Int. J. Fuzzy Syst. (2019) 21(2):407–420
https://doi.org/10.1007/s40815-018-0582-4

Linguistic Intuitionistic Fuzzy Group Decision Making Based on Aggregation Operators

Ruiping Yuan[1] · Jie Tang[2] · Fanyong Meng[2,3]

Received: 2 January 2018 / Revised: 26 August 2018 / Accepted: 31 October 2018 / Published online: 26 November 2018

Abstract This paper researches decision making with linguistic intuitionistic fuzzy variables. Several linguistic intuitionistic fuzzy operations are first defined. Then, several linguistic intuitionistic fuzzy aggregation operators are provided, including the linguistic intuitionistic fuzzy hybrid weighted arithmetical averaging operator, and the linguistic intuitionistic fuzzy hybrid weighted geometric mean operator. Considering the interactive characteristics between the weights of elements in a set, several linguistic intuitionistic fuzzy Shapley aggregation operators are presented, including the linguistic intuitionistic fuzzy hybrid Shapley arithmetical averaging operator, and the linguistic intuitionistic fuzzy hybrid Shapley geometric mean operator. To ensure the application reasonably, several desirable properties are discussed. When the weighting information is incompletely known, models for the optimal fuzzy and additive measures are constructed. After that, an approach to multi-criteria group decision making with linguistic intuitionistic fuzzy information is performed. Finally, a practical example about evaluating different types of engines is provided to illustrate the developed procedure.

✉ Fanyong Meng
mengfanyongtjie@163.com

[1] School of Information, Beijing Wuzi University, Beijing 101149, China

[2] School of Business, Central South University, Changsha 410083, Hunan, China

[3] School of Management and Economics, Nanjing University of Information Science and Technology, Nanjing 210044, China

Keywords Group decision making · Linguistic intuitionistic fuzzy variable · Aggregation operator · Shapley function

1 Introduction

In modern decision-making problems, uncertain and fuzzy information always exists. Considering this situation, researchers introduced Zadeh's fuzzy sets [41] into decision making and developed fuzzy decision-making theory. Atanassov [1] noted that Zadeh's fuzzy sets can only denote the preferred information of decision makers (DMs). When the DMs want to offer their preferred and non-preferred judgments simultaneously, Zadeh's fuzzy sets are helpless. Thus, Atanassov [1] further introduced the concept of intuitionistic fuzzy sets (IFSs) that are composed by a membership degree, a non-membership degree, and a hesitancy degree. After the original work of Atanassov [1], many decision-making methods with intuitionistic fuzzy information are proposed, such as methods based on intuitionistic fuzzy entropies [3, 30], methods based on intuitionistic fuzzy similarity measures [14, 20], and methods based on aggregation operators [17, 43]. Furthermore, Bao et al. [4] developed an intuitionistic fuzzy decision-making method based on prospect theory and evidential reasoning, and Krishankumar et al. [13] proposed an intuitionistic fuzzy PROMETHEE method. The applications of intuitionistic fuzzy decision making can be seen in the literature [5, 28, 38]. To denote the uncertain preferred and non-preferred judgments rather than exact ones, Atanassov and Gargov [2] further introduced the concept of interval-valued intuitionistic fuzzy sets that use intervals in [0, 1] to denote the uncertain preferred and non-preferred recognitions of the DMs, respectively. Because interval-valued intuitionistic fuzzy sets endow the DMs with more flexibility, many researchers devoted themselves to studying interval-valued intuitionistic fuzzy decision making. Taking interval-valued intuitionistic fuzzy decision making based on aggregation operators for example, interval-valued intuitionistic fuzzy aggregation operators can be classified into five types following the adopted operational laws: the interval-valued intuitionistic fuzzy Archimedean aggregation operator [35], the interval-valued intuitionistic fuzzy Frank aggregation operator [44], the interval-valued intuitionistic fuzzy Hamacher aggregation operator [15], the interval-valued intuitionistic fuzzy Einstein aggregation operator [36], and the interval-valued intuitionistic fuzzy Algebraic aggregation operator [16, 21, 37]. Note that the Einstein and Algebraic aggregation operators are two special cases of the Hamacher aggregation operators.

