Through the α-Cut Based Quadrilateral Fuzzy Number

The standard form of FLP in Eq. (8) is considered to be the quadrilateral fuzzy number $\tilde{\boldsymbol{B}} = \left(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i^*\right)$, as a consequence of increased and decreased availability of restrictions instead of the right triangle fuzzy number $\widetilde{\beta}_i$.Therefore, the general structure of the least lower, lower, upper bounds, and the most upper bounds of the optimal values are defined as follows.

The least lower bound ($\boldsymbol{Z}_l^*$)

$$\begin{aligned}&\text{Max}\,\mathbf{Z}_l^* = \sum_{j=1}^{n} c_j y_j\\&\text{Subject to}\ \sum_{j=1}^{n} a_{ij}x_j \le \beta_i - \varepsilon_i\\&\text{Where, } y_j \ge 0, i, j \in \mathbb{N}\end{aligned} \tag{40}$$

The lower bound $(\mathbf{Z_l})$

$$\begin{aligned}&\text{Max}\,\mathbf{Z}_l = \sum_{j=1}^{n} c_j y_j\\&\text{Subject to}\ \sum_{j=1}^{n} a_{ij}y_j \le \beta_i\\&\text{Where, } x_j \ge 0, i, j \in \mathbb{N}\end{aligned} \tag{41}$$

The upper bound $(\mathbf{Z_u})$

$$\begin{aligned}&\text{Max}\,\mathbf{Z}_u = \sum_{j=1}^{n} c_j y\\&\text{Subject to}\ \sum_{j=1}^{n} a_{ij}y \le \beta_i^*\\&\text{Where, } y_j \ge 0, i, j \in \mathbb{N}\end{aligned} \tag{42}$$

Now the most upper bound $(\mathbf{Z}_u^*)$

$$\begin{aligned}&\text{Max}\,\mathbf{Z}_u^* = \sum_{j=1}^{n} c_j y_j\\&\text{Subject to}\ \sum_{j=1}^{n} a_{ij}y_j \le \beta_i^* + \varepsilon_i^*\\&\text{Where, } y_j \ge 0, i, j \in \mathbb{N}\end{aligned} \tag{43}$$

Using the techniques of the simplex method to find the values of the optimization of these bounds, i.e.,z_u, z_l, z_u^* and z_l^* are the upper, lower, most upper, and least lower bounds.

5.1 Optimized Fuzzy LPP Model for α-Cut Based Quadrilateral Fuzzy Number

Here an optimized fuzzy LPP model based on an α-cut based quadrilateral fuzzy number is proposed in order to obtain the optimized values of these limits as defined below:

$$\begin{aligned}&\text{Max}\,Z = \gamma\\&\text{Subject to}\ \gamma(z_l - z_l^*) - \varsigma y \le -z_l^*\\&\gamma(\varepsilon_i) + \sum_{j=1}^{n} \alpha_{ij}y_j \le \beta_i\\&\gamma(z_u - z_l^*) - \varsigma y \le -z_l^*\\&\gamma(\varepsilon_i + \beta_i^* - \beta_i) + \sum_{j=1}^{n} \alpha_{ij}y_j \le \beta_i^*\\&\gamma(z_u^* - z_l^*) - \varsigma y \le -z_l^*\\&\gamma(\beta_i^* - \beta_i + \varepsilon_i^* + \varepsilon_i) + \sum_{j=1}^{n} \alpha_{ij}y_j \le \beta_i^* + \varepsilon_i^*\\&y_j > 0, i, j \in \mathbb{N}\end{aligned} \tag{44}$$

This fuzzy optimized LPP gives the membership grade for our initial LPP. Here λ represents the membership grade and z_u, z_l, z_u^* and z_l^* are the upper, lower, most upper, and least lower bounds.

ςy is the objective function of the initial LPP. The term with the summation sign represents the constraints of given LPP. ε_i^* and ε_i are the probabilistic increment and decrement, respectively, in the availability of the constraints. Figure 3 demonstrates the flow chart of the FLPP based on the proposed fuzzy number.

6 Numerical Experiment

The crisp optimization techniques are not good enough to illustrate the targeted optimum result in the fluctuated situation or feasible uncertainty in the following situations.

6.1 Situation 1 (Based on Right Triangle Fuzzy Number)

A company manufactures two products such as P_1 and P_2.Each product unit of P_1 requires twice the time required for each product of P_2. Total hours available are at least 300 h per day and may, as a result of special arrangements, be extended to 400 h per day. The supply of material is at least adequate for 200 units of both items, P_1 and P_2 per day, but according to the previous experience, the supply may likely be expanded to 300 units per day. The costs are \$5 on P_1 and \$3 on P_2. The question is, how many units of P_1 and P_2 should products be made every day to minimize the overall cost?

Let y_1 and y_2 denote the number of units of products P_1, P_2 made in one day, respectively. Then the problem can be formulated as the following fuzzy linear programming problems:

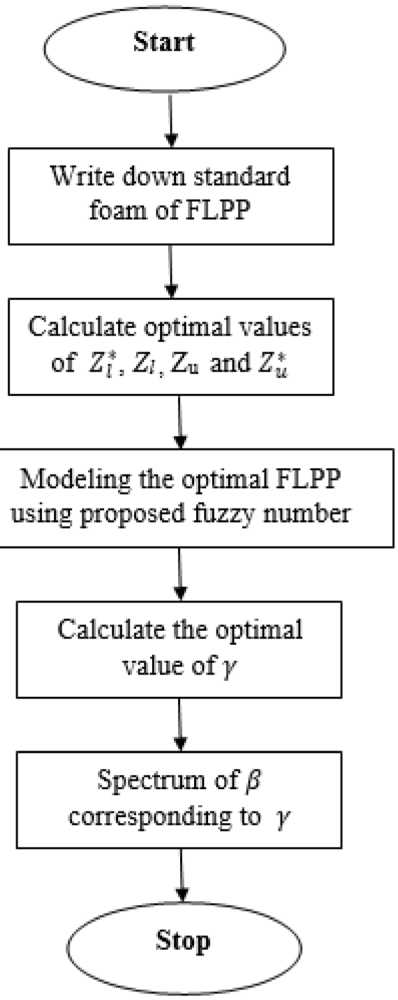

Fig. 3 Flow chart of the FLPP based on proposed fuzzy number

$$\begin{aligned}&\text{Min}\, z = 5y_1 + 3y_2 (\text{cost function})\\&\text{Subject to}\, y + y_2 \geq \widetilde{\beta_1} (\text{material constraint})\\&2y_1 + y_2 \geq \widetilde{\beta_2} (\text{working hours constraint})\\&y_1, y_2 \geq 0.\end{aligned} \tag{45}$$

where $\widetilde{\beta_1}$ is defined by

$$\widetilde{\beta_1} = \begin{cases} 1 & \text{When } y \leq 200 \\ \dfrac{300 - y}{100} & \text{When } 200 \leq y \leq 300 \\ 0 & \text{when } y \geq 300 \end{cases} \tag{46}$$

where $\widetilde{\beta_2}$ is defined by

$$\widetilde{\beta_2} = \begin{cases} 1 & \text{When } y \leq 300 \\ \dfrac{400 - y}{100} & \text{When } 300 \leq y \leq 400 \\ 0 & \text{when } y \geq 400 \end{cases} \tag{47}$$

The optimum lower limit value is $Z_l = 800$ with LPP (10), which is obtained by $y_1 = 100, y_2 = 100$ and the optimum upper limit value is $Z_u = 1100$ with when $y_1 = 100, y_2 = 200$ with LPP (11). The solution of FLPP (12), the minimum $\gamma = 0.5$ is obtained for $y_1 = 100, y_2 = 150 \Rightarrow Z_o = 950$.

If there exists an uncertain increment or decrement in the availability of constraint, the fuzzy linear programming problem using α-based quadrilateral fuzzy number, which is proposed to get proper optimization and with the membership grade, the credibility of optimization is also targeted.

