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Multi-parameter Portfolio Selection Model with Some Novel Score-Deviation Under Dual Hesitant Fuzzy Environment

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Abstract In risk investment, investors have to rely on uncertain information when it is difficult to obtain enough precise data. Dual hesitant fuzzy set (DHFS) is more applicable to deal with uncertain information because it involves membership degrees and non-membership degrees, which can validly describe positive and negative information, respectively. Although there has been research on decision-making based on the DHFS, the focus still remains on ranking the alternatives and choosing the best one, which cannot help investors to find the optimal portfolios. Therefore, to solve this problem, we mainly propose two novel portfolio selection models based on the DHFS in this paper. Firstly, we propose a Max-score dual hesitant fuzzy portfolio selection model with information preference (Model 3) for investors focusing on returns regardless of risks. Secondly, to consider the risks of portfolios, we improve Model 3 and develop a score-deviation dual hesitant fuzzy portfolio selection model with information preference and risk appetite (Model 5). Finally, a case study is conducted to highlight the effectiveness of the proposed models. A detailed sensitivity analysis and an efficient frontier analysis show that Model 5 can validly capture investors' information preferences and risk appetites. Furthermore, compared with the hesitant fuzzy portfolio model, Model 5 can offer more options to the investors with different information preferences.

Keywords Dual hesitant fuzzy set · Portfolio selection · Score-deviation · Information preference · Risk appetite

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1 Introduction

Markowitz's mean–variance model [1] forms the foundation of the modern portfolio theory, which focuses on the relationship between returns and risks. Based on Markowitz's theory, a lot of research has been carried out. Sharpe [2] simplified the mean–variance model by using stock market index. Mao [3] developed a mean-semi-variance model. Best and Hlouskova [4] took research on portfolio selection model with uncorrelated and bounded assets. Basak and Shapiro [5] explored the portfolio models with Value-at-Risk-Based risk management. However, most of these models above are based on precise data, which are sometimes difficult to obtain. Therefore, investors have to rely on qualitative data in real decision-making process.

To deal with uncertain information, fuzzy theories have been developed, such as the fuzzy set [6], the type-2 fuzzy set [7], the intuitionistic fuzzy set (IFS) [8], and the hesitant fuzzy set (HFS) [9]. Based on these fuzzy theories, some portfolio selection models have been proposed. Watada [10] studied the fuzzy portfolio selection and its application in decision-making. Tanaka and Guo [11] proposed a portfolio selection model based on upper and lower exponential possibility distributions. Deng and Pan [12] compared the multi-objective portfolio selection models based on intuitionistic fuzzy optimization. Zhou and Xu [13] proposed portfolio selection models for general investors and risk investors under hesitant fuzzy environment. Zhou et al. [14] developed a hesitant fuzzy portfolio selection model based on prospect theory to consider the psychological behaviors of experts.

Among fuzzy theories, the dual hesitant fuzzy set (DHFS) [15] is more applicable to describe uncertain information because it not only overcomes the IFS's limitation that only one membership degree and one non-membership degree cannot comprehensively describe information, but also solves the HFS's problem that membership degree is powerless to express both positive and negative information. Recently, the research on the DHFS has achieved great progress in three aspects. (1) Some basic concepts and operators are developed, including aggregation operators [16], distance and similarity measures [17], correlation coefficient measures [18] and entropy measures [19]. (2) Some extended theories based on the DHFS have been proposed, such as the probabilistic dual hesitant fuzzy set (PDHFS) [20], the dual hesitant fuzzy rough set (DHFRS) [21], and the dual hesitant bipolar fuzzy set (DHBFS) [22]. (3) Practical decision-making models have been developed, such as project assignment [23], teacher evaluation [24], and town selection for land policy [25].

The DHFS has also been applied in the investment area [26–28] to describe the information of stocks and help investors make investment decisions. However, it is worth noting that: (1) the models mentioned above mainly focus on selecting one stock or company to invest, but not on the portfolio selection under dual hesitant fuzzy environment. (2) Portfolio selection models based on the HFS [13] and IFS [29] have been properly proposed, but the similar model based on the DHFS cannot be found. It is well known that the DHFS is more valid in dealing with uncertain information. If the investment information is described by DHFSs and an investor wants to find an optimal portfolio, no applicable model can be used to satisfy the investor's requirement. Therefore, to solve this problem, it is necessary to develop some new portfolio selection models based on the DHFS.

To build portfolio selection models based on the DHFS, investors have to evaluate stocks according to some criteria. Different criteria usually have different importance degrees, so determining criteria weights is necessary. Generally, criteria weights can be divided into two categories: subjective weights and objective weights. Subjective weights are derived from subjective preference information on criteria provided by decision-makers [30, 31]. In contrast, objective weights are obtained from the original evaluation information. One representative method of objective weights is the entropy method [32, 33], in which the criteria with bigger entropy values will be assigned smaller weights. Under dual hesitant fuzzy environment, there has also been some research on criteria weights [34–36]. In financial environment, it is difficult for investors to provide valid preference information on criteria because of the lack of precise data. Therefore, to better make use of the original information of DHFSs, the entropy method [32] is extended to dual hesitant fuzzy environment and used to calculate the objective weights of criteria in this paper.

When the information of stocks has been well processed, portfolio selection models can be built based on the DHFS. In this paper, we mainly propose two novel models under the dual hesitant fuzzy environment. Firstly, we propose a Max-score portfolio selection model (Model 3) for investors focusing on returns regardless of risks. In Model 3, a parameter α is defined to describe investors' information preferences in terms of returns. Secondly, to consider the risks of portfolios, we improve Model 3 and develop a score-deviation portfolio selection model (Model 5) for investors with different information preferences and risk appetites. In Model 5, another two parameters ζ and β are defined to describe investors' risk appetites and information preferences in terms of risks, respectively. Finally, we conduct a case study to illustrate the effectiveness of the proposed models. A detailed sensitivity analysis and an efficient frontier analysis are conducted to show that Model 5 can validly capture investors' information preferences and risk appetites. Moreover, Model 5 is compared with the hesitant fuzzy portfolio selection model to highlight its wider application.

This paper is organized as follows. In Sect. 2, the basic definitions and operations of the DHFS are reviewed. In Sect. 3, the calculation method of criteria weights based on the DHFS is illustrated. In Sect. 4, we propose a Max-score dual hesitant fuzzy portfolio selection model with information preference. In Sect. 5, we develop a score-deviation dual hesitant fuzzy portfolio selection model with information preference and risk appetite. In Sect. 6, two construction processes of the proposed portfolio selection models are summarized. In Sect. 7, a case study is conducted to show the availability of the proposed models. Conclusions are obtained in Sect. 8.

2 Preliminaries

In this section, we briefly introduce some important concepts about the DHFS and the basic operations of DHFSs. Then, we explain the definitions of returns and risks under dual hesitant fuzzy environment.

2.1 Dual Hesitant Fuzzy Set

Definition 1 [15] Let X be a fixed set, a DHFS D on X is described as:

$$D = \{\langle x, h(x), g(x)\rangle | x \in X\}, \tag{1}$$

where $h(x)$ and $g(x)$ are two sets of values in $[0, 1]$, denoting the possible membership degrees and non-membership degrees of $x \in X$ to the set D, respectively, such that $\gamma \in h(x)$, $\eta \in g(x)$, $0 \le \gamma, \eta \le 1$, $\gamma^+ = \max\{\gamma | \gamma \in h(x)\}$, $\eta^+ = \max\{\eta | \eta \in g(x)\}$ and $0 \le \gamma^+ + \eta^+ \le 1$. For convenience, the pair $d = \langle h(x), g(x) \rangle$ is called a dual hesitant fuzzy element (DHFE) and is denoted by $d = \langle h, g \rangle$.

Definition 2 [15] Let $d = \langle h, g \rangle$ be a DHFE, the score function of d is

$$S = \frac{1}{\#h} \sum_{\gamma \in h} \gamma - \frac{1}{\#g} \sum_{\eta \in g} \eta, \tag{2}$$

where $\#h$ and $\#g$ are the numbers of elements in h and g, respectively. Let $S_h = \frac{1}{\#h} \sum_{\gamma \in h} \gamma$ and $S_g = \frac{1}{\#g} \sum_{\eta \in g} \eta$, where S_h is the mean of membership degrees and S_g is the mean of non-membership degrees, then

$$S = S_h - S_g. \tag{3}$$

Definition 3 [37] Let $d = \langle h, g \rangle$ be a DHFE. Denote

$$\mathrm{Std}_m = \sqrt{\frac{1}{\#h} \sum_{\gamma \in h} (\gamma - S_h)^2} \tag{4}$$

and

$$\mathrm{Std}_n = \sqrt{\frac{1}{\#g} \sum_{\eta \in g} (\eta - S_g)^2}, \tag{5}$$

where Std_m is the standard deviation of membership degrees of d and Std_n is the standard deviation of non-membership degrees of d. Std_m and Std_n reflect the degree of volatility when a decision-maker determines the values of elements in the DHFE. The larger the values of Std_m and Std_n, the more volatile the data determined by the decision-maker.

According to Definition 3, we define the deviation of the DHFE as follows.

Definition 4 Let $d = \langle h, g \rangle$ be a DHFE, the deviation function of d is

$$V = \frac{1}{\#h} \sum_{\gamma \in h} (\gamma - S_h)^2 + \frac{1}{\#g} \sum_{\eta \in g} (\eta - S_g)^2, \tag{6}$$

where $S_h = \frac{1}{\#h} \sum_{\gamma \in h} \gamma$ and $S_g = \frac{1}{\#g} \sum_{\eta \in g} \eta$. Let $V_h = \frac{1}{\#h} \sum_{\gamma \in h} (\gamma - S_h)^2$ and $V_g = \frac{1}{\#g} \sum_{\eta \in g} (\eta - S_g)^2$, where V_h is the deviation of membership degrees and V_g is the deviation of non-membership degrees, then

$$V = V_h + V_g. \tag{7}$$

Let d_1 and d_2 be two DHFEs, the comparison laws between them are defined as follows:

(1) If $S_{d_1} > S_{d_2}$, then d_1 is superior to d_2, denoted by $d_1 \succ d_2$.
(2) If $S_{d_1} < S_{d_2}$, then d_1 is inferior to d_2, denoted by $d_1 \prec d_2$.
(3) If $S_{d_1} = S_{d_2}$, then
  (I) if $V_{d_1} < V_{d_2}$, then $d_1 \succ d_2$;
  (II) if $V_{d_1} > V_{d_2}$, then $d_1 \prec d_2$;
  (III) if $V_{d_1} = V_{d_2}$, then d_1 is equivalent to d_2, denoted by $d_1 \sim d_2$.