Although (interval-valued) intuitionistic fuzzy sets are efficient to denote the preferred and non-preferred information of the DMs, they only permit the DMs to use real numbers or intervals in [0, 1] to express the quantitative judgments. However, in many situations, the DMs may want to give their qualitative preferences; namely, they use linguistic variables to express their judgments, such as "good", "bad", or "fair". Zadeh [42] first noted this issue and introduced the concept of linguistic variables. To facilitate the application, Herrera et al. [10] presented the concept of linguistic term sets to denote the possible values of linguistic variables. With the development of decision making with linguistic information, several extending forms based on different points of view are proposed, such as hesitant fuzzy linguistic term sets [29], interval intuitionistic uncertain linguistic fuzzy sets [18], linguistic hesitant fuzzy sets [22], uncertain linguistic hesitant fuzzy sets [23], and interval-valued intuitionistic uncertain linguistic sets [24]. To avoid information loss, there are two linguistic computing models: the 2-tuple linguistic representation model [6, 11] and the continuous linguistic representation model [39]. Dong et al. [9] proved the equivalence of these two representation models. Most of all researches about decision making with linguistic information are based on these two models. For instance, decision making based on 2-tuple linguistic representation model is studied in the literature [7, 25, 40], while decision making using the continuous linguistic representation model is discussed in the literature [26, 46, 47].

However, all of the above-mentioned linguistic variables cannot denote the preferred and non-preferred qualitative judgments of the DMs. To address this problem, Chen et al. [8] presented the concept of linguistic intuitionistic fuzzy variables (LIFVs) and defined several basic operations. Then, the authors defined two aggregation operators: the linguistic intuitionistic fuzzy weighted geometric (LIFWG) operator and the linguistic intuitionistic fuzzy hybrid geometric (LIFHG) operator, by which a group decision-making method with linguistic intuitionistic fuzzy information is presented. Garg and Kumar [10] introduced the set pair analysis (SPA) theory into LIFVs and proposed the linguistic connection number (LCN). Then, the authors developed an approach for decision making with LIFVs using the linguistic connection number ordered weighted geometric (LCNOWG) operator and the linguistic connection number hybrid geometric (LCNHG) operator. Furthermore, Liu and Liu [19] introduced a method for decision making with LIFVs using the scaled prioritized linguistic intuitionistic fuzzy weighted operator. Notably, this method needs to distinguish the categories of attributes. Moreover, Zhang et al. [45] offered an extended outranking approach for decision making with LIFVs.

After reviewing researches about decision making with LIFVs, we find that there are several limitations. The linguistic intuitionistic fuzzy hybrid operators [8, 10] do not satisfy idempotency and boundary, and methods in [8, 10] are based on the assumption that the weighting information is completely known. While methods in [19, 45] did not research group decision making. Furthermore, all of these methods are based on the assumption that there is no interaction between the weights of elements in a set. To avoid these issues, this paper continues to study linguistic intuitionistic fuzzy decision making based on aggregation operators. To do this, several linguistic intuitionistic fuzzy operational laws are defined. To calculate comprehensive LIFVs, several linguistic intuitionistic fuzzy aggregation operators are offered. Meanwhile, several linguistic intuitionistic fuzzy Shapley aggregation operators are provided to cope with the situations where there are interactive characteristics. Furthermore, models for determining the optimal fuzzy and additive measures are constructed to address decision making with incomplete weighting information. The organization is offered as follows:

Section 2 first reviews several basic concepts, including linguistic variables, LIFVs, and a ranking order. Then, it defines several linguistic intuitionistic fuzzy operational laws. Section 3 contains two parts. The first part defines several linguistic intuitionistic fuzzy aggregation operators based on additive measures, and the second part provides several linguistic intuitionistic fuzzy Shapley aggregation operators to reflect the interactions between the weights of elements. Section 4 builds several programming models to determine fuzzy and additive measures on the DM set, on the criteria set and on their ordered sets, respectively. Section 5 gives a group decision-making method with linguistic intuitionistic fuzzy information. Meanwhile, a practical example is offered to show the application of the new method.

