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Multiple Attribute Group Decision Making Method Based on Intuitionistic Fuzzy Einstein Interactive Operations

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Abstract The intuitionistic fuzzy numbers (IFNs) have been extensively studied in recent years. However, the traditional operational rules (ORs) of the IFNs still have some drawbacks in solving the practical decision-making problems. Einstein t-conorm and t-norm (TAT) are an important and typical class of the TAT, but the ORs for the IFNs based on the Einstein TAT (ETAT) cannot consider the interaction between the membership degree (MD) and the non-membership degree (N-MD), they may get the unreasonable evaluation results in some realistic decision-making situations. So this paper proposes some new Einstein interactive ORs for the IFNs, then, it further presents the intuitionistic fuzzy Einstein interactive weighted averaging (IFEIWA) operator to overcome above existing drawbacks, and some properties of this operator are proved. Simultaneously, in order to eliminate the effects of the existing biases of some decision experts in the process of evaluating attributes, this paper proposes the intuitionistic fuzzy Einstein interactive power averaging (IFEIPA) operator and the intuitionistic fuzzy Einstein interactive weighted power averaging (IFEIWPA) operator based on the revised power weighted averaging operator, and then gives their some desirable properties. Further, by using the IFEIPA operator and the IFEIWPA operator, this paper presents a novel method for the multi-attribute group decision making (MAGDM) problems to solve practical decision-making problems. Lastly, this paper uses some actual application examples to verify the applicability and validity of the proposed MAGDM method, and then demonstrates the superiority of novel method by detailed comparison analysis with other typical methods.

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1 Introduction

With the advancement of modern decision-making technology, multiple attribute group decision making (MAGDM) has been a hot and important decision research field, and it can rank finite alternatives according to the attribute evaluation values obtained by multiple experts. In recent years, MAGDM has received the attention from many scholars [27, 32, 42]. With the complexity of decision-making problems and the dynamic nature of decision-making environment, how to effectively describe the evaluation information has developed into an important issue that needs to be solved urgently in decision-making problems. Due to the uncertainty of people's cognition, it is not easy to express attribute evaluation information by real values, thus the form of fuzzy information had emerged. The intuitionistic fuzzy set (IFS) is extended from fuzzy sets (FS) [60], which was first proposed by Atanassov [3, 4] and has both the membership degree (MD) and the non-membership degree (N-MD). Since the IFSs are closer to people's evaluation habits and cognitive levels, they have been recognized by many scholars [15, 30, 38]. The researches on IFSs mainly include the following three aspects: (1) the basic theoretical research of intuitionistic fuzzy numbers (IFNs), such as the operational rules (ORs)

[16, 31], distance measure [45], similarity measure [5, 12, 15], comparison method [56], correlation measure [13, 23], information entropy measure [2, 19], and so on; (2) the researches on some extended traditional decision methods based on the IFS, such as TOPSIS [21], WASPAS [36], PPROMETHE [37] method, ELECTRE method [51, 52], DEMATEL method [1, 14], and so on; (3) the researches on the aggregation operators (AOs) for IFNs, such as Heronian mean AOs [26], Bonferroni mean AOs [28], Maclaurin symmetric mean AOs [29].

The AOs play an important role in information fusion. When we solve the problem of how to express information, we need to seek some tools to deal with the obtained information. In this case, the AOs have more advantages than some traditional approaches. This is because AOs can rank the alternatives based on the comprehensive evaluation values, while traditional approaches can only give a prioritization relationship based on some complex theory and related information measures, so the AOs are simpler and more intuitive than traditional approaches. Nowadays, AOs have gotten a lot of attentions and have also become a hot trend in the field of information fusion. The AOs are usually divided into two categories: (1) the AOs with different ORs. For IFNs, some common AOs are mainly using the algebraic ORs which are from the Archimedean TAT. Then some new ORs are from some new special cases of TAT, for instance, Einstein ORs [31, 49] have the typical characteristic which can give same smooth approximations, Hamacher ORs [24], Frank ORs [61], Dombi ORs [28] and Schweizer-Sklar ORs [32, 48] all have some flexible parameters. Based on above these operational laws of IFNs, Tan and Chen [46] proposed the generalized Archimedean intuitionistic fuzzy weighted averaging (GAIFWA) operator based on Archimedean TAT. Wang and Liu [49, 50] presented the intuitionistic fuzzy Einstein weighted averaging (IFEWA) operator and intuitionistic fuzzy geometric (IFEWG) operator, they are more suitable to the pessimistic decision makers (DMs). Liu et al. [28] and Liu and Wang [32] studied the Dombi AOs and Schweizer-Sklar AOs under intuitionistic fuzzy environment, respectively, and these operators all have a flexible and variable parameter, they can reduce to other AOs when the parameter takes different special values, so they are more universal and flexible. In a word, each special case of TAT has its own unique characteristics. (2) The AOs with functionality. In general, the AOs can aggregate some values to an integrated value and have a simple function to consider the weights of aggregated data. With the increasing complexity and diversity of decision-making environment, some new functional operators are proposed gradually. For example, Xu and Yager [57] presented the power AOs for IFNs which could eliminate the impact of extreme data on evaluation results by considering the mutual support of integrated input arguments. Yu and Wu [59] presented the Heronian mean (HM) operators which could capture the interrelationships between two interval valued IFNs (IVIFNs). Qin and Liu [40] presented the Maclaurin symmetric mean (MSM) operators which could capture the interrelationships of multiple integrated IFNs. However, when there are some abnormal data, these AOs considering the correlation of the integrated parameters will amplify the effect of these data, which will cause changes in the ranking results, i.e., these AOs are extremely sensitive to abnormal data. Further, Liu and Wang [32] extended the power AOs based on the Schweizer-Sklar ORs and presented Schweizer-Sklar power AOs, Liu [25] combined power operator with HM operator to presented the power Heronian AOs for IVIFNs, Teng et al. [47] and Liu et al. [27] combined power operator with MSM operator to deal with more complex fuzzy information under Pythagorean fuzzy environment and generalized orthopair fuzzy environment, respectively.

As discussed above, these existing researches have the following weaknesses. (1) For the ORs of IFNs (including their extended forms), they almost never consider the interaction between MD and N-MD of IFNs. So, these ORs are not enough to compute some special evaluation values whose MD or N-MD is zero. We take the Einstein operational laws of IFNs [49] as an example. Suppose $\tilde{\gamma}_1 = (0.3, 0)$, $\tilde{\gamma}_2 = (0.6, 0.1)$, $\tilde{\gamma}_3 = (0.5, 0.2)$ are three IFNs, we can get $\tilde{\gamma}_1 \oplus_e \tilde{\gamma}_2 = (0.763, 0)$ and $\tilde{\gamma}_1 \oplus_e \tilde{\gamma}_3 = (0.696, 0)$, we are easy to find that the N-MD of one IFN equals to zero and the others are not zero, the N-MD of their sum is equal to zero, i.e., if a MD or a N-MD of IFN is zero, it will play a decisive role in final computing result, but the other MDs or N-MDs do not influence the final computing result any more. Obviously, this is unreasonable. In addition, we know that the aggregated results of MD and N-MD are obtained independently from the MDs and the N-MDs, respectively. It is shown as Example 2 to explain this weakness in this paper. So, He et al. [16] and He and He [17] developed the interactive ORs of IFNs to solve the problems. Then, Garg [11] proposed the interactive operational laws based on Hamacher norms, but they still have some drawbacks, for example, they do not have the duality. (2) For functions of AOs, the AOs considering the correlation of the integrated parameters will amplify the effect of some abnormal data, and then cause unreasonable changes in ranking results. Although the PA operator can remove the influence of extreme values coming from the biased decision experts. However, when we adopted the original power weighted AOs in Xu [54] and Xu and Yager [57], their above functions may not be obvious. The specific analysis and explanation can be found in Sect. 6.3. In addition, some existing power AOs based some TAT operational laws didn't take into account the interaction between the MD and N-MD of IFNs, so they have some drawbacks in dealing with the real MAGDM problems. On the other hand, although some functional AOs use the interactive ORs for IFNs [29], not only can they not eliminate the effects of extreme evaluation values, but they can also amplify the effects of these values, the specific analysis and explanation can also be found in Sect. 6.3.

In order to solve the above weaknesses and deal with practical decision-making problems more effectively, we summarize the research results of the existing literature on IFNs, and the results are shown in Table 1. Considering the advantage of Einstein ORs which can give same smooth approximations and PA operator which can remove influences of extreme data, the goals and contribution of this paper are to (1) construct some Einstein interactive ORs for IFNs to overcome the first weakness mentioned above; (2) propose some novel revised power weighted AOs based on the Einstein interactive ORs for IFNs to overcome the second weakness mentioned above; (3) develop a novel MAGDM approach to solve complex actual decision problems by using the proposed operators. The advantages of the novel approach are mainly reflected in the following three aspects: (1) considering the interaction between MD and N-MD, and solving some existing problems when MD or N-MD of any an IFN equals to zero. For instance, in an addition operation of proposed Einstein interactive ORs for IFNs the computing result of N-MDs depends on both MDs and N-MDs of aggregated IFNs; (2) having the good smooth approximations to make the integrated result more robust and can reflect the DM's pessimistic attitude by Einstein operations; (3) eliminating the influences of unreasonable data coming from biased DMs and making the evaluation results more reasonable for complex decision-making applications.

The rest framework of this paper can be constructed as follows. In Sect. 2, a brief introduction about several fundamental conceptions of IFS, ETAT and PA operator is presented. In Sect. 3, some drawbacks of the existing Einstein AOs for IFNs are analyzed and the intuitionistic fuzzy Einstein interactive weighted averaging (IFEIWA) operator based on the Einstein interactive ORs is proposed. In Sect. 4, the intuitionistic fuzzy Einstein interactive weighted power averaging (IFEIWPA) operator based on the revised weighted PA operator is presented. In Sect. 5, a novel MAGDM approach is proposed. In Sect. 6, the effectiveness and superiority of presented approach are verified by some practical examples and detailed comparative analysis with the existing approaches. In Sect. 7, some conclusions are given.

2 Preliminaries

To better understand this article, some related concepts about IFSs, ETAT and PA operator are introduced in this part.

2.1 IFSs

Definition 1 [3, 4] Suppose $Y = \{y_1, y_2, \ldots, y_z\}$ is a universe of discourse. An IFS X in Y is represented by

Table 1 A summary on the relative characteristics of AOs for IFNs from some representative literature

Characteristic literature	Consider the interaction between MD and N-MD of IFN	Consider non algebraic ORs/offer similar smooth estimations	Eliminate the effect of extreme evaluation data	Sensitive to abnormal evaluation data (consider the correlation of input parameters)
Xu [54]	No	No/No	Yes	No
Wang and Liu [50]	No	Yes/Yes	No	No
Qin and Liu [40]	No	No/No	No	Yes
Zhang et al. [61]	No	Yes/No	Yes	No
He et al. [16]	Yes	No/No	No	No
Garg [11]	Yes	Yes/No	No	No
He and He [17]	Yes	No/No	No	Yes
Tan and Chen [46]	No	Yes/Yes	No	No
Liu [25]	No	Yes/Yes	No	Yes
Liu and Liu [29]	Yes	No/No	No	Yes
Liu et al. [28]	No	Yes/No	No	Yes
Wang and Liu [48]	No	Yes/No	No	Yes
Liu and Tang [31]	Yes	Yes/Yes	No	Yes
This paper	Yes	Yes/Yes	Yes	No

$$X = \{ <y, a_X(y), b_X(y) > | y \in Y\} \tag{1}$$

where, $a_X(y), b_X(y) \in [0,1]$ and $0 \le a_X(y) + b_X(y) \le 1$, $\forall y \in Y$. $a_X(y)$ and $b_X(y)$ are the MD and the N-MD. The indeterminacy degree $\pi(y) = 1 - a_X(y) - b_X(y)$, $\forall y \in Y$, and $0 \le \pi(y) \le 1$, $\forall y \in Y$.

Further, Xu and Xia [55] called the pair $(a_X(y), b_X(y))$ an IFN. For convenience, we use $\tilde{x} = (a_{\tilde{x}}, b_{\tilde{x}})$ to represent an IFN, where $a_{\tilde{x}} \in [0,1]$, $b_{\tilde{x}} \in [0,1]$ and $0 \le a_{\tilde{x}} + b_{\tilde{x}} \le 1$.

Suppose $\tilde{x}_1 = (a_{\tilde{x}_1}, b_{\tilde{x}_1})$ and $\tilde{x}_2 = (a_{\tilde{x}_2}, b_{\tilde{x}_2})$ are two IFNs, and $\lambda > 0$, then the ORs defined by Atanassov [4] are shown as below:

(i) $\tilde{x}_1 \oplus \tilde{x}_2 = (a_{\tilde{x}_1} + a_{\tilde{x}_2} - a_{\tilde{x}_1}a_{\tilde{x}_2}, b_{\tilde{x}_1}b_{\tilde{x}_2})$,
(ii) $\tilde{x}_1 \otimes \tilde{x}_2 = (a_{\tilde{x}_1}a_{\tilde{x}_2}, b_{\tilde{x}_1} + b_{\tilde{x}_2} - b_{\tilde{x}_1}b_{\tilde{x}_2})$,
(iii) $\lambda\tilde{x}_1 = \left(1 - (1 - a_{\tilde{x}_1})^{\lambda}, b_{\tilde{x}_1}^{\lambda}\right)$,
(iv) $\tilde{x}_1^{\lambda} = \left(a_{\tilde{x}_1}^{\lambda}, 1 - (1 - b_{\tilde{x}_1})^{\lambda}\right)$.

To compare the IFNs, the following definitions are presented.