2.2 Operations of DHFSs

Let $d_1 = \langle h_1, g_1 \rangle$ and $d_2 = \langle h_2, g_2 \rangle$ be two DHFEs, and $\lambda > 0$ be a parameter. The basic operations of them are defined as follows [15]:

1. $d_1 \oplus d_2 = \langle \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}, \cup_{\eta_1 \in g_1, \eta_2 \in g_2} \{\eta_1 \ \eta_2\} \rangle$;
2. $d_1 \otimes d_2 = \langle \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}, \ \cup_{\eta_1 \in g_1, \eta_2 \in g_2} \{\eta_1 + \ \eta_2 - \eta_1 \ \eta_2\} \rangle$;
3. $\lambda d_1 = \left\langle \cup_{\gamma_1 \in h_1} \{1 - (1 - \gamma_1)^\lambda\}, \cup_{\eta_1 \in g_1} \{\eta_1^\lambda\} \right\rangle$;
4. $d_1^\lambda = \left\langle \cup_{\gamma_1 \in h_1} \{\gamma_1^\lambda\}, \cup_{\eta_1 \in g_1} \{1 - (1 - \eta_1)^\lambda\} \right\rangle$.

So far, the score function, deviation function, comparison laws and operations of DHFEs have been well defined. In the next section, the definitions of returns and risks in dual hesitant fuzzy portfolios are explained.

2.3 Returns and Risks Under Dual Hesitant Fuzzy Environment

According to Markowitz's mean–variance model [1], the mean and variance of stock data represent the return and risk, respectively. However, statistics data are sometimes difficult to obtain and process in practical investment. For some new companies, there are even no useful data in the stock market, which makes it difficult to find the optimal investment proportions of new stocks. Therefore, proper definitions are needed to describe returns and risks under dual hesitant fuzzy environment.

Based on the model proposed by Zhou and Xu [13] and the definitions above, it can be found that Definition 2 is consistent with the definition of the mean. Meanwhile, Definition 4 describes the deviation degree from score value in a DHFE, which reflects the volatility degree of a decision-maker when he evaluates the stocks. The larger the value of deviation, the more volatile the data. Therefore, the score function of Eq. (3) and the deviation function of Eq. (7) are used to measure returns and risks, respectively, under dual hesitant fuzzy environment.

3 Calculation Method of Criteria Weights Based on Dual Hesitant Fuzzy Entropy

To better make use of the original information in DHFSs, we apply an entropy method to calculate the objective criteria weights in this paper. This method is similar to that proposed by Chen and Li [32]. The difference is that our method adopts dual hesitant fuzzy entropy, whereas the method of Chen and Li [32] adopts intuitionistic fuzzy entropy.

3.1 Entropy Measure of the DHFS

To determine the stability of the DHFS, Zhao and Xu [19] proposed an entropy measure. However, Ren et al. [25] pointed out that this entropy measure is not applicable when the numbers of elements in membership degree and non-membership degree are not equal, so they proposed a new entropy measure. In portfolio selection, the numbers of elements in membership degree and non-membership degree are generally different because of uncertainty. Therefore, the entropy [25] is used to calculate criteria weights in this paper.

Definition 5 [25] Let D be a DHFS, the entropy measure of D can be defined as:

$$E(D) = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{(\#h_{d_i})\cdot(\#g_{d_i})}\sum_{\gamma\in h_{d_i},\eta\in g_{d_i}}\frac{1-|\gamma-\eta|^{\lambda}+(1-\gamma-\eta)^{\lambda}}{2}\right), \tag{8}$$

where m is the number of DHFEs in D, $d_i = \langle h_{d_i}, g_{d_i}\rangle$ is a DHFE in D. In this paper, let $\lambda = 1$, then the entropy measure of the DHFE d_i can be denoted as:

$$E(d_i) = \frac{1}{(\#h_{d_i})\cdot(\#g_{d_i})}\sum_{\gamma\in h_{d_i},\eta\in g_{d_i}}\frac{1-|\gamma-\eta|+(1-\gamma-\eta)}{2}. \tag{9}$$

3.2 Calculation Process of Criteria Weights Based on Dual Hesitant Fuzzy Entropy

Assume that there are m alternatives $A_i(i = 1, 2, \ldots, m)$ and n criteria $C_j(i = 1, 2, \ldots, n)$. The dual hesitant fuzzy decision matrix M is

$$\begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{array}
\begin{array}{c} \begin{array}{ccccc} C_1 & \cdots & C_j & \cdots & C_n \end{array} \\
\begin{bmatrix}
\langle h_{11}, g_{11}\rangle & \cdots & \langle h_{1j}, g_{1j}\rangle & \cdots & \langle h_{1n}, g_{1n}\rangle \\
\vdots & \vdots & & \vdots & \vdots \\
\langle h_{i1}, g_{i1}\rangle & \cdots & \langle h_{ij}, g_{ij}\rangle & \cdots & \langle h_{in}, g_{in}\rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\langle h_{m1}, g_{m1}\rangle & \cdots & \langle h_{mj}, g_{mj}\rangle & \cdots & \langle h_{mn}, g_{mn}\rangle
\end{bmatrix} \end{array} \tag{10}$$

where $d_{ij} = \langle h_{ij}, g_{ij}\rangle$ is the performance value of A_i under C_j.

Step 1. Calculate the entropy value of each DHFE d_{ij}. Each performance value d_{ij} in the decision matrix M is then turned into an entropy value E_{ij} based on Eq. (9). The entropy matrix M_0 is

$$\begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{array}
\begin{array}{c} \begin{array}{ccccc} C_1 & \cdots & C_j & \cdots & C_n \end{array} \\
\begin{bmatrix}
E_{11} & \cdots & E_{1j} & \cdots & E_{1n} \\
\vdots & \vdots & & \vdots & \vdots \\
E_{i1} & \cdots & E_{ij} & \cdots & E_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
E_{m1} & \cdots & E_{mj} & \cdots & E_{mn}
\end{bmatrix} \end{array} \tag{11}$$

Step 2. Calculate the criteria weights by applying the following transformation:

$$k_j = \frac{1-\sum_{i=1}^{m}K_{ij}}{n-\sum_{j=1}^{n}\sum_{i=1}^{m}K_{ij}}, \tag{12}$$

where k_j is the weight value of the criterion $C_j$$\cdot$$K_{ij}$ is the normalized value of E_{ij} based on Eq. (13):

$$K_{ij} = \frac{E_{ij}}{\max_i(E_{ij})}. \tag{13}$$

There is another dual hesitant fuzzy entropy method proposed by Chen et al. [36], which normalizes E_{ij} based on $K_{ij} = \frac{E_{ij}}{m}$. It is obvious that $\sum_{i=1}^{m}K_{ij} = \frac{1}{m}\sum_{i=1}^{m}E_{ij}$ focuses on averaging the entropy values under each criterion. However, our method normalizes E_{ij} by using Eq. (13), which measures the closeness of E_{ij} to the maximum entropy value under the criterion C_j. In risk investment, the stock with the highest entropy value is the most unstable and can be an important reference point for decision-makers. Therefore, Eq. (13) is more suitable and adopted in this paper. Next, an example is given to illustrate the calculation process of our entropy method.

Example 1 Assume that there are two alternatives $A_i(i = 1, 2)$ and two criteria $C_j(i = 1, 2)$. The dual hesitant fuzzy decision matrix M is

$$\begin{array}{ccc} & C_1 & C_2 \\ A_1 & \left[\begin{array}{c} \langle(0.3,0.4),(0.5)\rangle \\ \langle(0.4,0.7),(0.1)\rangle \end{array}\right. & \left.\begin{array}{c} \langle(0.4),(0.2,0.1)\rangle \\ \langle(0.5,0.6),(0.3)\rangle \end{array}\right] \\ A_2 & & \end{array}$$

Step 1. The entropy matrix M_0 is

$$\begin{array}{ccc} & C_1 & C_2 \\ A_1 & \left[\begin{array}{c} 0.5 \\ 0.45 \end{array}\right. & \left.\begin{array}{c} 0.6 \\ 0.45 \end{array}\right] \\ A_2 & & \end{array}$$

where $E_{11} = \frac{1}{2} \cdot \frac{1-|0.3-0.5|+(1-0.3-0.5)+1-|0.4-0.5|+(1-0.4-0.5)}{2} = 0.5$ and other entropy values can be obtained similarly based on Eq. (9).

Step 2. The weight values of C_1 and C_2 are 0.5455 and 0.4545, respectively, where $k_1 = \frac{1}{2-3.65} \cdot [1 - (\frac{0.5}{0.5} + \frac{0.45}{0.5})] = 0.5455$ and $k_2 = \frac{1}{2-3.65} \cdot [1 - (\frac{0.6}{0.6} + \frac{0.45}{0.6})] = 0.4545$.

4 Max-Score Dual Hesitant Fuzzy Portfolio Selection Model with Information Preference

In Sect. 2, returns and risks under dual hesitant fuzzy environment have been well defined. In this section, we suppose that investors only want to obtain the maximum returns without taking into account the risks, so we propose a Max-score portfolio selection model with information preference under dual hesitant fuzzy environment.

4.1 Max-Score Dual Hesitant Fuzzy Portfolio Selection Model

In this section, we propose a Max-score dual hesitant fuzzy portfolio selection model. Assume that there are m new stocks $\{a_1, a_2, \ldots, a_i, \ldots, a_m\}$ and n criteria $\{C_1, C_2, \ldots, C_j, \ldots, C_n\}$. An investor wants to put a fund on these stocks, but cannot get enough quantitative data. Therefore, the investor collects qualitative data represented by a dual hesitant fuzzy matrix $M = [d_{ij}]_{m\times n}$, where $d_{ij} = \langle h_{ij}, g_{ij}\rangle (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ refers to the dual hesitant fuzzy information of the stock a_i with respect to the criterion C_j. Firstly, the criteria weights $k_j (j = 1, 2, \ldots, n)$ can be obtained based on the entropy method in Sect. 3. Then, the DHFEs of each stock are aggregated and the dual hesitant fuzzy matrix $M = [d_{ij}]_{m\times n}$ is transformed into an aggregated decision matrix $\overline{M} = [\overline{d_i}]_{m\times 1}$ based on Eq. (14), where $\overline{d_i} (i = 1, 2, \ldots, m)$ is the aggregated DHFE of the stock a_i.

$$\overline{d_i} = \langle \overline{h_i}, \overline{g_i} \rangle = \oplus_{j=1}^{n} k_j d_{ij} = \left\langle \cup_{\gamma_{ij}\in h_{ij}} \{1 - \prod_{j=1}^{n} (1-\gamma_{ij})^{k_j}\}, \cup_{\eta_{ij}\in g_{ij}} \left\{ \prod_{j=1}^{n} \eta_{ij}^{k_j} \right\} \right\rangle. \tag{14}$$

Finally, the optimal investment proportions can be obtained by using Model 1.