6.2 Situation 2 (Based on α-Cut Based Quadrilateral Fuzzy Number)

A company manufactures two products such as P_1 and P_2.Each product unit of P_1 requires twice the time required for each product of P_2. The total availability of working hours lies between 200 and 400 h per day. The degree of satisfaction in working hours depends on availability, from 200 to 260 h a day, from 260 to 300 h a day, and from 300 to 400 h a day. Material supplies range is from 100 to 300 units per day for both P_1 and P_2. However, the amount of material satisfaction is again dependent on availability, which is varying from 100 to 165 h a day, from 165 to around 200 h per day, and from 200 to 300 h per day. The cost margin is \$5 on P_1, and \$3 on P_2. The question is, how many units of P_1 and P_2 should products be made every day to minimize the overall cost?

Let y_1 and y_2 denote the number of units of products P_1, P_2 made in 1 day, respectively. Then the problem can be formulated as the following fuzzy linear programming problems:

$$\begin{aligned}&\text{Min}\, z = 5y_1 + 3y_2 (\text{cost function})\\&\text{Subject to}\, y + y_2 \geq \tilde{\boldsymbol{B}}_1 (\text{material constraint})\\&2y_1 + y_2 \geq \widetilde{\boldsymbol{B}}_2 (\text{working hours constraint})\\&y_1, y_2 \geq 0.\end{aligned} \tag{48}$$

where $\tilde{\boldsymbol{B}}_1$ is defined by

$$\tilde{\boldsymbol{B}}_1 = \begin{cases} \beta + \left(\dfrac{y - 165}{35}\right) \times \bar{\beta} & \text{When } 165 \leq y \leq 200 \\ \left(\dfrac{y - 100}{65}\right) \times \beta & \text{When } 100 \leq y \leq 165 \\ \dfrac{300 - y}{100} & \text{When } 200 \leq y \leq 300 \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

where $\tilde{\boldsymbol{B}}_2$ is defined by

$$\tilde{\boldsymbol{B}}_2 = \begin{cases} \beta + \left(\dfrac{y-260}{40}\right) \times \bar{\beta} & \text{When } 260 \le y \le 300 \\ \left(\dfrac{y-200}{60}\right) \times \beta & \text{When } 200 \le y \le 260 \\ \dfrac{400-y}{100} & \text{When } 300 \le y \le 400 \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

The membership grade for material costs $(\widetilde{\boldsymbol{B}}_1)$ and labor costs $(\tilde{\boldsymbol{B}}_2)$ is shown in Figs. 4 and 5, respectively.

Using the LPP (40) to (43), following optimal values of least lower, lower, upper, and greatest upper bounds are obtained.

$Z_l^* = 500 \;\; y_1 = 100, y_2 = 0,\;\; Z_l = 685 \;\; y_1 = 95\, y_2 = 70,\; Z_u = 800 \;\; y_1 = 100, y_2 = 100,$ and $Z_u^* = 1100 \;\; y_1 = 100, y_2 = 200.$

The solution of this LPP, the minimum $\gamma = 0.2357$ is obtained for $y_1 = 96.1783, y_2 = 53.5032 \Rightarrow Z_o = 641.4011.$

The following Fig. 6 shows the membership grade γ for the optimal cost of the company for the different values of β.

Figure 7 indicates the spectrum for the optimum value of $\beta \in [0.15, 0.24]$ corresponding to the grade of

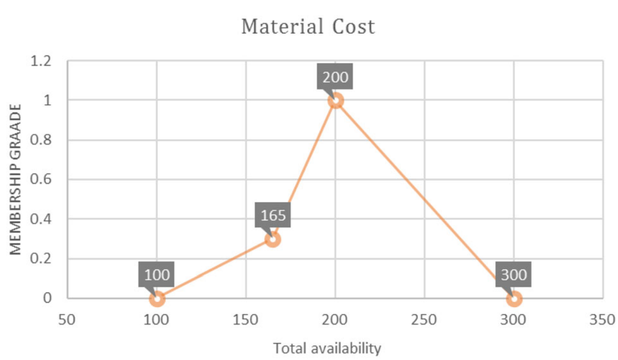

Fig. 4 Membership grade for material cost

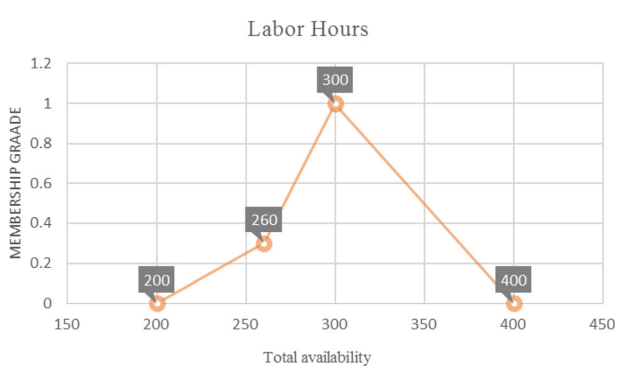

Fig. 5 Membership grade for labor cost

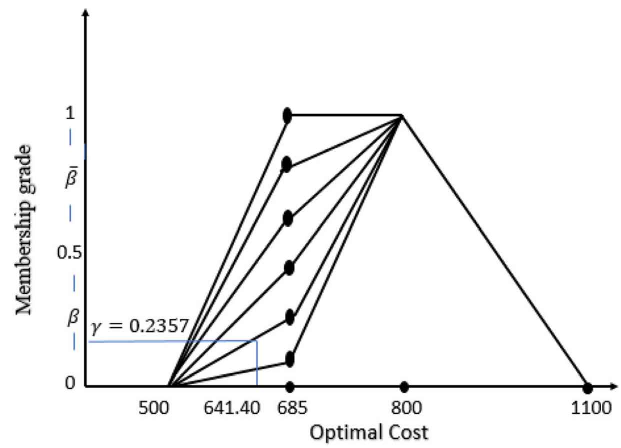

Fig. 6 Membership grade for optimal cost

membership γ, where the lines L1 and L2 depend on β. The lines L3 and L4 (optimal targeted value) are independent of β.

6.3 Data and Problem Identification

The data in Table 1 are from The Railway Industry in Kapurthala for the year 2010–2011. This data shows the manufacturing cost ('in lacs,' i.e., 100,000) of different types of constraints of coaches. Kapurthala Railway Industry was established in 1986. It is a coach manufacturing unit of Indian Railways and manufactured more than 30,000 passenger coaches of different types.

where C_1 is the labor, C_2 is the material, C_4 is the administrative overhead charge, C_5 is the factory overhead charges, C_6 is the township overhead charges, C_7 is the shop overhead charges, C_8 is the Performa charges, $C_9 =$ is the total cost and $C_{10} =$ is total cost (without Performa charges). All costs are in the number of lacs (Indian rupee).

In the year 2010–2011, the total production cost of different coaches is taken as an objective function, which is to be minimized concerning the constraints. As per the given data, the total availability of constraints $C_1, C_2, C_3, C_4, C_5, C_6, C_7,\ C_8, C_9,$ and C_{10}. are rupees 153.2, 2328.22, 256.56, 197.13, 41.23, 18.67, 513.61, 93.83, 3088.88, 2995.05, 1425, respectively.