Definition 2 [6] Suppose $\tilde{x} = (a_{\tilde{x}}, b_{\tilde{x}})$ is an IFN, then a score value (SV) S of $\tilde{x}$ is described as below:

$$S(\tilde{x}) = a_{\tilde{x}} - b_{\tilde{x}}, \tag{6}$$

Obviously, $S(\tilde{x}) \in [-1,1]$. The smaller the SV $S(\tilde{x})$ is, the smaller the IFN $\tilde{x}$ is.

Definition 3 [18] Suppose $\tilde{x} = (a_{\tilde{x}}, b_{\tilde{x}})$ is an IFN, then an accuracy value (AV) A of $\tilde{x}$ is described as below:

$$A(\tilde{x}) = a_{\tilde{x}} + b_{\tilde{x}}, \tag{7}$$

Obviously, $A(\tilde{x}) \in [0,1]$. The smaller the AV $A(\tilde{x})$ is, and the smaller $\tilde{x}$ is.

According to the above two definitions, a comparison method of IFNs was presented by Xu and Yager [56], which is described as below.

Definition 4 [56] Suppose $\tilde{x}_1 = (a_{\tilde{x}_1}, b_{\tilde{x}_1})$ and $\tilde{x}_2 = (a_{\tilde{x}_2}, b_{\tilde{x}_2})$ are any two IFNs, then

(1) If $S(\tilde{x}_1) > S(\tilde{x}_2)$, then $\tilde{x}_1 > \tilde{x}_2$
(2) If $S(\tilde{x}_1) = S(\tilde{x}_2)$, then
If $A(\tilde{x}_1) > A(\tilde{x}_2)$, then $\tilde{x}_1 > \tilde{x}_2$;
If $A(\tilde{x}_1) = A(\tilde{x}_2)$, then $\tilde{x}_1 = \tilde{x}_2$.

Definition 5 [44] Suppose $\tilde{x}_1 = (a_{\tilde{x}_1}, b_{\tilde{x}_1})$ and $\tilde{x}_2 = (a_{\tilde{x}_2}, b_{\tilde{x}_2})$ are any two IFNs, then the normalized Hamming distance between $\tilde{x}_1$ and $\tilde{x}_2$ is described as below:

$$d(\tilde{x}_1, \tilde{x}_2) = \frac{|a_{\tilde{x}_1} - a_{\tilde{x}_2}| + |b_{\tilde{x}_1} - b_{\tilde{x}_2}| + |\pi_{\tilde{x}_1} - \pi_{\tilde{x}_2}|}{2} \tag{8}$$

where $d(\tilde{x}_1, \tilde{x}_2) \in [0,1]$, $\pi_{\tilde{x}_1} = 1 - u_{\tilde{x}_1} - v_{\tilde{x}_1}$, $\pi_{\tilde{x}_2} = 1 - u_{\tilde{x}_2} - v_{\tilde{x}_2}$.

2.2 ETAT

The triangle-operators [43] are the intersection and union operators which can be represented by T-norm (T), and T-conorm (T^*), respectively. We can generate $T(u,v) = \xi^{-1}(\xi(u) + \xi(v))$ and $T^*(u,v) = \psi(\psi(u) + \psi(v))$ where $\xi(u)$ and $\psi(u)$ are monotonically decreasing function and increasing function, respectively, and meet $\xi(u) : (0,1] \to R^+$, $\xi^{-1}(u) : R^+ \to (0,1]$, $\lim_{u\to\infty} \xi^{-1}(u) = 0$, $\xi^{-1}(0) = 1$, $\psi(u) : (0,1] \to R^+$, $\psi^{-1}(u) : R^+ \to (0,1]$, $\lim_{u\to\infty} \psi^{-1}(u) = 1$ and $\psi^{-1}(0) = 0$.

According to the TAT, the generalized intersection and union for the IFNs were presented by Deschrijver and Kerre [8]. In addition, according to Klement and Mesiar [20], we can set $\psi(u) = \xi(1-u)$. Based on the T-norm $T(u,v)$ and T-conorm $T^*(u,v)$, the ORs of IFNs can be given by following definition.

Definition 6 [8] Suppose $\tilde{\gamma}_1 = (a_{\tilde{\gamma}_1}, b_{\tilde{\gamma}_1})$ and $\tilde{\gamma}_2 = (a_{\tilde{\gamma}_2}, b_{\tilde{\gamma}_2})$ are any two IFNs, then, the product and sum of IFNs based on the TAT are as follows:

$$\begin{aligned}\tilde{\gamma}_1 \otimes \tilde{\gamma}_2 &= \left(T(a_{\tilde{\gamma}_1}, a_{\tilde{\gamma}_2}), T^*(b_{\tilde{\gamma}_1}, b_{\tilde{\gamma}_2})\right) \\ &= \left(\xi^{-1}(\xi(a_{\tilde{\gamma}_1}) + \xi(a_{\tilde{\gamma}_2})), \psi^{-1}(\psi(b_{\tilde{\gamma}_1}) + \psi(b_{\tilde{\gamma}_2}))\right)\end{aligned} \tag{9}$$

$$\begin{aligned}\tilde{\gamma}_1 \oplus \tilde{\gamma}_2 &= \left(T^*(a_{\tilde{\gamma}_1}, a_{\tilde{\gamma}_2}), T(b_{\tilde{\gamma}_1}, b_{\tilde{\gamma}_2})\right) \\ &= \left(\psi^{-1}(\psi(a_{\tilde{\gamma}_1}) + \psi(a_{\tilde{\gamma}_2})), \xi^{-1}(\xi(b_{\tilde{\gamma}_1}) + \xi(b_{\tilde{\gamma}_2}))\right)\end{aligned} \tag{10}$$

In the following, we give the operations for IFNs from ETAT.

If we define $T(u,v) = \frac{uv}{1+(1-u)(1-v)}$ and $T^*(u,v) = \frac{u+v}{1+uv}$, and suppose $\tilde{\gamma}_1 = (a_{\tilde{\gamma}_1}, b_{\tilde{\gamma}_1})$ and $\tilde{\gamma}_2 = (a_{\tilde{\gamma}_2}, b_{\tilde{\gamma}_2})$ are two IFNs, $\lambda > 0$, then the ORs of IFNs based on ETAT are described as [43].

(1) $\tilde{\gamma}_1 \oplus \tilde{\gamma}_2 = \left(\frac{a_{\tilde{\gamma}_1} + a_{\tilde{\gamma}_2}}{1 + a_{\tilde{\gamma}_1}a_{\tilde{\gamma}_2}}, \frac{b_{\tilde{\gamma}_1}b_{\tilde{\gamma}_2}}{1 + (1 - b_{\tilde{\gamma}_1})(1 - b_{\tilde{\gamma}_2})}\right)$

(2) $\tilde{\gamma}_1 \otimes \tilde{\gamma}_2 = \left(\frac{a_{\tilde{\gamma}_1}a_{\tilde{\gamma}_2}}{1 + (1 - a_{\tilde{\gamma}_1})(1 - a_{\tilde{\gamma}_2})}, \frac{b_{\tilde{\gamma}_1} + b_{\tilde{\gamma}_2}}{1 + b_{\tilde{\gamma}_1}b_{\tilde{\gamma}_2}}\right)$

(3) $\lambda\tilde{\gamma}_1 = \left(\frac{(1 + a_{\tilde{\gamma}_1})^{\lambda} - (1 - a_{\tilde{\gamma}_1})^{\lambda}}{(1 + a_{\tilde{\gamma}_1})^{\lambda} + (1 - a_{\tilde{\gamma}_1})^{\lambda}}, \frac{2b_{\tilde{\gamma}_1}^{\lambda}}{(2 - b_{\tilde{\gamma}_1})^{\lambda} + b_{\tilde{\gamma}_1}^{\lambda}}\right) \lambda > 0$

(4) $\tilde{\gamma}_1^{\lambda} = \left(\frac{2a_{\tilde{\gamma}_1}^{\lambda}}{(2 - a_{\tilde{\gamma}_1})^{\lambda} + a_{\tilde{\gamma}_1}^{\lambda}}, \frac{(1 + b_{\tilde{\gamma}_1})^{\lambda} - (1 - b_{\tilde{\gamma}_1})^{\lambda}}{(1 + b_{\tilde{\gamma}_1})^{\lambda} + (1 - b_{\tilde{\gamma}_1})^{\lambda}}\right) \lambda > 0$

Theorem 1 *(Wang and Liu [49, 50]). Suppose $\tilde{\gamma} = (a_{\tilde{\gamma}}, b_{\tilde{\gamma}})$, $\tilde{\gamma}_1 = (a_{\tilde{\gamma}_1}, b_{\tilde{\gamma}_1})$ and $\tilde{\gamma}_2 = (a_{\tilde{\gamma}_2}, b_{\tilde{\gamma}_2})$ are any three IFNs, and $n, n_1, n_2 > 0$, then*

(1) $\tilde{\gamma}_1 \oplus \tilde{\gamma}_2 = \tilde{\gamma}_2 \oplus \tilde{\gamma}_1$,
(2) $\tilde{\gamma}_1 \otimes \tilde{\gamma}_2 = \tilde{\gamma}_2 \otimes \tilde{\gamma}_1$,
(3) $n(\tilde{\gamma}_1 \oplus \tilde{\gamma}_2) = n\tilde{\gamma}_1 \oplus n\tilde{\gamma}_2$,
(4) $n_1\tilde{\gamma} \oplus n_2\tilde{\gamma} = (n_1 + n_2)\tilde{\gamma}$,
(5) $\tilde{\gamma}^{n_1} \otimes \tilde{\gamma}^{n_2} = (\tilde{\gamma})^{n_1+n_2}$,
(6) $\tilde{\gamma}_1^n \otimes \tilde{\gamma}_2^n = (\tilde{\gamma}_1 \otimes \tilde{\gamma}_2)^n$.

2.3 The PA operator

The PA [58] can eliminate the effects of some extreme data, and it is defined as below:

Definition 7 [58] Suppose $x_\varepsilon(\varepsilon = 1, 2, \ldots, z)$ is a set of crisp numbers, the PA operator is defined as follows:

$$PA(x_1, x_2, \ldots, x_z) = \frac{\sum_{\varepsilon=1}^{z} (1 + TT(x_\varepsilon))x_\varepsilon}{\sum\limits_{\varepsilon=1}^{z} (1 + TT(x_\varepsilon))} \quad (21)$$

where

$$TT(x_\varepsilon) = \sum_{j=1, j\neq\varepsilon}^{z} \mathrm{Sup}(x_\varepsilon, x_j) \quad (22)$$

and $\mathrm{Sup}(x_\varepsilon, x_j)$ represents the support degree for x_ε from x_j, where

(a) $\mathrm{Sup}(x_\varepsilon, x_j) \in [0, 1]$; (b) $\mathrm{Sup}(x_\varepsilon, x_j) = \mathrm{Sup}(x_j, x_\varepsilon)$; (c) $\mathrm{Sup}(e, f) \geq \mathrm{Sup}(g, h)$, if $|e - f| < |g - h|$.

3 Analysis on the Drawbacks and Improvement of the Existing Einstein AOs for IFNs

In this part, the drawbacks of the existing Einstein AOs of IFNs will be analyzed and overcome. Firstly, we only give an analysis on the IFEWA operator in Wang and Liu [50] because it is representative, and the other AOs in Wang and Liu [49, 50], such as the IFEOWA operator, the *IFEHWA* operator and the dual form of these operators, such as the IFEWG operator, the *IFEOWG* operator, the *IFEHWG* operator, have the same drawbacks.

Definition 8 [50] Suppose $\tilde{\gamma}_\varepsilon = (a_{\tilde{\gamma}_\varepsilon}, b_{\tilde{\gamma}_\varepsilon})(\varepsilon = 1, 2, \ldots, z)$ is a collection of IFNs, and $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_z)^T$ is the weight vector of $\tilde{\gamma}_\varepsilon(\varepsilon = 1, 2, \ldots, z)$, which satisfies $\varpi_\varepsilon \in [0, 1]$ and $\sum\limits_{\varepsilon=1}^{z} \varpi_\varepsilon = 1$, then the intuitionistic fuzzy Einstein weighted averaging (IFEWA) operator is a mapping IFEWA: $M^z \to M$, where M is the set of all IFNs, then

$$\mathrm{IFEWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z) = \varpi_1\tilde{\gamma}_1 \oplus \varpi_2\tilde{\gamma}_2 \oplus \cdots \oplus \varpi_z\tilde{\gamma}_z = \left(\frac{\prod_{\varepsilon=1}^{z} (1 + a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{z} (1 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z} (1 + a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z} (1 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon}}, \frac{2\prod_{\varepsilon=1}^{z} b_{\tilde{\gamma}_\varepsilon}^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z} (2 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z} b_{\tilde{\gamma}_\varepsilon}^{\varpi_\varepsilon}} \right) \quad (23)$$

The drawbacks of IFEWA operator can be expounded by following some examples.

Example 1 Suppose $\tilde{\gamma}_1 = (0.6, 0)$, $\tilde{\gamma}_2 = (0.3, 0.2)$, $\tilde{\gamma}_3 = (0.5, 0.1)$ and $\tilde{\gamma}_4 = (0.3, 0.4)$ are four IFNs, and $\varpi_\varepsilon = (0.3, 0.2, 0.1, 0.4)^T$ is the weight vector of $(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4)$. Then based on Eq. (23), we can get

$$\mathrm{IFEWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4) = \left(\frac{\prod_{\varepsilon=1}^{4} (1 + a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{4} (1 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4} (1 + a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{4} (1 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon}}, \frac{2\prod_{\varepsilon=1}^{4} b_{\tilde{\gamma}_\varepsilon}^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4} (2 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{4} b_{\tilde{\gamma}_\varepsilon}^{\varpi_\varepsilon}} \right) = (0.421, 0).$$

From the aggregated result, we can find that although the N-MDs in $\tilde{\gamma}_2, \tilde{\gamma}_3$ and $\tilde{\gamma}_4$ are not zero, the N-MD of the aggregated result is still zero only because the N-MD of $\tilde{\gamma}_1$ equals to zero. Obviously, this is an unreasonable result. Further analysis shows that if there is only one N-MD of IFNs is zero, and then the N-MD of the aggregated result from z IFNs will be zero even if the N-MDs of the other $z - 1$ IFNs are not zero. This problem also exists for MDs. In short, when MD or N-MD is zero, it controls final computing result, but the other MDs or N-MDs do not affect the result any more. So, in this situation, the ranking results for all alternatives will be incorrect.

