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Aggregating Intuitionistic Fuzzy Preference Relations with Symmetrical Intuitionistic Fuzzy Bonferroni Mean Operators in Group Decision Making

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Abstract As a useful aggregation technique, the Bonferroni mean can capture the interrelationship between input arguments and has been a hot research topic, especially, in intuitionistic fuzzy environment. In this paper, it is pointed out by an example that the existing intuitionistic fuzzy Bonferroni mean (IFBM) operators fail to satisfy the need in group decision making with intuitionistic fuzzy preference relations (IFPRs). Then, symmetrical intuitionistic fuzzy Bonferroni mean (SIFBM) operator and weighted SIFBM operator are developed to settle the above issue and some desirable properties of them are provided. Furthermore, an acceptable group multiplicative consistency of the IFPRs is introduced and a novel algorithm is established to jointly and stepwisely reach the acceptable group multiplicative consistency and consensus of IFPRs in group decision making. Finally, numerical examples are given to illustrate the effectiveness of the proposed method and comparisons with the existing methods are made to demonstrate the advantages of the proposed method.

Keywords Symmetrical intuitionistic fuzzy Bonferroni mean operator · Group decision making · Acceptable group multiplicative consistency · Intuitionistic fuzzy preference relation

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1 Introduction

Group decision making problem with preference relations are to select the optimal alternative(s) from a given set of finite feasible alternatives by several decision makers has been widely used in various fields such as politics, social psychology, engineering, management, business and economics, etc. Among the procedure for group decision making with preference relations, checking and reaching the consistency and consensus are crucial without which unreasonable result could be derived.

Although preference relations with a certain degree have been extensively investigated [1–8], they don't always meet the real decision making problems, because the decision makers may not be able to provide their preferences for alternatives to a such certain degree due to lack of precise or sufficient level of knowledge related to the problems, or the difficulty in explaining explicitly the degree to which one alternative is better than others [9]. In these situations, there is usually a degree of uncertainty in providing their preferences over the alternatives considered, which makes the result of the preference process exhibits the characteristics of affirmation, negation and hesitation [9] and can be described with Atanassov's intuitionistic fuzzy sets [10] by simultaneously considering the degrees of membership and non-membership with hesitation index. Thus, IFPRs were introduced [9, 11]. At present, many scholars devoted themselves into group decision making with IFPRs. For aggregating the individual IFPRs into a collective one, the existing methods are mainly based on the simple intuitionistic fuzzy weighted geometric (SIFWG) operator [12] for the multiplicative consistency of IFPRs [12–14], intuitionistic fuzzy weighted arithmetic (IFWA) operator [15] for the additive consistency of IFPRs [16–19] and hyperbolic scale-based intuitionistic multiplicative weighted/power geometric operator for the hyperbolic scale-based intuitionistic multiplicative preference relations [20], because these operators could guarantee that if the individual IFPRs are all of acceptable consistency, then so is the aggregated result by these operators. Liao [12] also pointed that the symmetric intuitionistic fuzzy weighted averaging (SIFWA) operator [21] does not possess the above properties, which leads that it can not be used to aggregate the IFPRs in group decision making problems. Similar case happens to those in [22]. Many other intuitionistic fuzzy aggregation operators were also developed in intuitionistic environment [9, 23, 24, 26–31] to suit for different practical cases. For example, the SIFWA operator [21] was originally proposed for the case where the membership and non-membership information are treated fairly; the IFBM operators [23, 26, 27] were developed for capturing the interrelationship of the individual arguments.

For checking and reaching the consistency and consensus, various consistency and consensus were investigated. Xu [32] proposed a multiplicative consistency of IFPRs by replacing the operation in the multiplicative consistency of FPRs with a novel operation in intuitionistic fuzzy sets. Liao and Xu [24] improved the above definition in Ref. [32]. Wu and Chiclana [19] proposed a new multiplicative consistency of interval valued preference relations (IVPRs) and then defined another multiplicative consistency of IFPRs by the transformation function between the IVPRs and IFPRs. Motivated by the additive consistency of FPRs, Wang [42] defined an additive consistency for IFPRs, based on which Jin [33] proposed a multiplicative consistency by the geometric mean operator. Liao [34] reviewed most definitions of consistency in intuitionistic fuzzy environment. In practical application, the IFPRs provided by the decision makers are not always consistent or consensus, Liao [24] and Wan [13], respectively, provided a consistency index of IFPRs for reaching the acceptable consistency by modifying the whole IFPRs or a single element of IFPRs with a control parameter, and then reached the acceptable consensus. Xu [14] made a mathematical programming approach to jointly improve the consistency and consensus of IFPRs.

However, by analyzing the existing aggregation operators and the methods to check and reach the multiplicative consistency and consensus, we find that

(1) Some of the existing aggregation operators either could not fairly treat the membership and non-membership degrees in intuitionistic fuzzy sets [23, 26, 27] or could not capture the interrelationships of the individual arguments [12, 21, 22] which leads that they could be limited to solve the group decision making with IFPRs in some practical situations;
(2) When the consistency and consensus were reached by the existing methods [13, 14, 24], the original IFPRs provided by the decision makers were completely modified, which contradicts the principle of modification, that is, the improved preference relations should not only satisfy the acceptability requirement but also preserve the initial preference information as much as possible.

Inspired by the work in Refs. [12, 21–23, 26, 27], in the present paper, we introduce the (weighted) SIFBM operators and acceptable group multiplicative consistency to resolve the above problem.

The remainder of this paper is organized as follows: Sect. 2 briefly reviews some basic concepts and operations related to the IFBM operators and IFPRs. Particularly, an example is given to indicate that the existing IFBM operators are not suitable for aggregating IFPRs in group decision making. In Sect. 3, the SIFBM operator and the weighted SIFBM operator are developed, and some desirable properties of them are discussed. Section 4 develops an acceptable group multiplicative consistency and consensus for the IFPRs in group decision making, an algorithm to jointly check and reach the acceptable group multiplicative consistency and consensus is established using the partial order between the IFPRs which guarantees that the modification of individual IFPRs is made in a stepwise way and is apt to be accepted by the decision makers. Section 5 provides examples to illustrate the validity of the proposed method and demonstrates the advantages of the proposed method by comparing the proposed method with the existing ones. The main conclusions are drawn in Sect. 6.

2 Preliminaries

To make the presentation self-contained, in what follows, we review some basic concepts.

2.1 Intuitionistic Fuzzy Bonferroni Mean Operators

Definition 1 [35] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed universe. A fuzzy set on X is defined as $A = \{(x_i, \mu_A(x_i)) | x_i \in X\}$, where the function $\mu_A(x_i) \in [0, 1]$ is the membership degree of x_i to A in X.

Definition 2 [10] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed universe. An intuitionistic fuzzy set (IFS) A on X can be defined as:

$$A = \{(x_i, \mu_A(x_i), v_A(x_i)) | x_i \in X\}$$

where the function $\mu_A(x_i)$ and $v_A(x_i)$ denote the membership degree and non-membership degree of x_i to A in X with the condition $\mu_A(x_i), v_A(x_i) \in [0, 1]$ and $\mu_A(x_i) + v_A(x_i) \leq 1$. Furthermore, $\pi_A(x_i) = 1 - \mu_A(x_i) - v_A(x_i)$ is called an intuitionistic index (or hesitancy degree) of x_i to A.

For convenience, Xu [9] named the pair (μ_α, v_α) an intuitionistic fuzzy number (IFN) denoted as α with the condition $\mu_\alpha, v_\alpha \in [0, 1], \mu_\alpha + v_\alpha \leq 1$. Especially, if $\mu_\alpha + v_\alpha = 1$, then α reduces to $(\mu_\alpha, 1 - \mu_\alpha)$. Let $\alpha^c = (v_\alpha, \mu_\alpha)$ be the complement of α. $s(\alpha) = \mu_\alpha - v_\alpha$ is called the score of α and $h(\alpha) = \mu_\alpha + v_\alpha$ is called accuracy degree of α [36]. Based on the score function s and the accuracy function h, Xu and Yager [25] gave an order relation between IFNs as follows:

Definition 3 Let α and β be two IFNs.

(1) If $s(\alpha) < s(\beta)$, then $\alpha < \beta$
(2) If $s(\alpha) = s(\beta)$, then

- (a) if $h(\alpha) = h(\beta)$, then $\alpha = \beta$;
- (b) if $h(\alpha) < h(\beta)$, then $\alpha < \beta$.

Definition 4 [37–39] Let $p, q \geq 0$, and $a_i (i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$BM^{p,q}(a_1, a_2, \ldots, a_n) = \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{1}{n(n-1)} a_i^p a_j^q \right)^{\frac{1}{p+q}}, \tag{1}$$

then $BM^{p,q}$ is called a Bonferroni mean (BM) operator.

To reflect the interrelationship between the individual criterion and other criteria, Zhou [27] proposed the normalized weighted version of the BM operator.

Definition 5 [27] Let $p, q \geq 0$, and $a_i (i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$WBM^{p,q}(a_1, a_2, ..., a_n) = \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1 - \omega_i} a_i^p a_j^q \right)^{\frac{1}{p+q}}, \tag{2}$$

then $WBM^{p,q}$ is called a weighted BM operator, where $\omega_i \in [0, 1]$ and $\sum_{k=1}^{n} \omega_i = 1$.

Furthermore, to aggregate the IFNs, Xu [40] defined the following basic operations:

(1) $\alpha \odot \beta = (\mu_\alpha \mu_\beta, v_\alpha + v_\beta - v_\alpha v_\beta)$;
(2) $\alpha^\lambda = (\mu_\alpha^\lambda, 1 - (1 - v_\alpha)^\lambda), \lambda > 0$;
(3) $\alpha \oplus \beta = (\mu_\alpha + \mu_\beta - \mu_\alpha \mu_\beta, v_\alpha v_\beta)$;
(4) $\lambda\alpha = (1 - (1 - \mu_\alpha)^\lambda, v_\alpha{}^\lambda), \lambda > 0$.

Definition 6 [26] Let $p, q \geq 0$, and $\alpha_i (i = 1, 2, \ldots, n)$ be a collection of IFNs. If

$$IFBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\alpha_i^p \odot \alpha_j^q \right) \right)^{\frac{1}{p+q}}, \tag{3}$$

then $IFBM^{p,q}$ is called an intuitionistic fuzzy Bonferroni mean (IFBM) operator.

Associated with Xu's basic operations on the IFNs, it holds that

$$IFBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \mu_{\alpha_i}^p \mu_{\alpha_j}^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \; 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - v_{\alpha_i}\right)^p \left(1 - v_{\alpha_j}\right)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right). \tag{4}$$

2.2 Intuitionistic Fuzzy Preference Relations

Due to hesitancy and uncertainty, it may be difficult for decision makers to quantify their preference values with crisp numbers in group decision making problems, but they can be represented by intuitionistic fuzzy judgments in a pair comparison matrix.

Definition 7 [9] An intuitionistic fuzzy preference relation (IFPR) P on X is characterized by an intuitionistic fuzzy judgment matrix $P=(p_{ij})_{n\times n}$ with $p_{ij}=(\mu_{p_{ij}},v_{p_{ij}})$, where p_{ij} is an IFN, and $\mu_{p_{ij}}$ is the certainty degree to which alternative x_i is preferred to x_j, and $v_{p_{ij}}$ is the certainty degree to which alternative x_i is non-preferred to x_j, and $0\leq\mu_{p_{ij}}+v_{p_{ij}}\leq 1, \mu_{p_{ij}}=v_{p_{ji}}, v_{p_{ij}}=\mu_{p_{ji}}$, and $\mu_{p_{ii}}=v_{p_{ii}}=\frac{1}{2}$, for $i,j=1,2,\cdots,n$.