Model 1

$$\max R(W) = S(\oplus_{i=1}^{m} w_i \overline{d_i})$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} w_i \le 1 \\ w_i \ge 0, i = 1, 2, \ldots, m \end{cases}, \tag{15}$$

where $\oplus_{i=1}^{m} w_i \overline{d_i} = \langle \cup_{\overline{\gamma_i}\in \overline{h_i}} \{1 - \prod_{i=1}^{m} (1-\overline{\gamma_i})^{w_i}\}, \cup_{\overline{\eta_i}\in \overline{g_i}} \{\prod_{i=1}^{m} \overline{\eta_i}^{w_i}\} \rangle$ is the aggregated DHFE of a portfolio, $R(W)$ describes the portfolio return, $S(\oplus_{i=1}^{m} w_i \overline{d_i})$ is the score function of $\oplus_{i=1}^{m} w_i \overline{d_i}$ based on Eq. (3), k_j is the weight value of the criterion C_j, and w_i is the optimal investment proportion of the stock a_i.

Theorem 1 The constraint condition $\sum_{i=1}^{m} w_i = 1$ is equivalent to $\sum_{i=1}^{m} w_i \le 1$ in Model 1.

Proof Let $W_0 = \{w_i^*\}, i = 1, 2, \ldots, m$ be a feasible solution to Model 1 and $\sum_{i=1}^{m} w_i^* < 1$. Then, the return of Model 1 at W_0 is

$$R(W_0) = S\left(\left\langle \cup_{\overline{\gamma_i}\in \overline{h_i}} \left\{1 - \prod_{i=1}^{m} (1-\overline{\gamma_i})^{w_i^*}\right\}, \cup_{\overline{\eta_i}\in \overline{g_i}} \left\{\prod_{i=1}^{m} \overline{\eta_i}^{w_i^*}\right\} \right\rangle\right). \tag{16}$$

Next, we will show $W^* = (w_1^*, \ldots, w_{i-1}^*, 1 - \sum_{j\ne i} w_j^*, w_{i+1}^*, \ldots, w_m)$ is a better solution than W_0. Based on Eq. (3), we have

$$R(W^*) = S\left(\left\langle \cup_{\overline{\gamma_i}\in \overline{h_i}} \left\{1 - (1-\overline{\gamma_i})^{1-\sum_{j\ne i}^{m} w_j^*} \prod_{j\ne i}^{m} (1-\overline{\gamma_j})^{w_j^*}\right\}, \cup_{\overline{\eta_i}\in \overline{g_i}} \left\{\overline{\eta_i}^{1-\sum_{j\ne i}^{m} w_j^*} \prod_{j\ne i}^{m} \overline{\eta_j}^{w_j^*}\right\} \right\rangle\right). \tag{17}$$

Since $\sum_{i=1}^{m} w_i^* < 1$ and $0 \le \overline{\gamma_i}, \overline{\eta_i} \le 1$, we have $1 - \sum_{j\ne i}^{m} w_j^* > w_i^*$,

$$1-(1-\overline{\gamma_i})^{1-\sum_{j\neq i}^{m} w_j^*}\prod_{j\neq i}^{m}(1-\overline{\gamma_j})^{w_j^*}\geq 1-(1-\overline{\gamma_i})^{w_i^*}\prod_{j\neq i}^{m}(1-\overline{\gamma_j})^{w_j^*} \tag{18}$$

and

$$\overline{\eta_i}^{1-\sum_{j\neq i}^{m} w_j^*}\prod_{j\neq i}^{m}\overline{\eta_j}^{w_j^*}\leq \overline{\eta_i}^{w_i^*}\prod_{j\neq i}^{m}\overline{\eta_j}^{w_j^*}. \tag{19}$$

Therefore, $R(W^*)\geq R(W_0)$. Then, the constraint condition $\sum_{i=1}^{m} w_i\leq 1$ is equivalent to $\sum_{i=1}^{m} w_i=1$.

According to Theorem 1, Model 1 can be transformed into Model 2:

Model 2

$$\max R(W)=S(\oplus_{i=1}^{m} w_i\overline{d_i})$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{m} w_i=1\\ w_i\geq 0, i=1,2,\ldots,m\end{cases}. \tag{20}$$

For convenience, we denote $\oplus_{i=1}^{m} w_i\overline{d_i}$ as

$$D_R=\langle H,G\rangle, \tag{21}$$

where $H=\cup_{\overline{\gamma_i}\in\overline{h_i}}\{1-\prod_{i=1}^{m}(1-\overline{\gamma_i})^{w_i}\}$ and $G=\cup_{\overline{\eta_i}\in\overline{g_i}}\{\prod_{i=1}^{m}\overline{\eta_i}^{w_i}\}$. Let

$$R(W)=S\left(\left\langle \cup_{\overline{\gamma_i}\in\overline{h_i}}\{1-\prod_{i=1}^{m}(1-\overline{\gamma_i})^{w_i}\},\cup_{\overline{\eta_i}\in\overline{g_i}}\{\prod_{i=1}^{m}\overline{\eta_i}^{w_i}\}\right\rangle\right)=S_H(W)-S_G(W), \tag{22}$$

where

$$S_H(W)=\frac{1}{l_H}\sum_{u=1}^{l_H} s_u(W), \tag{23}$$

$$S_G(W)=\frac{1}{l_G}\sum_{e=1}^{l_G} s_e(W), \tag{24}$$

$s_u(W)=1-\prod_{i=1}^{m}(1-\overline{\gamma_i})^{w_i}$, $s_e(W)=\prod_{i=1}^{m}\overline{\eta_i}^{w_i}$, l_H is the number of elements in H and l_G is the number of elements in G.

Lemma 1 (Weierstrass' Theorem [38, Proposition A.8]) *Let T be a nonempty subset of $\mathbb{R}^n$, the n-dimensional Euclidean space, if $f:T\mapsto\mathbb{R}$ is upper semi-continuous at all points of T and T is closed and bounded, then there exists a vector $x\in T$ such that $f(x)=\sup_{z\in T}f(z)$.*

Remark 1 If $f:T\mapsto\mathbb{R}$ is a continuous function, then f is upper semi-continuous.

Theorem 2 *Model 2 has a globally optimal solution.*

Proof In Model 2, let $D_w=\{W=(w_1,\ldots,w_2,\ldots w_m)^T|\sum_{i=1}^{m} w_i=1\,,\,w_i\geq 0\}$. It is obvious that D_w is closed and bounded. In addition, since a function composed of a finite number of exponential functions and constants is continuous, $s_u(W)=1-\prod_{i=1}^{m}(1-\overline{\gamma_i})^{w_i}$ and $s_e(W)=\prod_{i=1}^{m}\overline{\eta_i}^{w_i}$ are continuous when $0\leq\overline{\gamma_i}, \overline{\eta_i}\leq 1$. Then, we can derive from Eqs. (22)–(24) that $R(W)$ is continuous on D_w. Therefore, Model 2 is well defined and has a globally optimal solution according to Lemma 1.

Theorem 3 *The objective function of Model 2 is a concave function.*

Proof To prove the theorem, we need to prove that the Hessian matrix of $R(W)$ is a negative semi-definite matrix. According to Eq. (22), if we can prove that the Hessian matrix of $S_H(W)$ is a negative semi-definite matrix and the Hessian matrix of $S_G(W)$ is a positive semi-definite matrix, we will prove the theorem. Consider the second-order mixed partial derivative of $R(W)$:

$$\frac{\partial^2 R(W)}{\partial w_i\partial w_j}=\frac{\partial^2 S_H(W)}{\partial w_i\partial w_j}-\frac{\partial^2 S_G(W)}{\partial w_i\partial w_j}=\frac{1}{l_H}\sum_{u=1}^{l_H}\frac{\partial^2 s_u(W)}{\partial w_i\partial w_j}-\frac{1}{l_G}\sum_{e=1}^{l_G}\frac{\partial^2 s_e(W)}{\partial w_i\partial w_j}. \tag{25}$$

Since $s_u(W)$ and $s_e(W)$ are all positive, we can just consider $\frac{\partial^2 s_u(W)}{\partial w_i\partial w_j}$ and $\frac{\partial^2 s_e(W)}{\partial w_i\partial w_j}$.

Firstly, consider the Hessian matrix of $s_u(W)$. Since

$$\frac{\partial^2 s_u(W)}{\partial w_i^2}=-(\ln(1-\overline{\gamma_i}))^2\prod_{i=1}^{m}(1-\overline{\gamma_i})^{w_i} \tag{26}$$

and

$$\frac{\partial^2 s_u(W)}{\partial w_i\partial w_j}=-\ln(1-\overline{\gamma_i})\ln(1-\overline{\gamma_j})\prod_{i=1}^{m}(1-\overline{\gamma_i})^{w_i}, \tag{27}$$

the Hessian matrix of $s_u(W)$ is

$$H_u=\begin{bmatrix}\frac{\partial^2 s_u(W)}{\partial w_1^2} & \cdots & \frac{\partial^2 s_u(W)}{\partial w_1\partial w_i} & \cdots & \frac{\partial^2 s_u(W)}{\partial w_1\partial w_m}\\ \vdots & \cdots & \vdots & \cdots & \vdots\\ \frac{\partial^2 s_u(W)}{\partial w_i\partial w_1} & \cdots & \frac{\partial^2 s_u(W)}{\partial w_i^2} & \cdots & \frac{\partial^2 s_u(W)}{\partial w_i\partial w_m}\\ \vdots & \cdots & \vdots & \cdots & \vdots\\ \frac{\partial^2 s_u(W)}{\partial w_m\partial w_1} & \cdots & \frac{\partial^2 s_u(W)}{\partial w_m\partial w_i} & \cdots & \frac{\partial^2 s_u(W)}{\partial w_m^2}\end{bmatrix}. \tag{28}$$

It is obvious that H_u can be transformed into:

$$H_u = AA^T \cdot (-\prod_{i=1}^{m}(1-\overline{\gamma_i})^{w_i}), \tag{29}$$

where $A = (\ln(1-\overline{\gamma_1}), \ldots, \ln(1-\overline{\gamma_i}), \ldots, \ln(1-\overline{\gamma_m}))^T$ and AA^T is a positive semi-definite matrix. Therefore, H_u is a negative semi-definite matrix.