2 Preliminary

To denote the qualitative judgments of the DMs rather than quantitative ones, linguistic variables introduced by Zadeh [42] are useful, which permit the DMs to apply linguistic variables to express their judgments, such as "fast", "slow", or "fair". To facilitate the application of linguistic variables, Herrera and Martinez [11] introduced linguistic term sets to denote linguistic variables. For example, a linguistic term set may be denoted as $S = \{s_i \mid i = 0, 1,\ldots, 2t\}$, where t is a positive integer. The linguistic term s_i represents a possible value for a linguistic variable. Furthermore, Herrera and Martinez [11] defined the following four properties of linguistic terms: (1) The set is ordered: $s_i > s_j$, if $i > j$; (2) Maximum operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$; (3) Minimum operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$; and (4) A negation operator: $neg(s_i) = s_j$ such that $j = 2t-i$.

For example, the linguistic term set S may be expressed as $S = \{s_0$: extremely bad, s_1: very bad, s_2: bad, s_3: fair, s_4: good, s_5: very good, s_6: extremely good$\}$. To preserve information, Xu [39] extended the discrete linguistic term set S to the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. For any $s_\alpha \in S_c$, if $s_\alpha \in S$, then it is called an original linguistic term. Otherwise, it is called a virtual linguistic term.

Let s_α and s_β be any two linguistic variables, then several of their operational laws are defined as follows [39]:

(1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;
(2) $s_\alpha \ominus s_\beta = s_{\alpha-\beta}$;
(3) $\lambda s_\alpha = s_{\lambda\alpha}$, $\lambda \in [0, 1]$;
(4) $\lambda(s_\alpha \oplus s_\beta) = \lambda s_\alpha \oplus \lambda s_\beta$, $\lambda \in [0, 1]$;
(5) $(\lambda_1 + \lambda_2)s_\alpha = \lambda_1 s_\alpha \oplus \lambda_2 s_\alpha$, $\lambda_1, \lambda_2 \in [0, 1]$.

To denote the preferred and non-preferred qualitative judgments of the DMs, Chen et al. [8] introduced linguistic intuitionistic fuzzy variables (LIFVs) in a similar way as intuitionistic fuzzy variables [1].

Definition 1 [8] A LIFV on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$ is expressed as $\tilde{s} = (s_\alpha, s_\beta)$, where s_α and s_β are the preferred and non-preferred qualitative degrees, respectively, and $s_\alpha \oplus s_\beta \leq s_{2t}$.

Considering the order relationship between LIFVs, Zhang et al. [45] offered the following ranking order

Definition 2 [45] Let $\tilde{s}_1 = (s_{\alpha_1}, s_{\beta_1})$ and $\tilde{s}_2 = (s_{\alpha_2}, s_{\beta_2})$ be any two LIFVs on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, their order relationship is defined as follows:

(1) $\tilde{s}_1 \geq \tilde{s}_2$ if $s_{\alpha_1} \geq s_{\alpha_2} \wedge s_{\beta_1} \leq s_{\beta_2} \wedge s_{2t-\alpha_1-\beta_1} \leq s_{2t-\alpha_2-\beta_2}$;
(2) $\tilde{s}_1 = \tilde{s}_2$ if $\tilde{s}_1 \geq \tilde{s}_2 \wedge \tilde{s}_1 \leq \tilde{s}_2$.

Next, we define several linguistic intuitionistic fuzzy operational laws to calculate the comprehensive LIFVs.