Let $X=\{x_1,x_2,\ldots,x_n\}$ and $E=\{e_1,e_2,\cdots,e_s\}$ be the set of alternatives under consideration and the set of s decision makers who are invited to evaluate the alternatives, respectively. In many cases, since the problem is very complicated or the decision makers are not familiar with the problem and thus they cannot give explicit preferences over the alternatives, it is suitable to use the IFNs, which express the preference information from three aspects: "preferred", "not preferred", and "indeterminate", to represent their opinions. Suppose that the decision maker e_l provides his/her preference values for the alternative x_i against the alternative x_j as $p_{ij}^{(l)}=(\mu_{p_{ij}^{(l)}},v_{p_{ij}^{(l)}})$ in which $\mu_{p_{ij}^{(l)}}$ denotes the degree to which the object x_i is preferred to the object x_j, $v_{p_{ij}^{(l)}}$ indicates the degree to which the object x_i is not preferred to the object x_j subject to $0\leq\mu_{p_{ij}^{(l)}}+v_{p_{ij}^{(l)}}\leq 1, \mu_{p_{ij}^{(l)}}=v_{p_{ji}^{(l)}}, v_{p_{ij}^{(l)}}=\mu_{p_{ji}^{(l)}}$, and $\mu_{p_{ii}^{(l)}}=v_{p_{ii}^{(l)}}=\frac{1}{2}$, for $i,j=1,2,\cdots,n$. The IFPR for the lth decision maker can be written as

$$P^{(l)}=\left(p_{ij}^{(l)}\right)_{n\times n}. \tag{5}$$

For a group decision making problem with s decision makers, we can obtain s individual IFPRs with the form in Eq. (5). To find the final result of the problem, it is needed to aggregate all these individual IFPRs into a collective one using an aggregation operator. However, the following example shows the aggregated result by the existing Bonferroni mean operators in intuitionistic fuzzy environment is not an IFPR in general.

Example 1 Let $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ be three IFPRs provided by three decision makers as listed in Table 1.

Aggregate these IFPRs $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ by the IFBM operator with $p=q=0.5$ [26] into P listed in Table 1 which is obviously not an IFPR. Similar cases happen for the other kinds of IFBM operators in Refs. [23, 27].

Definition 8 [41] An IFPR $P=(p_{ij})_{n\times n}$ with $p_{ij}=(\mu_{ij},v_{ij})$ is said to be multiplicatively consistent, if it satisfies the following multiplicative transitivity: $\mu_{ij}\mu_{jk}\mu_{ki}=v_{ij}v_{jk}v_{ki}, i,j,k=1,2,\ldots,n$.

With economy and social development, the decision making problems are becoming more complicated, uncertain and fuzzy than ever, crisp weights may not be able to rationally reflect the importance degrees of the alternatives in many group decision making problems. Thus, the intuitionistic fuzzy priority weight vector was introduced [42]. Suppose that $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ is an intuitionistic fuzzy priority weight vector of $P=(p_{ij})_{n\times n}$, where $\omega_i=(\mu_{\omega_i},v_{\omega_i})(i=1,2,\ldots,n)$ is an IFN, and $\mu_{\omega_i},v_{\omega_i}\in[0,1],\mu_{\omega_i}+v_{\omega_i}\leq 1$. μ_{ω_i} and v_{ω_i} can be interpreted as the membership degree and the non-membership degree of the importance of the alternative x_i ($i=1,2,\ldots,n$). An intuitionistic fuzzy priority weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ is said to be normalized if it satisfies the following requirements: $\sum_{i\neq j}\mu_{\omega_j}\leq v_{\omega_i}$ and $\sum_{i\neq j}v_{\omega_j}\leq\mu_{\omega_i}+n-2$, $i=1,2,\ldots,n$, where $\mu_{\omega_i},v_{\omega_i}\in(0,1],\mu_{\omega_i}+v_{\omega_i}\leq 1$. With the underlying normalized intuitionistic fuzzy priority weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$, a multiplicatively consistent IFPR $P=(p_{ij})_{n\times n}$ was established [24], where

$$p_{ij}=\begin{cases}(0.5,0.5), & i=j,\\ \left(\dfrac{2\mu_{\omega_i}}{\mu_{\omega_i}+\mu_{\omega_j}-v_{\omega_i}-v_{\omega_j}+2},\dfrac{2\mu_{\omega_j}}{\mu_{\omega_i}+\mu_{\omega_j}-v_{\omega_i}-v_{\omega_j}+2}\right), & i\neq j,\end{cases} \tag{6}$$

$\sum\limits_{i\neq j}\mu_{\omega_j}\leq v_{\omega_i}$ and $\sum\limits_{i\neq j}v_{\omega_j}\leq\mu_{\omega_i}+n-2$, $\mu_{\omega_i},v_{\omega_i}\in(0,1]$, $\mu_{\omega_i}+v_{\omega_i}\leq 1$ for all $i=1,2,\ldots,n$.

3 (Weighted) Symmetric Intuitionistic Fuzzy Bonferroni Mean Operators

The reason that the aggregated result of several individual IFPRs is not an IFPR is that the membership degrees and non-membership degrees are treated, respectively, with different operators from the existing IFBM operators. Thus, to capture the interrelationship of the individual IFPRs in group decision making, the (weighted) SIFBM operators are developed in this section.

3.1 Symmetric Intuitionistic Fuzzy Bonferroni Mean Operators

Let α,β be two IFNs, the following operations provided in Refs. [15, 21] are necessary to simulate the usual operations in the classical BM operator:

(1) $\alpha\otimes\beta=\left(\frac{\mu_\alpha\mu_\beta}{\mu_\alpha\mu_\beta+(1-\mu_\alpha)(1-\mu_\beta)},\frac{v_\alpha v_\beta}{v_\alpha v_\beta+(1-v_\alpha)(1-v_\beta)}\right)$;

(2) $\alpha^\lambda=\left(\frac{\mu_\alpha^\lambda}{\mu_\alpha^\lambda+(1-\mu_\alpha)^\lambda},\frac{v_\alpha^\lambda}{v_\alpha^\lambda+(1-v_\alpha)^\lambda}\right)$, $\lambda>0$;

(3) $\alpha\oplus\beta=\left(\min\{\mu_\alpha+\mu_\beta,1\},\max\{v_\alpha+v_\beta-1,0\}\right)$;

Table 1 Individual preference information from three decision makers in Example 1

$P^{(1)}$	(0.5, 0.5)	(0.5, 0.2)	(0.7, 0.1)	(0.5, 0.3)
	(0.2, 0.5)	(0.5, 0.5)	(0.6, 0.2)	(0.3, 0.6)
	(0.1, 0.7)	(0.2, 0.6)	(0.5, 0.5)	(0.3, 0.6)
	(0.3, 0.5)	(0.6, 0.3)	(0.6, 0.3)	(0.5, 0.5)
$P^{(2)}$	(0.5, 0.5)	(0.6, 0.1)	(0.8, 0.2)	(0.6, 0.3)
	(0.1, 0.6)	(0.5, 0.5)	(0.5, 0.1)	(0.3, 0.7)
	(0.2, 0.8)	(0.1, 0.5)	(0.5, 0.5)	(0.4, 0.6)
	(0.3, 0.6)	(0.7, 0.3)	(0.6, 0.4)	(0.5, 0.5)
$P^{(3)}$	(0.5, 0.5)	(0.6, 0.2)	(0.8, 0.1)	(0.7, 0.2)
	(0.2, 0.6)	(0.5, 0.5)	(0.6, 0.1)	(0.2, 0.7)
	(0.1, 0.8)	(0.1, 0.6)	(0.5, 0.5)	(0.2, 0.3)
	(0.2, 0.7)	(0.7, 0.2)	(0.3, 0.2)	(0.5, 0.5)
P	(0.5, 0.5)	(0.566, 0.166)	(0.767, 0.132)	(0.598, 0.267)
	(0.161, 0.568)	(0.5, 0.5)	(0.566, 0.132)	(0.264, 0.669)
	(0.128, 0.77)	(0.128, 0.568)	(0.5, 0.5)	(0.293, 0.51)
	(0.264, 0.605)	(0.666, 0.267)	(0.49, 0.301)	(0.5, 0.5)

(4) $\lambda\alpha = (\lambda\mu_\alpha, 1-\lambda(1-v_\alpha)), \lambda > 0.$

The following property of $\oplus$ is obvious.

Lemma 1 *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})(i = 1,2\cdots,n)$ be a collection of IFNs and $\omega = (\omega_1, \omega_1, \cdots \omega_n)$ such that $\sum_{i=1}^n \omega_i = 1$. Then*

$$\omega_1\alpha_1 \oplus \omega_2\alpha_2 \oplus \cdots \oplus \omega_n\alpha_n = \left(\sum_{i=1}^n \omega_i\mu_{\alpha_i}, \sum_{i=1}^n \omega_i v_{\alpha_i}\right).$$

Definition 9 Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})(i = 1,2\cdots,n)$ be a collection of IFNs. For any $p, q > 0$, if

$$SIFBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\alpha_i^p \otimes \alpha_j^q}{n(n-1)}\right)^{\frac{1}{p+q}}, \quad (7)$$

then $SIFBM^{p,q}$ is called a symmetrical intuitionistic fuzzy Bonferroni mean (SIFBM) operator.

Theorem 1 *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})(i = 1,2,\ldots,n)$ be a collection of IFNs. Then*

$$SIFBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\frac{A_\mu^{\frac{1}{p+q}}}{A_\mu^{\frac{1}{p+q}} + (n(n-1)-A_\mu)^{\frac{1}{p+q}}}, \frac{A_v^{\frac{1}{p+q}}}{A_v^{\frac{1}{p+q}} + (n(n-1)-A_v)^{\frac{1}{p+q}}}\right) \quad (8)$$

where $A_\mu = \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{\mu_{\alpha_i}^p\mu_{\alpha_j}^q}{\mu_{\alpha_i}^p\mu_{\alpha_j}^q + (1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q}$ *and* $A_v = \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{v_{\alpha_i}^p v_{\alpha_j}^q}{v_{\alpha_i}^p v_{\alpha_j}^q + (1-v_{\alpha_i})^p(1-v_{\alpha_j})^q}$.

Proof Using the proposed operations, we have

$$\alpha_i{}^p = \left(\frac{\mu_{\alpha_i}^p}{\mu_{\alpha_i}^p + (1-\mu_{\alpha_i})^p}, \frac{v_{\alpha_i}^p}{v_{\alpha_i}^p + (1-v_{\alpha_i})^p}\right), \alpha_j^q = \left(\frac{\mu_{\alpha_j}^q}{\mu_{\alpha_j}^q + (1-\mu_{\alpha_j})^q}, \frac{v_{\alpha_j}^q}{v_{\alpha_j}^q + (1-v_{\alpha_j})^q}\right),$$

and then

$$\alpha_i{}^p \otimes \alpha_j{}^q = \left(\frac{\mu_{\alpha_i}^p\mu_{\alpha_j}^q}{\mu_{\alpha_i}^p\mu_{\alpha_j}^q + (1-\mu_{\alpha_i})^p(1-\mu_{\alpha_j})^q}, \frac{v_{\alpha_i}^p v_{\alpha_j}^q}{v_{\alpha_i}^p v_{\alpha_j}^q + (1-v_{\alpha_i})^p(1-v_{\alpha_j})^q}\right).$$

By Lemma 1 with $\omega_i = \frac{1}{n(n-1)}$ for all $i = 1,2,\ldots,n(n-1)$ and the proposed operations, it holds that $\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\alpha_i^p\otimes\alpha_j^q}{n(n-1)} = \left(\frac{1}{n(n-1)}A_\mu, \frac{1}{n(n-1)}A_v\right)$, and then

$$\left(\frac{1}{n(n-1)}\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{n}\left(\alpha_i^p\otimes\alpha_j^q\right)\right)^{\frac{1}{p+q}}$$
$$=\left(\frac{1}{n(n-1)}A_\mu,\frac{1}{n(n-1)}A_v\right)^{\frac{1}{p+q}}$$
$$=\left(\frac{\left(\frac{A_\mu}{n(n-1)}\right)^{\frac{1}{p+q}}}{\left(\frac{A_\mu}{n(n-1)}\right)^{\frac{1}{p+q}}+\left(1-\frac{A_\mu}{n(n-1)}\right)^{\frac{1}{p+q}}},\frac{\left(\frac{A_v}{n(n-1)}\right)^{\frac{1}{p+q}}}{\left(\frac{A_v}{n(n-1)}\right)^{\frac{1}{p+q}}+\left(1-\frac{A_v}{n(n-1)}\right)^{\frac{1}{p+q}}}\right)$$
$$=\left(\frac{A_\mu^{\frac{1}{p+q}}}{A_\mu^{\frac{1}{p+q}}+\left(n(n-1)-A_\mu\right)^{\frac{1}{p+q}}},\frac{A_v^{\frac{1}{p+q}}}{A_v^{\frac{1}{p+q}}+\left(n(n-1)-A_v\right)^{\frac{1}{p+q}}}\right)$$

that is, the identity holds. □

Now, we provide some desirable properties of the SIFBM operator.