Secondly, consider the Hessian matrix of $s_e(W)$ similarly. Since

$$\frac{\partial^2 s_e(W)}{\partial w_i^2} = (\ln \overline{\eta}_i)^2 \prod_{i=1}^{m} \overline{\eta}_i^{w_i} \tag{30}$$

and

$$\frac{\partial^2 s_e(W)}{\partial w_i \partial w_j} = \ln \overline{\eta}_i \ln \overline{\eta}_j \prod_{i=1}^{m} \overline{\eta}_i^{w_i}, \tag{31}$$

the Hessian matrix of $s_e(W)$ is

$$H_e = BB^T \cdot \prod_{i=1}^{m} \overline{\eta}_i^{w_i}, \tag{32}$$

where $B = (\ln \overline{\eta}_1, \ldots, \ln \overline{\eta}_i, \ \ldots, \ln \overline{\eta}_m)^T$ and BB^T is a positive semi-definite matrix. Therefore, H_e is a positive semi-definite matrix and $-H_e$ is a negative semi-definite matrix. Finally, we can derive from Eq. (25) that the Hessian matrix of $R(W)$ is a negative semi-definite matrix, so the objective function of Model 2 is a concave function.

4.2 Max-Score Dual Hesitant Fuzzy Portfolio Selection Model with Information Preference

As discussed above, the membership degrees and non-membership degrees in the DHFS can represent positive and negative information, respectively. In practical investment, investors usually hold different attitudes to positive and negative information. Therefore, Model 3 is constructed based on Model 2 to incorporate investors' information preferences.

Model 3

$$\max R(W) = \alpha S_H(W) - (1-\alpha) S_G(W)$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \ldots, m \end{cases}, \tag{33}$$

where $0 \leq \alpha \leq 1$. The parameter α describes the investor's information preference and its value is determined by the investor. From the proof of Theorem 2, we can obtain that the function being maximized in Model 3 is continuous on D_w, so Model 3 has a globally optimal solution according to Lemma 1.

Corollary 1 *The constraint condition $\sum_{i=1}^{m} w_i \leq 1$ is equivalent to $\sum_{i=1}^{m} w_i = 1$ in Model 3.*

Proof Let $W_0 = \{w_i^*\}, i = 1, 2, \ldots, m$ be a feasible solution to Model 3 such that $\sum_{i=1}^{m} w_i^* < 1$, and $W^* = (w_1^*, \ldots, w_{i-1}^*, \ 1 - \sum_{j \neq i} w_j^*, w_{i+1}^*, \ldots, w_m)$. Since $0 \leq \alpha \leq 1$, we can derive from the proof of Theorem 1 that $\alpha S_H(W^*) \geq \alpha S_H(W_0)$, $(1-\alpha) S_G(W^*) \leq (1-\alpha) S_G(W_0)$ and $R(W^*) \geq R(W_0)$. Therefore, the constraint conditions $\sum_{i=1}^{m} w_i = 1$ and $\sum_{i=1}^{m} w_i \leq 1$ are equivalent.

Theorem 4 *The objective function of Model 3 is a concave function. Moreover, Model 3 is equivalent to a convex programming.*

Proof According to Eq. (25), it is obvious that the second-order mixed partial derivative of $R(W)$ in Model 3 is $\frac{\partial^2 R(W)}{\partial w_i \partial w_j} = \alpha \frac{\partial^2 S_H(W)}{\partial w_i \partial w_j} - (1-\alpha) \frac{\partial^2 S_G(W)}{\partial w_i \partial w_j}$. Since $0 \leq \alpha \leq 1$, we can derive from the proof of Theorem 3 that the Hessian matrix of $\alpha S_H(W)$ is a negative semi-definite matrix and the Hessian matrix of $(1-\alpha) S_G(W)$ is a positive semi-definite matrix. Then, the Hessian matrix of $R(W)$ in Model 3 is a negative semi-definite matrix. Therefore, the objective function of Model 3 is a concave function and $-R(W)$ is a convex function.

Note that Eq. (33) and Eq. (34) are equivalent.

$$\min -R(W) = (1-\alpha) S_G(W) - \alpha S_H(W)$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \ldots, m \end{cases} \tag{34}$$

Since the objective function is a convex function and the constraint conditions are linear functions, Eq. (34) is a convex programming. Therefore, Eq. (33) is equivalent to a convex programming and has some good properties similar to those of Eq. (34). For example, any local optimum is a global optimum according to optimization theory.

Property 1 *Different values of α in Model 3 represent investors' different information preferences in terms of returns.*

(1) If $0.5 < \alpha \leq 1$, investors pay more attention to positive information;
(2) If $0 \leq \alpha < 0.5$, investors pay more attention to negative information;
(3) If $\alpha = 0.5$, the importance degrees of positive and negative information are equal, and Model 3 is equivalent to Model 2;

(4) If $\alpha = 1$, Model 3 will be similar to the hesitant fuzzy portfolio selection model for general investors proposed by Zhou and Xu [13].

5 Score-Deviation Dual Hesitant Fuzzy Portfolio Selection Model with Information Preference and Risk Appetite

In Sect. 4, we have proposed the Max-score portfolio selection model with information preference (Model 3). However, Model 1, Model 2 and Model 3 only focus on maximizing the return, but ignore the risk in portfolio selection. It is well known that investors also want to avoid risks as much as they can in their pursuit of the maximum returns. Therefore, we improve Model 3 by considering both returns and risks.

5.1 Score-Deviation Dual Hesitant Fuzzy Portfolio Selection Model with Information Preference

For convenience, we firstly develop a bi-objective portfolio selection model with information preference, which is described as Model 4.

Model 4

$$\begin{aligned} &\max R(W) = \alpha S_H(W) - (1-\alpha) S_G(W) \\ &\min V(W) = \beta V_H(W) + (1-\beta) V_G(W) \\ &\text{s.t.} \begin{cases} \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \ldots, m \end{cases}, \end{aligned} \tag{35}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$,

$$V_H(W) = \frac{1}{l_H} \sum_{u=1}^{l_H} (s_u(W) - S_H(W))^2, \tag{36}$$

$$V_G(W) = \frac{1}{l_G} \sum_{e=1}^{l_G} (s_e(W) - S_G(W))^2, \tag{37}$$

and $V(W)$ describes the portfolio risk. The parameter α is the same as that in Model 3. Since investors' information preferences in terms of returns and risks may be different, the parameter β is defined to describe the investor's information preference in terms of risks. The value of β is determined by the investor.

Property 2 *Different values of β in Model 4 represent investors' different information preferences in terms of risks:*

(1) If $0.5 < \beta \leq 1$, investors pay more attention to positive information;

(2) If $0 \leq \beta < 0.5$, investors pay more attention to negative information;

(3) If $\beta = 0.5$, the importance degrees of positive and negative information are equal;

(4) If $\beta = 1$ and $\alpha = 1$, Model 4 will be similar to the hesitant fuzzy portfolio selection model for risk investors proposed by Zhou and Xu [13].

5.2 Score-Deviation Dual Hesitant Fuzzy Portfolio Selection Model with Information Preference and Risk Appetite

It is obvious that returns and risks are of the same importance in Model 4, which is not consistent with the consensus that investors usually have different risk appetites. Therefore, Model 5 is constructed based on Model 4 to incorporate investors' risk appetites.

Model 5

$$\begin{aligned} &\max R(W) = \alpha S_H(W) - (1-\alpha) S_G(W) \\ &\text{s.t} \begin{cases} V(W) \leq \zeta \\ \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \ldots, m \end{cases}, \end{aligned} \tag{38}$$

where $V(W) = \beta V_H(W) + (1-\beta) V_G(W)$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, ζ describes the investor's risk appetite, and $\zeta \in [V_{\min}, V_{\max}]$. When the value of β is given according to the investor's information preference, we have

$$\begin{aligned} &V_{\max} = \max V(W) \\ &\text{s.t} \begin{cases} \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \ldots, m \end{cases}, \end{aligned} \tag{39}$$

and

$$\begin{aligned} &V_{\min} = \min V(W) \\ &\text{s.t.} \begin{cases} \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \cdots, m \end{cases}, \end{aligned} \tag{40}$$

where $V_{\max}$ and $V_{\min}$ are the maximum risk value and the minimum risk value of the portfolio, respectively.

Theorem 5 *Model 5 has a globally optimal solution.*

Proof Let $T = \{W = (w_1, \ldots, w_2, \ldots, w_m)^T | V(W) = \beta V_H + (1-\beta) V_G \leq \zeta.\}$, where the value of ζ is given by the investor, and $D_w = \{W = (w_1, \ldots, w_2, \ldots, w_m)^T |$

$\sum_{i=1}^{m} w_i = 1\,,\, w_i \geq 0\}$. It is obvious that $T \cap D_w$ is bounded. Next, we will prove that $T \cap D_w$ is closed.

Let $\left\{W^k = (w_1^k, \ldots, w_2^k, \ldots, w_m^k)^T\right\}$ be an arbitrarily convergent sequence of elements in T such that $\lim_{k\to\infty} W^k = \overline{W}$. Since $s_u(W) = 1 - \prod_{i=1}^{m}(1-\overline{\gamma_i})^{w_i}$ and $s_e(W) = \prod_{i=1}^{m}\overline{\eta}_i^{w_i}$ are continuous when $0 \leq \overline{\gamma_i}, \overline{\eta_i} \leq 1$, we can derive from Eq. (36) and Eq. (37) that $V(W)$ is continuous. By mathematical analysis, it is easily verified that $V(\overline{W}) = \lim_{W\to\overline{W}} V(W) = \lim_{k\to\infty} V(W^k) \leq \zeta$. That is to say, $\overline{W} \in T$ and T is a closed set. Since D_w is closed, $T \cap D_w$ is closed. In addition, we can derive from the proof of Theorem 2 that the function being maximized in Model 5 is continuous on $T \cap D_w$, so Model 5 is well defined and has a globally optimal solution according to Lemma 1.

Property 3 *Different values of ζ in Model 5 represent investors' different risk appetites. By trisecting the range $[V_{\min}, V_{\max}]$, we can obtain that:*

(1) if $V_{\min} \leq \zeta \leq V_{\min} + \frac{1}{3}(V_{\max} - V_{\min})$, investors are risk-averse;
(2) if $V_{\min} + \frac{1}{3}(V_{\max} - V_{\min}) < \zeta \leq V_{\min} + \frac{2}{3}(V_{\max} - V_{\min})$, investors are risk-neutral;
(3) if $V_{\min} + \frac{2}{3}(V_{\max} - V_{\min}) < \zeta \leq V_{\max}$, investors are risk-seeking;
(4) if $\zeta = V_{\max}$, Model 5 will be the same as Model 3.

