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# Utilizing Linguistic Picture Fuzzy Aggregation Operators for Multiple-Attribute Decision-Making Problems

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**Abstract** The linguistic picture fuzzy set (LPFS) is an extension of the linguistic intuitionistic fuzzy set (LIFS), and can contain more information than the LIFS. In this paper, the degrees of positive, neutral and non-membership of PFSs are expressed in linguistic terms, which can more easily describe the uncertain and vague information existing in the real world. By combining the PFS and the linguistic term, we define the LPFS and propose operational rules for linguistic picture fuzzy numbers (LPFNs). We further propose weighted averaging and weighted geometric operators and discuss their properties. Additionally, we propose an approach to deal with a multiple-attribute group decision-making (MAGDM) problem based on the developed aggregation operators. Finally, we present an illustrative example to demonstrate the effectiveness and advantages of the developed method by comparing it with existing methods. In addition, our method can be utilized not only to solve problems with linguistic intuitionistic fuzzy numbers (LIFNs), but also to deal with problems with LPFNs, and is a generalization of a number of existing methods.

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## 1 Introduction

The concept of the picture fuzzy set (PFS), which was proposed by Cuong and Kreinovich [1], is basically a generalization of the intuitionistic fuzzy set. An attractive feature of the PFS is that it assigns to each element a degree of membership, neutral membership or non-membership. Because of the tendency for considerable hesitancy in human decision processes, PFSs have generally been applied to the field of decision-making. They can be applied in a directional fashion to human opinions such as "yes", "abstain", "no" and "refusal". Voting is a good example of this position, where voters may be divided into four groups: "vote for", "abstain", vote against" or "refusal to vote". The PFS has attracted the attention of many researchers in this area. Cuong [2] discussed characteristics of PFSs and also confirmed their distance measures. Cuong and Hai [3] defined the first fuzzy logic operators and implications for PFSs, and introduced principle operations for fuzzy derivation forms in PF logic. Cuong, Kreinovich and Ngan [4] examined the characteristics of picture fuzzy *t*-norm and *t*-conorm. Phong et al. [5] explored a certain configuration of picture fuzzy relations. Wei et al. [6–8] defined several procedures for computing the closeness between PFSs. To date, many researchers have developed models of PFSs conditions. For example, correlation coefficients of PFSs were proposed by Sing [9] and applied to clustering analysis. Son et al. [10] provided time arrangement calculation and temperature estimation based on the PFS domain. Son [11, 12] defined picture fuzzy separation measures, generalized picture fuzzy distance measures and picture fuzzy association measures, and combined them to tackle grouping examination under a PFSs condition. A novel fuzzy derivation structure on PFSs was defined by Son et al. [13] to improve the performance of the classical fuzzy inference system. Thong et al. [14, 15] utilized a PF clustering approach for complex data and particle swarm optimization. Wei [16] developed PF aggregation operators and applied them to multiple-attribute decision-making (MADM) problems for ranking enterprise resource planning (ERP) structures. Using the picture fuzzy weighted cross-entropy concept, Wei [17] studied a basic leadership technique and used this technique to rank the alternatives. Yang et al. [18] defined an adjustable soft discernibility matrix based on PFSs and tested it in decision-making. Garg [19] designed aggregation operators on PFSs and applied them to multi-criteria group decision-making (MCDM) problems. Peng et al. [20] proposed a PFS algorithm and tested it in decision-making. For other research on PFSs, readers are referred to [21–23]. To deal with multiple-attribute group decision-making (MAGDM) problems, Ashraf et al. [24] presented two techniques, aggregation operators and (ii) the technique for order of preference by similarity to ideal solution (TOPSIS) method, to aggregate picture fuzzy information. Bo and Zhang [25] studied operations of picture fuzzy relations including type-2 inclusion relation, type-2 union, type-2 intersection and type-2 complement operations, and also defined the anti-reflexive kernel, symmetric kernel, reflexive closure and symmetric closure of a picture fuzzy relation. Ashraf et al. [26] extended the structure of cubic sets to PFSs, and also defined the concept of positive-, neutral- and negative-internal and positive-, neutral- and negative-external cubic PFSs.

There are several methods for solving the MADM problem using linguistic picture fuzzy information. Linguistic ordered weighted average operators were developed by Bordogna et al. [27]. A multi-criteria linguistic decision-making model was presented by Rodriguez et al. [28] in which experts give their assessments by eliciting linguistic interpretation. Herrera et al. [29, 30] introduced a 2-tuple linguistic representation model to avoid the loss and misinterpretation of information in the linguistic information processing process. Martínez et al. [31] reviewed the use of the 2-tuple linguistic model for counting with words in decision-making, including its extensions, applications and challenges. Xu [32] defined a virtual linguistic label equivalent to the 2-tuple linguistic variable and introduced new aggregation operators including a linguistic weighted geometric averaging operator (LWGA), linguistic ordered weighted geometric averaging operator (LOWGA) and linguistic hybrid geometric averaging operator (LHGA). Xu [33] proposed the concept of an uncertain linguistic variable and defined the uncertain linguistic ordered weighted averaging operator and for more study about linguistic terms, we refer to [34–37].

Motivated by evidence that PFSs are particularly well suited for modeling estimated and vague information in real-world applications, the fundamental objective of this paper is to present various aggregation operators under the linguistic picture fuzzy environment, referred to as linguistic picture fuzzy aggregation operators, and their application in MADM problems. In this paper, new operational laws for the PFSs are defined, and their comparable aggregation operators, namely linguistic picture fuzzy weighted averaging, ordered weighted and hybrid averaging aggregation operators, are proposed. Useful properties are also studied and these proposed operators are applied to MADM problem to demonstrate the best alternative. In order to do so, we show the picture fuzzy numbers by linguistic variables using a linguistic term set. In the last section, we compare our results with those of Xu [33], in which a small difference occurs in the results for some reason. For example, the operations of the linguistic picture fuzzy numbers (LPFNs) are different from the operations of uncertain linguistic variable (ULVs). Secondly, we use the score and accuracy index to rank the LPFNs. However, Xu [33] compared each ULV with all ULVs, and then constructed a complementary matrix.

The remainder of the paper is organized as follows. In Sect. 2, we briefly discuss PFSs and the linguistic approach. In Sect. 3, we introduce the concept of LPFNs, which is a generalization of linguistic intuitionistic fuzzy numbers (LIFNs) for linguistic picture circumstances. In Sect. 4, we introduce some aggregation operators for LPFNs. We present the MADM method with LPFN assessments in Sect. 5. In Sect. 6, we illustrate an example, and in Sect. 7, we compare the results of this paper to the results obtained by Xu [33]. Conclusions are drawn in Sect. 8.

## 2 Preliminary

**Definition 1** ([1]) Let $R \neq 0$ be a universal set. Then a picture fuzzy set $J$ (PFS) in $R$ can be written as

$$J = \{\langle r, \mu_j(r), v_j(r), \eta_j(r)\rangle | r \in R\}$$

where the functions $\mu_j(r) : R \to [0,1], v_j(r) : R \to [0,1]$ and $\eta_j(r) : R \to [0,1]$ are the membership, neutral membership and non-membership degree of the set $J$, respectively, which satisfy the condition

$$0 \leq \mu_j(r) + v_j(r) + \eta_j(r) \leq 1, \forall r \in R.$$

For each $r \in R, \varpi_J(r) = 1 - \mu_j(r) - v_j(r) - \eta_j(r)$ is said to be the refusal degree of $J$.

If $v_j(r) = 0, \forall\, r \in R$, then set $J$ is reduced to an IFS, and if $\eta_j(r) = v_j(r) = 0$ for all $r \in R$, then set $J$ is reduced to an FS.