Proposition 1 (Monotonicity) Let $\alpha_i=\left(\mu_{\alpha_i},v_{\alpha_i}\right)$ *and* $\beta_i=\left(\mu_{\beta_i},v_{\beta_i}\right)$ *be two collections of the IFNs for* $i=1,2,\ldots,n$. *If* $\mu_{\alpha_i}\leq\mu_{\beta_i}$ *and* $v_{\alpha_i}\geq v_{\beta_i}$, *then* $SIFBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq SIFBM^{p,q}\left(\beta_1,\beta_2,\ldots,\beta_n\right)$.

Proof Since $\mu_{\alpha_i}\leq\mu_{\beta_i}$ and $v_{\alpha_i}\geq v_{\beta_i}$, we get

$$\frac{1}{1+\left(\frac{1}{\mu_{\alpha_i}}-1\right)^p\left(\frac{1}{\mu_{\alpha_j}}-1\right)^q}\leq\frac{1}{1+\left(\frac{1}{\mu_{\beta_i}}-1\right)^p\left(\frac{1}{\mu_{\beta_j}}-1\right)^q},$$

i.e.,

$$\frac{\mu_{\alpha_i}^p\mu_{\alpha_j}^q}{\mu_{\alpha_i}^p\mu_{\alpha_j}^q+\left(1-\mu_{\alpha_i}\right)^p\left(1-\mu_{\alpha_j}\right)^q}\leq\frac{\mu_{\beta_i}^p\mu_{\beta_j}^q}{\mu_{\beta_i}^p\mu_{\beta_j}^q+\left(1-\mu_{\beta_i}\right)^p\left(1-\mu_{\beta_j}\right)^q}.$$

Thus, $\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\mu_{\alpha_i}^p\mu_{\alpha_j}^q}{\mu_{\alpha_i}^p\mu_{\alpha_j}^q+\left(1-\mu_{\alpha_i}\right)^p\left(1-\mu_{\alpha_j}\right)^q}\leq\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\mu_{\beta_i}^p\mu_{\beta_j}^q}{\mu_{\beta_i}^p\mu_{\beta_j}^q+\left(1-\mu_{\beta_i}\right)^p\left(1-\mu_{\beta_j}\right)^q}$. We denote

$$A_\mu=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\mu_{\alpha_i}^p\mu_{\alpha_j}^q}{\mu_{\alpha_i}^p\mu_{\alpha_j}^q+\left(1-\mu_{\alpha_i}\right)^p\left(1-\mu_{\alpha_j}\right)^q}$$

and

$$B_\mu=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\mu_{\beta_i}^p\mu_{\beta_j}^q}{\mu_{\beta_i}^p\mu_{\beta_j}^q+\left(1-\mu_{\beta_i}\right)^p\left(1-\mu_{\beta_j}\right)^q}$$

for brevity, then we have $\frac{1}{1+\left(\frac{1}{\frac{1}{n(n-1)}A_\mu}-1\right)^{\frac{1}{p+q}}}\leq\frac{1}{1+\left(\frac{1}{\frac{1}{n(n-1)}B_\mu}-1\right)^{\frac{1}{p+q}}}$, that is, $\frac{A_\mu^{\frac{1}{p+q}}}{A_\mu^{\frac{1}{p+q}}+\left(n(n-1)-A_\mu\right)^{\frac{1}{p+q}}}\leq\frac{B_\mu^{\frac{1}{p+q}}}{B_\mu^{\frac{1}{p+q}}+\left(n(n-1)-B_\mu\right)^{\frac{1}{p+q}}}$.

Similarly, we have $\frac{A_v^{\frac{1}{p+q}}}{A_v^{\frac{1}{p+q}}+(n(n-1)-A_v)^{\frac{1}{p+q}}}\geq\frac{B_v^{\frac{1}{p+q}}}{B_v^{\frac{1}{p+q}}+(n(n-1)-B_v)^{\frac{1}{p+q}}}$, where $A_v=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{v_{\alpha_i}^p v_{\alpha_j}^q}{v_{\alpha_i}^p v_{\alpha_j}^q+\left(1-v_{\alpha_i}\right)^p\left(1-v_{\alpha_j}\right)^q}$ and $B_v=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{v_{\beta_i}^p v_{\beta_j}^q}{v_{\beta_i}^p v_{\beta_j}^q+\left(1-v_{\beta_i}\right)^p\left(1-v_{\beta_j}\right)^q}$.

Thus $SIFBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq SIFBM^{p,q}\left(\beta_1,\beta_2,\ldots,\beta_n\right)$. □

Proposition 2 (Commutativity) Let $\alpha_i=(\mu_{\alpha_i},v_{\alpha_i})(i=1,2,\ldots,n)$ *be a collection of the IFNs. Then* $SIFBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)=SIFBM^{p,q}\left(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n\right)$, *where* $\left(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n\right)$ *is any permutation of* $(\alpha_1,\alpha_2,\ldots,\alpha_n)$.

Proof It is trivial. □

Proposition 3 (Boundedness) Let $\alpha_i=(\mu_{\alpha_i},v_{\alpha_i})(i=1,2,\cdots n)$ *be a collection of the IFNs, Assume that* $\alpha^-=\left(\min_i\{\mu_{\alpha_i}\},\max_i\{v_{\alpha_i}\}\right)$ *and* $\alpha^+=(\max_i\{\mu_{\alpha_i}\},\min_i\{v_{\alpha_i}\})$. *Then* $\alpha^-\leq SIFBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq\alpha^+$.

Proof Assume that $\mu_m=\min_i\{\mu_{\alpha_i}\}\leq\mu_{\alpha_i}\leq\max_i\{\mu_{\alpha_i}\}=\mu_M$ and $v_m=\min_i\{v_{\alpha_i}\}\leq v_{\alpha_i}\leq\max_i\{v_{\alpha_i}\}=v_M$, for all i. It is obvious that

$$\mu_m \le \frac{1}{1+\left(\frac{1}{\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{1+\left(\frac{1}{\mu_{\alpha_i}}-1\right)^p\left(\frac{1}{\mu_{\alpha_j}}-1\right)^q}}-1\right)^{\frac{1}{p+q}}} \le \mu_M$$

and

$$v_m \le \frac{1}{1+\left(\frac{1}{\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{1+\left(\frac{1}{v_{\alpha_i}}-1\right)^p\left(\frac{1}{v_{\alpha_j}}-1\right)^q}}-1\right)^{\frac{1}{p+q}}} \le v_M.$$

Thus, $\alpha^- \le SIFBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \alpha^+$. □

Proposition 4 (Idempotency) If all $\alpha_i (i = 1, 2, \cdots n)$ *are equal, i.e.,* $\alpha_i = \alpha$, *for all* i, *then* $SIFBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha$.

Proof By Theorem 1, we have

$$\begin{aligned}
&SIFBM^{p,q}(\alpha_1, \alpha_2, \cdots \alpha_n)\\
&= \left(\frac{A_\mu^{\frac{1}{p+q}}}{A_\mu^{\frac{1}{p+q}} + (n(n-1)-A_\mu)^{\frac{1}{p+q}}}, \frac{A_v^{\frac{1}{p+q}}}{A_v^{\frac{1}{p+q}} + (n(n-1)-A_v)^{\frac{1}{p+q}}}\right)\\
&= \left(\begin{array}{c}\dfrac{\left(n(n-1)\dfrac{\mu_\alpha^{p+q}}{\mu_\alpha^{p+q}+(1-\mu_\alpha)^{p+q}}\right)^{\frac{1}{p+q}}}{\left(n(n-1)\dfrac{\mu_\alpha^{p+q}}{\mu_\alpha^{p+q}+(1-\mu_\alpha)^{p+q}}\right)^{\frac{1}{p+q}} + \left(n(n-1)\dfrac{(1-\mu_\alpha)^{p+q}}{\mu_\alpha^{p+q}+(1-\mu_\alpha)^{p+q}}\right)^{\frac{1}{p+q}}},\\ \dfrac{\left(n(n-1)\dfrac{v_\alpha^{p+q}}{v_\alpha^{p+q}+(1-v_\alpha)^{p+q}}\right)^{\frac{1}{p+q}}}{\left(n(n-1)\dfrac{v_\alpha^{p+q}}{v_\alpha^{p+q}+(1-v_\alpha)^{p+q}}\right)^{\frac{1}{p+q}} + \left(n(n-1)\dfrac{(1-v_\alpha)^{p+q}}{v_\alpha^{p+q}+(1-v_\alpha)^{p+q}}\right)^{\frac{1}{p+q}}}\end{array}\right)\\
&= \left(\frac{\mu_\alpha}{\mu_\alpha+1-\mu_\alpha}, \frac{v_\alpha}{v_\alpha+1-v_\alpha}\right) = (\mu_\alpha, v_\alpha) = \alpha.
\end{aligned}$$

Thus, the identity holds. □

In the following list, let us consider some special cases of the SIFBM operator by taking different values of the parameters p and q.

Case 1 If $p = 1, q = 0$, then $SIFBM^{1,0}(\alpha_1, \alpha_2, \cdots \alpha_n) = IFA(\alpha_1, \alpha_2, \cdots \alpha_n)$

Case 2 If $p + q = 1$, then

$$\begin{aligned}
&SIFBM^{p,q}(\alpha_1, \alpha_2, \cdots \alpha_n)\\
&= \left(\frac{A_\mu}{A_\mu + n(n-1) - A_\mu}, \frac{A_v}{A_v + n(n-1) - A_v}\right)\\
&= \left(\begin{array}{c}\dfrac{1}{n(n-1)}\sum_{\substack{i\neq j\\ i,j=1}}^{n}\dfrac{\mu_{\alpha_i}^p\mu_{\alpha_j}^q}{\mu_{\alpha_i}^p\mu_{\alpha_j}^q + (1-\mu_{\alpha_i})^p\left(1-\mu_{\alpha_j}\right)^q},\\ \dfrac{1}{n(n-1)}\sum_{\substack{i\neq j\\ i,j=1}}^{n}\dfrac{v_{\alpha_i}^p v_{\alpha_j}^q}{v_{\alpha_i}^p v_{\alpha_j}^q + (1-v_{\alpha_i})^p(1-v_{\alpha_j})^q}\end{array}\right)\\
&= \underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{IFA}}\left(SIFWG(\alpha_i, \alpha_j)\right).
\end{aligned}$$

Example 2 Assume that we have three IFNs, that is, $\alpha_1 = (0.1, 0.8)$, $\alpha_2 = (0.2, 0.7)$ and $\alpha_3 = (0.3, 0.6)$. We fuse these IFNs with the SIFBM operator. Using Theorem 1, we have

$$\begin{aligned}
A_\mu =& \frac{0.1^p0.2^q}{0.1^p0.2^q + 0.9^p0.8^q} + \frac{0.1^p0.3^q}{0.1^p0.3^q + 0.9^p0.7^q}\\
&+ \frac{0.2^p0.3^q}{0.2^p0.3^q + 0.8^p0.7^q} +\\
&\frac{0.1^q0.2^p}{0.1^q0.2^p + 0.9^q0.8^p} + \frac{0.1^q0.3^p}{0.1^q0.3^p + 0.9^q0.7^p}\\
&+ \frac{0.2^p0.3^q}{0.1^q0.2^p + 0.9^q0.8^p}
\end{aligned}$$

and

$$\begin{aligned}
A_v =& \frac{0.8^p0.7^q}{0.8^p0.7^q + 0.2^p0.3^q} + \frac{0.8^p0.6^q}{0.8^p0.6^q + 0.2^p0.4^q}\\
&+ \frac{0.7^p0.6^q}{0.7^p0.6^q + 0.3^p0.4^q} +\\
&\frac{0.8^q0.7^p}{0.8^q0.7^p + 0.2^q0.3^p} + \frac{0.8^q0.6^p}{0.8^q0.6^p + 0.2^q0.4^p}\\
&+ \frac{0.7^p0.6^q}{0.7^q0.6^p + 0.3^q0.4^p},
\end{aligned}$$

then

$$SIFBM^{p,q}(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{A_\mu^{\frac{1}{p+q}}}{A_\mu^{\frac{1}{p+q}} + (6-A_\mu)^{\frac{1}{p+q}}}, \frac{A_v^{\frac{1}{p+q}}}{A_v^{\frac{1}{p+q}} + (6-A_v)^{\frac{1}{p+q}}}\right).$$

Specially, $SIFBM^{0.5,0.5}(\alpha_1, \alpha_2, \alpha_3) = (0.19, 0.705)$, $SIFBM^{1,0}(\alpha_1, \alpha_2, \alpha_3) = (0.2, 0.7)$.