So far, we have proposed the Max-score dual hesitant fuzzy portfolio selection model with information preference and the score-deviation dual hesitant fuzzy portfolio selection model with information preference and risk appetite. In the next section, we summarize the portfolio selection process under dual hesitant fuzzy environment.

6 Portfolio Selection Process Under Dual Hesitant Fuzzy Environment

Assume that there are m new stocks $\{a_1, a_2, \ldots a_i, \ldots, a_m\}$ and n criteria $\{C_1, C_2, \ldots C_j, \ldots, C_n\}$. An investor wants to put a fund on these stocks, but cannot get enough quantitative data. Thus, the investor has to collect qualitative data from some experts, who evaluate the stocks based on the DHFS. The evaluation information of the stocks is represented by a dual hesitant fuzzy matrix $M = [d_{ij}]_{m\times n}$, where $d_{ij} = \langle h_{ij}, g_{ij}\rangle (i = 1,2,\ldots,m; j = 1,2,\cdots,n)$ is a DHFE describing the dual hesitant fuzzy information of the stock a_i with respect to the criterion C_j. To help the investor find the optimal investment proportions, a suitable portfolio selection process based on the DHFS is needed. Generally, portfolio selection process can be divided into two categories: Process I for investors focusing on returns regardless of risks and Process II for investors considering both returns and risks. Next, the details of the two processes under dual hesitant fuzzy environment are explained.

Process I According to the discussion above, it is obvious that Models 1-3 are suitable for Process I. However, it has been verified that (1) Model 1 is equivalent to Model 2 by Theorem 1. (2) Model 3 considers information preference, which is not considered in the other two models. (3) According to Property 1, Model 3 is equivalent to Model 2 when $\alpha = 0.5$. Therefore, Model 3 is a good improvement of the other two models and is the most suitable for Process I. The steps of Process I are as follows:

Step 1. Calculate the objective weight values $k_j(j = 1,2,\cdots,n)$ of the criteria based on the method mentioned in Sect. 3.
Step 2. Aggregate the information of each stock. Let $\{d_{ij}, j = 1,2,\cdots,n\}$ be a set of DHFEs under the stock a_i. Aggregate $\{d_{ij}, j = 1,2,\ldots n\}$ and the criteria weights $k_j(j = 1,2,\ldots,n)$ based on $\overline{d_i} = \langle\overline{h_i}, \overline{g_i}\rangle = \oplus_{j=1}^{n} k_j d_{ij} = \left\langle \cup_{\gamma_{ij}\in h_{ij}}\{1 - \prod_{j=1}^{n}(1-\gamma_{ij})^{k_j}\}, \cup_{\eta_{ij}\in g_{ij}}\{\prod_{j=1}^{n}\eta_{ij}^{k_j}\}\right\rangle$, where $\overline{d_i}(i = 1,2,\ldots,m)$ is the aggregated DHFE of the stock a_i. Then, the dual hesitant fuzzy matrix $M = [d_{ij}]_{m\times n}$ is transformed into an aggregated decision matrix $\overline{M} = [\overline{d_i}]_{m\times 1}$.
Step 3. Determine the value of α according to Property 1 and the investor's information preference in terms of returns. If the investor pays more attention to positive information, then $0.5 < \alpha \leq 1$. If the investor pays more attention to negative information, then $0 \leq \alpha < 0.5$. If there is no information preference, then $\alpha = 0.5$.
Step 4. Construct Model 3 based on Eq. (33) and calculate the optimal investment proportions $w_i(i = 1,2,\ldots,m)$.

Process II It is obvious that Model 4 and Model 5 are suitable for Process II. However, Model 5 considers the risk appetite, which is not considered in Model 4. Moreover, it is proved by Theorem 5 that Model 5 has a globally optimal solution. Therefore, Model 5 is the most suitable for Process II. The steps of Process II are as follows:

Step 1, **Step 2** and **Step 3** are the same as those in Process I.

Step 4. Determine the value of β according to Property 2 and the investor's information preference in terms of risks. If the investor pays more attention to positive information, then $0.5 < \beta \leq 1$. If the investor pays more attention to negative information, then $0 \leq \beta < 0.5$. If there is no information preference, then $\beta = 0.5$.

Step 5. Calculate the range of ζ. Since ζ describes the risk appetite and the deviation function measures the risk according to Sect. 2.3, we can calculate the range of deviation of the portfolio by using Eqs. (39) and (40), then obtain $\zeta \in [V_{\min}, V_{\max}]$.
Step 6. Determine the value of ζ according to Property 3 and the investor's risk appetite. If the investor is risk-seeking, then $V_{\min} + \frac{2}{3}(V_{\max} - V_{\min}) < \zeta \leq V_{\max}$. If the investor is risk-neutral, then $V_{\min} + \frac{1}{3}(V_{\max} - V_{\min}) < \zeta \leq V_{\min} + \frac{2}{3}(V_{\max} - V_{\min})$. If the investor is risk-averse, then $V_{\min} \leq \zeta \leq V_{\min} + \frac{1}{3}(V_{\max} - V_{\min})$.
Step 7. Construct Model 5 based on Eq. (38) and calculate the optimal investment proportions $w_i (i = 1, 2, \ldots, m)$.

It can be derived from Property 3 that Process II can also be used for the investor focusing on returns regardless of risks when $\zeta = V_{\max}$ in Step 6. However, to complete Process II, the investor has to determine the value of β in Step 4 and calculate the range of ζ in Step 5, which is unnecessary because the focus of the investor is not the risk. If the investor chooses Process I, he will not need to follow these unnecessary steps. Moreover, Model 3 is equivalent to a convex programming according to Theorem 4. Therefore, Process I is more convenient for the investors merely focusing on returns and their further analysis on the portfolio. In conclusion, Process I and Process II are well defined in this section. The flowchart of the two processes is shown in Fig. 1.

7 Case Study

In this section, we give an example of dual hesitant fuzzy portfolio selection to illustrate the availability of Model 3 and Model 5. Furthermore, a sensitivity analysis, an efficient frontier analysis and a comparison analysis are conducted to discuss the results.

7.1 Case and Calculation

The New Tertiary Board is a stock market in China, in which most of the stocks are newly listed. An investor wants to put a fund on four new stocks $\{a_i, i = 1, 2, 3, 4\}$ in this market. Since it is difficult to find sufficiently precise data of these new stocks, the DHFS is used to describe the information of the stocks. The settings of some parameters are described as follows:

(1) To make sure that each stock is invested, the investor determines that $w_i \geq 0.05$, $i = 1, 2, 3, 4$.
(2) If the investor is risk-seeking, then $\zeta = \zeta_s = V_{\min} + \frac{3}{4}(V_{\max} - V_{\min})$; if the investor is risk-neutral, then $\zeta = \zeta_n = V_{\min} + \frac{1}{2}(V_{\max} - V_{\min})$; if the investor is risk-averse, then $\zeta = \zeta_a = V_{\min} + \frac{1}{4}(V_{\max} - V_{\min})$.

Three qualitative criteria are used to evaluate the stocks (see Table 1).

The stocks are evaluated by three experienced experts and the evaluation information is represented by DHFEs $\{d_{ij}, i = 1, 2, 3, 4; j = 1, 2, 3\}$, where d_{ij} denotes the performance of the stock a_i under the criterion C_j. The higher the membership degrees, the more profitable the stock. The higher the non-membership degrees, the less profitable the stock. The dual hesitant fuzzy decision matrix $M = [d_{ij}]_{4\times 3}$ is presented in Table 2.

Step 1. Calculate the weight values of criteria based on the method mentioned in Sect. 3, and the results are shown in Table 3.
Step 2. Construct the aggregated decision matrix $\overline{M} = [\overline{d_i}]_{m\times 1}$ based on Eq. (14). The result is presented in Table 4,
where $0.4291 = 1 - \{(1-0.2)^{0.3126} \cdot (1-0.6)^{0.3444} \cdot (1-0.4)^{3430}\}$ and $0.2098 = 0.5^{0.3126} \cdot 0.1^{0.3444} \cdot 0.2^{3430}$ are the membership degree and non-membership degree in the aggregated DHFE of the stock a_1, respectively.
Step 3. Determine the value of α. Suppose the investor pays more attention to positive information in terms of returns and set $\alpha = 0.6$.

If the investor focuses on returns regardless of risks, according to Process I:

Step 4. Construct Model 3 and calculate the optimal investment proportions $w_i (i = 1, 2, 3, 4)$, then we obtain $w_1 = 0.5230$, $w_2 = 0.3770$, $w_3 = 0.05$, $w_4 = 0.05$.

If the investor considers both returns and risks, according to Process II:

Step 4. Determine the value of β. Suppose the investor pays more attention to negative information in terms of risks and set $\beta = 0.2$.
Step 5. Calculate the range of deviation based on Eqs. (39) and (40), then we obtain $[V_{\min}, V_{\max}] = [3.3794 \times 10^{-4}, 0.0021]$, $\zeta_a = 0.0008$, $\zeta_n = 0.0012$ and $\zeta_s = 0.0016$.
Step 6. Determine the value of ζ. Suppose the investor is risk-seeking and set $\zeta = \zeta_s = 0.0016$.
Step 7. Construct Model 5 and calculate the optimal investment proportions $w_i (i = 1, 2, 3, 4)$, then we obtain $w_1 = 0.5230$, $w_2 = 0.3770$, $w_3 = 0.05$, $w_4 = 0.05$.

In the next several sections, we mainly analyze the results of Model 5 to show the validity of the proposed models, because Model 5 is an improvement of Model 3 according to Property 3.