But they can be extended with some probabilistic increment, decrement, and reach to $(\beta_i - \varepsilon_i)$, β_i, β_i^*, $(\beta_i^* + \varepsilon_i^*)$. In this situation, we propose a newly constructed quadrilateral fuzzy LPP to minimize the total cost of production. Similarly, in certain situations, the total availability of any constraint can be inflexible from the one requirement to the other. Again, it can be intensified and declined by any probabilistic increment and decrement. We

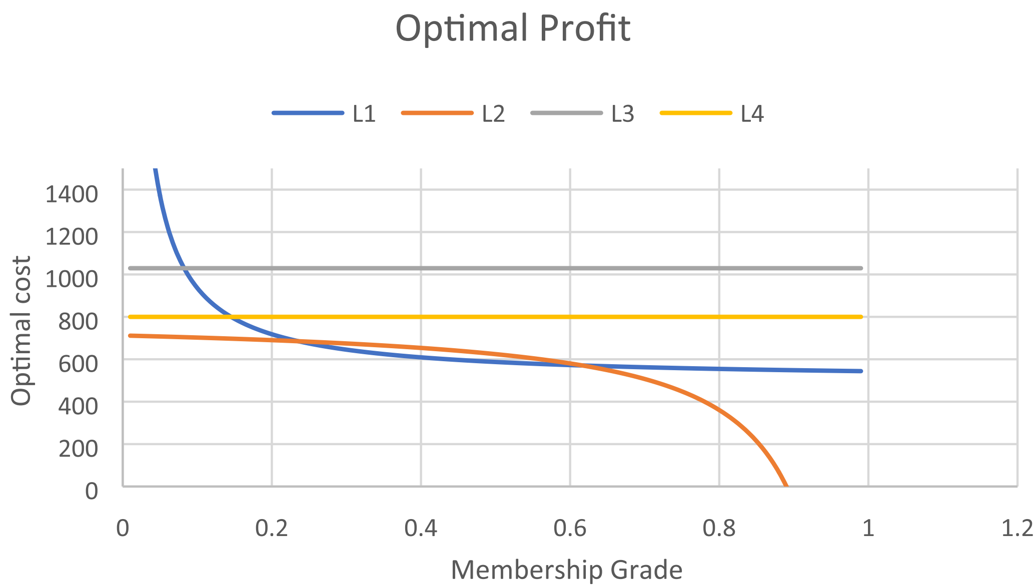

Fig. 7 Represent the β spectrum corresponding to the optimum value of γ

Table 1 Production cost of different coaches

Coaches type	C_1.	C_2.	C_3.	C_4.	C_5.	C_6.	C_7.	C_8.	C_9.	C_{10}.
SCN/AB	4.38	45.7	7.33	5.8	1.17	0.37	14.69	1.63	66.4	64.77
SLR/AB	4.07	41.99	6.81	5.39	1.09	0.34	13.63	1.89	61.58	59.69
GS/AB	4.04	44.49	6.77	5.35	1.08	0.36	13.57	2.37	64.47	62.10
MEMU/MC	9.88	211.93	16.55	13.09	2.65	1.74	34.04	8.27	264.12	255.85
MEMU/TC	4.10	49.13	6.87	5.43	1.10	0.40	13.80	2.13	69.17	67.04
ACCN/SG	7.38	94.06	12.36	9.78	1.98	0.77	24.89	4.14	130.48	126.34
RA SHELL	2.5	33.62	4.58	2.91	0.75	0.23	8.47	1.44	46.03	44.59
LWSCZ	8.11	124.32	14.35	9.14	2.35	0.85	26.68	5	164.11	159.11
LWSCZDA C	14.81	190.08	24.4	19.3	3.91	1.56	49.16	8.23	262.29	254.06
WGCWNA C	7.22	93.76	13.17	8.39	2.16	0.64	24.35	4.07	129.41	125.34
VPHX	3.05	37.29	5.55	3.54	0.91	0.25	10.26	1.57	52.16	50.59
FAC (LC)	9.18	156.57	15.27	12.08	2.45	1.28	31.09	5.96	202.79	196.83
ACCW(LC)	10.70	153.98	17.84	14.11	2.86	1.26	36.07	5.99	206.74	200.75
WGCB(LC)	6.38	110.86	10.59	8.38	1.70	0.91	21.58	3.35	142.17	138.82
EOG/LHB/ ACCB	10.38	184.34	16.98	13.43	2.72	1.51	34.64	7.17	236.53	229.36
EOG/LHB/ WLRRM	10.51	246.6	17.19	13.59	2.75	2.02	35.56	9.42	302.08	292.66
EOG/LHB/ ACCW	11.24	178.25	18.42	14.57	2.95	1.46	37.39	7.12	234.01	226.89
EOG/LHB/ ACCN	11.57	179.28	18.97	15.01	3.04	1.47	38.49	7.20	236.54	229.34
LGSLR	5.91	70.13	9.70	7.67	1.55	0.58	19.49	3.14	98.67	95.53
LGS(LC)	7.79	81.84	12.86	10.17	2.06	0.67	25.76	3.74	119.13	115.39
TOTAL	153.2	2328.22	256.5	197.1	41.2	18.6	513.61	93.83	3088.88	2995.05

Table 2 Probabilistic increments and decrements in the cost parameter

Cost parameters	$\beta_i - \varepsilon_i$	β_i	β_i^*	$\beta_i^* + \varepsilon_i^*$
Labour cost	145.54	150.7	155.7	160.86
Material cost	2211.809	2294.6	2361.84	2444.63
Factory overhead charges	243.732	251.98	261.14	269.13
Administrative overhead charge	187.2735	194.22	200.04	206.99
Township overhead charges	39.1685	40.48	41.98	43.29
Shop overhead charges	17.7365	18.44	18.9	19.60
Total overhead charges	487.9295	505.14	522.08	539.29
Performa charges	89.1385	92.39	95.27	98.5215

are also presenting the trapezoidal fuzzy LPP to minimize the total cost of production.

The membership grade is declined if there is an existence of a specific increment and decrement in the inflexible interval of essential availability. For example—$(\beta_i - \varepsilon_i) \sim \beta_i \sim \beta_i^* \sim (\beta_i^* + \varepsilon_i^*)$. The minimum production cost is targeted at most basic availability for all constraints and at least essential availability for Performa charge.

The fluctuation is given in Table 2.

6.4 Modeling for the System of Optimal Solution

Using the values shown in Tables 1 and 2, the standard form of fuzzy linear programming is defined (Appendix 3).

6.5 Numerical Results

Using the methodology mentioned, the modeling of production cost is being done, and the fuzzy quadrilateral numbers have been constructed. The optimized fuzzy linear programming problem has been constructed using the least lower, lower, upper, and greatest upper bounds, which are calculated. Demonstrating the membership grade of all constraints is represented by Eqs. (52) to (59) and graphically summarizing the constraints shown in Figs. 8, 9, 10, 11, 12, 13, 14 and 15 at the different values of the significance level.

Let B_{lab} be the membership grade for labor cost, and it varies as follows:

$$B_{\text{lab}} = \begin{cases} \beta + \left(\dfrac{y - 150.7}{5}\right) \times \bar{\beta} & \text{When } 150.70 \le y \le 155.7 \\ \left(\dfrac{y - 145.54}{5.16}\right) \times \beta & \text{When } 145.54 \le y \le 150.7 \\ \dfrac{160.86 - y}{5.16} & \text{When } 155.7 \le y \le 160.86 \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

Let B_{mat} be the membership grade for material cost, and it varies as follows:

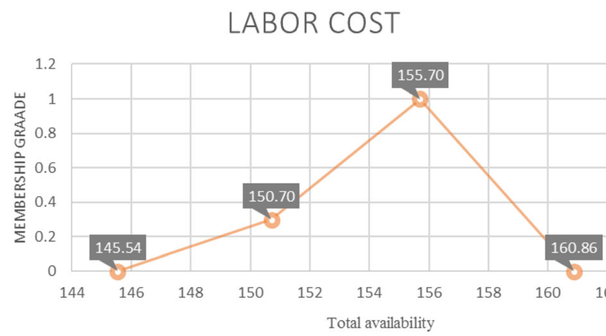

Fig. 8 Membership grade for labor cost

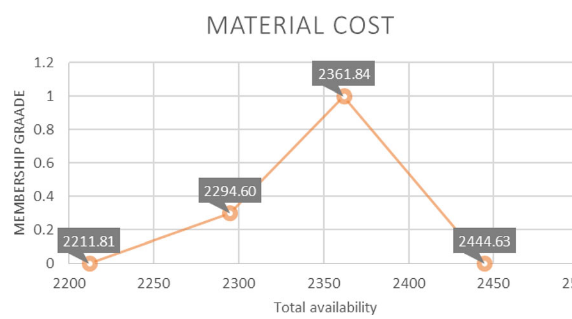

Fig. 9 Membership grade for material cost

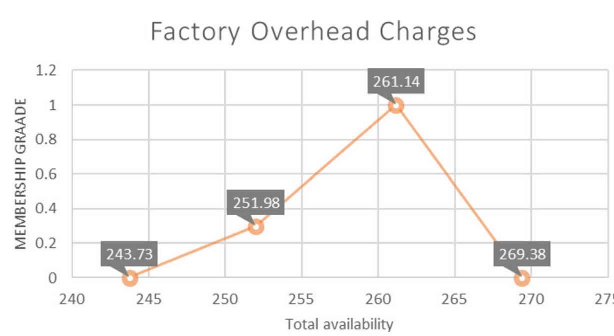

Fig. 10 Membership grade for factory overhead charges

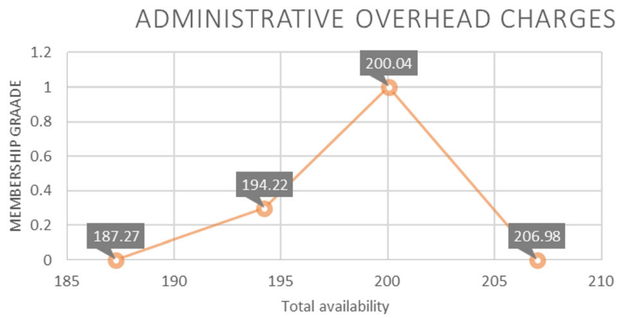

Fig. 11 Membership grade for administrative charges

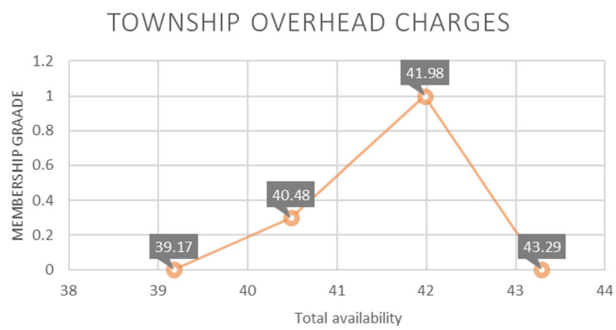

Fig. 12 Membership grade for township overhead charges

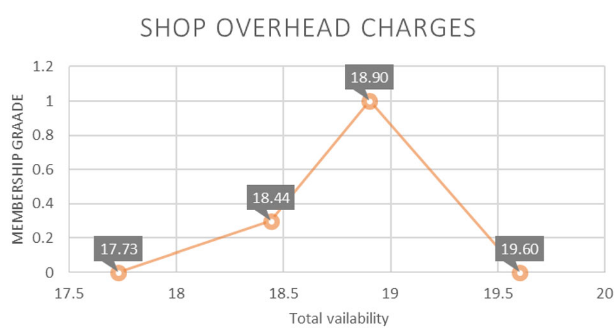

Fig. 13 Membership grade for shop overhead charges

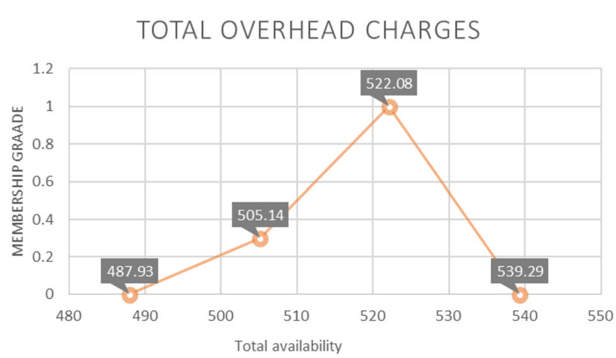

Fig. 14 Membership grade for total overhead charges

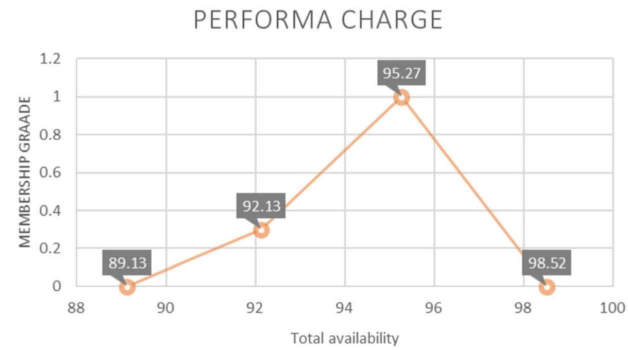

Fig. 15 Membership grade for Performa charge

$$B_{\text{mat}} = \begin{cases} \beta + \left(\dfrac{y - 2294.6}{67.24}\right) \times \bar{\beta} & \text{when } 2294.60 \leq y \leq 2361.84 \\ \left(\dfrac{y - 2211.81}{82.79}\right) \times \beta & \text{when } 2211.81 \leq y \leq 2294.60 \\ \dfrac{2444.63 - y}{82.79} & \text{when } 2361.84 \leq y \leq 2444.63 \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

Let B_{foh} be the membership grade for factory overhead charges, and it varies as follows:

$$B_{\text{foh}} = \begin{cases} \beta + \left(\dfrac{y - 251.98}{9.16}\right) \times \bar{\beta} & \text{when } 251.98 \leq y \leq 261.14 \\ \left(\dfrac{y - 243.73}{8.25}\right) \times \beta & \text{when } 243.73 \leq y \leq 251.98 \\ \dfrac{269.38 - y}{8.24} & \text{when } 261.14 \leq y \leq 269.38 \\ 0 & \textit{otherwise} \end{cases} \tag{53}$$

Let B_{aoh} be the membership grade for administrative overhead charges, and it varies as follows:

$$B_{\text{aoh}} = \begin{cases} \beta + \left(\dfrac{y - 194.22}{5.82}\right) \times \bar{\beta} & \text{when } 194.22 \leq y \leq 200.04 \\ \left(\dfrac{y - 187.27}{6.95}\right) \times \beta & \text{when } 187.27 \leq y \leq 194.22 \\ \dfrac{206.98 - y}{6.94} & \text{when } 200.04 \leq y \leq 206.98 \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

Let B_{tooh} be the membership grade for township overhead charges, and it varies as follows:

$$B_{\text{tooh}} = \begin{cases} \beta + \left(\dfrac{y - 40.48}{1.5}\right) \times \bar{\beta} & \text{when } 40.48 \leq y \leq 41.98 \\ \left(\dfrac{y - 39.19}{1.3115}\right) \times \beta & \text{when } 39.17 \leq y \leq 40.48 \\ \dfrac{41.98 - y}{1.31} & \text{when } 41.98 \leq y \leq 43.29 \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

Let B_{soh} be the membership grade for shop overhead charges, and it varies as folows:

$$B_{\mathrm{soh}} = \begin{cases} \beta + \left(\dfrac{y - 18.44}{0.46}\right) \times \bar{\beta} & \text{when } 18.44 \leq y \leq 18.90 \\ \left(\dfrac{y - 17.73}{0.71}\right) \times \beta & \text{when } 17.73 \leq y \leq 18.44 \\ \dfrac{18.90 - y}{0.70} & \text{when } 18.90 \leq y \leq 19.60 \\ 0 & \text{otherwise} \end{cases} \tag{56}$$

Let B_{toh} be the membership grade for total overhead charges, and it varies as follows:

$$B_{\mathrm{toh}} = \begin{cases} \beta + \left(\dfrac{y - 505.14}{16.94}\right) \times \bar{\beta} & \text{when } 505.14 \leq y \leq 522.08 \\ \left(\dfrac{y - 487093}{17.21}\right) \times \beta & \text{when } 487.93 \leq y \leq 505.14 \\ \dfrac{522.08 - y}{17.21} & \text{when } 522.08 \leq y \leq 539.29 \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

Let $\boldsymbol{B}_{\mathbf{prof}}$ be the membership grade for Performa charge, and it varies as follows:

$$B_{\mathrm{prof}} = \begin{cases} \beta + \left(\dfrac{y - 92.39}{2.88}\right) \times \bar{\beta} & \text{when } 92.39 \leq y \leq 95.27 \\ \left(\dfrac{y - 89.13}{3.26}\right) \times \beta & \text{when } 89.13 \leq y \leq 92.39 \\ \dfrac{95.27 - y}{3.26} & \text{when } 95.27 \leq y \leq 98.52 \\ 0 & \text{otherwise} \end{cases} \tag{58}$$

6.6 Optimal Results of a Numerical Experiment

The approach described the modeling of production cost, and quadrilateral fuzzy numbers for all cost parameters have derived. The least low, lower, upper, and greatest upper bounds are 2918.6, 3026.4, 3118, and 3225.8 rupees in lacs (Indian rupees), respectively. The optimized fuzzy linear programming problem (OFLPP) has been constructed using the least lower, lower, upper, and greatest upper bounds.