3.2 Weighted Symmetric Intuitionistic Fuzzy Bonferroni Mean Operators

In many practical situations, the considered criteria should be assigned different weights for different importance.

Thus, we define the weighted SIFBM operator in this section.

Definition 10 Let $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)(i = 1, 2\cdots, n)$ be a collection of IFNs. For any $p, q > 0$, if

$$WSIFBM_{\omega}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\underset{\substack{i,j=1 \\ j \neq i}}{\overset{n}{\oplus}} \frac{\omega_i \omega_j}{1-\omega_i} \left(\alpha_i^p \otimes \alpha_j^q \right) \right)^{\frac{1}{p+q}}, \tag{9}$$

then $WSIFBM_{\omega}^{p,q}$ is called a weighted SIFBM operator, where $\omega_i \in [0,1]$ and $\sum_{k=1}^{n} \omega_i = 1$.

Theorem 2 *Let $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)(i = 1, 2, \ldots, n)$ be a collection of IFNs. Then*

$$WSIFBM_{\omega}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{A_{\omega,\mu}^{\frac{1}{p+q}}}{A_{\omega,\mu}^{\frac{1}{p+q}} + \left(1 - A_{\omega,\mu}\right)^{\frac{1}{p+q}}}, \frac{A_{\omega,\upsilon}^{\frac{1}{p+q}}}{A_{\omega,\upsilon}^{\frac{1}{p+q}} + \left(1 - A_{\omega,v}\right)^{\frac{1}{p+q}}} \right), \tag{10}$$

where

$$A_{\omega,\mu} = \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \frac{\mu_{\alpha_i}^p \mu_{\alpha_j}^q}{\mu_{\alpha_i}^p \mu_{\alpha_j}^q + \left(1-\mu_{\alpha_i}\right)^p \left(1-\mu_{\alpha_j}\right)^q}$$

and

$$A_{\omega,v} = \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \frac{v_{\alpha_i}^p v_{\alpha_j}^q}{v_{\alpha_i}^p v_{\alpha_j}^q + \left(1-v_{\alpha_i}\right)^p \left(1-v_{\alpha_j}\right)^q}.$$

Proof The proof is similar to that of Theorem 1, so we omit it. □

The following properties of the weighted SIFBM operator are similar to those of the SIFBM operator, so we list them without proofs as follows:

Proposition 5 (Monotonicity) Let $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)$ *and* $\beta_i = \left(\mu_{\beta_i}, v_{\beta_i}\right)$ *where* $i = 1, 2, \ldots, n$ *be two collections of the IFNs. If* $\mu_{\alpha_i} \leq \mu_{\beta_i}$ *and* $v_{\alpha_i} \geq v_{\beta_i}$, *for all* $i = 1, 2, \ldots, n$, *then*

$$WSIFBM_{\omega}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq WSIFBM_{\omega}^{p,q}\left(\beta_1, \beta_2, \ldots, \beta_n\right).$$

Proposition 6 (Commutativity) Let $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)(i = 1, 2, \ldots, n)$ *be a collection of the IFNs. Then*

$$WSIFBM_{\omega}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = WSIFBM_{\omega}^{p,q}\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\right),$$

where $\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\right)$ *is any permutation of* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Proposition 7 (Boundedness) Let $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)(i = 1, 2, \cdots n)$ *be a collection of the IFNs, and we define* $\alpha^- = \left(\min_i\{\mu_{\alpha_i}\}, \max_i\{v_{\alpha_i}\}\right)$, $\alpha^+ = \left(\max_i\{\mu_{\alpha_i}\}, \min_i\{v_{\alpha_i}\}\right)$. *Then*

$$\alpha^- \leq WSIFBM_{\omega}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+.$$

Proposition 8 (Idempotency) If all $\alpha_i (i = 1, 2, \cdots n)$ *are equal, i.e.,* $\alpha_i = \alpha$, *for all* i, *then*

$$WSIFBM_{\omega}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha.$$

In the following part, some special cases of the weighted SIFBM operator by taking different values of the parameters p and q are given.

Case1. If $p = 1, q = 0$, then

$$WSIFBM_{\omega}^{1,0}(\alpha_1, \alpha_2, \cdots \alpha_n) = IFWA(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Case 2. If $p + q = 1$, then

$$\begin{aligned} & WSIFBM_{\omega}^{p,q}(\alpha_1, \alpha_2, \cdots \alpha_n) \\ & = \left(\frac{A_{\omega,\mu}}{A_{\omega,\mu} + 1 - A_{\omega,\mu}}, \frac{A_{\omega,v}}{A_{\omega,v} + 1 - A_{\omega,v}} \right) \\ & = \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{IFWA}} \left(SIFWG\left(\alpha_i, \alpha_j\right) \right). \end{aligned}$$

Example 3 In Example 2, assume that the weight vector $\omega = (0.3, 0.3, 0.4)$. Using Theorem 2, we have

$$\begin{aligned} A_{\omega,\mu} = & \frac{9}{70} \frac{0.1^p 0.2^q}{0.1^p 0.2^q + 0.9^p 0.8^q} + \frac{12}{70} \frac{0.1^p 0.3^q}{0.1^p 0.3^q + 0.9^p 0.7^q} \\ & + \frac{12}{70} \frac{0.2^p 0.3^q}{0.2^p 0.3^q + 0.8^p 0.7^q} + \\ & \frac{9}{70} \frac{0.1^q 0.2^p}{0.1^q 0.2^p + 0.9^q 0.8^p} + \frac{1}{5} \frac{0.1^q 0.3^p}{0.1^q 0.3^p + 0.9^q 0.7^p} \\ & + \frac{1}{5} \frac{0.2^p 0.3^q}{0.1^q 0.2^p + 0.9^q 0.8^p}, \end{aligned}$$

and

$$A_{\omega,v} = \frac{9}{70}\frac{0.8^p 0.7^q}{0.8^p 0.7^q + 0.2^p 0.3^q} + \frac{12}{70}\frac{0.8^p 0.6^q}{0.8^p 0.6^q + 0.2^p 0.4^q} + \frac{12}{70}\frac{0.7^p 0.6^q}{0.7^p 0.6^q + 0.3^p 0.4^q} + \frac{9}{70}\frac{0.8^q 0.7^p}{0.8^q 0.7^p + 0.2^q 0.3^p} + \frac{1}{5}\frac{0.8^q 0.6^p}{0.8^q 0.6^p + 0.2^q 0.4^p} + \frac{1}{5}\frac{0.7^p 0.6^q}{0.7^q 0.6^p + 0.3^q 0.4^p},$$

then $WSIFBM_{\omega}^{p,q}(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{A_{\omega,\mu}^{\frac{1}{p+q}}}{A_{\omega,\mu}^{\frac{1}{p+q}} + (1 - A_{\omega,\mu})^{\frac{1}{p+q}}}, \frac{A_{\omega,v}^{\frac{1}{p+q}}}{A_{\omega,v}^{\frac{1}{p+q}} + (1 - A_{\omega,v})^{\frac{1}{p+q}}} \right).$

4 Acceptably Group Multiplicative Consistency and Consensus for the IFPRs in Group Decision Making

For a group decision making problem with s decision makers, we can obtain s individual IFPRs with the form of Eq. (5). In order to find the final result of the problem, it is needed to aggregate these s individual IFPRs into a collective one. Before doing this, as presented in the introduction, we shall firstly check the consistency of each IFPR and make sure that all of them are consistent and consensus in group; otherwise the unreasonable results may be produced. Here, we use the following method to check and reach the acceptably group multiplicative consistency and consensus for the IFPRs in group decision making where deviation measure is a indispensable tool to measure the deviation between the IFPRs.

4.1 Acceptable Consensus for the IFPRs in Group Decision Making

Definition 11 Let P, Q be two IFPRs. Then, a partial order is defined between P and Q, denoted as $P \leq Q$, if it holds that $p_{ij} \leq q_{ij}$ (i.e., $\mu_{p_{ij}} \leq \mu_{q_{ij}}$ and $v_{p_{ij}} \leq v_{q_{ij}}$ [10]) for all $i<j$, $i,j = 1,2,\ldots,n$.

Definition 12 Let P, Q and R be three IFPRs; i.e., $P, Q, R \in \mathcal{IFPR}_{n\times n}$ which denote the set of the IFPRs with order n. Then, $D : \mathcal{IFPR}_{n\times n} \times \mathcal{IFPR}_{n\times n} \to [0,1]$ is called a deviation measure, if it possesses the following properties:

(1) $D(P,P) = 0$, for an IFPR P;
(2) $D(P,Q) = D(Q,P)$, for two IFPRs P and Q;
(3) $P \leq Q \leq R$ implies $D(P,Q) \vee D(Q,R) \leq D(P,R)$.

For two given IFPRs P, Q, a special deviation measure between P and Q can be given as follows:

$$D(P,Q) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(p_{ij}, q_{ij}), \tag{11}$$

where

$$d(p_{ij}, q_{ij}) = \frac{\left|\mu_{p_{ij}} - \mu_{q_{ij}}\right| + \left|v_{p_{ij}} - v_{q_{ij}}\right| + \left|\pi_{p_{ij}} - \pi_{q_{ij}}\right|}{2}. \tag{12}$$

Note that in the defined distance measure Eq. (11), we take the denominator as $n(n-1)$, but not $(n-1)(n-2)$ in Ref. [12]. Now, we provide the acceptable consensus as follows:

Definition 13 Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ and P be a collection of IFPRs. If $D(P^{(l)}, P) < b$ for a given threshold $b \in [0,1]$, then $P^{(l)}$ $(l = 1,\ldots,s)$ is said to be of acceptable consensus with respect to P.

If $P^{(1)} = P^{(2)} = \cdots = P^{(s)} = P$, then $D(P^{(l)}, P) = 0$, which implies a full consensus is reached. Otherwise, the bigger $D(P^{(l)}, P)$, the lower the consensus among these decision makers, at this time, the decision makers are requested to modify their IFPRs to reach the consensus. A concrete method to reach the acceptable consensus will be integrated into the procedure to reach the acceptable group multiplicative consistency in the following section.

4.2 Acceptable Group Multiplicative Consistency for the IFPRs in Group Decision Making

To assure that the aggregated individual IFPRs is of acceptably multiplicative consistency, we develop the following results:

Definition 14 Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ be a collection of IFPRs, and $\widetilde{P}$ be a multiplicatively consistent IFPR. For a given threshold $a \in [0,1]$, if $D(P^{(l)}, \widetilde{P}) \leq a$ for $l = 1,2,\ldots,s$, then $P^{(l)}$ is said to be of acceptably group multiplicative consistency with respect to $\widetilde{P}$.