In addition, from Eq. (23), we know that the MD and the N-MD of aggregated result are obtained independently from the MDs and the N-MDs of z IFNs, respectively. It is confirmed in Example 2.

Example 2 Suppose $\tilde{\gamma}_1 = (0.4, 0.2)$, $\tilde{\gamma}_2 = (0.3, 0.2)$, $\tilde{\gamma}_3 = (0.4, 0.1)$ and $\tilde{\gamma}_4 = (0.2, 0.3)$ are four IFNs, and $\varpi = (0.3, 0.2, 0.1, 0.4)^T$ is the weight vector of $(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4)$. Then based on Eq. (23), we can get

$$\mathrm{IFEWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4) = \left(\frac{\prod_{\varepsilon=1}^{4} (1 + a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{4} (1 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4} (1 + a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{4} (1 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon}}, \frac{2\prod_{\varepsilon=1}^{4} b_{\tilde{\gamma}_\varepsilon}^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4} (2 - a_{\tilde{\gamma}_\varepsilon})^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{4} b_{\tilde{\gamma}_\varepsilon}^{\varpi_\varepsilon}} \right) = (0.303, 0.221).$$

When $\tilde{\gamma}_3$ and $\tilde{\gamma}_4$ are changed to $\tilde{\gamma}_3 = (0.5, 0.1)$ and $\tilde{\gamma}_4 = (0.4, 0.3)$, then the aggregated result is

$$\begin{aligned}&\mathrm{IFEWA}(\tilde{\gamma}_1,\tilde{\gamma}_2,\tilde{\gamma}_3,\tilde{\gamma}_4)\\&=\left(\frac{\prod_{\varepsilon=1}^{4}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{4}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{4}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}},\frac{2\prod_{\varepsilon=1}^{4}b_{\tilde{\gamma}_\varepsilon}^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4}\left(2-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{4}b_{\tilde{\gamma}_\varepsilon}^{\varpi_\varepsilon}}\right)\\&=(0.391,0.221).\end{aligned}$$

Because $\tilde{\gamma}_3$ and $\tilde{\gamma}_4$ are only changed in MD, the aggregated result of MD is changed, however, the aggregated N-MD is unchanged, that is, the aggregated result cannot reflect the interaction between MD and N-MD of different IFNs. Obviously, MD and N-MD are fully independent in calculating process, this is not consistent with the actual decision situation. So, we cannot get the reasonable aggregated result.

Based on above analysis, we can find the existing operators, such as the IFEWA operator, the IFEOWA operator, the IFEHA operator, the IFEWG operator, the IFEOWG operator and the IFEHWG operator, are invalid to rank the alternatives in some situations, and they need to be improved to overcome these weaknesses.

To solve the above problems, He and He [17] proposed the following interactive ORs for IFNs based on the TAT.

$$\begin{aligned}\tilde{\gamma}_1\otimes\tilde{\gamma}_2=&\left(T^*(a_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_1},a_{\tilde{\gamma}_2}+b_{\tilde{\gamma}_2})-T^*(b_{\tilde{\gamma}_1},b_{\tilde{\gamma}_2}),T^*(b_{\tilde{\gamma}_1},b_{\tilde{\gamma}_2})\right)\\=&\left(\psi^{-1}(\psi(a_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_1})+\psi(a_{\tilde{\gamma}_2}+b_{\tilde{\gamma}_2}))\right.\\&\left.-\psi^{-1}(\psi(b_{\tilde{\gamma}_1})+\psi(b_{\tilde{\gamma}_2})),\psi^{-1}(\psi(b_{\tilde{\gamma}_1})+\psi(b_{\tilde{\gamma}_2}))\right)\end{aligned}\tag{24}$$

$$\begin{aligned}\tilde{\gamma}_1\oplus\tilde{\gamma}_2=&\left(T^*(a_{\tilde{\gamma}_1},a_{\tilde{\gamma}_2}),T^*(a_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_1},a_{\tilde{\gamma}_2}+b_{\tilde{\gamma}_2})-T^*(a_{\tilde{\gamma}_1},a_{\tilde{\gamma}_2})\right)\\=&\left(\psi^{-1}(\psi(a_{\tilde{\gamma}_1})+\psi(a_{\tilde{\gamma}_2})),\psi^{-1}(\psi(a_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_1})\right.\\&\left.+\psi(a_{\tilde{\gamma}_2}+b_{\tilde{\gamma}_2}))-\psi^{-1}(\psi(a_{\tilde{\gamma}_1})+\psi(a_{\tilde{\gamma}_2}))\right)\end{aligned}\tag{25}$$

Then according the ETAT, we can define some improved ORs for IFNs based on the Einstein operations.

Suppose $\tilde{\gamma}=\left(a_{\tilde{\gamma}},b_{\tilde{\gamma}}\right)$, $\tilde{\gamma}_1=\left(a_{\tilde{\gamma}_1},b_{\tilde{\gamma}_1}\right)$ and $\tilde{\gamma}_2=\left(a_{\tilde{\gamma}_2},b_{\tilde{\gamma}_2}\right)$ are any three IFNs, then the new interactive ORs based on Einstein operations are proposed as follows:

(1) $\tilde{\gamma}_1\oplus_E\tilde{\gamma}_2=\left(\frac{a_{\tilde{\gamma}_1}+a_{\tilde{\gamma}_2}}{1+a_{\tilde{\gamma}_1}a_{\tilde{\gamma}_2}},\frac{\left(a_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_1}\right)+\left(a_{\tilde{\gamma}_2}+b_{\tilde{\gamma}_2}\right)}{1+\left(a_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_1}\right)\left(a_{\tilde{\gamma}_2}+b_{\tilde{\gamma}_2}\right)}-\frac{a_{\tilde{\gamma}_1}+a_{\tilde{\gamma}_2}}{1+a_{\tilde{\gamma}_1}a_{\tilde{\gamma}_2}}\right)$,

(2) $\tilde{\gamma}_1\otimes_E\tilde{\gamma}_2=\left(\frac{\left(a_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_1}\right)+\left(a_{\tilde{\gamma}_2}+b_{\tilde{\gamma}_2}\right)}{1+\left(a_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_1}\right)\left(a_{\tilde{\gamma}_2}+b_{\tilde{\gamma}_2}\right)}-\frac{b_{\tilde{\gamma}_{\tilde{\gamma}_1}}+b_{\tilde{\gamma}_2}}{1+b_{\tilde{\gamma}_1}b_{\tilde{\gamma}_2}},\frac{b_{\tilde{\gamma}_1}+b_{\tilde{\gamma}_2}}{1+b_{\tilde{\gamma}_1}b_{\tilde{\gamma}_2}}\right)$,

(3) $\lambda\cdot_E\tilde{\gamma}=\left(\frac{\left(1+a_{\tilde{\gamma}}\right)^\lambda-\left(1-a_{\tilde{\gamma}}\right)^\lambda}{\left(1+a_{\tilde{\gamma}}\right)^\lambda+\left(1-a_{\tilde{\gamma}}\right)^\lambda},\frac{2\left(1-a_{\tilde{\gamma}}\right)^\lambda}{\left(1+a_{\tilde{\gamma}}\right)^\lambda+\left(1-a_{\tilde{\gamma}}\right)^\lambda}-\frac{2\left(1-\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^\lambda}{\left(1+\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^\lambda+\left(1-\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^\lambda}\right)$,

(4)

$$\tilde{\gamma}^{\wedge_E\lambda}=\left(\frac{2\left(1-b_{\tilde{\gamma}}\right)^\lambda}{\left(1+b_{\tilde{\gamma}}\right)^\lambda+\left(1-b_{\tilde{\gamma}}\right)^\lambda}-\frac{2\left(1-\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^\lambda}{\left(1+\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^\lambda+\left(1-\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^\lambda},\frac{\left(1+b_{\tilde{\gamma}}\right)^\lambda-\left(1-b_{\tilde{\gamma}}\right)^\lambda}{\left(1+b_{\tilde{\gamma}}\right)^\lambda+\left(1-b_{\tilde{\gamma}}\right)^\lambda}\right)\tag{29}$$

Then we can prove the ORs have the following properties.

Theorem 2 *Suppose* $\tilde{\gamma}_x=\left(a_{\tilde{\gamma}_x},b_{\tilde{\gamma}_x}\right)$ *and* $\tilde{\gamma}_z=\left(a_{\tilde{\gamma}_z},b_{\tilde{\gamma}_z}\right)$ *are any two IFNs, and* $n,n_1,n_2>0$, *then*

(1) $\tilde{\gamma}_x\oplus_E\tilde{\gamma}_z=\tilde{\gamma}_z\oplus_E\tilde{\gamma}_x$,
(2) $n\cdot_E\left(\tilde{\gamma}_x\oplus_E\tilde{\gamma}_z\right)=n\cdot_E\tilde{\gamma}_x\oplus_E n\cdot_E\tilde{\gamma}_z$,
(3) $n_1\cdot_E\tilde{\gamma}_z\oplus n_2\cdot_E\tilde{\gamma}_z=\left(n_1+n_2\right)\cdot_E\tilde{\gamma}_z$,
(4) $\tilde{\gamma}_x\otimes_E\tilde{\gamma}_z=\tilde{\gamma}_z\otimes_E\tilde{\gamma}_x$,
(5)

$$\left(\tilde{\gamma}_x\otimes_E\tilde{\gamma}_z\right)^{\wedge_E n}=\tilde{\gamma}_x^{\wedge_E n}\otimes_E\tilde{\gamma}_z^{\wedge_E n}\tag{34}$$

(6)

$$\tilde{\gamma}_z^{\wedge_E n_1}\otimes\tilde{\gamma}_z^{\wedge_E n_2}=\tilde{\gamma}_z^{\wedge_E(n_1+n_2)}\tag{35}$$

.

Example 3 Suppose According to the mathematical$\tilde{\gamma}_1=(0.3,0)$ and $\tilde{\gamma}_2=(0.4,0.3)$ are two IFNs, and $\mu=2$, then we can calculate

(1) $\tilde{\gamma}_1\oplus_E\tilde{\gamma}_2=\left(\frac{0.3+0.4}{1+0.3\times0.4},\frac{(0.3+0)+(0.4+0.3)}{1+(0.3+0)\times(0.4+0.3)}-\frac{0.3+0.4}{1+0.3\times0.4}\right)=(0.625,0.201)$,

(2) $\tilde{\gamma}_1\otimes_E\tilde{\gamma}_2=\left(\frac{(0.3+0)+(0.4+0.3)}{1+(0.3+0)(0.4+0.3)}-\frac{0+0.3}{1+0\times0.3},\frac{0+0.3}{1+0\times0.3}\right)=(0.526,0.300)$,

(3) $\mu\cdot_E\tilde{\gamma}_1=\left(\frac{(1+0.3)^2-(1-0.3)^2}{(1+0.3)^2+(1-0.3)^2},\frac{2\times(1-0.3)^2}{(1+0.3)^2+(1-0.3)^2}-\frac{2\times(1-0.3)^2}{(1+0.3)^2+(1-0.3)^2}\right)=(0.550,0)$,

$$\begin{aligned}\mu\cdot_E\tilde{\gamma}_2=&\left(\frac{(1+0.4)^2-(1-0.4)^2}{(1+0.4)^2+(1-0.4)^2},\frac{2\times(1-0.4)^2}{(1+0.4)^2+(1-0.4)^2}\right.\\&\left.-\frac{2\times(1-0.7)^2}{(1+0.7)^2+(1-0.7)^2}\right)=(0.690,0.250),\end{aligned}$$

(4)

$$\tilde{\gamma}_1^{\wedge_E\mu}=\left(\frac{2\times(1-0)^2}{(1+0)^2+(1-0)^2}-\frac{2\times(1-0.3)^2}{(1+0.3)^2+(1-0.3)^2},\frac{(1+0)^2-(1-0)^2}{(1+0)^2+(1-0)^2}\right)=(0.550,0)$$

$$\tilde{\gamma}_2^{\wedge_E\mu}=\left(\frac{2\times(1-0.3)^2}{(1+0.3)^2+(1-0.3)^2}-\frac{2\times(1-0.7)^2}{(1+0.7)^2+(1-0.7)^2},\frac{(1+0.3)^2-(1-0.3)^2}{(1+0.3)^2+(1-0.3)^2}\right)=(0.389,0.550)$$

From Example 3, we can know that although the N-MD of $\tilde{\gamma}_1$ is zero, but the N-MD of $\tilde{\gamma}_2$ is not equal to zero, so the N-MDs of $\tilde{\gamma}_1\oplus\tilde{\gamma}_2$ and $\tilde{\gamma}_1\otimes\tilde{\gamma}_2$ are also not equal to zero. Obviously, these results are reasonable.

In the following, this paper will propose the IFEIWA operator based on the Einstein interactive ORs of IFNs.