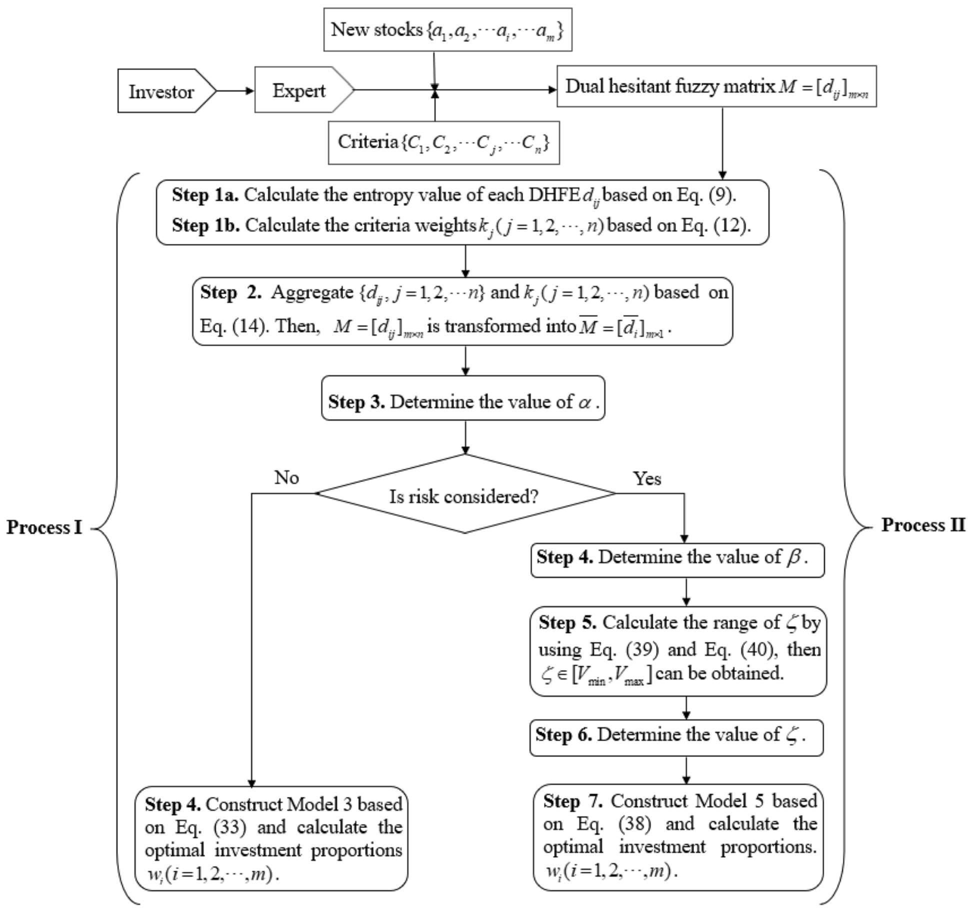

Fig. 1 The flowchart of the two portfolio selection processes

Table 1 Details of criteria

Criteria	Details
C_1	Ability of sustainable development, such as capability of sustainable profit making
C_2	Innovation capability, such as development of new technology
C_3	Reputation, such as the government support

Table 2 Dual hesitant fuzzy matrix M

Stocks	C_1	C_2	C_3
a_1	$\langle(0.2, 0.3, 0.4), (0.5, 0.6)\rangle$	$\langle(0.6, 0.7, 0.8), (0.1)\rangle$	$\langle(0.4, 0.5, 0.7), (0.2, 0.3)\rangle$
a_2	$\langle(0.4, 0.7), (0.2, 0.3)\rangle$	$\langle(0.7, 0.8), (0.1)\rangle$	$\langle(0.5, 0.6), (0.2, 0.3)\rangle$
a_3	$\langle(0.5, 0.6), (0.2, 0.4)\rangle$	$\langle(0.3, 0.5, 0.6), (0.2, 0.3, 0.4)\rangle$	$\langle(0.4), (0.1)\rangle$
a_4	$\langle(0.2, 0.3), (0.5, 0.6)\rangle$	$\langle(0.4, 0.5, 0.6), (0.2, 0.3)\rangle$	$\langle(0.6), (0.1, 0.2)\rangle$

7.2 Sensitivity Analysis

According to Model 5 and Process II, different results can be obtained when the parameters are set to different values. To better analyze the impacts of the parameters, we conduct a sensitivity analysis. Firstly, the impacts of the three parameters on returns and risks are discussed. Secondly, the changes of investment proportions are analyzed.

7.2.1 Impacts of the Parameters α, β and ζ on Returns and Risks

As mentioned in Property 3, under a given value of β, different values of ζ represent investors' different risk appetites. Let R be the maximum portfolio return in Model 5. In Table 5, there are values of R under different risk

Table 3 Weight values of criteria

Criteria	C_1	C_2	C_3
Weight values	0.3126	0.3444	0.3430

appetites and different values of α when $\beta = 0.5$. The result is discussed as follows:

(1) When the values of α and β are fixed, the higher the value of ζ in Model 5, the higher the value of R. It is reasonable that investors have to bear more loss if they want to make more profits.

(2) The value of R increases when the value of α increases under all kinds of risk appetites. That is because investors determining higher values of α generally prefer positive information which is related to a stock's ability to bring benefits. Therefore, higher values of α are more applicable for risk-seeking investors, which is reasonable because risk-seeking investors rely more on information about profits.

In Table 6, there are values of ζ under different risk appetites and different values of β when $\alpha = 0.5$. The result is discussed as follows:

(1) The value of ζ decreases when the value of β decreases under all kinds of risk appetites. That is because investors determining lower values of β generally prefer negative information that is related to a stock's potential to cause loss. Therefore, lower values of β are more applicable for risk-averse investors, which is reasonable because risk-averse investors rely more on information about loss.

(2) Combining the results in Tables 5 and 6, we can find that relatively higher values of α and lower values of β can help risk-neutral investors make more profits and avoid risks to some degree.

In conclusion, the parameters α and β capture investors' information preferences and the parameter ζ captures investors' risk appetites. In the next two subsections, we explore the impacts of the parameters α and β on investment proportions.

7.2.2 Impact of the Parameter α on Investment Proportions

As discussed in Sect. 7.2.1, higher values of α are more applicable for risk-seeking investors, so we compare the investment proportions for risk-seeking investors ($\zeta = \zeta_s$) under different values of α when $\beta = 0.5$ (see Fig. 2). In addition, we calculate the returns of stocks under different values of α by using Eq. (41) (see Fig. 3).

$$R_s^i = \alpha S_{\overline{h_i}} - (1-\alpha) S_{\overline{g_i}}, \tag{41}$$

where $\overline{d_i} = \langle \overline{h_i}, \overline{g_i} \rangle, i = 1, 2, 3, 4$, and R_s^i is the return of the stock a_i. For convenience, let R_s be the return of each stock. The result of Fig. 2 is discussed as follows:

(1) w_4 is the lowest and unchanged under different values of α, which is due to the lowest return of the stock a_4 (see Fig. 3). It is reasonable that stocks with lower returns are not preferred by investors.

(2) w_3 is higher than w_1 when $0.1 \le \alpha \le 0.2$; w_1 begins to increase when $\alpha > 0.2$ and it is higher than w_3 when $\alpha \ge 0.3$. The reason for this change can be found from Fig. 3, where the return of the stock a_3 is higher than that of the stock a_1 when $0.1 \le \alpha \le 0.2$, whereas it is lower than that of the stock a_1 when $\alpha \ge 0.3$.

(3) w_2 is higher than w_1 when $0.1 \le \alpha \le 0.5$, which is due to the higher return of the stock a_2 when $0.1 \le \alpha \le 0.5$ (see Fig. 3). Moreover, Table 7 shows that the mean of non-membership degrees of the

Table 4 The aggregated decision matrix $\overline{M}$

Stocks	Aggregated DHFEs
a_1	$\left\langle \begin{array}{l} (0.4291, 0.4637, 0.4524, 0.4782, 0.4830, 0.4856, 0.5041, 0.5098, \\ 0.5143, 0.5274, 0.5342, 0.5499, 0.5504, 0.5561, 0.5683, 0.5687, \\ 0.5776, 0.5886, 0.5890, 0.5924, 0.5949, 0.6090, 0.6139, 0.6274, \\ 0.6455, 0.6600, 0.6760), (0.2098, 0.2221, 0.2411, 0.2552) \end{array} \right\rangle$
a_2	$\left\langle \begin{array}{l} (0.3279, 0.3775, 0.4589, 0.4987, 0.5561, 0.5888, 0.6426, 0.6689), \\ (0.1575, 0.1788, 0.1810, 0.2055) \end{array} \right\rangle$
a_3	$\left\langle \begin{array}{l} (0.4023, 0.4426, 0.4677, 0.5036, 0.5071, 0.5403), \\ (0.1577, 0.1958, 0.1813, 0.2002, 0.2252, 0.2486) \end{array} \right\rangle$
a_4	$\left\langle \begin{array}{l} (0.4288, 0.4521, 0.4635, 0.4855, 0.5032, 0.5235), \\ (0.2100, 0.2223, 0.2414, 0.2556, 0.2663, 0.2820, 0.3062, 0.3242) \end{array} \right\rangle$

Table 5 Values of R under different risk appetites and different values of α when $\beta = 0.5$

α	Values of R		
	Risk-averse ($\zeta_a = 0.0016$)	Risk-neutral ($\zeta_n = 0.0027$)	Risk-seeking ($\zeta_s = 0.0038$)
0.10	− 0.1260	− 0.1218	− 0.1191
0.20	− 0.0562	− 0.0518	− 0.0489
0.30	0.0151	0.0193	0.0219
0.40	0.0885	0.0918	0.0931
0.50	0.1631	0.1643	0.1644
0.60	0.2377	0.2377	0.2377
0.70	0.3136	0.3136	0.3136
0.80	0.3908	0.3910	0.3910
0.90	0.4680	0.4685	0.4685

stock a_2 is lower than that of the stock a_1. As mentioned above, negative information has more impact on portfolios when $0.1 \leq \alpha \leq 0.5$, so the stock a_2 is preferred in portfolio selection when $0.1 \leq \alpha \leq 0.5$.

(4) w_1 increases while w_2 decreases substantially when $\alpha \geq 0.5$. The reason for these changes is that positive information has more impact on portfolio returns when $\alpha \geq 0.5$. Table 7 shows that the mean of membership degrees of the stock a_1 is higher than that of the stock a_2, so w_1 increases when $\alpha \geq 0.5$. Furthermore, w_1 is higher than w_2 when $\alpha \geq 0.6$. As discussed in Sect. 7.2.1, the models with higher values of α tend to choose the portfolios with higher returns. The return of the stock a_1 is higher than that of the stock a_2 when $\alpha \geq 0.6$ (see Fig. 3), so the stock a_1 is preferred when $\alpha \geq 0.6$. These results are consistent with the fact that risk-seeking investors rely more on positive information and prefer the stocks with higher returns.