The following Table 3 shows the solutions for an optimized value of least lower, lower, upper and most upper bounds and the optimized membership grade.

6.7 Analysis of Numerical Result

The production cost of RCF is minimized using the cost parameter. The optimum production cost has been obtained to get maximum membership grade. It shows that total production cost provided the highest credibility if the optimized cost is considered equal to the range of basic cost [3026, 3118]. The following equation and figure show the fuzzy number for optimized membership grade.

$$B_{\mathrm{Optimizedcost-I}} = \begin{cases} \beta + \left(\dfrac{y - 3026.4}{91.6}\right) \times \bar{\beta} & \text{when } 3026.4 \leq y \leq 3118 \\ \left(\dfrac{y - 2918.6}{107.8}\right) \times \beta & \text{when } 2918.6 \leq y \leq 3026.4 \\ \dfrac{3225.8 - y}{107.8} & \text{when } 3118 \leq y \leq 3225.8 \\ 0 & \text{otherwise} \end{cases} \tag{59}$$

In Fig. 16, L1 and L2 represented the range of optimal costs between [2918.6, 3026.4] and [3026.4, 3118], respectively, which are dependent on the different values of β. At the same time, L3 is the targeted optimal cost, i.e., 3118 and L4; the optimal cost lies between [3118, 3225.8], independent of β.

In Fig. 17, the line graph shows the performance in terms of optimized cost through these lines utilizing the different values of β of α-based quadrilateral fuzzy LPP.

From the given availably of the data, the basic targeted cost is in between lower and upper bounds, that is, L3 = 3118 unit, respectively. The proposed is a model using α-based quadrilateral fuzzy number; the performance of different values of $\beta \in [0.17, 0.30]$(degree of grade satisfaction) is observed to achieve the targeted cost, which lies between [3026.11, 3108.32].

7 Conclusions

In this article, a α-cut based quadrilateral fuzzy number is proposed with the proof of basic mathematical operations and shows the application of it in the fuzzy linear programming problems. This fuzzy number in R.H.S of the FLPP helps show the uncertainty in solution values due to chances of increment and decrement in the availability of different constraints. So, the description of a case of incertitude and the realistic model to extenuate the destruction in the optimization is shown. The comparative analysis of modeling and optimization of the production cost of the various coaches of RCF Kapurthala has been done through a α-cut based quadrilateral fuzzy linear programming problem. The credibility of optimized cost via a α-cut based quadrilateral fuzzy LPP model is examined. The total cost has been targeted to optimize the constraints of different expenses to construct the different types of coaches. The lower, least lower, upper, and most upper bounds have been calculated for the model, and then systems of optimized fuzzy LPP were constructed. The credibility of the model has been obtained and using these memberships grade, the minimum and greatest minimum cost have been exemplified. It is observed that the performance obtained by a α-cut based quadrilateral fuzzy LPP through the different values of $\beta \in [0.17, 0.30]$ shows the degree of satisfaction in ambiguous situations. Further,

Table 3 Optimized membership grade

Parametric variable	$(\beta_i - \varepsilon_i)$.	β_i.	β_i^*	$(\beta_i^* + \varepsilon_i^*)$.	γ.
y_1	9.4114	9.7992	9.9621	10.375	2.38816
y_2	0	0	0	0	0
y_3	0	0	0	0	0.587198
y_4	0	0	0	0	0
y_5	0	0	0	0	9.75224
y_6	0	0	0	0	0
y_7	0	0	0	0	0
y_8	1.8638	1.567	2.452	2.1101	0.0722392
y_9	0	0	0	0	0
y_{10}	0	0	0	0	2.47226
y_{11}	0	0	0	0	0
y_{12}	0	0	0	0	0
y_{13}	0	0	0	0	0
y_{14}	13.982	14.90	14.448	15.408	0
y_{15}	0	0	0	0	6.97911
y_{16}	0	0	0	0	0.51959
y_{17}	0	0	0	0	0
y_{18}	0	0	0	0	0
y_{19}	0	0	0	0	0
y_{20}	0	0	0	0	0
γ	0	0	0	0	0.299184
Z	2918.6	3026.4	3118	3225.8	$\gamma = 0.299184$

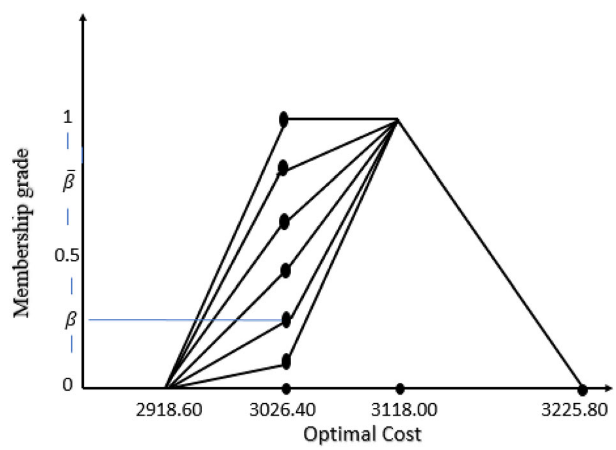

Fig. 16 Membership grade for optimal cost

these numbers are used for optimization through the other groups of FLPP. Also, by using different operations, the suggested methodology can be extended to include the study of uncertainty issues that can be used for further work.

Appendix 1

Let $\tilde{B_i^p}$ and $\tilde{B_i^q}$ be quadrilateral fuzzy numbers with different confidence levels such that $\beta^p \le \beta^q$. Take $\beta^s \in [\beta^p, \beta^q]$, i.e., $\beta^s = \beta^p$ then α^s-cut of $\tilde{B_i^p}$ and $\tilde{B_i^q}$ are

When $\alpha^p \le \beta^p, \alpha^p, \beta^p \ne 0$

$\tilde{B_i^p} = \left[\beta_i^p - \varepsilon_i^p, \beta_i^p - \varepsilon_i^p + \frac{\alpha^p \varepsilon_i^p}{\beta^p}\right) \cup \left[\beta_i^p, \beta_i^p + \overline{\alpha^p -} \quad \beta^p \overline{\beta^p} (\beta_i^{p*} - \beta_i^p)\right) \cup [\beta_i^{p*}, \alpha^p \varepsilon_i^{p*} + \quad (\beta_i^{p*} - \beta_i^p)]$ $\tilde{B_i^q} = [\beta_i^q - \varepsilon_i^q, \beta_i^q - \varepsilon_i^q + \frac{\alpha^q \varepsilon_i^q}{\beta^q}) \cup \left[\beta_i^q, \beta_i^q + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}} (\beta_i^{q*} - \beta_i^q)\right) \cup \quad [\beta_i^{q*}, \alpha^q \varepsilon_i^{q*} + (\beta_i^{q*} - \beta_i^q)]$

$$\begin{aligned}
\tilde{B_i^{q+}} &= \left[\beta_i^q - \varepsilon_i^q, \beta_i^q - \varepsilon_i^q + \frac{\alpha^q \varepsilon_i^{q+}}{\beta^q}\right) \\
&\quad \cup \left[\beta_i^q, \beta_i^q + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}} (\beta_i^{q*} - \beta_i^q)\right) \\
&\quad \cup [\beta_i^{q*}, \alpha^q \varepsilon_i^{q*} + (\beta_i^{q*} - \beta_i^q)] \; \beta_i^q - \varepsilon_i^{q+} \\
&= (\beta_i^q - \varepsilon_i^q) + \frac{\beta^s}{\beta^q} \varepsilon_i^q \Rightarrow \varepsilon_i^{q+} = \beta_i^q - (\beta_i^q - \varepsilon_i^q) - \frac{\beta^s}{\beta^q} \varepsilon_i^q
\end{aligned}$$