Definition 9 Suppose $\tilde{\gamma}_\varepsilon=\left(a_{\tilde{\gamma}_\varepsilon},b_{\tilde{\gamma}_\varepsilon}\right)$ $(\varepsilon=1,2\ldots,z)$ is a collection of the IFNs, and IFEIWA : $M^z\to M$, if

$$\mathrm{IFEIWA}(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_z)=\mathop{\oplus_E}_{\varepsilon=1}^{z}\varpi_\varepsilon\cdot_E\tilde{\gamma}_\varepsilon\tag{36}$$

where M stands for the collection of all IFNs, the weight vector of $(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z)$ is $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_z)^T$, and $0 \le \varpi_\varepsilon \le 1$, $\sum_{\varepsilon=1}^{z} \varpi_\varepsilon = 1$. Then the IFEIWA is the intuitionistic fuzzy Einstein interactive weighted averaging operator.

According to the Definition 9, we can get the following Theorem 3 about the aggregated result.

Theorem 3 *Suppose* $\tilde{\gamma}_\varepsilon = \left(a_{\tilde{\gamma}_\varepsilon}, b_{\tilde{\gamma}_\varepsilon}\right)$ $(\varepsilon = 1, 2\ldots, z)$ *is a collection of the IFNs, the aggregated result by Eq. (36) based on improved Einstein ORs of IFNs is still an IFN, and even*

$$\begin{aligned}
&\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \cdots, \tilde{\gamma}_z) \\
&= \left(\frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}, \right. \\
&\quad \frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \\
&\quad \left. - \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\right)
\end{aligned} \tag{37}$$

Proof We can give the following proof by the mathematical induction.

(1) When $z = 1$, obviously, Eq. (37) is kept.

(2) Suppose $z = \partial$ (37) is kept, i.e.,

$$\begin{aligned}
&\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_\partial) \\
&= \left(\frac{\prod_{\varepsilon=1}^{\partial}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{\partial}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{\partial}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{\partial}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}, \right. \\
&\quad \frac{2\prod_{\varepsilon=1}^{\partial}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{\partial}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{\partial}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \\
&\quad \left. - \frac{2\prod_{\varepsilon=1}^{\partial}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{\partial}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{\partial}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\right)
\end{aligned}$$

then when $z = \partial + 1$, we can get

$$\begin{aligned}
&\varpi_{\partial+1} \cdot_E \tilde{\gamma}_{\partial+1} \\
&= \left(\frac{\left(1+a_{\gamma_{\partial+1}}\right)^{\varpi_{\partial+1}} - \left(1-a_{\gamma_{\partial+1}}\right)^{\varpi_{\partial+1}}}{\left(1+a_{\gamma_{\partial+1}}\right)^{\varpi_{\partial+1}} + \left(1-a_{\gamma_{\partial+1}}\right)^{\varpi_{\partial+1}}}, \frac{2\left(1-a_{\gamma_{\partial+1}}\right)^{\varpi_{\partial+1}}}{\left(1+a_{\gamma_{\partial+1}}\right)^{\varpi_{\partial+1}} + \left(1-a_{\gamma_{\partial+1}}\right)^{\varpi_{\partial+1}}} \right. \\
&\quad \left. - \frac{2\left(1-\left(a_{\gamma_{\partial+1}}+b_{\gamma_{\partial+1}}\right)\right)^{\varpi_{\partial+1}}}{\left(1+\left(a_{\gamma_{\partial+1}}+b_{\gamma_{\partial+1}}\right)\right)^{\varpi_{\partial+1}} + \left(1-\left(a_{\gamma_{\partial+1}}+b_{\gamma_{\partial+1}}\right)\right)^{\varpi_{\partial+1}}}\right)
\end{aligned}$$

where, $A_\varepsilon = \left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}$, $B_\varepsilon = \left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}$, $C_\varepsilon = \left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}$, $D_\varepsilon = \left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}$ and $\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_\partial, \tilde{\gamma}_{\partial+1}) = \mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_\partial) \oplus_E \varpi_{\partial+1} \cdot_E \tilde{\gamma}_{\partial+1}$

$$\begin{aligned}
&= \left(\frac{\frac{\prod_{\varepsilon=1}^{\partial}A_\varepsilon - \prod_{\varepsilon=1}^{\partial}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial}A_\varepsilon + \prod_{\varepsilon=1}^{\partial}B_\varepsilon} + \frac{A_{\partial+1}-B_{\partial+1}}{A_{\partial+1}+B_{\partial+1}}}{1 + \left(\frac{\prod_{\varepsilon=1}^{\partial}A_\varepsilon - \prod_{\varepsilon=1}^{\partial}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial}A_\varepsilon + \prod_{\varepsilon=1}^{\partial}B_\varepsilon}\right)\left(\frac{A_{\partial+1}-B_{\partial+1}}{A_{\partial+1}+B_{\partial+1}}\right)}, \right. \\
&\quad \frac{\frac{\prod_{\varepsilon=1}^{\partial}A_\varepsilon - \prod_{\varepsilon=1}^{\partial}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial}A_\varepsilon + \prod_{\varepsilon=1}^{\partial}B_\varepsilon} + \frac{2\prod_{\varepsilon=1}^{\partial}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial}A_\varepsilon + \prod_{\varepsilon=1}^{\partial}B_\varepsilon} - \frac{2\prod_{\varepsilon=1}^{\partial}D_\varepsilon}{\prod_{\varepsilon=1}^{\partial}C_\varepsilon + \prod_{\varepsilon=1}^{\partial}D_\varepsilon} + \frac{A_{\partial+1}-B_{\partial+1}}{A_{\partial+1}+B_{\partial+1}} + \frac{2B_{\partial+1}}{A_{\partial+1}+B_{\partial+1}} - \frac{2D_{\partial+1}}{C_{\partial+1}+D_{\partial+1}}}{1 + \left(\frac{\prod_{\varepsilon=1}^{\partial}A_\varepsilon - \prod_{\varepsilon=1}^{\partial}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial}A_\varepsilon + \prod_{\varepsilon=1}^{\partial}B_\varepsilon} + \frac{2\prod_{\varepsilon=1}^{\partial}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial}A_\varepsilon + \prod_{\varepsilon=1}^{\partial}B_\varepsilon} - \frac{2\prod_{\varepsilon=1}^{\partial}D_\varepsilon}{\prod_{\varepsilon=1}^{\partial}C_\varepsilon + \prod_{\varepsilon=1}^{\partial}D_\varepsilon}\right)\left(\frac{A_{\partial+1}-B_{\partial+1}}{A_{\partial+1}+B_{\partial+1}} + \frac{2B_{\partial+1}}{A_{\partial+1}+B_{\partial+1}} - \frac{2D_{\partial+1}}{C_{\partial+1}+D_{\partial+1}}\right)} - \\
&\quad \left. \frac{\frac{\prod_{\varepsilon=1}^{\partial}A_\varepsilon - \prod_{\varepsilon=1}^{\partial}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial}A_\varepsilon + \prod_{\varepsilon=1}^{\partial}B_\varepsilon} + \frac{A_{\partial+1}-B_{\partial+1}}{A_{\partial+1}+B_{\partial+1}}}{1 + \left(\frac{\prod_{\varepsilon=1}^{\partial}A_\varepsilon - \prod_{\varepsilon=1}^{\partial}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial}A_\varepsilon + \prod_{\varepsilon=1}^{\partial}B_\varepsilon}\right)\left(\frac{A_{\partial+1}-B_{\partial+1}}{A_{\partial+1}+B_{\partial+1}}\right)}\right)
\end{aligned}$$

$$= \left(\frac{\prod_{\varepsilon=1}^{\partial+1}A_\varepsilon - \prod_{\varepsilon=1}^{\partial+1}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial+1}A_\varepsilon + \prod_{\varepsilon=1}^{\partial+1}B_\varepsilon}, \frac{2\prod_{\varepsilon=1}^{\partial+1}B_\varepsilon}{\prod_{\varepsilon=1}^{\partial+1}A_\varepsilon + \prod_{\varepsilon=1}^{\partial+1}B_\varepsilon} - \frac{2\prod_{\varepsilon=1}^{\partial+1}D_\varepsilon}{\prod_{\varepsilon=1}^{\partial+1}C_\varepsilon + \prod_{\varepsilon=1}^{\partial+1}D_\varepsilon}\right)$$

$$\begin{aligned}
&= \left(\frac{\prod_{\varepsilon=1}^{\partial+1}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{\partial+1}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{\partial+1}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{\partial+1}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}, \frac{2\prod_{\varepsilon=1}^{\partial+1}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{\partial+1}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{\partial+1}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \right. \\
&\quad \left. - \frac{2\prod_{\varepsilon=1}^{\partial+1}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{\partial+1}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{\partial+1}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\right)
\end{aligned}$$

So, when $l = \partial + 1$, Eq. (37) is kept.

(3) According to the mathematical induction and above two steps, we can obtain Eq. (37) holds for any ε. Then, we will prove that $\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z)$ is also an IFN.

Let $a = \frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}$ and

$b = \frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} - \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}$

Because $0 \le a_{\tilde{\gamma}_\varepsilon}, b_{\tilde{\gamma}_\varepsilon} \le 1$ and $a_{\tilde{\gamma}_\varepsilon} + b_{\tilde{\gamma}_\varepsilon} \le 1$,we have $1 - a_{\tilde{\gamma}_\varepsilon} \le 1 + a_{\tilde{\gamma}_\varepsilon}$, $1 - \left(a_{\tilde{\gamma}_\varepsilon} + b_{\tilde{\gamma}_\varepsilon}\right) \le 1 + \left(a_{\tilde{\gamma}_\varepsilon} + b_{\tilde{\gamma}_\varepsilon}\right)$. Further, $\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} \le \prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}$,and $\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} \le \prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}$.

So, $0 \le a = \frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \le 1$.
and that,

$$\begin{aligned}
&0 \le a + b \\
&= \frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \\
&\quad + \frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\tau=1}^{l}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\tau} + \prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \\
&\quad - \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}
\end{aligned}$$

$$= \frac{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} - \prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}} \le 1,$$

so,

$$0 \le b = \frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} - \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}} \le 1.$$

So the aggregated result of $\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z)$ can meet the following two conditions.

(i) $0 \le a \le 1, 0 \le b \le 1$;
(ii) $0 \le a + b \le 1$.

Thus, it is also an IFN and the proof of theorem 3 is completed.

Example 4 We can re-calculate Example 1 and Example 2 shown as follows.

For Example 1, we have

$$\begin{aligned}&\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4)\\ &= \left(\frac{\prod_{\varepsilon=1}^{4}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{4}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{4}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}},\right.\\ &\quad \frac{2\prod_{\varepsilon=1}^{4}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{4}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}\\ &\quad \left.-\frac{2\prod_{\varepsilon=1}^{4}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{4}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{4}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\right)\\ &= (0.421, 0.205).\end{aligned}$$

By the IFEIWA operator, we get the N-MD of the aggregated result from four IFNs is 0.205 and not 0, obviously, this is more reasonable.

For Example 2, we have

$\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4) = (0.303, 0.229)$.

When $\tilde{\gamma}_3$ and $\tilde{\gamma}_4$ change to $\tilde{\gamma}_3 = (0.5, 0.1)$ and $\tilde{\gamma}_4 = (0.4, 0.3)$, then the aggregated result is

$\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4) = (0.391, 0.234)$.

Obviously, with the change of MD of some integrated data, not only the MD of the aggregated result has changed, but also the N-MD of the aggregated result has changed simultaneously, this result is relatively more reasonable.

In the following, based on Theorem 3, we analyze the properties of the IFEIWA operator.

Theorem 4 *(Idempotency) Suppose* $\tilde{\gamma}_\varepsilon = \left(a_{\tilde{\gamma}_\varepsilon}, b_{\tilde{\gamma}_\varepsilon}\right)$ $(\varepsilon = 1, 2\ldots, z)$ *is a collection of the IFNs, if* $\tilde{\gamma}_\varepsilon = \tilde{\gamma} = \left(a_{\tilde{\gamma}}, b_{\tilde{\gamma}}\right), \varepsilon = 1, 2, \ldots, z$, *then*

$$\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z) = \tilde{\gamma} \tag{38}$$

Proof

$$\begin{aligned}&\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z)\\ &= \left(\frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}},\right.\\ &\quad \frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}\\ &\quad \left.-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\right)\\ &= \left(\frac{\left(1+a_{\tilde{\gamma}}\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}-\left(1-a_{\tilde{\gamma}}\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+a_{\tilde{\gamma}}\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-a_{\tilde{\gamma}}\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}},\right.\\ &\quad \frac{2\left(1-a_{\tilde{\gamma}}\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+a_{\tilde{\gamma}}\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-a_{\tilde{\gamma}}\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}\\ &\quad \left.-\frac{2\left(1-\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-\left(a_{\tilde{\gamma}}+b_{\tilde{\gamma}}\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}\right)\\ &= \left(a_{\tilde{\gamma}}, b_{\tilde{\gamma}}\right) = \tilde{\gamma}.\end{aligned}$$

Theorem 5 *(Monotonicity) Suppose* $\tilde{\gamma}_\varepsilon = \left(a_{\tilde{\gamma}_\varepsilon}, b_{\tilde{\gamma}_\varepsilon}\right)$ *and* $\tilde{\gamma}'_\varepsilon = \left(a'_{\tilde{\gamma}_\varepsilon}, b'_{\tilde{\gamma}_\varepsilon}\right)(\varepsilon = 1, 2, \ldots, z)$ *are two sets of IFNs. If* $a_{\tilde{\gamma}_\varepsilon} \ge a'_{\tilde{\gamma}_\varepsilon}, a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon} \le a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}$ *for all* ε, *then* $\mathrm{IFEIWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z) \ge \mathrm{IFEIWA}\left(\tilde{\gamma}'_1, \tilde{\gamma}'_2, \ldots, \tilde{\gamma}'_z\right)$.