Table 6 Values of ζ under different risk appetites and different values of β when $\alpha = 0.5$

β	Values of ζ		
	Risk-averse (ζ_a)	Risk-neutral (ζ_n)	Risk-seeking (ζ_s)
0.10	0.0005	0.0007	0.0009
0.20	0.0008	0.0012	0.0016
0.30	0.0011	0.0017	0.0023
0.40	0.0013	0.0022	0.0031
0.50	0.0016	0.0027	0.0038
0.60	0.0018	0.0031	0.0045
0.70	0.0021	0.0036	0.0051
0.80	0.0023	0.0041	0.0058
0.90	0.0025	0.0045	0.0065

In conclusion, the parameter α can capture investors' information preferences in terms of returns. When investors determine higher values of α, the stocks with higher mean values of membership degrees are preferred; when investors determine lower values of α, the stocks with lower mean values of non-membership degrees are preferred. Moreover, the impact of α on investment proportions is consistent with investors' preferences that the stocks with higher returns usually have higher investment proportions.

7.2.3 *Impact of the Parameter β on Investment Proportions*

As discussed in Sect. 7.2.1, lower values of β are more applicable for risk-averse investors. Therefore, we compare the investment proportions for risk-averse investors ($\zeta = \zeta_a$) under difference values of β when $\alpha = 0.5$ in this section (see Fig. 4). In addition, we calculate the risks of stocks under different values of β by using Eq. (42) (see Fig. 5).

$$V_s^i = \beta V_{\overline{h_i}} + (1 - \beta) V_{\overline{g_i}}, \tag{42}$$

where $\overline{d_i} = \langle \overline{h_i}, \overline{g_i} \rangle, i = 1, 2, 3, 4$, and V_s^i is the risk of the stock a_i. For convenience, let V_s be the risk of each stock. The results of Fig. 4 are discussed as follows:

(1) The stocks a_1 and a_2 occupy much larger investment proportions than the stocks a_3 and a_4 under different values of β. However, in Fig. 5, the risk of the stock a_2 is the highest under different values of β, and the risk of the stock a_1 is higher than those of the stocks a_3 and a_4 when $\beta \geq 0.3$. The reason for this strange result is that the objective of Model 5 is to maximize the portfolio return and the returns of the stocks a_1 and a_2 are much higher those of the stocks a_3 and a_4 (see Table 8).

(2) w_1 is lower than w_2 when $\beta \leq 0.2$, which is due to the lower return of the stock a_1(see Table 8). Moreover,

Table 7 Mean values of membership degrees and non-membership degrees of different stocks

Stocks	Mean of membership degrees	Mean of non-membership degrees
a_1	0.5537	0.2320
a_2	0.5149	0.1807
a_3	0.4773	0.2015
a_4	0.4761	0.2635

Table 8 The returns and deviations of stocks when $\alpha = 0.5$

Stocks	Returns	Deviations of membership degrees	Deviations of non-membership degrees
a_1	0.1544	0.0042	0.0005
a_2	0.1671	0.0149	0.0004
a_3	0.1379	0.0025	0.0010
a_4	0.1063	0.0012	0.0016

$\beta \le 0.2$ means that investors rely more on negative information. The deviation of non-membership degrees of the stock a_2 is lower than that of the stock a_1(see Table 8), so w_2 is higher.

(3) w_1 increases while w_2 decreases when $\beta \ge 0.2$. That is because the risk of the stock a_2 begins to increase and becomes much larger than that of the stock a_1 when $\beta \ge 0.2$ (see Fig. 5). It is reasonable that risk-averse investors will not invest most of their money into the stocks with too high risks. However, w_2 doesn't drop dramatically when $\beta \ge 0.5$. That is because the objective of Model 5 is to maximize the portfolio return, and the return of the stock a_2 is higher than that of the stock a_1(see Table 8).

(4) w_1 is still higher than w_2 and becomes larger than 0.5 when $\beta \ge 0.5$. That is because investors rely more on positive information when $\beta \ge 0.5$ and the deviation of membership degrees of the stock a_1 is lower than that of the stock a_2(see Table 8).

In conclusion, the parameter β can capture investors' preferences for stocks, but its impact on investment proportions depends on the returns of stocks because the objective of Model 5 is to maximize the portfolio return.

7.3 Efficient Frontier Analysis

In this section, we mainly analyze the efficient frontier of Model 5 when $\beta = 0.5$ and $\alpha = 0.5$. When $\beta = 0.5$, we have $[V_{\min}, V_{\max}] = [4.647 \times 10^{-4}, 0.0048]$ by using Eqs. (39) and (40). Based on Property 3, if the investor is risk-averse, then $4.647 \times 10^{-4} \le \zeta \le 0.0019$; if the investor is risk-neutral, then $0.0019 < \zeta \le 0.0034$; if the investor is risk-seeking, then $0.0034 < \zeta \le 0.0048$. To compare the returns under different risk levels, we calculate the value of R when the value of ζ changes from $V_{\min}$ to $V_{\max}$ and obtain the following conclusions from Fig. 6.

Table 9 Investment proportions under different risk appetites in Model 5 and the HFPSM

Risk appetite	Model	w_1	w_2	w_3	w_4
Risk-averse	HFPSM	**0.6969**	0.1789	0.0719	0.0523
	Model 5	**0.5139**	0.3861	0.0500	0.0500
Risk-neutral	HFPSM	**0.8500**	0.0500	0.0500	0.0500
	Model 5	0.2827	**0.6173**	0.0500	0.0500
Risk-seeking	HFPSM	**0.8500**	0.0500	0.0500	0.0500
	Model 5	0.2029	**0.6971**	0.0500	0.0500

The black bold numbers are the maximum investment proportions in the HFPSM, and the blue bold numbers are the maximum investment proportions in Model 5

(1) The returns of risk-seeking investors are higher than those of risk-neutral investors and risk-averse investors, which is reasonable because risk-seeking investors aim to make more profits.

(2) When $4.647 \times 10^{-4} \le \zeta \le 0.0019$, the value of R increases substantially, which is similar to that the investment proportions change substantially when $4.647 \times 10^{-4} \le \zeta \le 0.0019$ (see Fig. 7). From the result in Fig. 5, we can find that the stocks with lower risks have higher investment proportions, such as the stock a_1. It is reasonable that the investment proportions of risk-averse investors are more susceptible to risks and risk-averse investors prefer the stocks with lower risks.

(3) When $\zeta > 0.0034$, the value of R in Fig. 6 and the investment proportions in Fig. 7 are unchanged. According to Property 3, Model 5 will be the same as Model 3 if $\zeta = V_{\max}$. Therefore, the investment proportions of Model 3 are the same as those of Model 5 for risk-seeking investors.

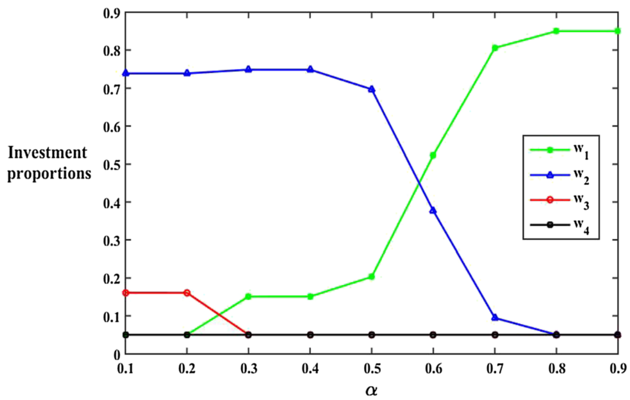

Fig. 2 Investment proportions for risk-seeking investors under different values of α when $\beta = 0.5$

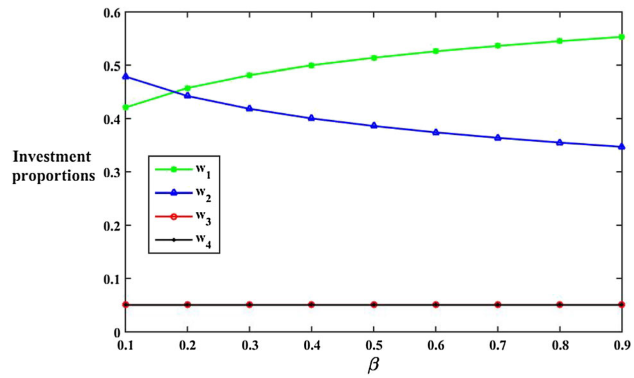

Fig. 4 Investment proportions for risk-averse investors under difference values of β when $\alpha = 0.5$

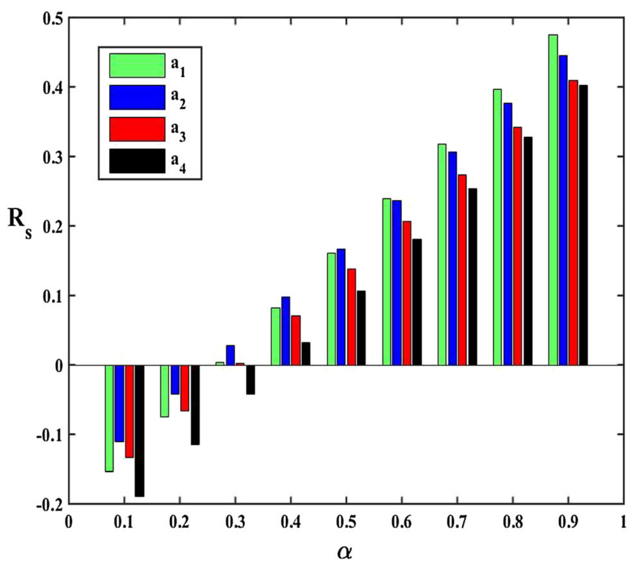

Fig. 3 The returns of stocks (R_s) under different values of α

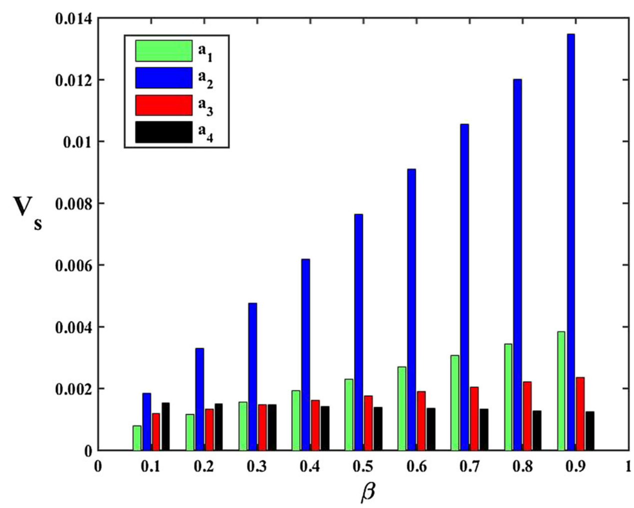

Fig. 5 The risks of stocks (V_s) under different values of β

(4) When the value of ζ increases, the values of R and w_2 both increase. Moreover, the investment proportions are unchanged and the stock a_2 makes up a large proportion of investment when $\zeta > 0.0034$ (see Fig. 7). That is because the return of the stock a_2 is the highest when $\alpha = 0.5$(see Table 8), and the risk of the stock a_2 is also the highest when $\beta = 0.5$ (see Fig. 5). It is well known that high-return stocks usually have high risks. Risk-seeking investors are less sensitive to risks, so it is reasonable for them to choose the stocks with higher risks for higher returns.