**Definition 2** ([38]) Let $R \neq 0$ be a universal set. The linguistic picture fuzzy set $J$ (LPFS) in $R$ can then be written as

$$J = \{\langle \ell_{\theta(r)}, \mu_j(r), v_j(r), \eta_j(r)\rangle | r \in R\}$$

where $\ell_{\theta(r)} \in L$ represents the linguistic term, and the functions $\mu_j(r) : R \to [0,1], v_j(r) : R \to [0,1]$ and $\eta_j(r) : R \to [0,1]$ are the membership, neutral membership and non-membership degree of the set $J$, respectively, which satisfy the condition

$$0 \leq \mu_j(r) + v_j(r) + \eta_j(r) \leq 1, \forall \mathrm{r} \in \mathrm{R}.$$

If $v_j(r) = 0,\ \forall\ r \in R$, then the picture linguistic set is reduced to the intuitionistic linguistic set [39].

**Definition 3** Let $L = \{\ell_i | i = 0, 1, \ldots, l\}$ be the linguistic term set with odd cardinality, where $\ell_i$ are the possible values of the linguistic variable and $l$ is a positive integer, i.e., a seven-linguistic-terms set $L$ could be assigned as [40]:

$$\begin{aligned} L &= (\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) \\ &= \{\text{very poor, poor, slightly poor, fair,} \\ &\quad \text{slightly good, good, very good}\}. \end{aligned}$$

**Definition 4** ([29]) Characteristics of the linguistic term set are:

1. The negation operator: neg $(L_i) = L_j$, where $j = l - 1$;
2. Be ordered: $\ell_i \leq \ell_j \iff i \leq j$;
3. Maximum operator: $\max(\ell_i, \ell_j) = \ell_i$ if $\ell_i \geq \ell_j$;
4. Minimum operator: $\min(\ell_i, \ell_j) = \ell_i$ if $\ell_i \leq \ell_j$.

## 3 Linguistic Picture Fuzzy Numbers

In real-life problems, it can be difficult to derive the degree of membership, neutral membership and non-membership of PFNs with perfect values. Therefore, the notion of linguistic picture fuzzy numbers (LPFNs) is introduced here, and we represent membership, neutral membership and non-membership in the form of linguistic terms.

**Definition 5** Let $\Gamma_{[0,l]}$ be the set of all LPFNs based on $L_{[0,l]}$ and $(\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j}) \in \Gamma_{[0,l]}$ $(j = 1, 2, 3\ldots, n)$. Then we define the following operation for the LPFNs as:

1. $(\ell_{\kappa_l}, \ell_{\xi_l}, \ell_{\delta_l}) \cup (\ell_{\kappa_m}, \ell_{\xi_m}, \ell_{\delta_m}) = \{\max(\ell_{\kappa_l}, \ell_{\kappa_m}), \min(\ell_{\xi_l}, \ell_{\xi_m}), \min(\ell_{\delta_l}, , \ell_{\delta_m})\}$;
2. $(\ell_{\kappa_l}, \ell_{\xi_l}, \ell_{\delta_l}) \cap (\ell_{\kappa_m}, \ell_{\xi_m}, \ell_{\delta_m}) = \{\min(\ell_{\kappa_l}, \ell_{\kappa_m}), \min(\ell_{\xi_l}, \ell_{\xi_m}), \max(\ell_{\delta_l}, , \ell_{\delta_m})\}$;
3. $(\ell_{\kappa_l}, \ell_{\xi_l}, \ell_{\delta_l})^c = (\ell_{\delta_l}, \ell_{\xi_l}, \ell_{\kappa_l})$.

**Theorem 1** Let $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}), (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})$ *and* $(\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})$ be the LPFNs. Then the following equalities always hold:

1. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cup (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) = (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \cup (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})$;
2. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cap (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) = (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \cap (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})$;
3. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cup [(\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \cup (\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})] = [(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cup (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})] \cup (\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})$;
4. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cap [(\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \cap (\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})] = [(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cap (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})] \cap (\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})$;
5. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cup [(\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \cap (\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})] = [(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cup (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})] \cap [(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cup (\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})]$;
6. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cap [(\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \cup (\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})] = [(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cap (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})] \cup [(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cap (\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3})]$;
7. $[(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cup (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})]^c = (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})^c \cap (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})^c$;
8. $[(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \cap (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})]^c = (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})^c \cup (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})^c$.

*Proof* Straightforward. □

**Definition 6** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j})$ $(j = 1, 2, 3\ldots, n)$ be the set of all LPFNs; then

$$z(F) = \kappa_j - \xi_j - \delta_j,\ \mathrm{and} g(\mathrm{F}) = \kappa_\mathrm{j} + \xi_\mathrm{j} + \delta_\mathrm{j}.$$

Then $z(F)$ and $g(F)$ are the linguistic score function and linguistic accuracy function of $F$, respectively.

**Definition 7** Let $F_1 = (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}), F_2 = (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})$ be two LPFNs. Then $z(F_1)$ and $z(F_2)$ are the score function of $F_1$ and $F_2$, and $g(F_1)$ and $g(F_2)$ are the accuracy function of $F_1$ and $F_2$. Then the following can be stated:

1. If $z(F_1) < z(F_2)$, then $F_1$ is smaller than $F_2$, denoted by $F_1 < F_2$;
2. If $z(F_1) > z(F_2)$, then $F_1$ is larger than $F_2$, denoted by $F_1 > F_2$;
3. If $z(F_1) = z(F_2)$,

a. If $g(F_1) < g(F_2)$, then $F_1$ is smaller than $F_2$, denoted by $F_1 < F_2$;
b. If $g(F_1) > g(F_2)$, then $F_1$ is larger than $F_2$, denoted by $F_1 > F_2$;
c. If $g(F_1) = g(F_2)$, then $F_1$ and $F_2$ have the same information, denoted by $F_1 = F_2$.

### 3.1 Operational Laws and Properties of Linguistic Picture Fuzzy Numbers

In this section, we define some basic operational laws for LPFNs and also define their properties.

**Definition 8** Let $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}), (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})$ and $(\ell_{\kappa_3}, \ell_{\xi_3}, \ell_{\delta_3}) \in \Gamma_{[0,l]}$, be the LPFNs with $\lambda \succ 0$. Then, the operational laws for the LPFNs are as follows:

1. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \oplus (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) = \left(\ell_{\kappa_1+\kappa_2-\frac{\kappa_1\kappa_2}{l}}, \ell_{\frac{\xi_1\xi_2}{l}}, \ell_{\frac{\delta_1\delta_2}{l}}\right);$
2. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \otimes (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) = \left(\ell_{\frac{\kappa_1\kappa_2}{l}}, \ell_{\xi_1+\xi_2-\frac{\xi_1\xi_2}{l}}, \ell_{\delta_1+\delta_2-\frac{\delta_1\delta_2}{l}}\right);$
3. $\lambda \cdot (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) = \left(\ell_{l-l\left(1-\frac{\kappa_1}{l}\right)^{\lambda}}, \ell_{l\left(\frac{\xi_1}{l}\right)^{\lambda}}, \ell_{l\left(\frac{\delta_1}{l}\right)^{\lambda}}\right);$
4. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})^{\lambda} = \left(\ell_{l\left(\frac{\kappa_1}{l}\right)^{\lambda}}, \ell_{l-l\left(1-\frac{\xi_1}{l}\right)^{\lambda}}, \ell_{l-l\left(1-\frac{\delta_1}{l}\right)^{\lambda}}\right)$

**Theorem 2** Let $(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta}), (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})$ and $(\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \in \Gamma_{[0,l]}$ be the LPFNs with $\lambda, \lambda_1, \lambda_2 > 0$. Then

1. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \oplus (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) = (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \oplus (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1});$
2. $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \otimes (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) = (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) \otimes (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1});$
3. $(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta}) \oplus \left[(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \oplus (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})\right] = [(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta}) \oplus (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})] \oplus (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2});$
4. $(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta}) \otimes \left[(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \otimes (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})\right] = [(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta}) \otimes (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})] \otimes (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2});$
5. $\lambda[(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \oplus (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})] = \lambda(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \oplus \lambda(\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2});$
6. $[(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \oplus (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})]^{\lambda} = (\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1})^{\lambda} \otimes (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2})^{\lambda};$
7. $\lambda_1(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta}) \oplus \lambda_2(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta}) = (\lambda_1 + \lambda_2)(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta});$
8. $(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta})^{\lambda_1} \otimes (\ell_{\kappa}, \ell_{\xi}, \ell_{\delta})^{\lambda_2} = (\ell_{\kappa}, \ell_{\xi}, \ell_{\delta})^{\lambda_1+\lambda_2};$
9. $\lambda_1[\lambda_2(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta})] = \lambda_1\lambda_2(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta});$
10. $[(\ell_{\kappa}, \ell_{\xi}, \ell_{\delta})^{\lambda_2}]^{\lambda_1} = (\ell_{\kappa}, \ell_{\xi}, \ell_{\delta})^{\lambda_1\lambda_2}.$

*Proof* Straightforward. □

## 4 Some New Aggregation Operators on Linguistic Picture Fuzzy Numbers

We introduced the linguistic picture fuzzy weighted averaging (LPFWA), linguistic picture fuzzy ordered weighted averaging (LPFOWA), and linguistic picture fuzzy hybrid averaging (LPFHA) operators using the defined operational laws for linguistic picture fuzzy information.

**Definition 9** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j})$ $(j = 1, 2, 3 \ldots, m) \in \Gamma_{[0,l]}$ be the set of all LPFNs. The LPFWA operator is a mapping $\Gamma^m_{[0,l]} \to \Gamma_{[0,l]}$, such that:

$$\mathrm{LPFWA}_w(F_1, F_2, \ldots, F_m) = w_1F_1 \oplus w_2F_2 \oplus \ldots \oplus w_mF_m$$

where $w = (w_1, w_2, \ldots, w_m)^T$ are the weighted vectors of $F_j(\, j = 1, 2, \ldots m)$, such that $0 \le w_j \le 1, \sum_{j=1}^{m} w_j = 1$.

**Theorem 3** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j})$ $(j = 1, 2, 3 \ldots, m) \in \Gamma_{[0,l]}$ be the set of all LPFNs and $w = (w_1, w_2, \ldots, w_m)^T$ be the weighted vectors of $F_j(\, j = 1, 2, \ldots m)$, such that $0 \le w_j \le 1, \sum_{j=1}^{m} w_j = 1$. Then

$$\mathrm{LPFWA}_w(F_1, F_2, \ldots, F_m) = \left(\ell_{l-l\prod_{j=1}^{m}\left(1-\frac{\kappa_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m}\left(\frac{\xi_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m}\left(\frac{\delta_j}{l}\right)^{w_j}}\right)$$

*Proof* By utilizing the technique of mathematical induction on $m$, we have the following for $m = 2$, since

$$\begin{aligned}\mathrm{LPFWA}_w(F_1, F_2) &= w_1F_1 \oplus w_2F_2 \\ &= \left(\ell_{l-l\left(1-\frac{\kappa_1}{l}\right)^{w_1}}, \ell_{l\left(\frac{\xi_1}{l}\right)^{w_1}}, \ell_{l\left(\frac{\delta_1}{l}\right)^{w_1}}\right) \\ &\quad \oplus \left(\ell_{l-l\left(1-\frac{\kappa_2}{l}\right)^{w_2}}, \ell_{l\left(\frac{\xi_2}{l}\right)^{w_1}}, \ell_{l\left(\frac{\delta_2}{l}\right)^{w_2}}\right) \\ &= (\ell_u, \ell_v, \ell_w)\end{aligned}$$

where

$$\begin{aligned}u &= \left[l - l\left(1 - \frac{\kappa_1}{l}\right)^{w_1}\right] + \left[l - l\left(1 - \frac{\kappa_2}{l}\right)^{w_2}\right] \\ &\quad - \frac{\left[l - l\left(1 - \frac{\kappa_1}{l}\right)^{w_1}\right]\left[l - l\left(1 - \frac{\kappa_1}{l}\right)^{w_2}\right]}{l} \\ &= l - l\left(1 - \frac{\kappa_1}{l}\right)^{w_1}\left(1 - \frac{\kappa_2}{l}\right)^{w_2} \\ &= l - l\prod_{j=1}^{2}\left(1 - \frac{\kappa_j}{l}\right)^{w_j}\end{aligned}$$

$$\begin{aligned}v &= \frac{\left[l\left(\frac{\xi_1}{l}\right)^{w_1}\right]\left[l\left(\frac{\xi_2}{l}\right)^{w_2}\right]}{l} = l\left(\frac{\xi_1}{l}\right)^{w_1}\left(\frac{\xi_2}{l}\right)^{w_2} \\ &= l\prod_{j=1}^{2}\left(\frac{\xi_j}{l}\right)^{w_j}\end{aligned}$$

and

$$w = \frac{\left[l\left(\frac{\delta_1}{l}\right)^{w_1}\right]\left[l\left(\frac{\delta_2}{l}\right)^{w_2}\right]}{l} = l\left(\frac{\delta_1}{l}\right)^{w_1}\left(\frac{\delta_2}{l}\right)^{w_2} = l\prod_{j=1}^{2}\left(\frac{\delta_j}{l}\right)^{w_j}$$

So, we have proved that the result is true for $m = 2$. Now, suppose that the result is true for $m - 1$, i.e.,

$$\mathrm{LPFWA}_w(F_1, F_2, \ldots, F_{m-1}) = \left(\ell_{l-l\prod_{j=1}^{m-1}\left(1-\frac{\kappa_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m-1}\left(\frac{\xi_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m-1}\left(\frac{\delta_j}{l}\right)^{w_j}}\right)$$

Now, we have to prove for $m$,

$$\begin{aligned}\mathrm{LPFWA}_w(F_1, F_2, \ldots, F_m) &= w_1F_1 \oplus w_2F_2 \oplus \ldots \oplus w_{m-1}F_{m-1} \oplus w_mF_m \\ &= \left(\ell_{l-l\prod_{j=1}^{m-1}\left(1-\frac{\kappa_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m-1}\left(\frac{\xi_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m-1}\left(\frac{\delta_j}{l}\right)^{w_j}}\right) \\ &\oplus \left(\ell_{l-l\prod_{j=1}^{m}\left(1-\frac{\kappa_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m}\left(\frac{\xi_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m}\left(\frac{\delta_j}{l}\right)^{w_j}}\right) \\ &= (\ell_{u_m}, \ell_{v_m}, \ell_{w_m})\end{aligned}$$

where

$$\begin{aligned}u_m &= \left[l - l\prod_{j=1}^{m-1}\left(1-\frac{\kappa_j}{l}\right)^{w_j}\right] + \left[l - l\left(1-\frac{\kappa_m}{l}\right)^{w_m}\right] - \frac{\left[l - l\prod_{j=1}^{m-1}\left(1-\frac{\kappa_1}{l}\right)^{w_1}\right]\left[l - l\left(1-\frac{\kappa_m}{l}\right)^{w_m}\right]}{l} \\ &= l - l\prod_{j=1}^{m}\left(1-\frac{\kappa_j}{l}\right)^{w_j}\end{aligned}$$

$$v_m = \frac{l\prod_{j=1}^{m}\left(\frac{\xi_j}{l}\right)^{w_j}.\left[l\left(\frac{\xi_m}{l}\right)^{w_m}\right]}{l} = l\prod_{j=1}^{m}\left(\frac{\xi_j}{l}\right)^{w_j}$$

and

$$w_m = \frac{l\prod_{j=1}^{m}\left(\frac{\delta_j}{l}\right)^{w_j}.\left[l\left(\frac{\delta_m}{l}\right)^{w_m}\right]}{l} = l\prod_{j=1}^{m}\left(\frac{\delta_j}{l}\right)^{w_j}$$