Proof

(1) Let $g(\zeta) = \frac{1-\zeta}{1+\zeta}, \zeta \in [0, 1]$, then $g'(\xi) = \frac{-2}{(1+\xi)^2} < 0$; so $g(\zeta)$ is a decreasing function. Because $a_{\tilde{\gamma}_\varepsilon} \ge a'_{\tilde{\gamma}_\varepsilon}$, for all ε, then $\frac{1-a_{\tilde{\gamma}_\varepsilon}}{1+a_{\tilde{\gamma}_\varepsilon}} \le \frac{1-a'_{\tilde{\gamma}_\varepsilon}}{1+a'_{\tilde{\gamma}_\varepsilon}}$. Because the weight vector of $\tilde{\gamma}_\varepsilon$ is $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_z)^T$, and $0 \le \varpi_\varepsilon \le 1$, $\sum_{\varepsilon=1}^{z}\varpi_\varepsilon = 1$, then have $\left(\frac{1-a_{\tilde{\gamma}_\varepsilon}}{1+a_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon} \le \left(\frac{1-a'_{\tilde{\gamma}_\varepsilon}}{1+a'_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}$. Thus

$$\prod_{\varepsilon=1}^{z}\left(\frac{1-a_{\tilde{\gamma}_\varepsilon}}{1+a_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon} \le \prod_{\varepsilon=1}^{z}\left(\frac{1-a'_{\tilde{\gamma}_\varepsilon}}{1+a'_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}$$

$$\Rightarrow \frac{2}{1+\prod_{\varepsilon=1}^{z}\left(\frac{1-a_{\tilde{\gamma}_\varepsilon}}{1+a_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}} - 1 \ge \frac{2}{1+\prod_{\varepsilon=1}^{z}\left(\frac{1-a'_{\tilde{\gamma}_\varepsilon}}{1+a'_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}} - 1$$

$$\Rightarrow \frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \geq \frac{\prod_{\varepsilon=1}^{z}\left(1+a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}$$

.

(2) Let $h(\zeta)=\frac{1+\zeta}{1-\zeta}, \zeta\in[0,1]$, then $h'(\zeta)=\frac{2}{(1-\zeta)^2}>0$; thus, $h(\zeta)$ is an increasing function. Because $a_{\tilde{\gamma}_\varepsilon}\geq a'_{\tilde{\gamma}_\varepsilon}$, for all ε, then $\frac{1+a_{\tilde{\gamma}_\varepsilon}}{1-a_{\tilde{\gamma}_\varepsilon}}\geq\frac{1+a'_{\tilde{\gamma}_\varepsilon}}{1-a'_{\tilde{\gamma}_\varepsilon}}$. Suppose $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_z)^T$ is the weight vector of $\tilde{\gamma}_\varepsilon$ and $0\leq\varpi_\varepsilon\leq1$, $\sum_{\varepsilon=1}^{z}\varpi_\varepsilon=1$, then we have $\left(\frac{1+a_{\tilde{\gamma}_\varepsilon}}{1-a_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}\geq\left(\frac{1+a'_{\tilde{\gamma}_\varepsilon}}{1-a'_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}$. Thus

$$\prod_{\varepsilon=1}^{z}\left(\frac{1+a_{\tilde{\gamma}_\varepsilon}}{1-a_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}\geq\prod_{\varepsilon=1}^{z}\left(\frac{1+a'_{\tilde{\gamma}_\varepsilon}}{1-a'_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}$$

$$\Rightarrow \frac{1}{1+\prod_{\varepsilon=1}^{z}\left(\frac{1+a_{\tilde{\gamma}_\varepsilon}}{1-a_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}}\leq\frac{1}{1+\prod_{\varepsilon=1}^{z}\left(\frac{1+a'_{\tilde{\gamma}_\varepsilon}}{1-a'_{\tilde{\gamma}_\varepsilon}}\right)^{\varpi_\varepsilon}}$$

$$\Rightarrow \frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}\leq\frac{2\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}$$

In a similar way, we can get

$$\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}} \geq \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}$$

Further, we have

$$\frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} - \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}} \leq \frac{2\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} - \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}$$

So, according to (1) (2), we can get

$$\frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} - \left(\frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} - \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\right)$$

$$\geq \frac{\prod_{\varepsilon=1}^{z}\left(1+a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} - \left(\frac{2\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a'_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} - \frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a'_{\tilde{\gamma}_\varepsilon}+b'_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\right)$$

Therefore, by Definition 4, we can get $IFEIWA(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_z)\geq IFEIWA(\tilde{\gamma}'_1,\tilde{\gamma}'_2,\ldots,\tilde{\gamma}'_z)$.

Theorem 6 *(Boundedness) Suppose* $\tilde{\gamma}_\varepsilon=\left(a_{\tilde{\gamma}_\varepsilon},b_{\tilde{\gamma}_\varepsilon}\right)$ $(\varepsilon=,1,2,\ldots z)$ *is a collections of IFNs, and* $\tilde{\gamma}^-=\left(\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon}),\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})-\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)$, $\tilde{\gamma}^+=\left(\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon}),\max\left(0,\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})-\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)\right)$,

Then we have $\tilde{\gamma}^-\leq IFEIWA(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_l)\leq\tilde{\gamma}^+$.

Proof Based on Theorem 5, we have

(1) For the MD of IFEIWA$(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_z)$, we get

$$\frac{\prod_{\varepsilon=1}^{z}\left(1+\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}} \leq \frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \leq \frac{\prod_{\varepsilon=1}^{z}\left(1+\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}}$$

$$\Rightarrow \frac{\left(1+\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}-\left(1-\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-\min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}} \leq \frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \leq \frac{\left(1+\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}-\left(1-\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-\max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}$$

$$\Rightarrow \min_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon}) \leq \frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}} \leq \max_{1\leq\varepsilon\leq z}(a_{\tilde{\gamma}_\varepsilon})$$

(2) For the N-MD of IFEIWA$(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_z)$, we get

$$\begin{aligned}
&\frac{2\prod_{\varepsilon=1}^{z}\left(1-\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}}\\
&-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\varpi_\varepsilon}}\\
&\le\frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}\\
&-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\\
&\le\frac{2\prod_{\varepsilon=1}^{z}\left(1-\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\varpi_\varepsilon}}\\
&-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\varpi_\varepsilon}}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\frac{2\left(1-\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}\\
&-\frac{2\left(1-\left(\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+\left(\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-\left(\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}\\
&\le\frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}\\
&-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\\
&\le\frac{2\left(1-\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}\\
&-\frac{2\left(1-\left(\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}{\left(1+\left(\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}+\left(1-\left(\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})\right)\right)^{\sum_{\varepsilon=1}^{z}\varpi_\varepsilon}}\\
&\Rightarrow\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})-\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})
\end{aligned}$$

$$\begin{aligned}
&\le\frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}\\
&-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\\
&\le\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})-\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})
\end{aligned}$$

Because $\mathrm{IFEIWA}(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_z)$ is an IFN, we have

$$\frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\ge 0,$$

then,

$$\begin{aligned}
&\max\left(0,\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})-\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon})\right)\\
&\le\frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\varpi_\varepsilon}}\\
&-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varpi_\varepsilon}}\\
&\le\max_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon})-\min_{1\le\varepsilon\le z}(a_{\tilde{\gamma}_\varepsilon}).
\end{aligned}$$

According to steps (1), (2) and Definition 4, we have $\tilde{\gamma}^-\le\mathrm{IFEIWA}(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_z)\le\tilde{\gamma}^+$.

4 Intuitionistic fuzzy Einstein interactive power average operators

In the above section, we have proposed the IFEIWA operator to overcome the shortcomings of some existing AOs for IFNs. However, the IFEIWA operator only has the ability of aggregating n IFNs to one IFN. Considering PA operator has the advantages of relieving the influences of some unreasonable data coming from the biased DMs, it is necessary to apply PA to deal with IFNs by the Einstein interactive operational laws. So, we will present following some new AOs for IFNs.

4.1 The intuitionistic fuzzy Einstein interactive power averaging (IFEIPA) operator

Definition 10 Suppose $\tilde{\gamma}_\varepsilon=(a_{\tilde{\gamma}_\varepsilon},b_{\tilde{\gamma}_\varepsilon})(\varepsilon=1,2,\ldots,z)$ is a collection of IFNs, and $\mathrm{IFEIPA}:M^z\to M$, if

$$\mathrm{IFEIPA}(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_z)=\frac{\oplus_E{}_{\varepsilon=1}^{z}((1+TT(\tilde{\gamma}_\varepsilon))\cdot_E\tilde{\gamma}_\varepsilon)}{\sum_{\varepsilon=1}^{z}(1+TT(\tilde{\gamma}_\varepsilon))}\tag{39}$$

where M is the set of all IFNs, and

$$TT(\tilde{\gamma}_\varepsilon)=\sum_{j=1,j\ne\varepsilon}^{z}Sup(\tilde{\gamma}_\varepsilon,\tilde{\gamma}_j)\tag{40}$$

expresses the support degree of the ε th IFN from all the other IFNs, $\mathrm{Sup}(\tilde{\gamma}_\varepsilon,\tilde{\gamma}_j)$ is the support degree for $\tilde{\gamma}_\varepsilon$ from $\tilde{\gamma}_j$. Then the IFEIPA stands for the intuitionistic fuzzy Einstein interactive power averaging operator.

Theorem 7 *Suppose $\tilde{\gamma}_\varepsilon=(a_{\tilde{\gamma}_\varepsilon},b_{\tilde{\gamma}_\varepsilon})(\varepsilon=1,2,\ldots,z)$ is a collection of IFNs, the aggregation result of the IFEIPA operator from Eq. (39) is still an IFN, and has*

$$\begin{aligned}
&\mathrm{IFEIPA}(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_z)\\
&=\left(\frac{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\omega_\varepsilon}-\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\omega_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\omega_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\omega_\varepsilon}},\right.\\
&\frac{2\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\omega_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+a_{\tilde{\gamma}_\varepsilon}\right)^{\omega_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-a_{\tilde{\gamma}_\varepsilon}\right)^{\omega_\varepsilon}}\\
&\left.-\frac{2\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\omega_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1+\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\omega_\varepsilon}+\prod_{\varepsilon=1}^{z}\left(1-\left(a_{\tilde{\gamma}_\varepsilon}+b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\omega_\varepsilon}}\right)
\end{aligned}\tag{41}$$

Proof Let $\omega_\varepsilon=\frac{(1+TT(\tilde{\gamma}_\varepsilon))}{\sum_{\varepsilon=1}^{z}(1+TT(\tilde{\gamma}_\varepsilon))}$, and use it to replace ϖ_τ of Eq. (37), we can get Eq. (41).

Obviously, the IFEIWA operator includes the IFEIPA operator.

Then, we analyze the properties of the IFEIPA operator.

Theorem 8 (Idempotency) *Suppose* $\tilde{\gamma}_\varepsilon = \tilde{\gamma} = (a_{\tilde{\gamma}}, b_{\tilde{\gamma}})$ *for all* ε, *then* $\text{IFEIPA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z) = \tilde{\gamma}$.

Proof It is the same as Theorem 4 (omitted here).

Theorem 9 (Boundedness) *Suppose* $\tilde{\gamma}_\varepsilon = \left(a_{\tilde{\gamma}_\varepsilon}, b_{\tilde{\gamma}_\varepsilon}\right)$ $(\varepsilon = 1, 2, \ldots, z)$ *is a collections of IFNs, and* $\tilde{\gamma}^- = \left(\min_{1 \le \varepsilon \le z}(a_{\tilde{\gamma}_\varepsilon}), \max_{1 \le \varepsilon \le z}(a_{\tilde{\gamma}_\varepsilon} + b_{\tilde{\gamma}_\varepsilon}) - \min_{1 \le \varepsilon \le z}(a_{\tilde{\gamma}_\varepsilon})\right)$, $\tilde{\gamma}^+ = \left(\max_{1 \le \varepsilon \le z}(a_{\tilde{\gamma}_\varepsilon}), \max\left(0, \min_{1 \le \varepsilon \le z}(a_{\tilde{\gamma}_\varepsilon} + b_{\tilde{\gamma}_\varepsilon}) - \max_{1 \le \varepsilon \le z}(a_{\tilde{\gamma}_\varepsilon})\right)\right)$, *Then we have* $\tilde{\gamma}^- \le \text{IFEIPA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z) \le \tilde{\gamma}^+$.

Proof It is the same as Theorem 6 (omitted here).

4.2 The intuitionistic fuzzy Einstein interactive weighted power averaging (IFEIWPA) operator

Although the IFEIPA operator can eliminate the effect of unreasonable data, it doesn't take into account the weight of the aggregated data. In many real decision applications, the weight vector of the aggregated data can directly affect the selection of alternatives. However, the original power weighted operators in Xu [54] and Xu and Yager [57] have some drawbacks, that is, they use the initial weight twice cause a weakening of the power operator itself. Thus, it is necessary to develop the IFEIWPA operator by the revised power weighted averaging operator, and it is presented as below.

Definition 11 Suppose $\tilde{\gamma}_\varepsilon = (a_{\tilde{\gamma}_\varepsilon}, b_{\tilde{\gamma}_\varepsilon})(\varepsilon = 1, 2, \ldots, z)$ is a collection of IFNs, then $\text{IFEIPWA} : M^z \to M$, if

$$\text{IFEIPWA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z) = \frac{\oplus_E{}_{\varepsilon=1}^{z}(\varpi_\varepsilon(1 + TT(\tilde{\gamma}_\varepsilon)) \cdot_E \tilde{\gamma}_\varepsilon)}{\sum_{\varepsilon=1}^{z} \varpi_\varepsilon(1 + TT(\tilde{a}_\varepsilon))} \tag{42}$$

where M is the collection of all IFNs, the weight vector of $\tilde{\gamma}_\varepsilon(\varepsilon = 1, 2, \ldots, z)$ is $W = (\varpi_1, \varpi_2, \ldots, \varpi_z)^T$ and satisfies $\varpi_\varepsilon \ge 0$ and $\sum_{\varepsilon=1}^{z} \varpi_\varepsilon = 1$. $TT(\tilde{\gamma}_\varepsilon) = \sum_{j=1, j \ne \varepsilon}^{z} \text{Sup}(\tilde{\gamma}_\varepsilon, \tilde{\gamma}_j)$ expresses the support degree of the ε th IFN from all the other IFNs. It should be noted that $TT(\tilde{\gamma}_\varepsilon) = \sum_{j=1, j \ne \varepsilon}^{z} \varpi_j \text{Sup}(\tilde{\gamma}_\varepsilon, \tilde{\gamma}_j)$ is the original power weighted operators. Then the IFEIWPA stands for the intuitionistic fuzzy Einstein interactive weighted power averaging operator.