Let $\tilde{B_i^s} = \tilde{B_i^p} + \tilde{B_i^{q+}} = \{y | y \in \tilde{B_\alpha^s}\} \forall \alpha^s \in [0, 1]$. Here $\tilde{B_\alpha^s} = \left[\tilde{B_\alpha^{sL}}(\alpha^s), \tilde{B_\alpha^{sU}}(\alpha^s)\right]$ be its α^s-cuts such that $\tilde{B_\alpha^{sL}}(\alpha^s) = \tilde{B_i^{pL}}(\alpha^p) + \tilde{B_i^{qL}}(\alpha^q)$ and $\tilde{B_\alpha^{sU}} \; (\alpha^s) = \tilde{B_i^{pU}}(\alpha^p) + \tilde{B_i^{qU}}(\alpha^q)$ i.e.

$\tilde{B_\alpha^s} = [\tilde{B_i^{pL}}(\alpha^p) + \tilde{B_i^{qL}}(\alpha^q), \tilde{B_i^{pU}} \; (\alpha^p) + \tilde{B_i^{qU}}(\alpha^q)]$

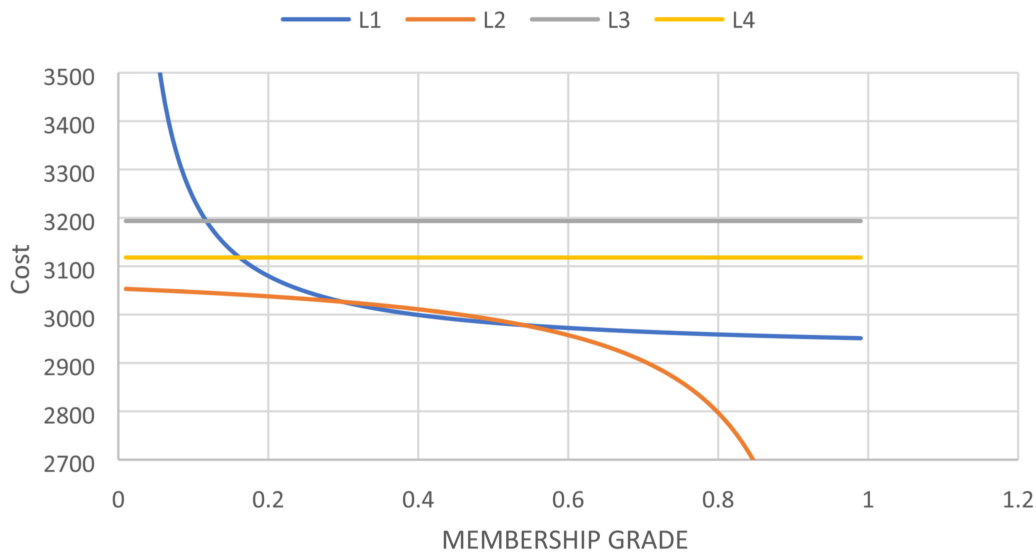

Fig. 17 Representation of the β spectrum corresponding to the optimum value γ

$$\tilde{B}^s_\alpha = I^s_1 \cup I^s_2 \cup I^s_3, \textit{where}$$

$$I^s_1 = \left[\beta^p_i - \varepsilon^p_i + \beta^q_i - \varepsilon^q_i, \beta^p_i - \varepsilon^p_i + \frac{\alpha^p \varepsilon^p_i}{\beta^p} + \beta^q_i - \varepsilon^q_i + \frac{\alpha^q \varepsilon^{q+}_i}{\beta^q}\right)$$

$$I^s_2 = \left[\beta^p_i + \beta^q_i, \beta^p_i + \frac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}}(\beta^{p*}_i - \beta^p_i) + \beta^q_i + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}}(\beta^{q*}_i - \beta^q_i)\right]$$

$$I^s_3 = [\beta^{p*}_i + \beta^{q*}_i, (\beta^{p*}_i + \varepsilon^{p*}_i) - \alpha^p \varepsilon^{p*}_i + (\beta^{q*}_i + \varepsilon^{q*}_i) - \alpha^q \varepsilon^{q*}_i]$$

Now

$$I^s_1 = \left[\beta^p_i + \beta^q_i - (\varepsilon^p_i + \varepsilon^q_i), \beta^p_i + \beta^q_i - (\varepsilon^p_i + \varepsilon^q_i) + \frac{\alpha^p \varepsilon^p_i}{\beta^p} + \frac{\alpha^q \varepsilon^{q+}_i}{\beta^{q+}}\right)$$

$$\alpha^p = \alpha^q = \alpha^s$$

$$I^s_1 = \left[\beta^p_i + \beta^q_i - (\varepsilon^p_i + \varepsilon^q_i), \beta^p_i + \beta^q_i - (\varepsilon^p_i + \varepsilon^q_i) + \alpha^s \left(\frac{\varepsilon^p_i}{\beta^p} + \frac{\varepsilon^{q+}_i}{\beta^q}\right)\right)$$

$$\beta^p_i + \beta^q_i - (\varepsilon^p_i + \varepsilon^q_i) + \alpha^s \left(\frac{\varepsilon^p_i}{\beta^p} + \frac{\varepsilon^{q+}_i}{\beta^q}\right) - y = 0$$

$$f^U_{B^s}(y) = \frac{y - (\beta^p_i - \varepsilon^p_i) + (\beta^q_i - \varepsilon^q_i)}{\left(\frac{\varepsilon^p_i}{\beta^p} + \frac{\varepsilon^{q+}_i}{\beta^{q+}}\right)}$$

$$\textit{when } \beta^p = \beta^{q+} = \beta^s$$

$$g^U_{B^s}(y) = \beta^s \left(\frac{y - [(\beta^p_i - \varepsilon^p_i) + (\beta^q_i - \varepsilon^q_i)]}{\varepsilon^p_i + \varepsilon^{q+}_i}\right)$$

$$g^U_{B^s}(y) = \beta^s \left(\frac{y - [(\beta^p_i - \varepsilon^p_i) + (\beta^q_i - \varepsilon^q_i)]}{\varepsilon^p_i + \beta^q_i - (\beta^q_i - \varepsilon^q_i) - \frac{\beta^r}{\beta^p}\varepsilon^q_i}\right)$$

$$g^U_{B^s}(y) = \beta^s \left(\frac{y - [(\beta^p_i - \varepsilon^p_i) + (\beta^q_i - \varepsilon^q_i)]}{\beta^p_i + \beta^q_i - \frac{\beta^s}{\beta^p}\varepsilon^q_i - [(\beta^p_i - \varepsilon^p_i) + (\beta^q_i - \varepsilon^q_i)]}\right)$$

$$g^U_{B^s}(y) = \beta^s \left(\frac{y - (\beta^s_i - \varepsilon^s_i)}{\beta^p_i + \beta^q_i - \frac{\beta^s}{\beta^p}\varepsilon^q_i - (\beta^s_i - \varepsilon^s_i)}\right)$$

where $\beta^s_i = \beta^p_i + \beta^q_i, \varepsilon^s_i = \varepsilon^p_i + \varepsilon^q_i$

$$(\beta^s_i - \varepsilon^s_i) \le y \le \beta^p_i + \beta^q_i - \frac{\beta^s}{\beta^p}\varepsilon^q_i$$