Definition 3 Let $\tilde{s}_1 = (s_{\alpha_1}, s_{\beta_1})$ and $\tilde{s}_2 = (s_{\alpha_2}, s_{\beta_2})$ be any two LIFVs on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, their operational laws are defined as follows:

(1) $\lambda \tilde{s}_1 = (s_{\lambda\alpha_1}, s_{\lambda\beta_1})$, $\lambda \in [0, 1]$;
(2) $\tilde{s}_1^\lambda = \left(s_{\alpha_1^\lambda}, s_{\beta_1^\lambda}\right)$, $\lambda \in [0, 1]$;
(3) $\lambda_1 \tilde{s}_1 \oplus \lambda_2 \tilde{s}_2 = (s_{\lambda_1\alpha_1+\lambda_2\alpha_2}, s_{\lambda_1\beta_1+\lambda_2\beta_2})$, $\lambda_1, \lambda_2 \in [0, 1]$ $\wedge \lambda_1 + \lambda_2 \leq 1$;
(4) $\tilde{s}_1^{\lambda_1} \otimes \tilde{s}_2^{\lambda_2} = \left(s_{\alpha_1^{\lambda_1} \times \alpha_2^{\lambda_2}}, s_{\beta_1^{\lambda_1} \times \beta_2^{\lambda_2}}\right)$, $\lambda_1, \lambda_2 \in [0, 1] \wedge \lambda_1$ $+\lambda_2 \leq 1$.

From Definition 3, we can check that their results are still LIFVs in S_c.

Definition 2 shows that Zhang's ranking method for LIFVs needs the preferred, non-preferred, and hesitant linguistic values of LIFVs to satisfy the defined relationship simultaneously. This ranking method in fact compares three-dimensional vectors formed by LIFVs. Thus, many situations are incomparable. For example, let $\tilde{s}_1 = (s_4, s_5)$ and $\tilde{s}_2 = (s_3, s_4)$ be two LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 8]\}$. Because $s_4 > s_3$ and $s_5 > s_4$, we cannot derive their order relationship following Zhang's ranking method. However, we can easily check that $\tilde{s}_1$ is better than $\tilde{s}_2$. Considering this situation, we applied the ranking order introduced in the literature [8]:

Definition 4 [8] Let $\tilde{s} = (s_\alpha, s_\beta)$ be a LIFV on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, the score function is defined as: $S(\tilde{s}) = \alpha - \beta$, and the accuracy function is defined as: $A(\tilde{s}) = \alpha + \beta$.

Let $\tilde{s}_1 = (s_{\alpha_1}, s_{\beta_1})$ and $\tilde{s}_2 = (s_{\alpha_2}, s_{\beta_2})$ be any two LIFVs on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, their order relationship is defined as follows:

If $S(\tilde{s}_1) \geq S(\tilde{s}_2)$, then $\tilde{s}_1 \geq \tilde{s}_2$;

If $S(\tilde{s}_1) = S(\tilde{s}_2)$, then $\begin{cases} A(\tilde{s}_1) > A(\tilde{s}_2), \tilde{s}_1 > \tilde{s}_2 \\ A(\tilde{s}_1) = A(\tilde{s}_2), \tilde{s}_1 = \tilde{s}_2 \end{cases}$.

Property 1 *Let $\tilde{s}_1 = (s_{\alpha_1}, s_{\beta_1})$ and $\tilde{s}_2 = (s_{\alpha_2}, s_{\beta_2})$ be any two LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, $\tilde{s}_1 = \tilde{s}_2$ if and only if $s_{\alpha_1} = s_{\alpha_2}$ and $s_{\beta_1} = s_{\beta_2}$.*

Proof When $\tilde{s}_1 = \tilde{s}_2$, we have $S(\tilde{s}_1) = S(\tilde{s}_2)$ and $A(\tilde{s}_1) = A(\tilde{s}_2)$, namely, $\alpha_1 - \beta_1 = \alpha_2 - \beta_2$ and $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$. Thus, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. On the other hand, if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, we easily derive $\tilde{s}_1 = \tilde{s}_2$. □

3 Several Linguistic Intuitionistic Fuzzy Aggregation Operators

On the basis of the defined linguistic intuitionistic fuzzy operations, this section introduces several linguistic intuitionistic fuzzy aggregation operators. Following the adopted weighting information, we classify them into two types. The first type uses additive measures, while the second type applies fuzzy measures.