Hence, the result is true for any $m$, i.e.,

$$\mathrm{LPFWA}_w(F_1, F_2, \ldots, F_m) = \left(\ell_{l-l\prod_{j=1}^{m}\left(1-\frac{\kappa_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m}\left(\frac{\xi_j}{l}\right)^{w_j}}, \ell_{l\prod_{j=1}^{m}\left(\frac{\delta_j}{l}\right)^{w_j}}\right)$$

Proved. □

The LPFWA operator has the following properties which can be easily proved:

1. Commutativity: Let $F_j \in \Gamma_{[0,l]}$ $(j = 1, 2, \ldots, m)$,and $\sigma(1), \sigma(2), \ldots \sigma(m)$ is a permutation of (1, 2, …, $m$); then

   $$\mathrm{LPFWA}_w(F_1, F_2, \ldots, F_m) = \mathrm{LPFWA}_{w/}(F_{\sigma(1)}, F_{\sigma(2)}, \ldots, F_{\sigma(m)})$$

   where $\sigma$ is any permutation on the set (1, 2, …, $m$).
2. Monotonicity: Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j}), F_j^* = (\ell_{\kappa_j^*}, \ell_{\xi_j^*}, \ell_{\delta_j^*}) \in \Gamma_{[0,l]}$, if $\kappa_j \le \kappa_j^*, \xi_j \ge \xi_j^*, \delta_j \ge \delta_j^*$ $(j = 1, 2, \ldots, m)$, then

   $$\mathrm{LPFWA}_w(F_1, F_2, \ldots, F_m) \le \mathrm{LPFWA}_w(F_1^*, F_2^*, \ldots, F_m^*).$$
3. Boundary: Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j}) \in \Gamma_{[0,l]}$ $(j = 1, 2, \ldots, m)$,and $\kappa^- = \min(\kappa_j)$, $\kappa^+ = \max(\kappa_j)$, $\xi^- = \min(\xi_j)$, $\xi^+ = \max(\xi_j)$, $\delta^- = \min(\delta_j)$ and $\delta^+ = \max(\delta_j)$; then

   $$(\ell_{\kappa^+}, \ell_{\xi^+}, \ell_{\delta^+}) \le \mathrm{LPFWA}_w(F_1, F_2, \ldots, F_m) \le (\ell_{\kappa^-}, \ell_{\xi^-}, \ell_{\delta^-})$$
4. Idempotency: Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j}) \in \Gamma_{[0,l]}$ $(j = 1, 2, \ldots, m)$ and $F_j = F$, always for any $j$; then

   $$\mathrm{LPFWA}_w(F_1, F_2, \ldots, F_m) = F.$$

**Definition 10** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j})$ $(j = 1, 2, 3\ldots, m) \in \Gamma_{[0,l]}$ be the set of all LPFNs. The linguistic picture fuzzy ordered weighted averaging (LPFOWA) operator is a mapping $\Gamma_{[0,l]}^m \to \Gamma_{[0,l]}$,with associated weight vector $\psi = (\psi_1, \psi_2, \ldots, \psi_m)^T$, $0 \le \psi_j \le 1$, $\sum_{j=1}^m \psi_j = 1$ such that

$$\mathrm{LPFOWA}_w(F_1, F_2, \ldots, F_m) = w_1F_{\sigma(1)} \oplus w_2F_{\sigma(2)} \oplus \ldots, w_mF_{\sigma(m)}$$

where $\sigma(1), \sigma(2), \ldots \sigma(m)$ is a permutation of (1, 2, …, $m$), such that $F_{\sigma(j-1)} \ge F_{\sigma(j)}$ for all $j = (1, 2, \ldots, m)$.

**Definition 11** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j})$ $(j = 1, 2, 3\ldots, m) \in \Gamma_{[0,l]}$ be the set of all LPFNs. The linguistic picture fuzzy hybrid weighted averaging (LPFHWA) operator is a mapping $\Gamma^m_{[0,l]} \to \Gamma_{[0,l]}$,with associated weight vector $\psi = (\psi_1, \psi_2, \ldots, \psi_m)^T$, $0 \le \psi_j \le 1$, $\sum_{j=1}^m \psi_j = 1$ such that

$$\mathrm{LPFHWA}_{w,\psi}(F_1, F_2, \ldots, F_m) = w_1 F^{/}_{\sigma(1)} \oplus w_2 F^{/}_{\sigma(2)} \oplus \ldots, w_m F^{/}_{\sigma(m)}$$

The weight vectors of $F_j (j = 1, 2, \ldots m)$, are $w = (w_1, w_2, \ldots, w_m)^T$, such that $0 \le w_j \le 1, \sum_{j=1}^m w_j = 1$, and $F^{/}_j = m w_j F_j$, $m$ is the balancing coefficient, $\sigma(1), \sigma(2), \ldots \sigma(m)$ is a permutation of (1, 2, …, $m$), such that $F^{/}_{\sigma(j-1)} \ge F^{/}_{\sigma(j)}$ for all $j = 1, 2, \ldots, m$.

*Example 1* Assume that $F_1 = (\ell_4, \ell_2, \ell_1), F_2 = (\ell_3, \ell_4, \ell_1), F_3 = (\ell_5, \ell_1, \ell_2)$ and $F_4 = (\ell_2, \ell_3, \ell_3) \in \Gamma_{[0,8]}$ are the LPFNs and weight vector $w = (0.4, 0.1, 0.2, 0.3)^T$ of the $F_j( j = 1, 2, 3, 4)$, and $\psi = (0.2, 0.3, 0.3, 0.2)^T$ is the position weighted vector.

According to the definition, we have $\lambda(\ell_\kappa, \ell_\xi, \ell_\delta) = \left(\ell_{\ell - \ell\left(1 - \frac{\kappa}{\ell}\right)^\lambda}, \ell_{\ell\left(\frac{\xi}{\ell}\right)^\lambda}, \ell_{\ell\left(\frac{\delta}{\ell}\right)^\lambda}\right)$ Thus, we have

$$F^{/}_1 = 4 \times 0.4(\ell_4, \ell_2, \ell_1) = (\ell_{5.36}, \ell_{0.87}, \ell_{0.28}),$$
$$F^{/}_2 = 4 \times 0.1(\ell_3, \ell_2, \ell_1) = (\ell_{1.37}, \ell_{4.59}, \ell_{3.48})$$
$$F^{/}_3 = 4 \times 0.2(\ell_5, \ell_1, \ell_2) = (\ell_{4.35}, \ell_{1.52}, \ell_{2.64})$$
$$F^{/}_4 = 4 \times 0.3(\ell_2, \ell_3, \ell_3) = (\ell_{2.34}, \ell_{2.47}, \ell_{2.47})$$

We find the linguistic score and accuracy values of each argument and then rank these arguments: $F^{/}_i (i = 1, 2, 3, 4)$: $z(F^{/}_1) = 4.21$, $z(F^{/}_2) = -6.7$, $z(F^{/}_3) = 0.19$ and $z(F^{/}_4) = -2.6$. The ranking order of the arguments $F^{/}_i (i = 1, 2, 3, 4)$, according to the values of score index, in descending order, are given as:

$$F^{/}_1 \succ F^{/}_3 \succ F^{/}_4 \succ F^{/}_2$$

Now aggregate as

$$\begin{aligned} &= 0.2F^{/}_1 \oplus 0.3F^{/}_3 \oplus 0.3F^{/}_4 \oplus 0.2F^{/}_2 \\ &= 0.2((\ell_{5.36}, \ell_{0.87}, \ell_{0.28}) \oplus 0.3(\ell_{4.35}, \ell_{1.52}, \ell_{2.64}) \\ &\quad \oplus 0.3(\ell_{2.34}, \ell_{2.47}, \ell_{2.47}) + \oplus 0.2(\ell_{1.37}, \ell_{6.06}, \ell_{3.48}) \\ &= (\ell_{3.34}, \ell_{2.57}, \ell_{2.70}) \end{aligned}$$

Further, we propose the linguistic picture fuzzy weighted geometric (LPFWG) operator, linguistic picture fuzzy ordered weighted geometric (LPFOWG) operator and linguistic picture fuzzy hybrid geometric (LPFHG) operator.