Specially, if $W = \left(\frac{1}{z}, \frac{1}{z}, \ldots, \frac{1}{z}\right)^T$, the IFEIWPA operator should reduce to the IFEIPA operator.

Theorem 10 *Suppose* $\tilde{\gamma}_\varepsilon = (a_{\tilde{\gamma}_\varepsilon}, b_{\tilde{\gamma}_\varepsilon})(\varepsilon = 1, 2, \ldots, z)$ *is a collection of IFNs, the aggregated result of IFEIWPA operator from Eq. (42) is represented by*

$$\begin{aligned}
&\text{IFEIWPA}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_z) \\
&= \left(\frac{\prod_{\varepsilon=1}^{z}\left(1 + a_{\tilde{\gamma}_\varepsilon}\right)^{\varphi_\varepsilon} - \prod_{\varepsilon=1}^{z}\left(1 - a_{\tilde{\gamma}_\varepsilon}\right)^{\varphi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1 + a_{\tilde{\gamma}_\varepsilon}\right)^{\varphi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1 - a_{\tilde{\gamma}_\varepsilon}\right)^{\varphi_\varepsilon}}, \right. \\
&\quad \frac{2\prod_{\varepsilon=1}^{z}\left(1 - a_{\tilde{\gamma}_\varepsilon}\right)^{\varphi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1 + a_{\tilde{\gamma}_\varepsilon}\right)^{\varphi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1 - a_{\tilde{\gamma}_\varepsilon}\right)^{\varphi_\varepsilon}} \\
&\quad \left. - \frac{2\prod_{\varepsilon=1}^{z}\left(1 - \left(a_{\tilde{\gamma}_\varepsilon} + b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varphi_\varepsilon}}{\prod_{\varepsilon=1}^{z}\left(1 + \left(a_{\tilde{\gamma}_\varepsilon} + b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varphi_\varepsilon} + \prod_{\varepsilon=1}^{z}\left(1 - \left(a_{\tilde{\gamma}_\varepsilon} + b_{\tilde{\gamma}_\varepsilon}\right)\right)^{\varphi_\varepsilon}} \right)
\end{aligned} \tag{43}$$

where $\varphi_\varepsilon = \frac{\varpi_\varepsilon(1 + TT(\tilde{\gamma}_\varepsilon))}{\sum_{\varepsilon=1}^{z} \varpi_\varepsilon(1 + TT(\tilde{\gamma}_\varepsilon))}$.

Obviously, we use φ_ε to replace ϖ_ε of Eq. (36), we can get Eq. (43).

Proof It is the same as Theorem 3.

Note that the IFEIWPA operator only has the property of boundedness.

5 The novel MAGDM method based on the IFEIWA operator and the IFEIPWA operator

In this section, a novel MAGDM method based on the IFEIWA operator and the IFEIPWA operator is developed. The algorithm of this new method concludes two main phases: one is to deal with the decision matrices of multiple experts, the other is to deal with the comprehensive decision matrix. Because some decision-making experts may have subjective bias and give the extreme attribute evaluation values for some alternatives, we can use the IFEIPWA operator to integrate evaluation information coming from multiple experts to eliminate the influence of some extreme data, and get a comprehensive evaluation decision matrix. Moreover, we select the IFEIWA operator to process the comprehensive evaluation matrix, this process can effectively deal with the situation where the MD or the N-MD of evaluation data is zero. The MAGDM problem is described as follows.

For a MAGDM problem, there is a collection of alternatives $\{X_1, X_2, \ldots, X_m\}$ can be evaluated based on p attributes $\{A_1, A_2, \ldots, A_p\}$, and there is a group of experts $\{E_1, E_2, \ldots, E_z\}$ who are invited to evaluate these alternatives. The ϖ_s is the weight of the expert E_s, and $\varpi_s \ge 0 (s = 1, 2, \ldots, z)$, $\sum_{s=1}^{z} \varpi_s = 1$. The w_j is the weight vector of the attribute A_j, and $w_j \ge 0\ (j = 1, 2, \ldots, p)$, $\sum_{j=1}^{p} w_j = 1$. $R^s = \left[\tilde{r}_{ij}^s\right]_{m \times p}$ is the decision matrix expressed by IFN $\tilde{r}_{ij}^s$ which is the evaluation value of attribute A_j for alternative X_i from expert E_s. Then, we can rank all alternatives based on the given information.

The steps of presented method are shown as below (note that $i=1,2,\ldots,m;\ j=1,2,\ldots,p;\ s,e=1,2,\ldots,z,\ s\neq e$ in the following formulas).

5.1 Phase One: Standardize and Aggregate Evaluation Information from Multiple Experts

Step 1 Standardize the decision information

Usually, there are two types of attribute indicators in the actual decision-making, one is the benefit type, that is, the bigger the better, the other is the cost type, that is, the smaller the better. To eliminate the impact of different attribute types, we need to standardize them by the following formula (44):

$$\tilde{r}_{ij}^{s}=\begin{cases}\left\langle u_{ij}^{s},v_{ij}^{s}\right\rangle & A_j \text{ is benefit type}\\ \left\langle v_{ij}^{s},u_{ij}^{s}\right\rangle & A_j \text{ is cost type}\end{cases} \tag{44}$$

Step 2 Calculate the support $\mathrm{Sup}\left(\tilde{r}_{ij}^{s},\tilde{r}_{ij}^{e}\right)$ by formula (45)

$$\mathrm{Sup}\left(\tilde{r}_{ij}^{s},\tilde{r}_{ij}^{e}\right)=1-d\left(\tilde{r}_{ij}^{s},\tilde{r}_{ij}^{e}\right). \tag{45}$$

Step3 Calculate the support $TT\left(\tilde{r}_{ij}^{s}\right)$ and the weights φ_{ij}^{s} by formula (46) and (47)

$$TT\left(\tilde{r}_{ij}^{s}\right)=\sum_{e=1,s\neq e}^{z}\mathrm{Sup}\left(\tilde{r}_{ij}^{s},\tilde{r}_{ij}^{e}\right). \tag{46}$$

and have

$$\varphi_{ij}^{s}=\frac{\varpi_s\left(1+TT\left(\tilde{r}_{ij}^{s}\right)\right)}{\sum_{s=1}^{z}\varpi_s\left(1+TT\left(\tilde{r}_{ij}^{s}\right)\right)}. \tag{47}$$

Step 4 Aggregate all expert matrices $R^s=\left[\tilde{r}_{ij}^{s}\right]_{m\times p}$ into a comprehensive decision matrix $R=\left[\tilde{r}_{ij}\right]_{m\times p}$ by IFEIPWA operator

$$R_{ij}=\mathrm{IFEIPWA}(\tilde{r}_{ij}^{1},\tilde{r}_{ij}^{2},\ldots,\tilde{r}_{ij}^{z}), \tag{48}$$

5.2 Phase Two: Deal with the Comprehensive Decision Matrix and Rank Alternatives

Step 5 Calculate all comprehensive attribute results for each alternative to get the comprehensive evaluation value R_i by the IFEIWA operator

$$R_i=\mathrm{IFEIWA}(\tilde{r}_{i1},\tilde{r}_{i2},\ldots,\tilde{r}_{im}), \tag{49}$$

Step 6 Calculate the SV $S(R_i)$ and the AV $A(R_i)$ for each alternative $X_m(i=1,2,\ldots,m)$. It is worth noting that if we can rank according to the SV $S(R_i)$, we do not need to calculate the AV $A(R_i)$.

Step 7 Rank all alternatives based on the Definition 4 and the SV of $S(R_i)$ and $A(R_i)$. The larger the value of $S(R_i)$, the better the ranking order of alternative X_i. If the value of $S(R_i)$ is the same, compare the obtained AV $A(R_i)$ and the larger the value of $A(R_i)$, the better the ranking order of alternative X_i.

6 Application examples

In order to apply the novel MAGDM method to practical applications, we adopt it to deal with some practical examples. Moreover, we use other existing methods to solve these practical examples and do a detailed comparative analysis of their ranking results, and then give the confirmation of effectiveness and superiority of the proposed method.

6.1 The application of the proposed MAGDM method

Example 5 [29] This is a MAGDM problem about the distribution schemes selection. There are five possible routes as alternatives which are $\{X_1,X_2,X_3,X_4,X_5\}$, and three decision-making experts $\{E_1,E_2,E_3\}$ are going to evaluate these alternatives based on four attributes, including A_1: the risk response evaluation; A_2: the transportation cost evaluation; A_3: the transportation convenience evaluation; A_4: the environmental evaluation. The weight vector of three experts is $\varpi=(0.35,0.40,0.25)^T$ and the weight vector of four attribute is $w=(0.2,0.1,0.3,0.4)^T$. The evaluation information is represented by decision matrices $R^s=\left[\tilde{r}_{ij}^{s}\right]_{5\times 4}(s=1,2,3)$, where $\tilde{r}_{ij}^{s}$ stands for the evaluation result of attribute A_j from alternative X_i obtained by expert E_s, which is an IFN. This practical problem ultimately seeks an optimal choice from five possible routes. The specific evaluation data are shown in Tables 2, 3 and 4.

The specific steps to solve this distribution schemes selection problem are as follows:

Step 1. Standardize the decision matrix

Since these four attribute are all benefit type, the standardized operation can be omitted in this step.

Step 2. Calculate the support $\mathrm{Sup}\left(\tilde{r}_{ij}^{s},\tilde{r}_{ij}^{e}\right)=1-d\left(\tilde{r}_{ij}^{s},\tilde{r}_{ij}^{e}\right)$, $(i=1,2,\ldots,5;\ j=1,2,\ldots,4;\ s,e=1,2,3,\ s\neq e)$ by Eq. (45). For convenience, we represent $\mathrm{Sup}\left(\tilde{r}_{ij}^{s},\tilde{r}_{ij}^{e}\right)$ as $S^{se}(s,e=1,2,3,s\neq e)$. Then, we can get

Table 2 The decision matrix R^1 given by expert E_1

	A_1	A_2	A_3	A_4
X_1	$(0.5, 0.4)$	$(0.6, 0.3)$	$(0.3, 0.6)$	$(0.5, 0.4)$
X_2	$(0.6, 0.3)$	$(0.6, 0.3)$	$(0.6, 0.2)$	$(0.6, 0.3)$
X_3	$(0.5, 0.4)$	$(0.2, 0.6)$	$(0.6, 0.2)$	$(0.4, 0.4)$
X_4	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.5, 0.4)$	$(0.4, 0.4)$
X_5	$(0.4, 0.3)$	$(0.7, 0.2)$	$(0.4, 0.5)$	$(0.4, 0.5)$

Table 3 The decision matrix R^2 given by expert E_2

	A_1	A_2	A_3	A_4
X_1	$(0.4, 0.2)$	$(0.6, 0.2)$	$(0.5, 0.4)$	$(0.5, 0.3)$
X_2	$(0.5, 0.3)$	$(0.6, 0.2)$	$(0.6, 0.2)$	$(0.5, 0.4)$
X_3	$(0.4, 0.4)$	$(0.3, 0.5)$	$(0.5, 0.4)$	$(0.4, 0.2)$
X_4	$(0.5, 0.4)$	$(0.7, 0.2)$	$(0.5, 0.2)$	$(0.7, 0.2)$
X_5	$(0.6, 0.3)$	$(0.7, 0.2)$	$(0.4, 0.2)$	$(0.6, 0.2)$

Table 4 The decision matrix R^3 given by expert E_3

	A_1	A_2	A_3	A_4
X_1	$(0.4, 0.5)$	$(0.5, 0.2)$	$(0.5, 0.3)$	$(0.5, 0.2)$
X_2	$(0.5, 0.4)$	$(0.5, 0.3)$	$(0.6, 0.2)$	$(0.7, 0.2)$
X_3	$(0.4, 0.5)$	$(0.4, 0.4)$	$(0.5, 0.3)$	$(0.6, 0.3)$
X_4	$(0.5, 0.3)$	$(0.4, 0.5)$	$(0.5, 0.4)$	$(0.5, 0.3)$
X_5	$(0.6, 0.2)$	$(0.5, 0.3)$	$(0.4, 0.4)$	$(0.6, 0.3)$

$$S^{12} = S^{21} = \begin{bmatrix} 0.700 & 0.900 & 0.800 & 0.900 \\ 0.900 & 0.900 & 1.000 & 0.900 \\ 0.900 & 0.900 & 0.800 & 0.800 \\ 0.800 & 0.900 & 0.800 & 0.700 \\ 0.800 & 1.000 & 0.700 & 0.700 \end{bmatrix},$$

$$S^{13} = S^{31} = \begin{bmatrix} 0.900 & 0.800 & 0.700 & 0.800 \\ 0.900 & 0.900 & 1.000 & 0.900 \\ 0.900 & 0.800 & 0.900 & 0.800 \\ 0.900 & 0.700 & 1.000 & 0.900 \\ 0.800 & 0.800 & 0.900 & 0.800 \end{bmatrix},$$

$$S^{23} = S^{32} = \begin{bmatrix} 0.700 & 0.900 & 0.900 & 0.900 \\ 0.900 & 0.900 & 1.000 & 0.800 \\ 0.900 & 0.900 & 0.900 & 0.700 \\ 0.900 & 0.700 & 0.800 & 0.800 \\ 0.900 & 0.800 & 0.800 & 0.900 \end{bmatrix}$$