3.1 Linguistic Intuitionistic Fuzzy Aggregation Operators Based on Additive Measures

Definition 5 Let $\tilde{s}_i = \left(s_{\alpha_i}, s_{\beta_i}\right)$, $i = 1, 2, \ldots, n$, be a set of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, the linguistic intuitionistic fuzzy weighted arithmetical averaging (LIFWAA) operator is defined as:

$$\mathrm{LIFWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \oplus_{i=1}^{n} \omega_i \tilde{s}_i = \left(s_{\sum_{i=1}^{n} \omega_i \alpha_i}, s_{\sum_{i=1}^{n} \omega_i \beta_i}\right) \quad (1)$$

and the linguistic intuitionistic fuzzy weighted geometric mean (LIFWGM) operator is defined as:

$$\mathrm{LIFWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \otimes_{i=1}^{n} \tilde{s}_i^{\omega_i} = \left(s_{\Pi_{i=1}^{n} \alpha_i^{\omega_i}}, s_{\Pi_{i=1}^{n} \beta_i^{\omega_i}}\right) \quad (2)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a weighting vector on $\{\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\}$ such that $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$ for all $i = 1, 2, \ldots, n$.

Considering the importance of the ordered positions to reduce the influence of extreme values, the linguistic intuitionistic fuzzy ordered weighted arithmetical averaging (LIFOWAA) operator and the linguistic intuitionistic fuzzy ordered weighted geometric mean (LIFOWGM) operator are defined as follows:

Definition 6 Let $\tilde{s}_i = \left(s_{\alpha_i}, s_{\beta_i}\right)$, $i = 1, 2, \ldots, n$, be a set of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, the LIFOWAA operator is defined as:

$$\mathrm{LIFOWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \oplus_{j=1}^{n} w_j \tilde{s}_{(j)} = \left(s_{\sum_{j=1}^{n} w_j \alpha_{(j)}}, s_{\sum_{j=1}^{n} w_j \beta_{(j)}}\right) \quad (3)$$

and the LIFOWGM operator is defined as:

$$\mathrm{LIFOWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \otimes_{j=1}^{n} \tilde{s}_{(j)}^{w_j} = \left(s_{\Pi_{j=1}^{n} \alpha_{(j)}^{w_j}}, s_{\Pi_{j=1}^{n} \beta_{(j)}^{w_j}}\right) \quad (4)$$

where $(\cdot)$ is a permutation on $N = \{1, 2, \ldots, n\}$ such that $\tilde{s}_{(1)} \leq \tilde{s}_{(2)} \leq \cdots \leq \tilde{s}_{(n)}$, and $w = (w_1, w_2, \ldots, w_n)$ is a weighting vector on the ordered position set N such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \geq 0$ for all $j = 1, 2, \ldots, n$.