**Definition 12** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j}) \quad (j = 1, 2, 3 \ldots, m) \in \Gamma_{[0,l]}$ be the set of all LPFNs. The linguistic picture fuzzy weighted geometric (LPFWG) operator is a mapping $\Gamma^m_{[0,l]} \to \Gamma_{[0,l]}$, such that:

$$\mathrm{LPFWG}_w(F_1, F_2, \ldots, F_m) = F_1^{w_1} \otimes F_2^{w_2} \otimes \ldots \otimes F_m^{w_m}$$

where $w = (w_1, w_2, \ldots, w_m)^T$ are the weight vectors of $F_j (j = 1, 2, \ldots m)$, such that $0 \le w_j \le 1, \sum_{j=1}^m w_j = 1$.

**Theorem 4** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j}) \quad (j = 1, 2, 3 \ldots, m) \in \Gamma_{[0,l]}$ be the set of all LPFNs and $w = (w_1, w_2, \ldots, w_m)^T$ be the weighted vectors of $F_j( j = 1, 2, \ldots m)$, such that $0 \le w_j \le 1, \sum_{j=1}^m w_j = 1$. Then

$$\mathrm{LPFWG}_w(F_1, F_2, \ldots, F_m) = \left( \ell_{l \prod_{j=1}^m \left(\frac{\kappa_j}{l}\right)^{w_j}}, \ell_{l - l \prod_{j=1}^m \left(1 - \frac{\xi_j}{l}\right)^{w_j}}, \ell_{l - l \prod_{j=1}^m \left(1 - \frac{\xi_j}{l}\right)^{w_j}} \right)$$

*Proof* The proof is the same as proof 4. □

**Definition 13** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j}) \quad (j = 1, 2, 3 \ldots, m) \in \Gamma_{[0,l]}$ be the set of all LPFNs. The linguistic picture fuzzy ordered weighted geometric (LPFOWG) operator is a mapping $\Gamma^m_{[0,l]} \to \Gamma_{[0,l]}$, with the associated weight vector $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)^T$, where $0 \le \zeta_j \le 1, \sum_{j=1}^m \zeta_j = 1$, such that

$$\mathrm{LPFOWG}_\zeta(F_1, F_2, \ldots, F_m) = F^{\zeta_1}_{\sigma(1)}, F^{\zeta_2}_{\sigma(2)}, \ldots, F^{\zeta_m}_{\sigma(m)}$$

where $\sigma(1), \sigma(2), \ldots \sigma(m)$ is a permutation of (1, 2, …, $m$), such that $F^{/}_{\sigma(j-1)} \ge F^{/}_{\sigma(j)} \forall j = 1, 2, \ldots, m$.

**Definition 14** Let $F_j = (\ell_{\kappa_j}, \ell_{\xi_j}, \ell_{\delta_j}) \ (j = 1, 2, 3 \ldots, m) \in \Gamma_{[0,l]}$ be the set of all LPFNs. The linguistic picture fuzzy hybrid weighted geometric (LPFHWG) operator is a mapping $\Gamma^m_{[0,l]} \to \Gamma_{[0,l]}$, with the associated weight vector $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)^T$, with $0 \le \zeta_j \le 1, \sum_{j=1}^m \zeta_j = 1$, such that

$$\mathrm{LPFHG}_{w,\zeta}(F_1, F_2, \ldots, F_m) = (F^{/}_{\sigma(1)})^{\zeta_1}, (F^{/}_{\sigma(2)})^{\zeta_2}, \ldots, (F^{/}_{\sigma(m)})^{\zeta_m}$$

where $w = (w_1, w_2, \ldots, w_m)^T$ is the weight vector of $F_j = (1, 2, \ldots, m)$, with $0 \le w_j \le 1, \sum_{j=1}^m w_j = 1$, and $F^{/}_j = F_j^{m w_j}, m$ is the balancing coefficient, where $\sigma(1), \sigma(2), \ldots \sigma(m)$ is a permutation of (1, 2, …, $m$), satisfying $F^{/}_{\sigma(j-1)} \ge F^{/}_{\sigma(j)} \ \forall j = 1, 2, \ldots, m$.

*Example 2* Assume that $F_1=(\ell_5,\ell_2,\ell_1), F_2=(\ell_4,\ell_3,\ell_1), F_3=(\ell_3,\ell_2,\ell_3)$ and $F_4=(\ell_2,\ell_3,\ell_3)\in\Gamma_{[0,8]}$ are the LPFNs and $w=(0.4,0.1,0.2,0.3)^T$ is the weight vector of the $F_j(\ j=(1,2,3,4)$, and $\zeta=(0.2,0.3,0.3,0.2)^T$ is the position weighted vector.

Since, according to the definition, we have $(\ell_\kappa,\ell_\xi,\ell_\delta)^\lambda=\left(\ell_{l\left(\frac{\kappa}{l}\right)^\lambda},\ell_{l-l\left(1-\frac{\xi}{l}\right)^\lambda},\ell_{l-l\left(1-\frac{\delta}{l}\right)^\lambda}\right)$ we have

$$F_1^{/}=(\ell_5,\ell_2,\ell_1)^{4\times0.4}=(\ell_{3.77},\ell_{2.95},\ell_{1.53})$$
$$F_2^{/}=(\ell_4,\ell_3,\ell_1)^{4\times0.1}=(\ell_{1.93},\ell_{1.37},\ell_{0.41})$$
$$F_3^{/}=(\ell_3,\ell_2,\ell_3)^{4\times0.2}=(\ell_{4.34},\ell_{1.64},\ell_{2.50})$$
$$F_4^{/}=(\ell_2,\ell_3,\ell_3)^{4\times0.3}=(\ell_{6.48},\ell_{3.44},\ell_{2.33})$$

We find the linguistic score and accuracy values of each argument and then rank these arguments: $F_i^{/}(i=1,2,3,4)$: $z(F_1^{/})=-0.71$, $z(F_2^{/})=0.15$, $z(F_3^{/})=0.20$ and $z(\aleph_4^{/})=0.71$. Now we rank the arguments $\aleph_i=(1,2,3,4)$, according the values $z(F_i^{/})(i=(1,2,3,4)$, in descending order as:

$$F_4^{/}\geq\ F_3^{/}\geq\ F_2^{/}\geq F_1^{/}$$

Now,

$$\begin{aligned}&=(F_4^{/})^{0.2}\otimes(\ F_3^{/})^{0.3}\otimes(F_2^{/})^{0.3}\otimes(F_1^{/})^{0.2}\\&=(\ell_{6.48},\ell_{3.44},\ell_{2.33})^{0.2}\otimes(\ell_{4.34},\ell_{1.64},\ell_{2.50})^{0.3}\\&\quad\otimes\ (\ell_{1.93},\ell_{1.37},\ell_{0.41})^{0.3}\otimes(\ell_{3.77},\ell_{2.95},\ell_{1.53})^{0.2}\\&=(\ell_{3.54},\ell_{1.99},\ell_{1.27})\end{aligned}$$

## 5 An Approach to Group Decision-Making with Linguistic Picture Fuzzy Information

Let $H=(h_1,h_2,\ldots,h_m)$ be a distinct set of $m$ probable alternatives and $Y=(y_1,y_2,\ldots,y_n)$ be a finite set of $n$ criteria, where $h_i$ indicates the ith alternatives and $y_j$ indicates the jth criteria. Let $D=(d_1,d_2,\ldots,d_t)$ be a finite set of $k$ experts, where $d_k$ indicates the $k$th expert. The expert $d_k$ supplies an appraisal of an alternative $h_i$ on an attribute $y_j$ as an LPFN $r_{ij}^k$ $(i=1,2,\ldots,m;j=1,2,\ldots,n)$ according to a predefined linguistic term set $\ell$. The expert information is represented by the linguistic picture fuzzy decision matrices $R_k=(r_{ij}^k)_{m\times n}(k=1,2,\ldots,p)$.