Now

$$I^s_2 = \left[\beta^p_i + \beta^q_i, \beta^p_i + \frac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}}(\beta^{p*}_i - \beta^p_i) + \beta^q_i + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}}(\beta^{q*}_i - \beta^q_i)\right]$$

$$I^s_2 = \left[\beta^p_i + \beta^q_i, \beta^p_i + \beta^q_i + \frac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}}(\beta^{p*}_i - \beta^p_i) + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}}(\beta^{q*}_i - \beta^q_i)\right]$$

here $\dfrac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}} = \dfrac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}} = \dfrac{\overline{\alpha^s} - \beta^s}{\overline{\beta^s}}$

$$I^s_2 = \left[\beta^p_i + \beta^q_i, \beta^p_i + \beta^q_i + (\overline{\alpha^s} - \beta^s)\left(\frac{\beta^{p*}_i - \beta^p_i}{\overline{\beta^p}} + \frac{\beta^{q*}_i - \beta^q_i}{\overline{\beta^q}}\right)\right]$$

$$\beta^p_i + \beta^q_i + (\overline{\alpha^s} - \beta^s)\left(\frac{\beta^p_i - \beta^p_i}{\overline{\beta^p}} + \frac{\beta^q_i - \beta^q_i}{\overline{\beta^q}}\right) - y = 0$$

when $\overline{\beta^p} = \overline{\beta^q} = \overline{\beta^s}$

$$g^{U}_{B^s}(y) = \overline{\alpha^s} - \beta^s = \left(\frac{y-(\beta_i^p+\beta_i^q)}{\beta_i^{p*}-\beta_i^p+\beta_i^{q*}-\beta_i^q}\right)\times\overline{\beta^s}$$

$$\overline{\alpha^s} = \beta^s + \left(\frac{y-(\beta_i^p+\beta_i^q)}{\beta_i^{p*}-\beta_i^p+\beta_i^{q*}-\beta_i^q}\right)\times\overline{\beta^s} \Rightarrow \overline{\alpha^s}$$
$$= \beta^s + \left(\frac{y-\beta_i^s}{\beta_i^{s*}-\beta_i^s}\right)\times\overline{\beta^s}$$

where $\beta_i^{s*} = \beta_i^{p*}+\beta_i^{q*}, \beta_i^s = \beta_i^p+\beta_i^q$

$$I_3^s = [\beta_i^{p*}+\beta_i^{q*}, (\beta_i^{p*}+\varepsilon_i^{p*}) - \alpha^p\varepsilon_i^{p*} + (\beta_i^{q*}+\varepsilon_i^{q*}) - \alpha^q\varepsilon_i^{q*}]$$

$$\alpha^p = \alpha^q = \alpha^s$$

$$I_3^s = [\beta_i^{p*}+\beta_i^{q*}, (\beta_i^{p*}+\varepsilon_i^{p*}) + (\beta_i^{q*}+\varepsilon_i^{q*}) - \alpha^s(\varepsilon_i^{p*}+\varepsilon_i^{q*})]$$

$$y - (\beta_i^{p*}+\varepsilon_i^{p*}) - (\beta_i^{q*}+\varepsilon_i^{q*}) + \alpha^s(\varepsilon_i^{p*}+\varepsilon_i^{q*}) = 0$$

$$g^{U}_{B^s}(y) = \alpha^s = \left(\frac{(\beta_i^{p*}+\varepsilon_i^{p*}) + (\beta_i^{q*}+\varepsilon_i^{q*}) - y}{\varepsilon_i^{p*}+\varepsilon_i^{q*}}\right) \Rightarrow g^{U}_{B^s}(y)$$
$$= \left(\frac{(\beta_i^{s*}+\varepsilon_i^{s*}) - y}{\varepsilon_i^{s*}}\right)$$

where, $\beta_i^{s*} = \beta_i^{p*}+\beta_i^{q*}$, $\varepsilon_i^{s*} = \varepsilon_i^{p*}+\varepsilon_i^{q*}$

$$B_i^s(y) = \begin{cases} \beta^s + \left(\frac{y-\beta_i^s}{\beta_i^{s*}-\beta_i^s}\right)\times\overline{\beta^s} & \text{When } \beta_i^s \le y \le \beta_i^{s*} \\ \beta^s\left(\frac{y-(\beta_i^s-\varepsilon_i^s)}{\beta_i^p+\beta_i^q-\frac{\beta^s}{\beta^p}\varepsilon_i^q-(\beta_i^s-\varepsilon_i^s)}\right) & \text{When } \beta_i^s-\varepsilon_i^s \le y \le \beta_i^p+\beta_i^q-\frac{\beta^s}{\beta^p}\varepsilon_i^q \\ \frac{(\beta_i^{s*}+\varepsilon_i^{s*})-y}{\varepsilon_i^{s*}} & \text{When } \beta_i^{s*} \le y \le \beta_i^{s*}+\varepsilon_i^{s*} \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p-\varepsilon_i^p) + (\beta_i^q-\varepsilon_i^q).$$

$$\beta_i^s = \beta_i^p + \beta_i^q - \frac{\beta^s}{\beta^p}\varepsilon_i^q.$$

$$\beta_i^{s*} = \beta_i^{p*}+\beta_i^{q*}.$$

$$\beta_i^{s*}+\varepsilon_i^{s*} = (\beta_i^{p*}+\varepsilon_i^{p*}) + (\beta_i^{q*}+\varepsilon_i^{q*}).$$

Appendix 2

Proof As the quadrilateral membership functions of $\tilde{B}_i^p$ and $\tilde{B}_i^q$ are given in Eqs. (17) and (18), respectively. Thus, in order to find the membership of $\tilde{B}_i^s = \tilde{B}_i^p \times \tilde{B}_i^q = (\beta_i^s - \varepsilon_i^s, \beta_i^s, \beta_i^{s*}, \beta_i^{s*}+\varepsilon_i^{s*})$

Let $\tilde{B}_i^s = \tilde{B}_i^p \times \tilde{B}_i^{q+} = \{y | y \in \tilde{B}_\alpha^s\}\, \forall \alpha^s \in [0,1]$. Here $\tilde{B}_\alpha^s = [\tilde{B}_\alpha^{sL}(\alpha^s), \tilde{B}_\alpha^{sU}(\alpha^s)]$ be its α^s-cuts such that $\tilde{B}_\alpha^{sL}(\alpha^s) = \tilde{B}_i^{pL}(\alpha^p)\times\tilde{B}_i^{qL}(\alpha^q)$ and $\tilde{B}_\alpha^{sU}(\alpha^s) = \tilde{B}_i^{pU}(\alpha^p)\times\tilde{B}_i^{qU}(\alpha^q)$ i.e

$$\tilde{B}_\alpha^s = [\tilde{B}_i^{pL}(\alpha^p)\times\tilde{B}_i^{qL}(\alpha^q), \tilde{B}_i^{pU}(\alpha^p)\times\tilde{B}_i^{qU}(\alpha^q)]$$

$\tilde{B}_\alpha^s = I_1^s \cup I_2^s \cup I_3^s$, where

$$I_1^s = \left[(\beta_i^p-\varepsilon_i^p)(\beta_i^q-\varepsilon_i^q), \left\{\beta_i^p-\varepsilon_i^p+\frac{\alpha^p\varepsilon_i^p}{\beta^p}\right\}\left\{\beta_i^q-\varepsilon_i^q+\frac{\alpha^q\varepsilon_i^{q+}}{\beta^q}\right\}\right)$$

$$I_2^s = \left[\beta_i^p\beta_i^q, \left\{\beta_i^p+\frac{\overline{\alpha^p}-\beta^p}{\overline{\beta^p}}(\beta_i^{p*}-\beta_i^p)\right\}\left\{\beta_i^q+\frac{\overline{\alpha^q}-\beta^q}{\overline{\beta^q}}(\beta_i^{q*}-\beta_i^q)\right\}\right]$$

$$I_3^s = [\beta_i^{p*}\beta_i^{q*}, \{(\beta_i^{p*}+\varepsilon_i^{p*})-\alpha^p\varepsilon_i^{p*}\}\{(\beta_i^{q*}+\varepsilon_i^{q*})-\alpha^q\varepsilon_i^{q*}\}]$$