In conclusion, the efficient frontier of our proposed Model 5 is reasonable. In the next section, we compare Model 5 with other portfolio selection models to highlight its superiority.

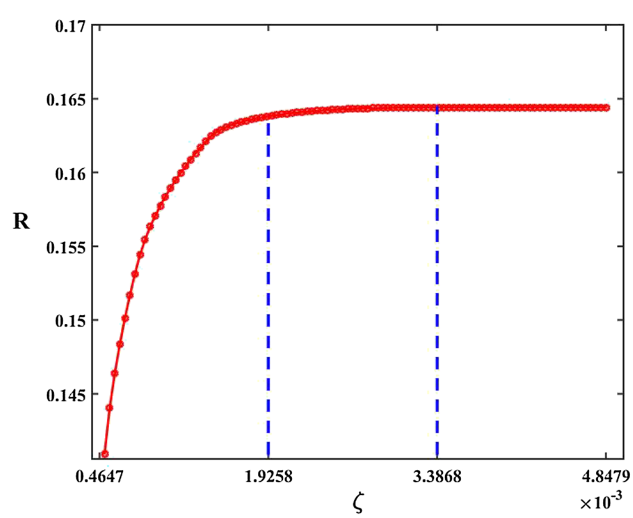

Fig. 6 The efficient frontier of Model 5 when $\beta = 0.5$ and $\alpha = 0.5$

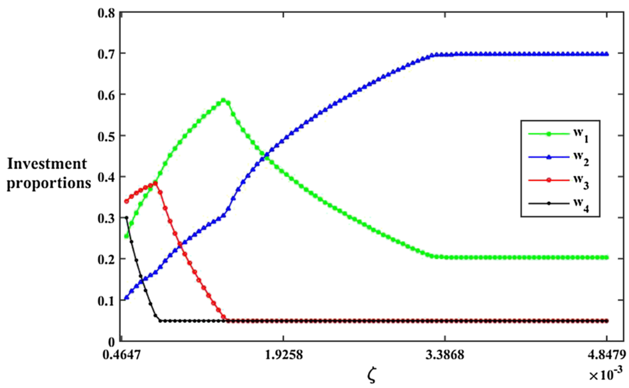

Fig. 7 Investment proportions under different values of ζ when $\beta = 0.5$ and $\alpha = 0.5$

7.4 Comparison Between Model 5 and the Hesitant Fuzzy Portfolio Selection Model

As mentioned above, the DHFS is a good improvement of the HFS. Therefore, Model 5 is compared with the hesitant fuzzy portfolio selection model (HFPSM) proposed by Zhou and Xu [13] to show the superiority of Model 5.

7.4.1 Theoretical Comparison Between Model 5 and the HFPSM

According to the mathematical symbol of the hesitant fuzzy element (HFE) proposed by Xia and Xu [39], the set H in $D_R = \langle H, G \rangle$ in Eq. (21) can be seen as a HFE. Moreover, based on the score function [39] of the HFE, the score function of H is equivalent to $S_H(W)$ based on Eq. (23). According to the deviation function [40] of the HFE, the deviation function of H is

$$\overline{V_H}(W) = \frac{1}{l_H} \sum_{u=1}^{l_H} \sqrt{\left(s_u(W) - S_H(W)\right)^2}. \tag{43}$$

In the hesitant fuzzy portfolio selection model [13], the score and deviation of the HFE are used to measure returns and risks, respectively. Therefore, the HFPSM for risk investors [13] is represented as follows.

HFPSM

$$\max F(W) = S_H(W)$$
$$\text{s.t} \begin{cases} \overline{V_H}(W) \leq \zeta \\ \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \ldots, m \end{cases}, \tag{44}$$

where $F(W)$ describes the portfolio return and ζ describes the investor's risk appetite. $\zeta \in [V_{\min}, V_{\max}]$, where

$$V_{\max} = \max \overline{V_H}(W)$$
$$\text{s.t} \begin{cases} \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \ldots, m \end{cases} \tag{45}$$

and

$$V_{\min} = \min \overline{V_H}(W)$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} w_i = 1 \\ w_i \geq 0, i = 1, 2, \ldots, m \end{cases}. \tag{46}$$

According to Zhou and Xu [13]: if the investor is risk-seeking, then $\zeta = V_{\max}$; if the investor is risk-neutral, then $\zeta = V_{\min} + \frac{2}{3}(V_{\max} - V_{\min})$; if the investor is risk-averse, then $\zeta = V_{\min} + \frac{1}{3}(V_{\max} - V_{\min})$. By comparing Model 5 and the HFPSM, we can obtain that:

(1) Model 5 and the HFPSM both consider returns and risks, which are both applicable for investors with different risk appetites. When $\beta = 1$ and $\alpha = 1$, Model 5 is similar to the HFPSM.

(2) Mode 5 considers information preference, which is not considered in the HFPSM. That is because Model 5 is constructed based on the DHFS, which can describe positive and negative information more comprehensively than the HFS. Therefore, Model 5 has a wider application than the HFPSM.

(3) In the HFPSM, there are only three values of ζ for investors to choose. However, according to Property 3, the range $[V_{\min}, V_{\max}]$ in Model 5 is divided into three intervals, which include all the values of ζ. Investors usually determine different values of ζ based on their risk appetites, so the setting of ζ in Model 5 is more reasonable.

7.4.2 Empirical Comparison Between Model 5 and the HFPSM

In this section, we compare Model 5 and the HFPSM based on the information and settings of parameters in case study. For convenience, we take the membership degrees of DHFEs in Tables 2 and 4 as the membership degrees of HFEs. Firstly, the investment proportions of the HFPSM and Model 5 under different risk appetites when $\alpha = 0.5$ and $\beta = 0.5$ are presented in Table 9. The following conclusions are obtained from Table 9.

(1) For risk-averse investors, w_1 is the highest in the HFPSM, because the mean of membership degrees

of the stock a_1 is the highest (see Table 7). In Model 5, w_1 is smaller than that in the HFPSM, while w_2 is larger than that in the HFPSM. The reason for this difference is that Model 5 considers the negative information of stocks. In Table 7, the mean of non-membership degrees of the stock a_1 is higher than that of the stock a_2, so w_1 decreases while w_2 increases.

(2) For risk-neutral investors and risk-seeking investors in Model 5, w_2 increases while w_1 decreases, which is similar to that in conclusion (1). Moreover, the stock a_2 makes up a large proportion of investment for risk-seeking investors in Model 5, which can be explained by the highest return of the stock a_2 (see Table 8).

(3) For risk-neutral investors and risk-seeking investors in the HFPSM, the stock a_1 makes up a main proportion of investment, which is different from that in conclusion (2). This is because the HFPSM only considers membership degrees and the mean of membership degrees of the stock a_1 is the highest (see Table 7). It is reasonable that investors who aim to make more profits will prefer the stocks with higher mean values of membership degrees in the HFPSM.

In conclusion, Model 5 can help investors with different information preferences find the optimal portfolios, which cannot be achieved by the HFPSM. Therefore, our proposed Model 5 is a good improvement of the HFPSM.

7.5 Utilities of the Proposed Models

According to the analyses above, the utilities of Model 3 and Model 5 are summarized as follows.

(1) The proposed models can help investors to find the optimal portfolios when precise data are difficult to obtain. Model 3 is applicable for investors focusing on returns regardless of risks, whereas Model 5 is applicable for investors considering both returns and risks.

(2) The parameter α in Model 3 and Model 5 can reflect investors' information preferences in terms of returns. Moreover, the change of α is consistent with investors' preferences for stocks, and higher values of α are more applicable for risk-seeking investors.

(3) The parameter β in Model 5 can reflect investors' information preferences in terms of risks. In addition, lower values of β are more applicable for risk-averse investors. However, the impact of β on investment proportions depends on the return of each stock because the objective of Model 5 is to maximize the portfolio return.

(4) The parameter ζ in Model 5 can describe investors' risk appetites. The higher the value of ζ, the higher the corresponding portfolio return. Therefore, Model 5 is applicable for risk-averse investors, risk-neutral investors and risk-seeking investors.

(5) Compared with the hesitant fuzzy portfolio selection model, Model 5 can offer more options to the investors with different information preferences.

8 Conclusions

DHFS can validly describe uncertain information. Although there have been some decision-making models based on the DHFS, most of the research merely focuses on ranking the alternatives and choosing the best one. If the investment information is described by DHFSs and investors hope to find the optimal portfolios, the existing models are inapplicable. Therefore, we propose some novel portfolio selection models under dual hesitant fuzzy environment to solve this problem. The main contributions of this paper are concluded as follows.

(1) To make use of positive and negative information in the DHFS, the parameter α is defined in Model 3 and Model 5 to capture investors' information preferences in terms of returns, and the parameter β is defined in Model 5 to capture investors' information preferences in terms of risks.

(2) The Max-score dual hesitant fuzzy portfolio selection model with information preference (Model 3) is proposed to help the investors focusing on returns regardless of risks to find the optimal portfolios. Moreover, it is proved that Model 3 is equivalent to a convex programming.

(3) To consider the risks of portfolios, the score-deviation dual hesitant fuzzy portfolio selection model with information preference and risk appetite (Model 5) is developed. In Model 5, another parameter ζ is defined to capture investors' risk appetites.

(4) The sensitivity analysis, efficient frontier analysis and comparison analysis with the hesitant fuzzy portfolio selection model are conducted to highlight the utilities of the proposed models and the impacts of the parameters.

However, there are some limitations of the models. For example, the relationship among the parameters in Model 5 may have influence on the investment proportions, but the theoretical analysis on the relationship is not carried out in this paper because of the complexity. This is also one of the topics in our future study. Research on portfolio selection under dual hesitant fuzzy environment is still at an early stage. There is a lot of work to do in the future and we will keep on researching the portfolio selection in this area.

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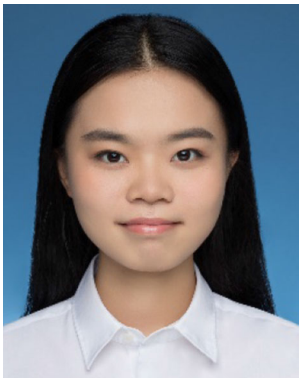

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