Assume that $w_i(i=1,2,\ldots,m)$ is the weight vector of the attribute $y_j$ such that $0\leq w_j\ \leq1,\sum_{j=1}^n w_j=1$, and $\psi=(\psi_1,\psi_2,\ldots,\psi_m)$ is the weight vector of the decision-makers $d_j$ such that $\psi_j\leq1,\sum_{j=1}^n\psi_j=1$.

Step 1: Taking the decision information from the given matrix $R_k$, and using the LPFWA operator, the individual total linguistic picture fuzzy preference value $r_i^k$ of the alternative $h_i$ is derived as follows:

$$r_i^k=\mathrm{LPFWA}_w(r_{i1}^k,r_{i2}^k,\ldots,r_{i3}^k),\quad(i=1,2,\ldots,m;k=1,2,\ldots,t)$$

where $w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of the attribute.

Step 2: Due to the separate total linguistic intuitionistic fuzzy preference value $w_i^k$ of alternative $h_i$ $(i=1,2,\ldots,m;k=1,2,\ldots,t)$. Using the LPFHA operator with associated weight vector $\psi=(\psi_1,\psi_2,\ldots,\psi_m)^T$: $r_i^k=\mathrm{LPFHA}_{w,\psi}(r_{i1}^1,r_{i2}^2,\ldots,r_{i3}^t)$, $(i=1,2,\ldots,m)$ Collect the cumulative total linguistic picture fuzzy preference value $r_i$ of the alternative $h_i$ $(i=1,2,\ldots,m)$, where the weight vector of the decision-maker is $\psi=(\psi_1,\psi_2,\ldots,\psi_t)^T$.

Step 3: We find the linguistic score function $z(r_i)$ and the linguistic accuracy function $g(r_i)$ of the cumulative overall linguistic preference value $h_i$ $(i=1,2,\ldots,m)$.

Step 4: By the definition, rank the alternatives $h_i$ $(i=1,2,\ldots,m)$ and choose the best one.

## 6 Numerical Example

A construction company wants to find the best universal supplier for one of its frequently critical sections used in a gathering operation. Assume that $H=(h_1,h_2,h_3,h_4)$ is the set of possible international suppliers (i.e., alternatives) under consideration and $Y=(y_1,y_2,y_3,y_4,y_5)$ is the set of criteria, where $y_i$ $(i=1,2,..,5)$ represent the "total cost of the product", "quality of the product", "service performance of supplier", "supplier's profile" and "risk factor", respectively. According to the linguistic term set, the four alternatives $h_i$ $(i=1,2,3,4)$ are to be charged: $\ell=\ell_0=$ extremely poor, $\ell_1=$ very poor, $\ell_2=$ poor, $\ell_3=$ slightly poor, $\ell_4=$ fair, $\ell_5=$ slightly good, $\ell_6=$ good, $\ell_7=$ very good, $\ell_8=$ extremely good by four decision-makers $d_k(k=1,2,3,4)$ under the given criteria, and construct the linguistic picture fuzzy decision matrices $R_k=(r_{ij}^k)_{4\times5}$ as listed in Tables 1, 2, 3 and 4, respectively.

**Table 1** Decision matrix $R_1$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $h_1$ | $\langle\ell_6,\ell_1,\ell_1\rangle$ | $\langle\ell_5,\ell_1,\ell_2\rangle$ | $\langle\ell_6,\ell_1,\ell_1\rangle$ | $\langle\ell_3,\ell_2,\ell_3\rangle$ | $\langle\ell_1,\ell_4,\ell_1\rangle$ |
| $h_2$ | $\langle\ell_5,\ell_1,\ell_2\rangle$ | $\langle\ell_3,\ell_2,\ell_1\rangle$ | $\langle\ell_2,\ell_1,\ell_4\rangle$ | $\langle\ell_4,\ell_2,\ell_2\rangle$ | $\langle\ell_4,\ell_3,\ell_1\rangle$ |
| $h_3$ | $\langle\ell_3,\ell_1,\ell_3\rangle$ | $\langle\ell_6,\ell_1,\ell_1\rangle$ | $\langle\ell_4,\ell_2,\ell_1\rangle$ | $\langle\ell_5,\ell_2,\ell_1\rangle$ | $\langle\ell_4,\ell_1,\ell_2\rangle$ |
| $h_4$ | $\langle\ell_1,\ell_3,\ell_4\rangle$ | $\langle\ell_3,\ell_3,\ell_2\rangle$ | $\langle\ell_2,\ell_1,\ell_3\rangle$ | $\langle\ell_5,\ell_2,\ell_1\rangle$ | $\langle\ell_2,\ell_2,\ell_4\rangle$ |

**Table 2** Decision matrix $R_2$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $h_1$ | $\langle \ell_6, \ell_1, \ell_1 \rangle$ | $\langle \ell_5, \ell_1, \ell_2 \rangle$ | $\langle \ell_4, \ell_2, \ell_2 \rangle$ | $\langle \ell_3, \ell_2, \ell_3 \rangle$ | $\langle \ell_3, \ell_4, \ell_1 \rangle$ |
| $h_2$ | $\langle \ell_3, \ell_3, \ell_2 \rangle$ | $\langle \ell_3, \ell_4, \ell_1 \rangle$ | $\langle \ell_2, \ell_4, \ell_2 \rangle$ | $\langle \ell_3, \ell_2, \ell_3 \rangle$ | $\langle \ell_1, \ell_1, \ell_5 \rangle$ |
| $h_3$ | $\langle \ell_3, \ell_1, \ell_3 \rangle$ | $\langle \ell_1, \ell_5, \ell_1 \rangle$ | $\langle \ell_2, \ell_2, \ell_2 \rangle$ | $\langle \ell_2, \ell_3, \ell_3 \rangle$ | $\langle \ell_3, \ell_2, \ell_1 \rangle$ |
| $h_4$ | $\langle \ell_4, \ell_2, \ell_2 \rangle$ | $\langle \ell_1, \ell_4, \ell_1 \rangle$ | $\langle \ell_5, \ell_1, \ell_2 \rangle$ | $\langle \ell_5, \ell_2, \ell_1 \rangle$ | $\langle \ell_3, \ell_2, \ell_3 \rangle$ |

**Table 3** Decision matrix $R_3$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $h_1$ | $\langle \ell_4, \ell_2, \ell_1 \rangle$ | $\langle \ell_5, \ell_1, \ell_2 \rangle$ | $\langle \ell_3, \ell_2, \ell_3 \rangle$ | $\langle \ell_2, \ell_3, \ell_2 \rangle$ | $\langle \ell_6, \ell_1, \ell_1 \rangle$ |
| $h_2$ | $\langle \ell_2, \ell_3, \ell_2 \rangle$ | $\langle \ell_1, \ell_4, \ell_2 \rangle$ | $\langle \ell_2, \ell_3, \ell_2 \rangle$ | $\langle \ell_3, \ell_3, \ell_1 \rangle$ | $\langle \ell_1, \ell_1, \ell_5 \rangle$ |
| $h_3$ | $\langle \ell_5, \ell_1, \ell_2 \rangle$ | $\langle \ell_2, \ell_5, \ell_1 \rangle$ | $\langle \ell_2, \ell_3, \ell_2 \rangle$ | $\langle \ell_3, \ell_1, \ell_3 \rangle$ | $\langle \ell_4, \ell_2, \ell_1 \rangle$ |
| $h_4$ | $\langle \ell_6, \ell_1, \ell_1 \rangle$ | $\langle \ell_3, \ell_2, \ell_3 \rangle$ | $\langle \ell_4, \ell_3, \ell_1 \rangle$ | $\langle \ell_4, \ell_2, \ell_2 \rangle$ | $\langle \ell_3, \ell_1, \ell_3 \rangle$ |