Now,

$$I_1^s = \left[(\beta_i^p-\varepsilon_i^p)(\beta_i^q-\varepsilon_i^q), \left\{\beta_i^p-\varepsilon_i^p+\frac{\alpha^p\varepsilon_i^p}{\beta^p}\right\}\left\{\beta_i^q-\varepsilon_i^q+\frac{\alpha^q\varepsilon_i^{q+}}{\beta^q}\right\}\right)$$

$\alpha^p = \alpha^q = \alpha^s$ and $\beta^p = \beta^{q+} = \beta^s$

$$\left\{\beta_i^p-\varepsilon_i^p+\frac{\alpha^p\varepsilon_i^p}{\beta^p}\right\}\left\{\beta_i^q-\varepsilon_i^q+\frac{\alpha^q\varepsilon_i^{q+}}{\beta^q}\right\} - y = 0$$

$$(\alpha^s)^2\frac{\varepsilon_i^p(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^s\beta^q}} + \alpha^s\left\{\frac{(\beta_i^p-\varepsilon_i^p)(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^q}} + \frac{(\beta_i^q-\varepsilon_i^q)\varepsilon_i^p}{\overline{\beta^s}}\right\} + (\beta_i^p - \varepsilon_i^p)(\beta_i^q-\varepsilon_i^q) - y = 0$$

Take $M_2 = \frac{\varepsilon_i^p(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^s\beta^q}}$, $K_2 = \frac{(\beta_i^p-\varepsilon_i^p)(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^q}} + \frac{(\beta_i^q-\varepsilon_i^q)\varepsilon_i^p}{\overline{\beta^s}}$, and $N_2 = (\beta_i^p-\varepsilon_i^p)(\beta_i^q-\varepsilon_i^q)\,(\alpha^s)^2M_2 + \alpha^sK_2 + N_2 - y = 0.$

$$\alpha^s = \frac{-K_2 \pm \sqrt{K_2^2+4M_2(y-N_2)}}{2M_2}$$

Similarly, $In\, I_2^s = \left[\beta_i^p\beta_i^q, \left\{\beta_i^p+\frac{\overline{\alpha^p}-\beta^p}{\overline{\beta^p}}(\beta_i^{p*}-\beta_i^p)\right\}\left\{\beta_i^q+\frac{\overline{\alpha^q}-\beta^q}{\overline{\beta^q}}(\beta_i^{q*}-\beta_i^q)\right\}\right]$

$$\alpha^s = \beta^s + \frac{-K_1+\sqrt{K_1^2+4M_1(y-N_1)}}{2M_1}$$

And

$$I_3^s = [\beta_i^{p*}\beta_i^{q*}, \{(\beta_i^{p*}+\varepsilon_i^{p*})-\alpha^p\varepsilon_i^{p*}\}\{(\beta_i^{q*}+\varepsilon_i^{q*})-\alpha^q\varepsilon_i^{q*}\}]$$

$$\alpha^s = \frac{K_3+\sqrt{K_3^2+4M_3(y-N_3)}}{2M_3}$$

Appendix 3

Objective Function

Let $y_1, y_2, \ldots, y_{20}$ be variables for different constraints.

Minimize $Z = 66.4y_1 + 61.58y_2 + 64.47y_3 + 264.12y_4 + 69.17y_5 + 130.48y_6 + 46.03y_7 + 164.11y_8 + 262.29y_9 + 129.41y_{10} + 52.16y_{11} + 202.79y_{12} + 206.74y_{13} + 142.17y_{14} + 236.53y_{15} + 302.08y_{16} + 234.01y_{17} + 236.54y_{18} + 98.67y_{19} + 119.13y_{20}.$

Subjected to constraints:

$4.38y_1 + 4.07y_2 + 4.04y_3 + 9.88y_4 + 4.10y_5 + 7.38y_6 + 2.5y_7 + 8.11y_8 + 14.81y_9 + 7.22y_{10} + 3.05y_{11} + 9.18 y_{12} + 10.70y_{13} + 6.38y_{14} + 10.38y_{15} + 10.51y_{16} + 11.24y_{17} + 11.57y_{18} + 5.91y_{19} + 7.79y_{20} \geq B_{\text{lab}}$.

$45.7y_1 + 41.99y_2 + 44.49y_3 + 211.93y_4 + 49.13y_5 + 94.06y_6 + 33.62y_7 + 124.32y_8 + 190.08y_9 + 93.76y_{10} + 37.29y_{11} + 156.57y_{12} + 153.98y_{13} + 110.86y_{14} + 184.34y_{15} + 246.6y_{16} + 178.25y_{17} + 179.28y_{18} + 70.13y_{19} + 81.84y_{20} \geq B_{\text{mat}}$.

$7.33y_1 + 6.81y_2 + 6.77y_3 + 16.55y_4 + 6.87y_5 + 12.36y_6 + 4.58y_7 + 14.35y_8 + 24.4y_9 + 13.17y_{10} + 5.55y_{11} + 15.27y_{12} + 17.84y_{13} + 10.59y_{14} + 16.98y_{15} + 17.19y_{16} + .18.42y_{17} + 18.97y_{18} + 9.70y_{19} + 12.86y_{20} \geq B_{\text{foh}}$.

$5.8y_1 + 5.39y_2 + 5.35y_3 + 13.09y_4 + 5.43y_5 + 9.78y_6 + 2.91y_7 + 9.14y_8 + 19.3y_9 + 8.39y_{10} + 3.54y_{11} + 12.08y_{12} + 14.11y_{13} + 8.38y_{14} + 13.43y_{15} + 13.59y_{16}- + 14.57y_{17} + 15.01y_{18} + 7.67y_{19} + 10.17y_{20} \geq B_{\text{aoh}}$.

$1.17y_1 + 1.09y_2 + 1.08y_3 + 2.65y_4 + 1.10y_5 + 1.98y_6 + 0.75y_7 + 2.35y_8 + 3.91y_9 + 2.16y_{10} + 0.91y_{11} + 2.45y_{12} + 2.86y_{13} + 1.70y_{14} + 2.72y_{15} + 2.75y_{16} + 2.95y_{17} + 3.04y_{18} + 1.55y_{19} + 2.06y_{20} \geq B_{\text{tooh.}}$

$0.37y_1 + 0.34y_2 + 0.36y_3 + 1.74y_4 + 0.40y_5 + 0.77y_6 + 0.23y_7 + 0.85y_8 + 1.56y_9 + 0.64y_{10} + 0.25y_{11} + 1.28y_{12} + 1.26y_{13} + 0.91y_{14} + 1.51y_{15} + 2.02y_{16} + 1.46y_{17} + 1.47y_{18} + 0.58y_{19} + 0.67y_{20} \geq B_{\text{soh.}}$

$14.69y_1 + 13.63y_2 + 13.57y_3 + 34.04y_4 + 13.80y_5 + 24.89y_6 + 8.47y_7 + 26.68y_8 + 49.16y_9 + 24.35y_{10} + 10.26y_{11} + 31.09y_{12} + 36.07y_{13} + 21.58y_{14} + 34.64y_{15} + 35.56y_{16} + 37.39y_{17} + 38.49y_{18} + 19.49y_{19} + 25.76y_{20} \geq B_{\text{toh.}}$

$1.63y_1 + 1.89y_2 + 2.37y_3 + 8.27y_4 + 2.13y_5 + 4.14y_6 + 1.44y_7 + 5y_8 + 8.23y_9 + 4.07y_{10} + 1.57y_{11} + 5.96y_{12} + 5.99y_{13} + 3.35y_{14} + 7.17y_{15} + 9.42y_{16} + 7.12y_{17} + 7.20y_{18} + 3.14y_{19} + 3.74 \geq \leq B_{\text{proof.}}$

For all $y_i \geq 0$.

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