Definitions 5 and 6 show that the LIFWAA and LIFWGM operators only give the importance of LIFVs, while the LIFOWAA and LIFOWGM operators only consider the weights of the ordered positions. To show these two aspects simultaneously, we further offer the linguistic intuitionistic fuzzy hybrid weighted arithmetical averaging (LIFHWAA) operator and the linguistic intuitionistic fuzzy hybrid weighted geometric mean (LIFHWGM) operator as follows:

Definition 7 Let $\tilde{s}_i = \left(s_{\alpha_i}, s_{\beta_i}\right)$, $i = 1, 2, \ldots, n$, be a set of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, the LIFHWAA operator is defined as:

$$\begin{aligned}\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \oplus_{j=1}^{n} \frac{w_j \omega_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}} \tilde{s}_{(j)} \\ &= \left(s_{\sum_{j=1}^{n} \frac{w_j \omega_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}} \alpha_{(j)}}, s_{\sum_{j=1}^{n} \frac{w_j \omega_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}} \beta_{(j)}}\right)\end{aligned} \quad (5)$$

and the LIFHWGM operator is defined as:

$$\begin{aligned}\mathrm{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \otimes_{j=1}^{n} \tilde{s}_{(j)}^{\frac{w_j \omega_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}}} \\ &= \left(s_{\Pi_{j=1}^{n} \alpha_{(j)}^{\frac{w_j \omega_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}}}}, s_{\Pi_{j=1}^{n} \beta_{(j)}^{\frac{w_j \omega_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}}}}\right)\end{aligned} \quad (6)$$

where $(\cdot)$ is a permutation on N such that $\omega_{(1)} \tilde{s}_{(1)} \leq \omega_{(2)} \tilde{s}_{(2)} \leq \cdots \leq \omega_{(n)} \tilde{s}_{(n)}$, and the other notations as shown in Definitions 5 and 6.

Theorem 1 *Let $\tilde{s}_i = \left(s_{\alpha_i}, s_{\beta_i}\right)$, $i = 1, 2, \ldots, n$, be a set of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, their aggregated values using the LIFHWAA and LIFHWGM operators are still IVIFVs.*

Proof Following Eqs. (5) and (6), we can easily derive the conclusions. □

Next, we discuss several desirable properties of the above-defined aggregation operators.

Theorem 2 *Let $\tilde{s}_i = \left(s_{\alpha_i}, s_{\beta_i}\right)$ and $\tilde{t}_i = \left(s_{\mu_i}, s_{v_i}\right)$, $i = 1, 2, \ldots, n$, be any two collections of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$.*

(1) ***Commutativity.*** *Let* $\tilde{s}'_i = \left(s_{\alpha'_i}, s_{\beta'_i}\right)$ *be a permutation of* $\tilde{s}_i$, *i = 1, 2, …, n, then*

$$\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \mathrm{LIFHWAA}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n)$$
$$\mathrm{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \mathrm{LIFHWGM}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n)$$

(2) ***Idempotency.*** *If all LIFVs* $\tilde{s}_i$, *i = 1, 2, …, n, are equal, i.e.,* $\tilde{s}_i = \tilde{s} = (s_\alpha, s_\beta)$ *for all i, then*

$$\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}$$
$$\mathrm{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}$$

(3) ***Comonotonicity.*** *If* $\tilde{s}_i$ *and* $\tilde{t}_i$ *are comonotonic, namely,* $\tilde{s}_{(1)} \le \tilde{s}_{(2)} \le \cdots \le \tilde{s}_{(n)}$ *if and only if* $\tilde{t}_{(1)} \le \tilde{t}_{(2)} \le \ldots \le \tilde{t}_{(n)}$ *for all i, where* $(\cdot)$ *is a permutation on N = {1, 2, …, n} such that* $\tilde{s}_{(j)}$ *and* $\tilde{t}_{(j)}$ *are the jth least values of* $\tilde{s}_i$ *and* $\tilde{t}_i$, *i = 1, 2, …, n, respectively. Then,*

$$\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \mathrm{LIFHWAA}(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$$
$$\mathrm{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \mathrm{LIFHWGM}(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$$

(4) ***Boundary.*** *Let* $\tilde{s}^- = \left(s_{\min_{i\in N}\alpha_i}, s_{\max_{i\in N}\beta_i}\right)$ *and* $\tilde{s}^+ = \left(s_{\max_{i\in N}\alpha_i}, s_{\min_{i\in N}\beta_i}\right)$, *then*

$$\tilde{\alpha}^- \le \mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \tilde{\alpha}^+ \text{ and } \tilde{\alpha}^- \le \mathrm{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \tilde{\alpha}^+$$

where N = {1, 2, …, n}.