**Table 4** Decision matrix $R_4$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $h_1$ | $\langle \ell_3, \ell_2, \ell_1 \rangle$ | $\langle \ell_4, \ell_1, \ell_2 \rangle$ | $\langle \ell_2, \ell_3, \ell_3 \rangle$ | $\langle \ell_2, \ell_3, \ell_2 \rangle$ | $\langle \ell_1, \ell_6, \ell_1 \rangle$ |
| $h_2$ | $\langle \ell_3, \ell_3, \ell_1 \rangle$ | $\langle \ell_1, \ell_5, \ell_2 \rangle$ | $\langle \ell_2, \ell_1, \ell_3 \rangle$ | $\langle \ell_3, \ell_2, \ell_1 \rangle$ | $\langle \ell_2, \ell_1, \ell_5 \rangle$ |
| $h_3$ | $\langle \ell_5, \ell_1, \ell_2 \rangle$ | $\langle \ell_2, \ell_2, \ell_4 \rangle$ | $\langle \ell_2, \ell_3, \ell_2 \rangle$ | $\langle \ell_6, \ell_1, \ell_1 \rangle$ | $\langle \ell_4, \ell_2, \ell_2 \rangle$ |
| $h_4$ | $\langle \ell_4, \ell_1, \ell_2 \rangle$ | $\langle \ell_3, \ell_3, \ell_1 \rangle$ | $\langle \ell_4, \ell_3, \ell_1 \rangle$ | $\langle \ell_2, \ell_4, \ell_2 \rangle$ | $\langle \ell_2, \ell_2, \ell_3 \rangle$ |

Step 1: Assume that the weight vector of the criteria is $w = (w_1, w_2, w_3, w_4, \ w_5)^T = (0.25, 0.2, 0.15, 0.18, 0.22)^T$. Now we can find the individual total preference value $r_1^1$ of candidate $y_1$by mixing the decision matrix $R_1$ and the weight vector of the criteria with the LPFWA operator, which is derived as follows:

$$\begin{aligned} r_1^1 &= LPFWA_w(r_{11}^1, r_{12}^1, r_{13}^1, r_{14}^1, r_{15}^1) \\ &= 0.25(\ell_6, \ell_1, \ell_1) \\ &\quad \oplus 0.2(\ell_5, \ell_1, \ell_2) \oplus 0.15(\ell_6, \ell_1, \ell_1) \\ &\quad \oplus 0.18(\ell_3, \ell_2, \ell_3) \oplus 0.22(\ell_3, \ell_4, \ell_1) \\ &= (\ell_{4.67}, \ell_{1.48}, \ell_{1.39}) \end{aligned}$$

Similarly, we have

$$\begin{aligned} r_2^1 &= (\ell_{3.92}, \ell_{1.60}, \ell_{1.59}), & r_3^1 &= (\ell_{4.77}, \ell_{1.21}, \ell_{1.46}) \\ r_4^1 &= (\ell_{3.02}, \ell_{2.12}, \ell_{2.53}), & r_1^2 &= (\ell_{4.56}, \ell_{1.64}, \ell_{1.40}) \\ r_2^2 &= (\ell_{2.62}, \ell_{2.41}, \ell_{2.26}), & r_3^2 &= (\ell_{2.50}, \ell_{2.12}, \ell_{1.72}) \\ r_4^2 &= (\ell_{3.80}, \ell_{2.04}, \ell_{1.64}), & r_1^3 &= (\ell_{4.44}, \ell_{1.60}, \ell_{1.48}) \\ r_2^3 &= (\ell_{1.81}, \ell_{2.51}, \ell_{2.14}), & r_3^3 &= (\ell_{3.64}, \ell_{1.85}, \ell_{1.62}) \\ r_4^3 &= (\ell_{4.31}, \ell_{1.48}, \ell_{1.82}), & r_1^4 &= (\ell_{2.68}, \ell_{3.07}, \ell_{1.52}) \\ r_2^4 &= (\ell_{2.46}, \ell_{2.05}, \ell_{1.91}) & r_3^4 &= (\ell_{4.22}, \ell_{1.55}, \ell_{2.05}) \\ r_4^4 &= (\ell_{3.09}, \ell_{2.57}, \ell_{1.72}) \end{aligned}$$

Step 2: Suppose the weight vector of four professionals is $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T = (0.25, 0.3, 0.2, 0.25)^T$. Using the (LPFHA) operator with correlated weight vector $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T = (0.15, 0.35, 0.35, 0.15)^T$ : $r_i = LPFHA_{\psi,\zeta}(r_i^1, r_i^2, r_i^3, r_i^4)(i = 1, 2, 3, 4)$ we calculate the separate total linguistic picture fuzzy preference values $r_i^k$ $(k = 1, 2, 3, 4)$ and obtain the cumulative total preference value $w_i$ of alternative $h_i(i = 1, 2, 3, 4)$. By $4\psi_1 r_1^1 = (\ell_{4.67}, \ell_{1.48}, \ell_{1.39})$, $4\psi_2 r_1^2 = (\ell_{5.09}, \ell_{1.19}, \ell_{1.00})$, $4\psi_3 r_1^3 = (\ell_{3.81}, \ell_{2.21}, \ell_{2.07})$, $4\psi_4 r_1^4 = (\ell_{2.68}, \ell_{3.07}, \ell_{1.52})$ and $4\psi_2 r_1^2 \succ 4\psi_1 r_1^1 \succ 4\psi_4 r_1^4 \succ 4\psi_3 r_1^3$.

Step 3: Find the linguistic score index $z(F)(i = 1, 2, 3, 4)$ of the cumulative total preference value $r_i(i = 1, 2, 3, 4)$ as follows: $z(r_1) = 0.62, z(r_2) = -0.48, \ z(r_3) = 0.29, z(r_4) = -0.50$. Rank $r_i$, according to the value of $z(r_i)(i = 1, 2, 3, 4)$ in descending order:

$$r_1 \succ r_3 \succ r_2 \succ r_4$$

We can obtain

$$\begin{aligned} r_1 &= LPFHA_{\psi,\zeta}(r_1^1, r_1^2, r_1^3, r_1^4) \\ &= 0.15(\ell_{5.09}, \ell_{1.19}, \ell_{1.00}) \oplus 0.35(\ell_{4.67}, \ell_{1,48}, \ell_{1.39}) \\ &\quad \oplus 0.35(\ell_{2.68}, \ell_{3.07}, \ell_{1.52}) \oplus 0.15(\ell_{3.81}, \ell_{2.21}, \ell_{2.07}) \\ &= (\ell_{4.02}, \ell_{1.96}, \ell_{1.44}) \end{aligned}$$

Similarly, we have

$$\begin{aligned} r_2 &= LPFHA_{\psi,\zeta}(r_2^1, r_2^2, r_2^3, r_2^4) \\ &= 0.15(\ell_{3.92}, \ell_{1.60}, \ell_{1.59}) \oplus 0.35(\ell_{3.03}, \ell_{1.90}, \ell_{1.70}) \\ &\quad \oplus 0.35(\ell_{2.46}, \ell_{2.05}, \ell_{1.91}) \oplus 0.15(\ell_{1.49}, \ell_{3.17}, \ell_{2.79}) \\ &= (\ell_{3.36}, \ell_{1.99}, \ell_{1.85}) \\ r_4 &= LPFHA_{\psi,\zeta}(r_4^1, r_4^2, r_4^3, r_4^4) \\ &= 0.15(\ell_{4.35}, \ell_{1.55}, \ell_{1.19}) \oplus 0.35(\ell_{3.70}, \ell_{2.07}, \ell_{2.44}) \\ &\quad \oplus 0.35(\ell_{3.09}, \ell_{2.57}, \ell_{1.72}) \oplus 0.15(\ell_{3.02}, \ell_{2.12}, \ell_{2.53}) \\ &= (\ell_{3.55}, \ell_{2.12}, \ell_{1.93}) \end{aligned}$$

Step 4: Choose the best alternative $h_i$, according to $r_i(i = 1, 2, 3, 4)$:

$$h_1 \succ h_3 \succ h_2 \succ h_4$$

which shows that the best alternative is $h_1$.