Particularly, if $s = 1$, then $P^{(1)}$ is said to be of acceptably multiplicative consistency with respect to $\widetilde{P}$. Based on the normalized intuitionistic fuzzy priority weight vector, a multiplicatively consistent IFPR $\widetilde{P} = (\widetilde{p}_{ij})_{n\times n}$ can be established with the Lehmer mean operator [43], where

$$\widetilde{p}_{ij} = \begin{cases} (0.5, 0.5), & i = j, \\ \left(\dfrac{\mu_{\omega_i} + \lambda \dfrac{\pi_{\omega_i}^{\gamma}}{\pi_{\omega_i}^{\gamma-1} + \pi_{\omega_j}^{\gamma-1}}}{\mu_{\omega_i} + \mu_{\omega_j} + L(\pi_{\omega_i}, \pi_{\omega_j}, \gamma)}, \dfrac{\mu_{\omega_j} + \lambda \dfrac{\pi_{\omega_j}^{\gamma}}{\pi_{\omega_i}^{\gamma-1} + \pi_{\omega_j}^{\gamma-1}}}{\mu_{\omega_i} + \mu_{\omega_j} + L(\pi_{\omega_i}, \pi_{\omega_j}, \gamma)} \right), & i \neq j, \end{cases} \tag{13}$$

$\lambda \in [0,1]$, $L(\pi_{\omega_i},\pi_{\omega_j},\gamma)=\frac{\pi_{\omega_i}^{\gamma}+\pi_{\omega_j}^{\gamma}}{\pi_{\omega_i}^{\gamma-1}+\pi_{\omega_j}^{\gamma-1}}$, $\gamma \in (-\infty,+\infty)$ is an aggregated result of π_{ω_i} and π_{ω_j} for the intuitionistic fuzzy priority weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$. Obviously, Eq. (13) not only makes a full consideration on the membership degrees and non-membership degrees of ω_i and ω_j, but also on their hesitancy degrees, when constructing a multiplicatively consistent IFPR.

Theorem 3 *The matrix $\widetilde{P}=(\widetilde{p}_{ij})_{n\times n}$ is a multiplicatively consistent IFPR, where $\widetilde{p}_{ij}$ $(i,j=1,2,\ldots,n)$ are defined in Eq.* (13).

Proof Firstly, we prove that $\widetilde{P}$ is an IFPR. It is obvious that $\frac{\mu_{\omega_i}+\lambda\frac{\pi_{\omega_i}^{\gamma}}{\pi_{\omega_i}^{\gamma-1}+\pi_{\omega_j}^{\gamma-1}}}{\mu_{\omega_i}+\mu_{\omega_j}+L(\pi_{\omega_i},\pi_{\omega_j},\gamma)}, \frac{\mu_{\omega_j}+\lambda\frac{\pi_{\omega_j}^{\gamma}}{\pi_{\omega_i}^{\gamma-1}+\pi_{\omega_j}^{\gamma-1}}}{\mu_{\omega_i}+\mu_{\omega_j}+L(\pi_{\omega_i},\pi_{\omega_j},\gamma)} \in [0,1]$, $\frac{\mu_{\omega_i}+\lambda\frac{\pi_{\omega_i}^{\gamma}}{\pi_{\omega_i}^{\gamma-1}+\pi_{\omega_j}^{\gamma-1}}}{\mu_{\omega_i}+\mu_{\omega_j}+L(\pi_{\omega_i},\pi_{\omega_j},\gamma)} + \frac{\mu_{\omega_j}+\lambda\frac{\pi_{\omega_j}^{\gamma}}{\pi_{\omega_i}^{\gamma-1}+\pi_{\omega_j}^{\gamma-1}}}{\mu_{\omega_i}+\mu_{\omega_j}+L(\pi_{\omega_i},\pi_{\omega_j},\gamma)} \leq 1$ and $\mu_{\widetilde{p}_{ij}}=\nu_{\widetilde{p}_{ji}}, \nu_{\widetilde{p}_{ij}}=\mu_{\widetilde{p}_{ji}}$. Thus, $\widetilde{P}$ is an IFPR. Using Eq. (13), we have

$$\begin{aligned}
&\mu_{\widetilde{p}_{ij}}\mu_{\widetilde{p}_{jk}}\mu_{\widetilde{p}_{ki}}\\
&=\frac{\mu_{\omega_i}+\lambda\frac{\pi_{\omega_i}^{\gamma}}{\pi_{\omega_i}^{\gamma-1}+\pi_{\omega_j}^{\gamma-1}}}{\mu_{\omega_i}+\mu_{\omega_j}+L(\pi_{\omega_i},\pi_{\omega_j},\gamma)}\frac{\mu_{\omega_j}+\lambda\frac{\pi_{\omega_j}^{\gamma}}{\pi_{\omega_j}^{\gamma-1}+\pi_{\omega_k}^{\gamma-1}}}{\mu_{\omega_j}+\mu_{\omega_k}+L(\pi_{\omega_j},\pi_{\omega_k},\gamma)}\\
&\quad\frac{\mu_{\omega_k}+\lambda\frac{\pi_{\omega_k}^{\gamma}}{\pi_{\omega_k}^{\gamma-1}+\pi_{\omega_i}^{\gamma-1}}}{\mu_{\omega_k}+\mu_{\omega_i}+L(\pi_{\omega_k},\pi_{\omega_i},\gamma)}\\
&=\frac{\mu_{\omega_j}+\lambda\frac{\pi_{\omega_j}^{\gamma}}{\pi_{\omega_i}^{\gamma-1}+\pi_{\omega_j}^{\gamma-1}}}{\mu_{\omega_i}+\mu_{\omega_j}+L(\pi_{\omega_i},\pi_{\omega_j},\gamma)}\frac{\mu_{\omega_k}+\lambda\frac{\pi_{\omega_k}^{\gamma}}{\pi_{\omega_j}^{\gamma-1}+\pi_{\omega_k}^{\gamma-1}}}{\mu_{\omega_j}+\mu_{\omega_k}+L(\pi_{\omega_j},\pi_{\omega_k},\gamma)}\\
&\quad\frac{\mu_{\omega_i}+\lambda\frac{\pi_{\omega_i}^{\gamma}}{\pi_{\omega_k}^{\gamma-1}+\pi_{\omega_i}^{\gamma-1}}}{\mu_{\omega_k}+\mu_{\omega_i}+L(\pi_{\omega_k},\pi_{\omega_i},\gamma)}\\
&=\nu_{\widetilde{p}_{ij}}\nu_{\widetilde{p}_{jk}}\nu_{\widetilde{p}_{ki}}.
\end{aligned}$$

Thus, $\widetilde{P}$ is multiplicatively consistent, which completes the proof. □

When $\lambda=0$ and $\gamma=1$, Eq. (13) reduces to Liao's formula [41], i.e., Eq. (6). Next, we develop some properties of the IFPRs with respect to the SIMBM operator and the acceptably group multiplicative consistency as follows:

Proposition 9 *Let $P^{(1)},P^{(2)},\ldots,P^{(s)}$ be a collection of IFPRs. Then,*

$$P=\left(SIFBM^{p,q}(p_{ij}^{(1)},\ldots,p_{ij}^{(s)})\right)_{n\times n} \tag{14}$$

is an IFPR.

Proof It is obvious that $SIFBM^{p,q}(p_{ij}^{(1)},\ldots,p_{ij}^{(s)})$ is an IFN for all $i,j=1,2,\ldots,n$. Using idempotency of the SIFBM operator, we get $p_{ii}=(0.5,0.5)$. By Theorem 1, it holds that

$$\begin{aligned}
&SIFBM^{p,q}(p_{ij}^{(1)},\ldots,p_{ij}^{(s)})^c\\
&=\left(\frac{A_\nu^{\frac{1}{p+q}}}{A_\nu^{\frac{1}{p+q}}+(n(n-1)-A_\nu)^{\frac{1}{p+q}}},\frac{A_\mu^{\frac{1}{p+q}}}{A_\mu^{\frac{1}{p+q}}+\left(n(n-1)-A_\mu\right)^{\frac{1}{p+q}}}\right)\\
&=SIFBM^{p,q}(p_{ji}^{(1)},\ldots,p_{ji}^{(s)}).
\end{aligned}$$

Thus, P is an IFPR. □

Proposition 10 *Let $P^{(1)},P^{(2)},\ldots,P^{(s)}$ be a collection of IFPRs. Then, $P^-=(p_{ij}^-)_{n\times n}$ and $P^+=(p_{ij}^+)_{n\times n}$ defined by*

$$p_{ij}^-=\begin{cases}(\min_{l=1}^{s}\{\mu_{p_{ij}^{(l)}}\},\max_{l=1}^{s}\{\nu_{p_{ij}^{(l)}}\}), & i<j,\\ (0.5,0.5), & i=j,\\ (\max_{l=1}^{s}\{\nu_{p_{ij}^{(l)}}\},\min_{l=1}^{s}\{\mu_{p_{ij}^{(l)}}\}), & i>j,\end{cases} \tag{15}$$

$$p_{ij}^+=\begin{cases}(\max_{l=1}^{s}\{\mu_{p_{ij}^{(l)}}\},\min_{l=1}^{s}\{\nu_{p_{ij}^{(l)}}\}), & i<j,\\ (0.5,0.5), & i=j,\\ (\min_{l=1}^{s}\{\nu_{p_{ij}^{(l)}}\},\max_{l=1}^{s}\{\mu_{p_{ij}^{(l)}}\}), & i>j,\end{cases} \tag{16}$$

are IFPRs.

Proof We only prove that P^- is an IFPR, and P^+ can be proved in a similar way. It is obvious that $\min_{l=1}^{s}\{\mu_{p_{ij}^{(l)}}\}$ and $\max_{l=1}^{s}\{\nu_{p_{ij}^{(l)}}\}\in[0,1]$. Without loss of the generality, we assume that $i<j$, $\min_{l=1}^{s}\{\mu_{p_{ij}^{(l)}}\}=\mu_{p_{ij}^{(l_0)}}$ and $\max_{l=1}^{s}\{\nu_{p_{ij}^{(l)}}\}=\nu_{p_{ij}^{(l_1)}}$. Then, $\mu_{p_{ij}^{(l_1)}}+\nu_{p_{ij}^{(l_1)}}\leq 1$ and $\nu_{p_{ij}^{(l_1)}}\geq\nu_{p_{ij}^{(l_0)}}$, thus, $\mu_{p_{ij}^{(l_1)}}+\nu_{p_{ij}^{(l_0)}}\leq 1$; that is, p_{ij}^- is an IFN. It follows immediately from Definition 7 that P^- is an IFPR. □

Proposition 11 *Let $P^{(1)},P^{(2)},\ldots,P^{(s)}$ be a collection of IFPRs and P, P^-, P^+ be IFPRs defined in Eqs.* (14), (15) *and* (16), *respectively. Then, $P^-\leq P\leq P^+$.*

Proof It follows immediately from the boundedness of the SIFBM operator. □

Theorem 4 *Let $P^{(1)},P^{(2)},\ldots,P^{(s)}$ be a collection of IFPRs and K be a multiplicatively consistent IFPR* *satisfying* $K \le P^{(l)}$ *for all* $l = 1, 2, \ldots, s$. *If* P^{+} *defined in Eq.* (16) *is of acceptable group multiplicative consistency with respect to* K, *then* $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ *and* P *are of acceptable group multiplicative consistency with respect to* K.