Step 3. Calculate the support $TT\left(\tilde{r}_{ij}^{s}\right)$ $(i = 1, 2, \ldots, 5;\ j = 1, 2, \ldots, 4;\ s, e = 1, 2, 3,\ s \neq e)$ and the weights φ_{ij}^{s} by Eqs. (46) and (47), respectively. For convenience, we represent $TT\left(\tilde{r}_{ij}^{s}\right)$ as TT^{s} $(s = 1, 2, 3)$ and represent φ_{ij}^{s} as φ^{s} $(s = 1, 2, 3)$. Then, we can get

$$TT^{1} = \begin{bmatrix} 1.600 & 1.700 & 1.500 & 1.700 \\ 1.800 & 1.800 & 2.000 & 1.800 \\ 1.800 & 1.700 & 1.700 & 1.600 \\ 1.700 & 1.600 & 1.800 & 1.600 \\ 1.600 & 1.800 & 1.600 & 1.500 \end{bmatrix},$$

$$TT^{2} = \begin{bmatrix} 1.400 & 1.800 & 1.700 & 1.800 \\ 1.800 & 1.800 & 2.000 & 1.700 \\ 1.800 & 1.800 & 1.700 & 1.500 \\ 1.700 & 1.600 & 1.600 & 1.500 \\ 1.700 & 1.800 & 1.500 & 1.600 \end{bmatrix},$$

$$TT^{3} = \begin{bmatrix} 1.600 & 1.700 & 1.600 & 1.700 \\ 1.800 & 1.800 & 2.000 & 1.700 \\ 1.800 & 1.700 & 1.800 & 1.500 \\ 1.800 & 1.400 & 1.800 & 1.700 \\ 1.700 & 1.600 & 1.700 & 1.700 \end{bmatrix}.$$

and

$$\varphi^{1} = \begin{bmatrix} 0.361 & 0.345 & 0.336 & 0.345 \\ 0.350 & 0.350 & 0.350 & 0.358 \\ 0.350 & 0.345 & 0.347 & 0.359 \\ 0.347 & 0.357 & 0.360 & 0.352 \\ 0.341 & 0.356 & 0.352 & 0.338 \end{bmatrix},$$

$$\varphi^{2} = \begin{bmatrix} 0.381 & 0.409 & 0.415 & 0.409 \\ 0.400 & 0.400 & 0.400 & 0.395 \\ 0.400 & 0.409 & 0.396 & 0.394 \\ 0.396 & 0.408 & 0.382 & 0.387 \\ 0.405 & 0.407 & 0.387 & 0.402 \end{bmatrix},$$

$$\varphi^{3} = \begin{bmatrix} 0.258 & 0.246 & 0.250 & 0.246 \\ 0.250 & 0.250 & 0.250 & 0.247 \\ 0.250 & 0.246 & 0.257 & 0.247 \\ 0.257 & 0.235 & 0.257 & 0.261 \\ 0.253 & 0.236 & 0.261 & 0.261 \end{bmatrix}.$$

Step 4 Aggregate all expert matrices $R^{s} = \left[\tilde{r}_{ij}^{s}\right]_{5\times 4}$ $(s = 1, 2, 3)$ into a comprehensive matrix $R = \left[\tilde{r}_{ij}\right]_{5\times 4}$ by the IFEIPWA operator, shown as follows in Table 5.

Step 5 Calculate the comprehensive evaluation value R_i $(i = 1, 2, \ldots, 5)$ by the IFEIWA operator, shown as follows

$R_1 = (0.478, 0.366), R_2 = (0.582, 0.281),$

$R_3 = (0.462, 0.360), R_4 = (0.541, 0.308), R_5 = (0.513, 0.327)$

Table 5 The comprehensive decision matrix R

	A_1	A_2	A_3	A_4
X_1	$(0.437, 0.389)$	$(0.577, 0.248)$	$(0.437, 0.444)$	$(0.500, 0.324)$
X_2	$(0.537, 0.331)$	$(0.576, 0.266)$	$(0.600, 0.200)$	$(0.591, 0.309)$
X_3	$(0.436, 0.431)$	$(0.292, 0.508)$	$(0.536, 0.311)$	$(0.454, 0.320)$
X_4	$(0.536, 0.311)$	$(0.605, 0.267)$	$(0.500, 0.346)$	$(0.557, 0.290)$
X_5	$(0.538, 0.287)$	$(0.660, 0.222)$	$(0.400, 0.391)$	$(0.539, 0.329)$

Step 5 Calculate the SV $S(R_i)$.

$S(R_1) = 0.112, S(R_2) = 0.301, S(R_3) = 0.102,$
$S(R_4) = 0.233, S(R_5) = 0.186.$

Step 6 Give the ranking result of all alternatives. Because $S(R_2) > S(R_4) > S(R_5) > S(R_3) > S(R_1)$, we can get

$X_2 \succ X_4 \succ X_5 \succ X_3 \succ X_1$

So, X_2 is the best route.

6.2 The effectiveness test and the DM's attitude analysis

To test the effectiveness of the presented method, we employ some representative MAGDM methods to solve the above practical Example 5, and then compare their ranking results with that obtained by the proposed method. These representative MAGDM methods include the extended method based on the intuitionistic fuzzy weighted averaging (IFWA) operator in Xu [53], the method based on the intuitionistic fuzzy power weighted averaging (IFPWA) operator and the IFWA operator in Xu [54], the method based on the intuitionistic fuzzy weighted Archimedean Heronian aggregation (IFWAHA) operator in Liu and Chen [26] (for comparison, we adopt the Einstein ORs for this method). The results of comparison are presented in Table 6 and Fig. 1.

From Table 5, although these used methods are different, they all can get the same ranking results for the problem of distribution schemes, i.e., $X_2 \succ X_4 \succ X_5 \succ X_1 \succ X_3$. This shows that the proposed method is effective and available. From Fig. 1, we can find that as we adopt the extended method based on IFWA operator [53], the method based on IFPWA and IFWA operators [54], the method based on IFWAHA operator [26] and the proposed method in sequence, the SVs of each alternative obtained by those methods are gradually decreasing. Further analysis finds that the SVs of each alternative obtained by the method based on IFWAHA operator [26] and the proposed method are smaller than those obtained by the extended method based on IFWA operator [53] and the method based on IFPWA and IFWA operators [54], especially the result of the new method. This is because the method based on IFWAHA operator [26] and the proposed method all adopt the intuitionistic fuzzy Einstein AOs which have the same smooth approximations. Combined with the Corollary 4 in Wang and Liu [50], we think the proposed method

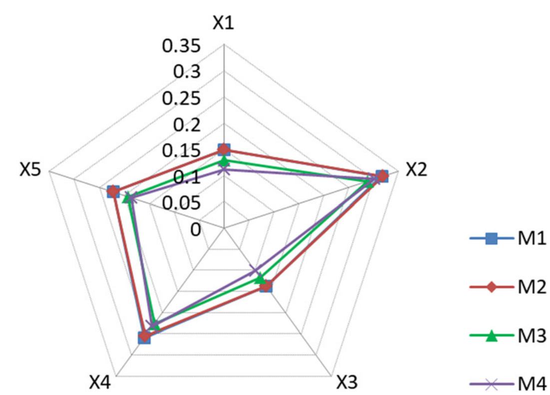

Fig. 1 The SVs of all alternatives from different methods

Table 6 The comparison of the ranking orders for different methods

Methods AOs	SVs	Ranking results
M1:The extended method based on IFWA operator [53]	$S(\tilde{Q}_1) = 0.149,\ S(\tilde{Q}_2) = 0.316,\ S(\tilde{Q}_3) = 0.138,$ $S(\tilde{Q}_4) = 0.259,\ S(\tilde{Q}_5) = 0.221$	$X_2 \succ X_4 \succ X_5 \succ X_1 \succ X_3$
M2: The method based on IFPWA and IFWA operators [54]	$S(\tilde{Q}_1) = 0.149,\ S(\tilde{Q}_2) = 0.318,\ S(\tilde{Q}_3) = 0.138,$ $S(\tilde{Q}_4) = 0.255,\ S(\tilde{Q}_5) = 0.221$	$X_2 \succ X_4 \succ X_5 \succ X_1 \succ X_3$
M3:The method based on IFWAHA operator [26]	$S(\tilde{Q}_1) = 0.129,\ S(\tilde{Q}_2) = 0.288,\ S(\tilde{Q}_3) = 0.117,$ $S(\tilde{Q}_4) = 0.226,\ S(\tilde{Q}_5) = 0.192$	$X_2 \succ X_4 \succ X_5 \succ X_1 \succ X_3$
M4:The proposed method based on IFEIPWA and IFEIWA operators	$S(\tilde{Q}_1) = 0.112, S(\tilde{Q}_2) = 0.301, S(\tilde{Q}_3) = 0.102,$ $S(\tilde{Q}_4) = 0.233, S(\tilde{Q}_5) = 0.186$	$X_2 \succ X_4 \succ X_5 \succ X_1 \succ X_3$

Table 7 The new decision matrix $\tilde{R}^3$

	A_1	A_2	A_3	A_4
X_1	$(0.4, 0.5)$	$(0.5, 0.2)$	$(0.5, 0.3)$	$(0.5, 0.2)$
X_2	$(0.5, 0.4)$	$(0.5, 0.3)$	$(0.6, 0.2)$	$(0.7, 0.2)$
X_3	$(0.4, 0.5)$	$(0.4, 0.4)$	$(0.5, 0.3)$	$(0.6, 0.3)$
X_4	$(0.5, 0.3)$	$(0.4, 0.5)$	$(0.5, 0.4)$	$(0.5, 0.3)$
X_5	$(0.6, 0.2)$	$(0.5, 0.3)$	$(0.9, 0.1)$	$(0.9, 0.1)$

more able to reflect decision makers' pessimistic attitude than the other methods.

6.3 Further Comparative Analysis with Other Methods

To further demonstrate the superiority of the proposed method, we conduct some detailed comparative analysis with other existing methods from different aspects by following practical examples. The existing methods include the extended method based on IFWA operator [53], the method based on IFPWA and IFWA operators [54], the method based on IFWAHA operator [26] and the method based on the intuitionistic fuzzy weighted interaction Maclaurin symmetric mean (IFWIMSM) operator [29] (We set k=3 in this method).

Example 6 In some practical applications, some experts may have personal bias, who may give some extreme evaluation data, i.e., too high or too low. In this case, the abnormal data will affect the decision result. In order to explain this situation, we can change the Example 5, assume that expert E_3 is too preferred to alternative X_5, so we replace $\tilde{r}^3_{53}$ and $\tilde{r}^3_{54}$ with an excessively high value $(0.9, 0.1)$, and the new decision matrix $\tilde{R}^3$ is presented in Table 7.

We use above five methods to solve the Example 6, the compare results are described in Table 8 and Fig. 2.

From Table 8 and Fig. 2, it can be found that too high evaluation value seriously affect the SVs of alternative X_5 (they have a sharp increase) and affect the ranking results obtained by the extended method based on IFWA operator [53], the method based on IFPWA and IFWA operators [54], the method based on IFWAHA operator [26] and the method based on IFWIMSM operator [29]. Their ranking orders are changed from $X_2 \succ X_4 \succ X_5 \succ X_1 \succ X_3$ to $X_5 \succ X_2 \succ X_4 \succ X_1 \succ X_3$, i.e., the ranking orders of alternative X_5 ascend from the third to the first. These ranking results are unreasonable, because they are extremely vulnerable to extreme values. But the new method still has a reasonable ranking for all alternatives, the results are still $X_2 \succ X_4 \succ X_5 \succ X_1 \succ X_3$, and these ranking results are same as that obtained from original Example 5. The reason can be interpreted as the presented method can use PA operator to eliminate the effects of extreme data coming from biased DMs. For the evaluation of the same attribute under the same alternative, once a DM gives an evaluation value that is too high or too low compared to the value

Table 8 The comparison of the ranking orders for different methods

Methods	SVs	Ranking results
M1:The extended method based on IFWA operator [53]	$S(\tilde{Q}_1) = 0.149,\ S(\tilde{Q}_2) = 0.316,\ S(\tilde{Q}_3) = 0.138,$ $S(\tilde{Q}_4) = 0.259,\ S(\tilde{Q}_5) = 0.394$	$X_5 \succ X_2 \succ X_4 \succ X_1 \succ X_3$
M2:The method based on IFPWA and IFWA operators [54]	$S(\tilde{Q}_1) = 0.149,\ S(\tilde{Q}_2) = 0.318,\ S(\tilde{Q}_3) = 0.138,$ $S(\tilde{Q}_4) = 0.255,\ S(\tilde{Q}_5) = 0.395$	$X_5 \succ X_2 \succ X_4 \succ X_1 \succ X_3$
M3:The method based on IFWAHA operator [26]	$S(\tilde{Q}_1) = 0.129,\ S(\tilde{Q}_2) = 0.288,\ S(\tilde{Q}_3) = 0.117,$ $S(\tilde{Q}_4) = 0.226,\ S(\tilde{Q}_5) = 0.333$	$X_5 \succ X_2 \succ X_4 \succ X_1 \succ X_3$
M4:The method based on IFWIMSM operator [29]	$S(\tilde{Q}_1) = -0.737,\ S(\tilde{Q}_2) = -0.714,\ S(\tilde{Q}_3) = -0.740,$ $S(\tilde{Q}_4) = -0.722,\ S(\tilde{Q}_5) = -0.586$	$X_5 \succ X_2 \succ X_4 \succ X_1 \succ X_3$
M5:The proposed method IFEIPWA and IFEIWA	$S(\tilde{Q}_1) = 0.112, S(\tilde{Q}_2) = 0.301, S(\tilde{Q}_3) = 0.102,$ $S(\tilde{Q}_4) = 0.233, S(\tilde{Q}_5) = 0.228$	$X_2 \succ X_4 \succ X_5 \succ X_1 \succ X_3$