## 7 A Comparison Analysis to MADM with Uncertain Linguistic Information

The concept of ULOWA and ULHA operators with uncertain linguistic information was introduced by Xu [33]. These operators are used to solve the problem of evaluating university faculty for tenure and promotion [33]. According to Xu [33], practical use involves the assessment of university faculty for tenure and promotion. In this problem, we use the criteria: $h_1$ : teaching, $h_2$ : research, $h_3$ : service. The alternatives (faculty members) $y_j (j = 1,2,3,4,5)$ are to be evaluated with linguistic terms $\ell = \{\ell_0 =$ extremely poor, $\ell_1 =$ very poor, $\ell_2 =$ poor, $\ell_3 =$ slightly poor, $\ell_4 =$ fair, $\ell_5 =$ slightly good, $\ell_6 =$ good, $\ell_7 =$ very good, $\ell_8 =$ extremely good by four decision-makers $j_k (k = 1,2,3,4)$, with the weight vector $w = (0.24, 0.26, 0.23, 0.27)^T$ under these three criteria. Xu [33] used two operators, the ULOWA operator and the ULHA), with associated weight vectors $\psi = (0.3, 0.4, 0.3)^T$, and $\zeta = (0.2, 0.3, 0.3, 0.2)^T$, respectively, to obtain the cumulative total preference value of the alternative. We construct a conclusive matrix and then rank the alternatives as: $y_3 \succ y_2 \succ y_1 \succ y_4 \succ y_5$.

First, transforming the uncertain linguistic decision information into the LIFN forms, i.e., the ULV $[\ell_6, \ell_7]$ in $\Gamma_{[0,8]}$ can be taken over from the LIFNs $(\ell_6, \ell_1)$. The following value transformations are shown as $R_k = (r^k_{ij})_{3\times5} (k = 1,2,3,4)$, where $r^k_{ij}$ takes the form of LIFNs. The results obtained from the alternative $y_j$ with respect to attribute $h_i$,and the decision-maker $d_k$, are listed in Tables 5, 6, 7 and 8, respectively.

**Table 5** Decision matrix $R_1$

| | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ |
|---|---|---|---|---|---|
| $y_1$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_4,\ell_3\rangle$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_7,\ell_0\rangle$ |
| $y_2$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_3,\ell_3\rangle$ | $\langle\ell_5,\ell_1\rangle$ |
| $y_3$ | $\langle\ell_5,\ell_1\rangle$ | $\langle\ell_6,\ell_0\rangle$ | $\langle\ell_6,\ell_0\rangle$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_6,\ell_1\rangle$ |

**Table 6** Decision matrix $R_2$

| | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ |
|---|---|---|---|---|---|
| $y_1$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_4,\ell_2\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_6,\ell_0\rangle$ | $\langle\ell_7,\ell_0\rangle$ |
| $y_2$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_5,\ell_1\rangle$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_6,\ell_1\rangle$ |
| $y_3$ | $\langle\ell_4,\ell_3\rangle$ | $\langle\ell_6,\ell_0\rangle$ | $\langle\ell_6,\ell_0\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_4,\ell_1\rangle$ |

**Table 7** Decision matrix $R_3$

| | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ |
|---|---|---|---|---|---|
| $y_1$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_4,\ell_2\rangle$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_5,\ell_2\rangle$ |
| $y_2$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_4,\ell_3\rangle$ | $\langle\ell_5,\ell_2\rangle$ |
| $y_3$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_4,\ell_1\rangle$ |

**Table 8** Decision matrix $R_4$

| | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ |
|---|---|---|---|---|---|
| $y_1$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_6,\ell_0\rangle$ | $\langle\ell_4,\ell_2\rangle$ | $\langle\ell_4,\ell_2\rangle$ |
| $y_2$ | $\langle\ell_6,\ell_0\rangle$ | $\langle\ell_6,\ell_0\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_5,\ell_1\rangle$ |
| $y_3$ | $\langle\ell_5,\ell_1\rangle$ | $\langle\ell_6,\ell_1\rangle$ | $\langle\ell_7,\ell_0\rangle$ | $\langle\ell_5,\ell_2\rangle$ | $\langle\ell_4,\ell_3\rangle$ |

To collect the separate total preference value of the alternative, we use the LIFOA operator with associated weight vector $w = (0.3, 0.4, 0.3)^T$, after which we use the weight vector of experts $\psi(0.24, 0.26, 0.23, 0.27)^T$ and the LIFHA operator with an associated weight vector $\zeta = (0.2, 0.3, 0.3, 0.2)^T$ to obtain the cumulative total preference value $r_j$ of the alternative $h_i$,which are the following:

$$\begin{aligned} r_1 &= \langle\ell_{5.75}, \ell_0\rangle, \\ r_2 &= \langle\ell_{5.62}, \ell_0\rangle, \\ r_3 &= \langle\ell_{6.05}, \ell_0\rangle, \\ r_4 &= \langle\ell_{5.58}, \ell_0\rangle, \\ r_5 &= \langle\ell_{5.40}, \ell_0\rangle \end{aligned}$$

Since $z(r_1) = 5.75$,$z(r_2) = 5.62$, $z(r_3) = 6.05$, $z(r_4) = 5.58$,$z(r_5) = 5.40$,and $z(r_3) \succ z(r_1) \succ z(r_2) \succ z(r_4) \succ z(r_5)$, the ranking is $h_3 \succ h_1 \succ h_2 \succ h_4 \succ h_5$.

The ranking of the results obtained in this paper is slightly different from the ranking of the results obtained by Xu [33]. The difference occurs in the ranking order of $h_1$and $h_3$, i.e., $h_3 \succ h_1$ by the former and $h_1 \succ h_3$ by the latter. This difference occurs for the following reasons:

1. The main difference between this paper and Xu's paper [33] is that the operations of LPFNs are remarkably different from the operations of ULVs defined by Xu [33]; i.e, the operation of the addition of LPFNs as $(\ell_{\kappa_1}, \ell_{\xi_1}, \ell_{\delta_1}) \oplus (\ell_{\kappa_2}, \ell_{\xi_2}, \ell_{\delta_2}) = \left(\ell_{\kappa_1+\kappa_2-\frac{\kappa_1\kappa_2}{l}}, \ell_{\frac{\xi_1\xi_2}{l}}, \ell_{\frac{\delta_1\delta_2}{l}}\right)$, and the operation of the addition of ULVs by Xu [33] as $[\ell_{\kappa_1}, \ell_{\xi_1}] \oplus [\ell_{\kappa_2}, \ell_{\xi_2}] = [\ell_{\kappa_1+\kappa_2}, \ell_{\xi_1+\xi_2}]$, where $\ell_{\kappa_1}$, $\ell_{\xi_1}, \ell_{\kappa_2}, \ell_{\xi_2} \in \ell_{[0,l]}$. The addition operation of ULVs is not closed, i.e., $\ell_{\kappa_1+\kappa_2}$ and $\ell_{\xi_1+\xi_2}$ may not belong to $\ell_{[0,l]}$.
2. To rank the LPFNs, we used the score function and accuracy function method in this paper. However, the method used by Xu [33] involved analyzing each ULV with all ULVs and then constructing a complementary matrix.

## 8 Conclusion

The concept of picture fuzzy set was proposed by Cuong and Kreinovich, and has become an accepted mathematical mechanism to deal with ambiguity. The linguistic path shows a qualitative facet as linguistic values by means of linguistic variables. To define the ambiguity and uncertainty of the natural world, we can use linguistic variables, which can provide us with greater flexibility. In this paper, we have introduced the notion of linguistic picture fuzzy sets by assimilating picture fuzzy sets and linguistic access. We have defined some operations on picture fuzzy linguistic variables and given their proof. Furthermore, we have discussed the linguistic picture fuzzy operators and applied these operators on multiple-group decision-making problems in which criteria values take the form of linguistic picture fuzzy information. Finally, we have solved a multi-criteria group decision-making problem using the LPFWA and LPFHA operators.

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