Proof Since $P^{(l)}$ is an IFPR satisfying $K \le P^{(l)}$, it holds that $K \le P^{-} \le P^{(l)}, P \le P^{+}$. Because P^{+} is of acceptable group multiplicative consistency with respect to K; i.e., $D(P^{+}, K) < a$, we have $D(P, K) \le D(P^{+}, K) < a$ and $D(P^{(l)}, K) \le D(P^{+}, K) < a$; i.e. $P^{(l)}, P$ are of acceptable group multiplicative consistency with respect to K. □

4.3 An Algorithm to Jointly Check and Reach the Acceptably Group Multiplicative Consistency and Consensus

By Theorem 4, to obtain a reasonable decision result from the collective IFPR, it is required that all the individual IFPRs provided by decision makers should be of acceptable group multiplicative consistency. Unfortunately, due to the vagueness inherent in the human thinking, it is not easy for decision makers to provide such individual IFPRs in practical group decision making problems. Thus, how to repair the consistency of IFPRs is very important in decision making procedure. To circumvent this issue, an iterative algorithm is designed in the sequel.

$$\textbf{Model 1}: \min D(P^{+(h)}, \widetilde{P^{+}}^{(h)})$$

$$s.t.\begin{cases} \mu_{\widetilde{p^{+}}_{ij}^{(h)}} \le \mu_{p^{-}{}_{ij}^{(h)}},\ \nu_{\widetilde{p^{+}}_{ij}^{(h)}} \ge \nu_{p^{-}{}_{ij}^{(h)}}, \\ \mu_{\omega_i^{(h)}}, \nu_{\omega_i^{(h)}} \in [0,1],\ \mu_{\omega_i^{(h)}} + \nu_{\omega_i^{(h)}} \le 1, \\ \sum_{j=1, i\neq j} \mu_{\omega_j^{(h)}} \le \nu_{\omega_i^{(h)}},\ \sum_{j=1, i\neq j} \nu_{\omega_j^{(h)}} \le \mu_{\omega_i^{(h)}} + n - 2, \\ \gamma^{(h)} \in (-\infty, +\infty), \\ \lambda^{(h)} \in [0,1]. \end{cases}$$

$$\textbf{Model 2}: \min D(P^{+(h+1)}, \widetilde{P^{+}}^{(h+1)})$$

$$s.t.\begin{cases} \mu_{\widetilde{P^{+}}^{(h)}} \le \mu_{\widetilde{P^{+}}^{(h+1)}},\ \nu_{\widetilde{P^{+}}^{(h)}} \ge \nu_{\widetilde{P^{+}}^{(h+1)}}; \\ \mu_{\widetilde{P^{+}}^{(h+1)}} \le \mu_{P^{-(h+1)}},\ \nu_{\widetilde{P^{+}}^{(h+1)}} \ge \nu_{P^{-(h+1)}}; \\ \mu_{P^{+(h+1)}} \le \mu_{P^{+(h)}},\ \nu_{P^{+(h+1)}} \ge \nu_{P^{+(h)}}; \\ \mu_{\omega_i^{(h+1)}}, \nu_{\omega_i^{(h+1)}} \in [0,1],\ \mu_{\omega_i^{(h+1)}} + \nu_{\omega_i^{(h+1)}} \le 1; \\ \sum_{j=1, i\neq j} \mu_{\omega_j^{(h+1)}} \le \nu_{\omega_i^{(h+1)}},\ \sum_{j=1, i\neq j} \nu_{\omega_j^{(h+1)}} \le \mu_{\omega_i^{(h+1)}} + n - 2, \\ \gamma^{(h)} \in (-\infty, +\infty), \\ \lambda^{(h)} \in [0,1]. \end{cases}$$

Note that although Yang et al. [44] investigated the multiplicative consistency threshold of an IFPR which varies with the order of the IFPR, where the investigated multiplicative consistency [45] is different from that in the present paper, we can not guarantee the rationality of their threshold used in this paper. Thus, the value of threshold is taken as 0.1 following Ref. [2].

Algorithm 1 Check and reach the acceptably group multiplicative consistency

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Input: The original IFPRs $P^{(l,0)}$ for $l = 1 : s$, iteration $h \leftarrow 0$ and the threshold $a \leftarrow 0.1$
Output: The modified IFPRs $P^{(l,h)}$ for $l = 1 : s$

1. $P^{-(h)}$ and $P^{+(h)}$ are computed by Eqs. (15) and (16)
2. $\widetilde{P^{+}}^{(h)}$ is assumed by Eq. (13) and computed by Model 1, that is, $D(P^{+(h)}, \widetilde{P^{+}}^{(h)})$ is minimal such that $\widetilde{P^{+}}^{(h)} \le P^{-(h)}$

while $D(P^{(1,h)}, \widetilde{P^{+}}^{(h)}) \ge a$ **or** $\cdots$ **or** $D(P^{(s,h)}, \widetilde{P^{+}}^{(h)}) \ge a$ **do**

3. select those $P^{(l,h)}$ satisfying $D(P^{(l,h)}, \widetilde{P^{+}}^{(h)}) \ge a$ with $l_k, k = 1 : t$
4. Find the position in $P^{(l_k,h)}$ such that the distance between the IFN $p_{ij}^{(l_k,h)}$ and the corresponding IFN $\widetilde{p_{ij}^{+}}^{(h)}$ in $\widetilde{P^{+}}^{(h)}$ is maximal for $i < j, k = 1, \cdots, t$.
5. $P^{(l,h+1)} \leftarrow P^{(l,h)}$ for $l = 1 : s$ besides the IFN $p_{ij}^{(l_k,h)}$ in the position provided by Step 4. as an unknown
6. $P^{-(h+1)}$ and $P^{+(h+1)}$ are computed by Eqs. (15) and (16) and $\widetilde{P^{+}}^{(h+1)}$ is assumed by Eq. (13); the unknown IFN in Step 5. and $\widetilde{P^{+}}^{(h+1)}$ are computed by Model 2, that is, $D(P^{+(h+1)}, \widetilde{P^{+}}^{(h+1)})$ is minimal such that $\widetilde{P^{+}}^{(h)} \le \widetilde{P^{+}}^{(h+1)} \le P^{-(h+1)}$

$h \leftarrow h + 1$

end while

Theorem 5 *Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ be a collection of IFPRs and $\{P^{+(h)}\}$ and $\{\widetilde{P^{+(h)}}\}$ be the sequences of IFPRs generated from Algorithm I. Then it holds that*

$$\lim_{h\to\infty} D(P^{+(h)}, \widetilde{P^{+(h)}}) = 0. \tag{17}$$

Proof Using Definition 12, we have $D(P^{+(h)}, \widetilde{P^{+(h)}}) \geq 0$, and hence the sequence $D(P^{+(h)}, \widetilde{P^{+(h)}})$ has a lower bound. By Step 3 in Algorithm I, we get $P^{+(h)} \leq P^{+(h+1)} \leq \widetilde{P^{+(h+1)}} \leq \widetilde{P^{+(h)}}$. Then, $D(P^{+(h+1)}, \widetilde{P^{+(h_1)}}) \leq D(P^{+(h)}, \widetilde{P^{+(h)}})$, that is, the sequence $D(P^{+(h)}, \widetilde{P^{+(h)}})$ is monotone decreasing for any h. Thus, $\lim_{h\to\infty} D(P^{+(h)}, \widetilde{P^{+(h)}}) = 0$. □

Theorem 5 shows that Algorithm I is convergent.

Example 4 For given three IFPRs $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ as in Table 2, a detailed procedure is given to check and modify the acceptable group multiplicative consistency of the three IFPRs with the mathematical software Sagemath as follows:

Step 0 Denote the original IFPR $P^{(l)}$ as $P^{(l,h)}$ for $l = 1, 2, \ldots, s$, $h = 0$ and the threshold $a = 0.1$.

Step 1 Construct the IFPRs $P^{+(0)}$ and $P^{-(0)}$ by Eqs. (16) and (15), and then compute the multiplicatively consistent IFPR $\widetilde{P^{+(0)}}$ by Model 1, we have

$$\widetilde{P^{+(0)}} = \begin{pmatrix} (0.5,\ 0.5) & (0.5,\ 0.305) & (0.67,\ 0.2) \\ (0.305,\ 0.5) & (0.5,\ 0.5) & (0.47,\ 0.23) \\ (0.2,\ 0.67) & (0.23,\ 0.47) & (0.5,\ 0.5) \end{pmatrix},$$

with $\gamma^{(0)} = 129.904$ and $\lambda^{(0)} = 0.001$.

Step 2 Compute the deviation measures $D(P^{(l,0)}, \widetilde{P^{+(0)}})$ between the IFPRs $P^{(l,0)}$ and $\widetilde{P^{+(0)}}$ by Eq. (11) for $l = 1, 2, 3$, we get $D(P^{(1,0)}, \widetilde{P^{+(0)}}) = 0.1117$, $D(P^{(2,0)}, \widetilde{P^{+(0)}}) = 0.1219$ and $D(P^{(3,0)}, \widetilde{P^{+(0)}}) = 0.1219$. Thus, they all need to be modified, go to next step.

Step 3 Compute the distance measures $d(p_{ij}^{(l_k,0)}, \widetilde{p^{+}}_{ij}^{(0)})$ $(i<j$ and $k = 1, 2, 3)$ between the elements $p_{ij}^{(l_k,0)}$ in $P_{ij}^{(l_k,0)}$ and $\widetilde{p^{+}}_{ij}^{(0)}$ in $\widetilde{P^{+(0)}}$, respectively. We take $l_0 0 = 2$, $i_0^{(0)} = 1$ and $j_0^{(0)} = 2$ such that $d(p_{i_0^{(0)} j_0^{(0)}}^{(l_0,0)}, \widetilde{p^{+}}_{i_0^{(0)} j_0^{(0)}}^{(0)}) = \max\{d(p_{ij}^{(l_k,0)}, \widetilde{p^{+}}_{ij}^{(0)}) | i<j, k = 1, 2, 3\}$.

Step 4 Modify the IFPR $P^{(2,0)}$ as variable $P^{(2,1)}$ with $p_{12}^{(2,1)} = (0.5865, 0.248)$ determined by Model 2, and the multiplicatively consistent IFPR

$$\widetilde{P^{+(1)}} = \begin{pmatrix} (0.5,\ 0.5) & (0.5,\ 0.305142) & (0.683672,\ 0.2) \\ (0.305142,\ 5) & (0.5,\ 0.5) & (0.491657,\ 0.2) \\ (0.2,\ 0.683672) & (0.2,\ 0.491657) & (0.5,\ 0.5) \end{pmatrix},$$

with $\gamma^{(1)} = 5.3356$ and $\lambda^{(1)} = 0.6185$.

Step 5 Compute the deviation measures $D(P^{(l,1)}, \widetilde{P^{+(1)}})$ between the IFPRs $P^{(l,1)}$ and $\widetilde{P^{+(1)}}$ by Eq. (11), we get $D(P^{(1,1)}, \widetilde{P^{+(1)}}) = 0.104$, $D(P^{(2,1)}, \widetilde{P^{+(1)}}) = 0.068$ and $D(P^{(31)}, \widetilde{P^{+(1)}}) = 0.11$, that is, $P^{(2,1)}$ is of acceptably group multiplicative consistency with respect to $\widetilde{P^{+(1)}}$ and $P^{(2,1)}$ and $P^{(3,1)}$ are not. Thus, they needs to be modified, go to next step.

Step 6 Compute the distance measures $d(p_{ij}^{(l_k,1)}, \widetilde{p^{+}}_{ij}^{(1)})$ $(i<j$ and $k = 1, 3)$ between the elements $p_{ij}^{(l_k,1)}$ in $P_{ij}^{(l_k,1)}$ and $\widetilde{p^{+}}_{ij}^{(1)}$ in $\widetilde{P^{+(1)}}$, respectively. We take $l_0 1 = 3$, $i_0^{(1)} = 1$ and $j_0^{(1)} = 3$ such that $d(p_{i_0^{(1)} j_0^{(1)}}^{(l_0 1)}, \widetilde{p^{+}}_{i_0^{(1)} j_0^{(1)}}^{(1)}) = \max\{d(p_{ij}^{(l_k 1)}, \widetilde{p^{+}}_{ij}^{(1)}) | i<j, k = 1, 3\}$.

Step 7 Modify the IFPR $P^{(3,1)}$ as variable $P^{(3,2)}$ with $p_{13}^{(3,2)} = (0.7799, 0.1206)$ determined by Model 2, and the multiplicatively consistent IFPR

$$\widetilde{P^{+(2)}} = \begin{pmatrix} (0.5, 0.5) & (0.5, 0.305143) & (0.683672, 0.2) \\ (0.305143, 0.5) & (0.5, 0.5) & (0.491659, 0.2) \\ (0.2, 0.683672) & (0.2, 0.491659) & (0.5, 0.5) \end{pmatrix},$$

with $\gamma^{(2)} = 5.3354$ and $\lambda^{(2)} = 0.6185$.