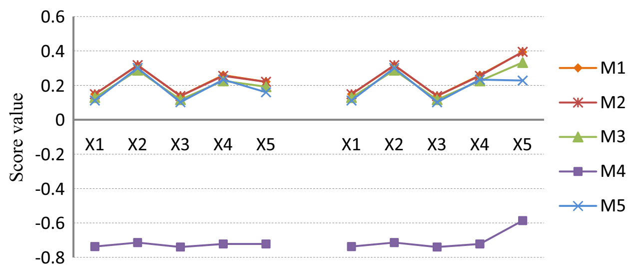

Fig. 2 The SVs of all alternatives for Examples 5 and 6 based on different methods (Note that left part of the figure is for Example 5, the right part of the figure is for Example 6)

given by other DMs, the PA operator can give these anomalous data a small weight through the corresponding mechanism, thus eliminating their influence on the evaluation results. It should be noted that the method based on IFPWA and IFWA operators [54] also uses the PA operator, but it still gets a unreasonable ranking result, the reason is that power weighted averaging operator in Xu [54] uses the original weight twice in its step 2 (when they appear in the calculation of $T(\tilde{r}_{ij}^{(k)})$ and the calculation of $\xi_{ij}^{(k)}$, respectively.), then weaken the function of adjusting weight. The following experimental results precisely verify this point. For the weight matrix of expert E_3, it was originally 0.25. Based on the Step 2 of the method based on IFPWA and IFWA operators [54] and the Step 4 of the proposed method, when we adopt the method based on IFPWA and IFWA operators [54], the weight matrix of expert E_3 is φ_1^3, when we adopt the proposed method, the weight matrix of E_3 is φ_2^3, they are as follows:

$$\varphi_1^3 = \begin{bmatrix} 0.267 & 0.261 & 0.263 & 0.261 \\ 0.263 & 0.263 & 0.264 & 0.260 \\ 0.263 & 0.261 & 0.268 & 0.259 \\ 0.268 & 0.252 & 0.268 & 0.270 \\ 0.266 & 0.253 & \underline{0.248} & \underline{0.256} \end{bmatrix}, \ \varphi_2^3 = \begin{bmatrix} 0.258 & 0.246 & 0.250 & 0.246 \\ 0.250 & 0.250 & 0.250 & 0.247 \\ 0.250 & 0.246 & 0.257 & 0.247 \\ 0.257 & 0.235 & 0.257 & 0.261 \\ 0.253 & 0.236 & \underline{0.233} & \underline{0.241} \end{bmatrix}$$

We can see that the weights of attribute values $\tilde{r}_{53}^3$, and $\tilde{r}_{54}^3$ are 0.248 and 0.256 when we adopt the method based on IFPWA and IFWA operators [54], and the weights of attribute values $\tilde{r}_{53}^3$ and $\tilde{r}_{54}^3$ are 0.233 and 0.241 when we adopt the presented method. Obviously, the results coming from the presented method are significantly less than the original weight 0.25, the results coming from the method based on IFPWA and IFWA operators [54] have a greater weight than original weight 0.25. So, the method based on IFPWA and IFWA operators [54] does not fully give the extreme value a small weight. By the above fact, the result obtained by the new method is more in line with the fact that expert E_3 is a biased expert and he/she gives some unreasonable extreme evaluating values, so they should get a smaller weight. However, the method based on IFPWA and IFWA operators [54], the method based on IFWAHA operator [26] and the method based on IFWIMSM operator [29] all adopt the original weight 0.25, the method based on IFPWA and IFWA operators [54] uses some unreasonable weights, which are not reasonable and realistic. Therefore, the presented method has a function that can eliminate the effects of unreasonable evaluating data of the alternatives from biased DM, i.e., the presented method is more preferable than the other four methods in dealing with the anomalous data.

In some practical cases, it is inevitable that the MD or N-MD of some evaluation values is 0. For such problems, some existing methods are not able to be effectively used, and often lead to some wrong evaluation results. The following Example 7 and its results can explain this situation. In order to be able to simplify the calculation steps under the premise of effectively explaining the problem, we suppose that there is only one expert on the decision-making problem.

Example 7 With the rapid development of e-commerce, online shopping has become the preferred channel for people. In order to expand the scale and business, an electric business company wants to build a large logistics storage center, there are five locations X_1, X_2, X_3, X_4, X_5. Evaluate these locations through the following four attribute indicators, A_1: the economic performance; A_2: the traffic condition; A_3: the environmental impact; A_4: the service level, and their weight vector is $w = (0.25, 0.25, 0.25, 0.25)^T$. The decision information is presented as matrix R in Table 9.

In order to be able to solve the above problem, we simplify the method based on IFWAHA operator [26] and the method based on IFWIMSM operator [29] by omitting the step 2, respectively. Then we adopt the extended method based on IFWA operator [53], the method based on IFWAHA operator [26] and the method based on IFWIMSM operator [29] and the proposed method based on the IFEIWA operator to deal with the Example 7. The ranking results are described in Table 10.

From Table 10, we can find that the extended method based on IFWA operator [53] and the method based on IFWAHA operator [26] cannot distinguish the orders between the location X_2 and location X_3. Obviously, their ranking results are unreasonable. This reason can be interpreted as the N-MDs of $\tilde{r}_{24}$ and $\tilde{r}_{33}$ are all equal to zero, then the N-MDs of the aggregated results for the location X_2 and location X_3 are also zero. In this case, other

Table 9 The decision matrix R coming from Example 7

	A_1	A_2	A_3	A_4
X_1	$(0.6, 0.2)$	$(0.5, 0.4)$	$(0.5, 0.3)$	$(0.7, 0.1)$
X_2	$(0.6, 0.1)$	$(0.4, 0.3)$	$(0.5, 1)$	$(0.6, 0.0)$
X_3	$(0.4, 0.4)$	$(0.5, 0.4)$	$(0.6, 0.0)$	$(0.6, 0.2)$
X_4	$(0.5, 0.4)$	$(0.4, 0.2)$	$(0.5, 0.1)$	$(0.3, 0.2)$
X_5	$(0.4, 0.3)$	$(0.5, 0.3)$	$(0.4, 0.5)$	$(0.5, 0.2)$

Table 10 The comparison of the ranking orders for different methods

Methods	SVs $S(\tilde{Q}_i)$	Ranking results
M1:The extended method based on IFWA operator [53]	$S(\tilde{Q}_1) = 0.363,\ S(\tilde{Q}_2) = 0.532,\ S(\tilde{Q}_3) = 0.532,\ S(\tilde{Q}_4) = 0.231,\ S(\tilde{Q}_5) = 0.144$	$X_2 = X_3 \succ X_1 \succ X_4 \succ X_5$
M2:the method based on IFWAHA operator [26]	$S(\tilde{Q}_1) = 0.346,\ S(\tilde{Q}_2) = 0.530,\ S(\tilde{Q}_3) = 0.530,\ S(\tilde{Q}_4) = 0.219,\ S(\tilde{Q}_5) = 0.137$	$X_2 = X_3 \succ X_1 \succ X_4 \succ X_5$
M3: the method based on IFWIMSM operator [29]	$S(\tilde{Q}_1) = -0.566,\ S(\tilde{Q}_2) = -0.557,\ S(\tilde{Q}_3) = -0.594,\ S(\tilde{Q}_4) = -0.620,\ S(\tilde{Q}_5) = -0.657$	$X_2 \succ X_1 \succ X_3 \succ X_4 \succ X_5$
M4:The proposed method IFEIWA operator	$S(\tilde{Q}_1) = 0.331, S(\tilde{Q}_2) = 0.407, S(\tilde{Q}_3) = 0.262, S(\tilde{Q}_4) = 0.165, S(\tilde{Q}_5) = 0.111$	$X_2 \succ X_1 \succ X_3 \succ X_4 \succ X_5$

N-MDs of attribute values from locations X_2 and X_3 except $\tilde{r}_{22}$ and $\tilde{r}_{34}$ do not work for ranking results. However, the method based on IFWIMSM operator [29] and the presented method can give a reasonable ranking result, because they all use the interactive AOs that can avoid the problem by considering interaction between MD and N-MD of the IFNs. In addition, for the ranking result from the method based on IFWIMSM operator [29], it can be found that the SVs are quite close, that is, once individual data changes slightly, there will be a change in ranking, so this method is extremely unstable. Therefore, the proposed method can overcome the drawbacks of these methods.

6.4 The advantages of the proposed MAGDM method

In the following, we will elaborate on the advantages of presented method by comparing with other typical methods in detail.

(1) Compare with the extended method based on IFWA operator [53] and the method based on IFPWA and IFWA operators [54]. Firstly, these two methods all use the IFWA operator. From the aggregated results of Example 7, we can see that as long as there is a N-MD equals to zero in the extended method based on IFWA operator [53], the SV will has a sharp increase because the N-MD of the aggregated result will be zero. Obviously, this is an unreasonable result. So, in some situations in which N-MD of an attribute value is zero, the ranking results for all alternatives will be incorrect. Secondly, the weighted PA operator in Xu [54] is not corrected. In some cases, it cannot get some reasonable weight because it uses two original weights when it calculates the adjusted weights, and weakens the power operator's function that can assign extreme values with smaller weights. However, the proposed method can avoid these problems by using the Einstein interaction operator and revised power weighted averaging operator for IFNs. So, the proposed method is more reasonable than these two methods in solving practical complex decision problems.

(2) Compare with the method based on IFWAHA operator [26]. Firstly, the method based on IFWAHA operator [26] and the proposed method all use the Einstein AOs, they can give same smooth approximations and reflect the decision maker's pessimistic attitude more effectively, so these methods are more suitable for risk-preferred DMs. Secondly, the method based on IFWAHA operator [26] is impossible to deal with the problem that MD or N-MD is 0. The proposed method can solve this problem effectively by using intuitionistic fuzzy interaction AOs.

Thirdly, the method based on IFWAHA operator [26] is easily affected by some extreme values. When the biased DMs give some very high or very low evaluation values, they will have a serious impact on the ranking results obtained by the method based on IFWAHA operator [26]. In this case, if we use the IFEIWPA operator from the proposed method to consider the mutual support of different attribute values, it will give a smaller weight for these un-normal values, thus it can remove the effect of the extreme data on the ranking results. This advantage is crucial in the complex and volatile environment of decision-making, because different DMs have different prejudice in different environment. DM.

(3) Compare with the method based on IFWIMSM operator [29]. Firstly, these two methods all adopt the interaction ORs to process intuitionistic fuzzy evaluation information, they effectively avoid the problem of unreasonable calculation caused by the MD or N-MD being zero. Secondly, the method based on IFWIMSM operator [29] uses the MSM operator, it is very unstable and its ranking results are highly susceptible to data change. In addition, in some cases, the SVs of alternatives obtained by this method are very close and they are not easy to be distinguished. However, the proposed method is relatively more stable by the IFEIWA operator and IFEIPWA operators. Thirdly, the new method has the function that can eliminate the impact of extreme evaluation values on evaluation results, but the method based on IFWIMSM operator [29] does not have this function. Contrary, the method based on IFWIMSM operator [29] takes into account the correlation between experts and experts, as well as between attributes and attributes. So, it may amplify the influence of extreme values on ranking results, and then get unreasonable evaluation results. This further illustrates that their method is highly unstable. Therefore, the presented method can be more suitable for solving practical decision application problems.

7 Conclusions

AO is an important tool to process decision-making information, especially in intuitionistic fuzzy environment. Recently, a lot of methods based on the AOs have been put forward to solve the MAGDM problems. However, these methods still have some limitations when faced with some special practical problems. For instance, under the intuitionistic fuzzy environment, some AOs based on the traditional ORs cannot solve the problem which MD or N-MD equals to zero. Based on this case, we have presented some new Einstein interactive ORs to solve this problem and then have presented the IFEIWA operator. In addition, in real decision-making, some DMs can give some biased evaluation values by their preferences, in order to remove the impact of these abnormal data, we have presented the IFEIWPA operator based on the revised weighted PA operator. Further, we have presented a new MAGDM method based on the IFEIWA and the IFEIWPA operators and applied them to solve the realistic decision problems. The advantages of the presented method are mainly reflected in the following three aspects: (1) can consider the interaction between MD and N-MD, and solve the existing problem which the MD or N-MD of any an IFN equals to zero; (2) can give the good smooth approximations and reflect the decision maker's pessimistic attitude more effectively by using Einstein AOs, and be more suitable for risk-preferred decision makers; (3) can remove the influences of unreasonable data coming from biased DMs and get more stable evaluation results. Finally, the feasibility and superiority are confirmed by some practical examples and some detailed experimental comparisons.

Considering the superiority of new interactive ORs, they can be developed by some other TATs in further research, such as Dombi TAT [27, 28], Frank TAT [35], Schweizer-Sklar TAT [48]. In addition, these proposed AOs can also be extended to other fuzzy environments, such as pythagorean fuzzy sets [47], q-rung orthopair fuzzy sets [27, 28, 33, 34]. In order to be able to apply this new method better to actual decision-making cases, we can apply it to practical application field, such as performance evaluation problems [7, 22], ecological environment quality assessment [41], supplier selection problems [32] and logistics site selection problems [39]. In addition, because Intuitionistic fuzzy T-sets (IFTSs) are more oriented to real life than IFSs [9, 10], we will also extend the proposed methods to IFTSs.

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