Step 8 Compute the deviation measures $D(P^{(l,2)}, \widetilde{P^{+(2)}})$ between the IFPRs $P^{(l,2)}$ and $\widetilde{P^{+(2)}}$ by Eq. (11) for $l = 1, 2, 3$, we get

Table 2 Individual preference information from three decision makers in Example 4

$P^{(1)}$	(0.5, 0.5)	(0.5, 0.2)	(0.7, 0.1)
	(0.2, 0.5)	(0.5, 0.5)	(0.6, 0.2)
	(0.1, 0.7)	(0.2, 0.6)	(0.5, 0.5)
$P^{(2)}$	(0.5, 0.5)	(0.6, 0.1)	(0.7, 0.2)
	(0.1, 0.6)	(0.5, 0.5)	(0.5, 0.1)
	(0.2, 0.7)	(0.1, 0.5)	(0.5, 0.5)
$P^{(3)}$	(0.5, 0.5)	(0.6, 0.2)	(0.8, 0.1)
	(0.2, 0.6)	(0.5, 0.5)	(0.6, 0.1)
	(0.1, 0.8)	(0.1, 0.6)	(0.5, 0.5)

$D(P^{(1,2)}, \widetilde{P+}^{(2)}) = 0.1045$, $D(P^{(2,2)}, \widetilde{P+}^{(2)}) = 0.0676$ and $D(P^{(3,2)}, \widetilde{P+}^{(2)}) = 0.1032$, that is, $P^{(2,2)}$ is of acceptablely multiplicative consistency with respect to $\widetilde{P+}^{(2)}$ and $P^{(1,2)}$ and $P^{(3,2)}$ are not. Thus, they needs to modify, go to next step.

Step 9 Compute the distance measures $d(p_{ij}^{(l_k,2)}, \widetilde{p+}_{ij}^{(2)})$ $(i<j$ and $k = 1,3)$ between the elements $p_{ij}^{(l_k,2)}$ in $P_{ij}^{(l_k,2)}$ and $\widetilde{p+}_{ij}^{(2)}$ in $\widetilde{P+}^{(2)}$, respectively. We take $l_0, 2 = 1$, $i_0^{(1)} = 2$ and $j_0^{(1)} = 3$ such that $d(p_{i_0^{(2)} j_0^{(2)}}^{(l_0,2)}, \widetilde{p+}_{i_0^{(2)} j_0^{(2)}}^{(2)}) = \max\{d(p_{ij}^{(l_k,2)}, \widetilde{p+}_{ij}^{(2)}) | i<j, k = 1,3\}$.

Step 10 Modify the IFPR $P^{(1,2)}$ as variable $P^{(1,3)}$ with $p_{23}^{(1,3)} = (0.4999, 0.1668)$ determined by Model 2 , and the multiplicatively consistent IFPR

$$\widetilde{P+}^{(3)} = \begin{pmatrix} (0.5, 0.5) & (0.5, 0.301665) & (0.683672, 0.2) \\ (0.301665, 0.5) & (0.5, 0.5) & (0.5, 0.171286) \\ (0.2, 0.683672) & (0.171286, 0.5) & (0.5, 0.5) \end{pmatrix},$$

with $\gamma^{(3)} = 11.7112$ and $\lambda^{(3)} = 0.6034$.

Step 11 Compute the deviation measures $D(P^{(l,3)}, \widetilde{P+}^{(3)})$ between the IFPRs $P^{(l,3)}$ and $\widetilde{P+}^{(3)}$ by Eq. (11) for $l = 1,2,3$, we get $D(P^{(1,3)}, \widetilde{P+}^{(3)}) = 0.0687$, $D(P^{(2,3)}, \widetilde{P+}^{(3)}) = 0.058$ and $D(P^{(3,3)}, \widetilde{P+}^{(3)}) = 0.0993$, that is, $P^{(1,3)}$, $P^{(2,3)}$ and $P^{(3,3)}$ are of acceptably group multiplicative consistency with respect to $\widetilde{P+}^{(3)}$.

Step 12 Output all the IFPRs $P^{(l,3)}$ for $l = 1,2,3$ as in Table 3.

Note that we have $\widetilde{P+}^{(h)} \leq P^{-(h)}$ in Step 2 and $P^{-(h)} \leq P^{+(h)}$ in Proposition 11. Thus, $\widetilde{P+}^{(h)} \leq P^{-(h)} \leq P^{+(h)}$, and hence $\lim_{h\to\infty} D(P^{+(h)}, P^{-(h)}) = 0$ from Theorem 5 which leads that the deviation $D(P^{(l)}, P)$ is getting smaller and smaller with the increasing of h, where P is determined by Proposition 9, that is, while the acceptable group multiplicative consistency is being reached, the acceptable consensus is also doing at the same time. Thus, the proposed algorithm possesses the following advantages:

(1) individual IFPRs are modified by their elements one by one, not done on the whole which is apt to be accepted by the decision makers;

(2) the acceptable group multiplicative consistency and consensus of individual IFPRs are modified simultaneously.

Example 5 We continue Example 4 to check the acceptable consensus with the mathematical software Sagemath as follows:

Step 13 Aggregate the IFPRs $P^{(1,3)}$, $P^{(2,3)}$ and $P^{(3,3)}$ into a collective IFPR $P^{(pq)}$ with the SIFBM operator (8). We determine p, q by

$$\max_{p,q} \max\{D(P^{(1,3)}, P^{(p,q)}), D(P^{(2,3)}, P^{(p,q)}), D(P^{(3,3)}, P^{(p,q)})\}$$
$$s.t.\ p, q \geq 0.$$

We have $p = 0$ and $q = 1.913$ and the maximal value is $0.0633 < 0.1$. Thus, $P^{(1,3)}$, $P^{(2,3)}$ and $P^{(3,3)}$ are of acceptable consensus with respect to $P^{(p,q)}$ for all $p, q \geq 0$ by Definition

Table 3 The modified individual preference information in Example 4

$P^{(1,3)}$	(0.5, 0.5)	(0.5, 0.2)	(0.7, 0.1)
	(0.2, 0.5)	(0.5, 0.5)	(0.4999, 0.1668)
	(0.1, 0.7)	(0.1668, 0.4999)	(0.5, 0.5)
$P^{(2,3)}$	(0.5, 0.5)	(0.5865, 0.248)	(0.7, 0.2)
	(0.248, 0.5865)	(0.5, 0.5)	(0.5, 0.1)
	(0.2, 0.7)	(0.1, 0.5)	(0.5, 0.5)
$P^{(3,3)}$	(0.5, 0.5)	(0.6, 0.2)	(0.7799, 0.1206)
	(0.2, 0.6)	(0.5, 0.5)	(0.6, 0.1)
	(0.1206, 0.7799)	(0.1, 0.6)	(0.5, 0.5)
$P^{(0.5,0.5)}$	(0.5, 0.5)	(0.5625, 0.2154)	(0.728, 0.1363)
	(0.2154, 0.5625)	(0.5, 0.5)	(0.5336, 0.1199)
	(0.1363, 0.728)	(0.1199, 0.5336)	(0.5, 0.5)

13. Without loss of the generality, we take $p = q = 0.5$, the collective IFPR $P^{(0.5,0.5)}$ in Table 3 and $D(P^{(1,3)}, P^{(0.5,0.5)}) = 0.063$, $D(P^{(2,3)}, P^{(0.5,0.5)}) = 0.058$ and $D(P^{(3,3)}, P^{(0.5,0.5)}) = 0.052$;

Step 14 Compute the intuitionistic fuzzy priority weight vector of $P^{(0.5,0.5)}$ by Model 1, we have $w_1 = (0.479, 0.34)$, $w_2 = (0.171, 0.569)$ and $w_3 = (0.091, 0.831)$ with $\gamma = 1.047$ and $\lambda = 0.124$. Thus, $s(w_1) = 0.138$, $s(w_2) = -0.398$ and $s(w_3) = -0.74$, that is, $x_1 > x_2 > x_3$.

5 Illustrative Example and Comparative Analysis

5.1 Illustrative Example

To illustrate the proposed algorithm to acceptable group multiplicative consistency and consensus for the IFPRs in group decision making, we provide an example as follows:

Example 6 Consider a group decision making problem concerning the selection of the international exchange doctoral students (adapted from [42]). Assume that a committee including three experts (decision makers) has been set up to assess applications, and the weights for the experts are 0.2, 0.5 and 0.3, respectively. Without loss of generality, assume that four students x_1, x_2, x_3 and x_4 are short listed as potential candidates. Each expert d_k ($k = 1, 2, 3$) is required to conduct pairwise comparisons for the four candidates, resulting in the following IFPRs in Table 4.

A detailed procedure is provided by the mathematical software Sagemath to derive the ranking as follows:

(1) Check and reach the acceptable group multiplicative consistency of the IFPRs $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ by Algorithm I. After 17 iterations, we obtain the modified IFPRs $P^{(1,17)}$, $P^{(2,17)}$, $P^{(3,17)}$ and the corresponding multiplicatively consistent IFPR $\widetilde{P+}^{(17)}$ in Table 5. Using Eq. (11), we have $D(P^{(2,17)}, \widetilde{P+}^{(17)}) = 0.0835$, $D(P^{(3,17)}, \widetilde{P+}^{(17)}) = 0.0827$ and $D(P^{(3,17)}, \widetilde{P+}^{(17)}) = 0.0827$. Thus, they are all of acceptable group multiplicative consistency.

(2) Aggregate the IFPRs $P^{(1,17)}$, $P^{(2,17)}$ and $P^{(3,17)}$ into a collective IFPRs $P^{(17)}$ with the weighted SIFBM operator and then determine p, q by

$$\min \frac{D(P^{(1,17)}, P^{(17)}) + D(P^{(2,17)}, P^{(17)}) + D(P^{(3,17)}, P^{(17)})}{3}$$
$$s.t.\ p, q \geq 0,$$

we have $p = 0$ and $q = 0.965$, the collective IFPRs $P^{(17)}$ in Table 5 and $D(P^{(1,17)}, P^{(17)}) = 0.063$, $D(P^{(2,17)}, P^{(17)}) = 0.059$ and $D(P^{(3,17)}, P^{(17)}) = 0.042$. Thus, they are of acceptable consensus by Definition 13.

(3) Compute the intuitionistic fuzzy priority weight vector of $P^{(17)}$ by Model 1, we have $w_1 = (0.15, 0.632)$, $w_2 = (0.182, 0.545)$, $w_3 = (0.071, 0.628)$ and $w_4 = (0.228, 0.613)$ with $\gamma = 2.021$ and $\lambda = 0.567$. Thus, $s(w_1) = -0.482$, $s(w_2) = -0.363$, $s(w_3) = -0.557$ and $s(w_4) = -0.384$, that is, $x_2 > x_4 > x_1 > x_3$.

Table 4 Individual preference information from three decision makers in Example 6

$P^{(1)}$	(0.5, 0.5)	(0.35, 0.55)	(0.4, 0.35)	(0.55, 0.35)
	(0.55, 0.35)	(0.5, 0.5)	(0.7, 0.1)	(0.6, 0.2)
	(0.35, 0.4)	(0.1, 0.7)	(0.5, 0.5)	(0.55, 0.3)
	(0.35, 0.55)	(0.2, 0.6)	(0.3, 0.55)	(0.5, 0.5)
$P^{(2)}$	(0.5, 0.5)	(0.55, 0.25)	(0.65, 0.2)	(0.35, 0.55)
	(0.25, 0.55)	(0.5, 0.5)	(0.4, 0.25)	(0.55, 0.3)
	(0.2, 0.65)	(0.25, 0.4)	(0.5, 0.5)	(0.6, 0.2)
	(0.55, 0.35)	(0.3, 0.55)	(0.2, 0.6)	(0.5, 0.5)
$P^{(3)}$	(0.5, 0.5)	(0.6, 0.3)	(0.75, 0.15)	(0.6, 0.2)
	(0.3, 0.6)	(0.5, 0.5)	(0.45, 0.2)	(0.6, 0.2)
	(0.15, 0.75)	(0.2, 0.45)	(0.5, 0.5)	(0.4, 0.4)
	(0.2, 0.6)	(0.2, 0.6)	(0.4, 0.4)	(0.5, 0.5)

Table 5 The modified individual preference information in Example 6

$P^{(1,17)}$	(0.5, 0.5)	(0.35, 0.55)	(0.4, 0.35)	(0.419, 0.419)
	(0.55, 0.35)	(0.5, 0.5)	(0.537, 0.339)	(0.528, 0.345)
	(0.35, 0.4)	(0.339, 0.537)	(0.5, 0.5)	(0.474, 0.34)
	(0.419, 0.419)	(0.345, 0.528)	(0.4, 0.474)	(0.5, 0.5)
$P^{(2,17)}$	(0.5, 0.5)	(0.435, 0.388)	(0.459, 0.332)	(0.35, 0.55)
	(0.388, 0.435)	(0.5, 0.5)	(0.4, 0.25)	(0.55, 0.3)
	(0.332, 0.459)	(0.25, 0.4)	(0.5, 0.5)	(0.432, 0.334)
	(0.55, 0.35)	(0.3, 0.55)	(0.334, 0.432)	(0.5, 0.5)
$P^{(3,17)}$	(0.5, 0.5)	(0.343, 0.433)	(0.445, 0.303)	(0.393, 0.426)
	(0.433, 0.343)	(0.5, 0.5)	(0.474, 0.225)	(0.51, 0.305)
	(0.303, 0.445)	(0.225, 0.474)	(0.5, 0.5)	(0.4, 0.4)
	(0.426, 0.393)	(0.305, 0.51)	(0.4, 0.4)	(0.5, 0.5)
$\widetilde{P^{+}}^{(17)}$	(0.5, 0.5)	(0.304, 0.55)	(0.4, 0.41)	(0.35, 0.55)
	(0.55, 0.304)	(0.5, 0.5)	(0.4, 0.367)	(0.475, 0.348)
	(0.41, 0.4)	(0.367, 0.4)	(0.5, 0.5)	(0.4, 0.4)
	(0.55, 0.35)	(0.348, 0.475)	(0.4, 0.4)	(0.5, 0.5)
$P^{(17)}$	(0.5, 0.5)	(0.377, 0.462)	(0.433, 0.33)	(0.388, 0.465)
	(0.462, 0.377)	(0.5, 0.5)	(0.472, 0.276)	(0.53, 0.318)
	(0.33, 0.433)	(0.276, 0.472)	(0.5, 0.5)	(0.438, 0.377)
	(0.465, 0.388)	(0.318, 0.53)	(0.377, 0.438)	(0.5, 0.5)

5.2 Comparative Analysis

In this section, we make a comparison with the existing work. In the procedure of group decision problem, the aggregation operators, checking and reaching consistency and consensus are crucial, we make a comparative analysis from these three aspects as follows:

(1) We compare the existing intuitionistic fuzzy aggregation operators from three aspects: (S) suitability for aggregating IFPRs; (DP) possess a desirable properties, that is, if the individual IFPRs are all of acceptable consistency, then so is the aggregated result by these operators. (CI) capture the interrelationship between these IFPRs, the results are listed in Table 6.

(2) Definition and a formula of multiplicative consistency in intuitionistic fuzzy environment were proposed in [12], but the formula has its disadvantage, that is, it pays much attention to the membership degree and non-membership degree of the priority vector, and neglects the hesitancy degree of them which is also basic in intuitionistic fuzzy numbers [9]. In this paper, a new formula which can reflect the three parts of an IFN is provided where two parameters are included and takes Liao's formula as a special case, and hence a multiplicatively consistent IFPR constructed from the parameterized formula of multiplicative consistency could be more close to the given IFPR than that from the formula in Ref. [12]. For a given IFPR, it could be of acceptably multiplicative consistency with respect to the multiplicatively consistent IFPR using the proposed formula, but it could be not of acceptably multiplicative consistency with that in [12], and hence it should be modified. Thus, the proposed formula could preserve the original preference information in this sense and is better than that in [12].

Example 7 For a given IFPR P in Table 7. Compute the model 1 with $\gamma = 2, \lambda = 0.5$ and $\gamma = 1, \lambda = 0$ (that is, Liao's formula), respectively, two multiplicatively consistent IFPRs can be provided by the mathematical software Sagemath as follows: Taking the threshold $a = 0.1$, we have $D(P, \widetilde{P}_{2,0.5}) = 0.0967 < 0.1$ and $D(P, \widetilde{P}_{1,0}) = 0.1055 > 0.1$. Thus, P is of acceptably multiplicative consistency with respect to $\widetilde{P}_{2,0.5}$, but it needs to be modified with respect to $\widetilde{P}_{1,0}$ by Liao [12]'s consistency formula.

(3) For consistency reaching, Liao [41] dealt with it by iteration with a control parameter, but how to choose a control parameter is not discussed. In fact, it could be time consuming or difficult for a decision maker to provide suitable values to meet the needs; Xu [14] substituted the initial IFPRs with an acceptably multiplicative consistent IFPR proposed by a optimized model. Obviously, the initial preference information is completely modified by the

Table 6 Comparative analysis to various aggregation operators with IFPRs in GDM

Operators	S	DP	CI
IFA operator [15]	No	No	No
IFHWA operator [24]	No	No	No
IFWG operator [25]	No	No	No
IFWA operator [9]	No	No	No
IFWA operator [15]	Yes	No	No
SIFWA operator [21, 22]	Yes	No	No
SIFWG operator [12]	Yes	Yes	No
IFBM operator [26]	No	No	Yes
NWIFBM operator [27]	No	No	Yes
IFBM operator [23]	No	No	Yes
The proposed operator	Yes	Yes	Yes

two methods in Refs. [14, 41]. While the proposed method modifies the IFPR in a stepwise way, some initial preference could be preserved.

(4) For consensus reaching, the methods in [14, 41] are different from that in this paper. Liao's method stresses the interaction with the decision makers; Xu considered the requirement of consensus as a constraint condition in the optimized model of the consistency reaching. Roughly, the proposed method in this paper is similar to that of Xu. In spite of this, it seems to be more easy to be achieved than Xu's consensus, because while the acceptably multiplicative consistency is reached, the consensus is naturally reached by the proposed method without any constraint conditions. Next, we investigate Example 6 by Xu's method as follows:

Step 1 Take the threshold $a = b = 0.1$, we check the consistency and consensus by the consistency index CI and consensus index CM proposed in [14], we have $CI(P^{(1)}) = 0.0388 < 0.1$, $CI(P^{(2)}) = 0.0971 < 0.1$, $CI(P^{(3)}) = 0.0191 < 0.1$ and $CM(P^{(1)}, P^{(2)}, P^{(3)}) = 0.1833 > 0.1$. Thus, they are of acceptably multiplicative consistency, but not acceptably consensus, so they needs to be modified.

Step 2 Modify these IFPRs by the model (that is, Eq. (28)) proposed in [14], we still denote the modified IFPRs as $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ in Table 8. At this time, we have $CI(P^{(1)}) = 0.06475 < 0.1$, $CI(P^{(2)}) = 0.09995 < 0.1$, $CI(P^{(3)}) = 0.03758 < 0.1$ and $CM(P^{(1)}, P^{(2)}, P^{(3)}) = 0.09683 < 0.1$. Both the consistency and the consensus are reached.

(1) Aggregate the IFPRs $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ into a collective one P listed in Table 8 using the SIFWG operator [12] and calculate the intuitionistic fuzzy priority weight vector ω of P by Model 1 with $\gamma = 1$ and $\lambda = 0$ (equivalent to the Liao [24]'s model). We get $\omega = ((0.179, 0.768), (0.276, 0.571), (0.155, 0.693), (0.151, 0.815))$.

(2) Rank the intuitionistic fuzzy priority weight vector ω by Definition 3, we have $s(x_2) = -0.295 > s(x_3) = -0.538 > s(x_1) = -0.589 > s(x_4) = -0.663$. Although the ranking is different from that of the proposed method, the optimal one by two methods is x_2.

Table 7 IFPRs for Example 7

P	(0.5, 0.5)	(0.7,0.2)	(0.6,0.2)	(0.3,0.5)
	(0.2, 0.7)	(0.5, 0.5)	(0.4,0.2)	(0.5,0.4)
	(0.2, 0.6)	(0.2, 0.4)	(0.5, 0.5)	(0.1,0.7)
	(0.5, 0.3)	(0.4, 0.5)	(0.7, 0.1)	(0.5, 0.5)
$\widetilde{P}_{2,0.5}$	(0.5, 0.5)	(0.617, 0.283)	(0.6, 0.2)	(0.3, 0.535)
	(0.283, 0.617)	(0.5, 0.5)	(0.4, 0.268)	(0.14, 0.666)
	(0.2, 0.6)	(0.268, 0.4)	(0.5, 0.5)	(0.066, 0.711)
	(0.535, 0.3)	(0.666, 0.14)	(0.711, 0.066)	(0.5, 0.5)
$\widetilde{P}_{1,0}$	(0.5, 0.5)	(0.577, 0.323)	(0.6, 0.2)	(0.273, 0.636)
	(0.323, 0.577)	(0.5, 0.5)	(0.4, 0.238)	(0.165, 0.686)
	(0.2, 0.6)	(0.238, 0.4)	(0.5, 0.5)	(0.1, 0.7)
	(0.636, 0.273)	(0.686, 0.165)	(0.7, 0.1)	(0.5, 0.5)

Table 8 The modified IFPRs by Xu [14]'s method in Example 7

$P^{(1)}$	(0.5, 0.5)	(0.438, 0.462)	(0.721, 0.033)	(0.55, 0.35)
	(0.462, 0.438)	(0.5, 0.5)	(0.564, 0.201)	(0.6, 0.2)
	(0.033, 0.721)	(0.201, 0.564)	(0.5, 0.5)	(0.55, 0.3)
	(0.35, 0.55)	(0.2, 0.6)	(0.3, 0.55)	(0.5, 0.5)
$P^{(2)}$	(0.5, 0.5)	(0.507, 0.3)	(0.737, 0.031)	(0.39, 0.51)
	(0.3, 0.507)	(0.5, 0.5)	(0.428, 0.222)	(0.561, 0.28)
	(0.031, 0.737)	(0.222, 0.428)	(0.5, 0.5)	(0.56, 0.24)
	(0.51, 0.39)	(0.28, 0.561)	(0.24, 0.56)	(0.5, 0.5)
$P^{(3)}$	(0.5, 0.5)	(0.597, 0.303)	(0.739, 0.031)	(0.593, 0.224)
	(0.303, 0.597)	(0.5, 0.5)	(0.45, 0.2)	(0.6, 0.2)
	(0.031, 0.739)	(0.2, 0.45)	(0.5, 0.5)	(0.441, 0.359)
	(0.224, 0.593)	(0.2, 0.6)	(0.359, 0.441)	(0.5, 0.5]
P	(0.5, 0.5)	(0.504, 0.349)	(0.733, 0.032)	(0.479, 0.405)
	(0.349, 0.504)	(0.5, 0.5)	(0.473, 0.211)	(0.581, 0.24)
	(0.032, 0.733)	(0.211, 0.473)	(0.5, 0.5)	(0.533, 0.282)
	(0.405, 0.479)	(0.24, 0.581)	(0.282, 0.533)	(0.5, 0.5]

6 Conclusions

In the present paper, we introduced the (weighted) SIFBM operators and acceptable group multiplicative consistency to derive the ranking for group decision making with IFPRs. The proposed method possesses the following advantages:

1. The proposed SIFBM operator not only can capture the interrelationship of the individual arguments, but also can fairly treat the membership and non-membership degrees of the intuitionistic fuzzy preference values of the IFPRs in group decision making to guarantee that the aggregated result is still an IFPR;
2. The proposed formula on constructing a multiplicatively consistent IFPR makes a full consideration on the components of the IFN and can guarantee the deviation between a given IFPR and the corresponding multiplicatively consistent IFPR is smaller than that by the existing formula;
3. The proposed algorithm simplifies the procedure to check and reach the acceptably multiplicative consistency and consensus, because the acceptable consensus could always be reached, while the acceptably multiplicative consistency is done.

Although the proposed algorithm possesses some advantages, some issues should be further investigated, for example, for different multiplicative consistencies, different thresholds should be provided to assure the reasonability of ranking which is left as the future work.

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Compliance with Ethical Standards

Conflicts of interest The authors declare that they have no conflict of interest.

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