Proof For (1): We have

$$\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \left(s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\alpha_{(j)}}, s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\beta_{(j)}}\right) = \left(s_{\sum_{j=1}^n \frac{w_j\omega'_{(j)}}{\sum_{j=1}^n w_j\omega'_{(j)}}\alpha'_{(j)}}, s_{\sum_{j=1}^n \frac{w_j\omega'_{(j)}}{\sum_{j=1}^n w_j\omega'_{(j)}}\beta'_{(j)}}\right) = \mathrm{LIFHWAA}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n).$$

Similarly, we have $\mathrm{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \mathrm{LIFHWGM}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n)$.

For (2): We have

$$\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \left(s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\alpha_{(j)}}, s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\beta_{(j)}}\right) = \left(s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\alpha}, s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\beta}\right) = (s_\alpha, s_\beta) = \tilde{s}$$

and

$$\mathrm{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \left(s_{\prod_{j=1}^n \alpha_{(j)}^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}}}, s_{\prod_{j=1}^n \beta_{(j)}^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}}}\right) = \left(s_{\alpha^{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}}}, s_{\beta^{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}}}\right) = (s_\alpha, s_\beta) = \tilde{s}$$

For (3): From $\tilde{s}_{(i)} \le \tilde{s}_{(j)}$ and $\tilde{t}_{(i)} \le \tilde{t}_{(j)}$ for any *i*, *j* = 1, 2, …, *n*, we obtain $\omega_{(i)}\tilde{s}_{(i)} \le \omega_{(j)}\tilde{s}_{(j)} \Leftrightarrow \omega_{(i)}\tilde{t}_{(i)} \le \omega_{(j)}\tilde{t}_{(j)}$ and $\omega_{(i)}\tilde{s}_{(i)} \ge \omega_{(j)}\tilde{s}_{(j)} \Leftrightarrow \omega_{(i)}\tilde{t}_{(i)} \ge \omega_{(j)}\tilde{t}_{(j)}$. Taking the first case for example, if $\omega_{(i)}\tilde{s}_{(i)} \le \omega_{(j)}\tilde{s}_{(j)}$, we have

$$LS(\omega_{(i)}\tilde{s}_{(i)}) \le LS(\omega_{(j)}\tilde{s}_{(j)}) \text{ or } LS(\omega_{(i)}\tilde{s}_{(i)}) = LS(\omega_{(j)}\tilde{s}_{(j)}) \text{ and } LA(\omega_{(j)}\tilde{s}_{(j)}) \le LA(\omega_{(j)}\tilde{s}_{(j)})$$

by which we derive $s_{\omega_{(i)}(\alpha_{(i)}-\beta_{(i)})} \le s_{\omega_{(j)}(\alpha_{(j)}-\beta_{(j)})}$ or $s_{\omega_{(i)}(\alpha_{(i)}-\beta_{(i)})} = s_{\omega_{(j)}(\alpha_{(j)}-\beta_{(j)})}$ and $s_{\omega_{(i)}(\alpha_{(i)}+\beta_{(i)})} \le s_{\omega_{(j)}(\alpha_{(j)}+\beta_{(j)})}$. Thus,

$$s_{\alpha_{(i)}-\beta_{(i)}} \le s_{\alpha_{(j)}-\beta_{(j)}} \text{ or } s_{\alpha_{(i)}-\beta_{(i)}} = s_{\alpha_{(j)}-\beta_{(j)}} \text{ and } s_{\alpha_{(i)}+\beta_{(i)}} \le s_{\alpha_{(j)}+\beta_{(j)}}.$$

Following the order relationship of $\tilde{s}_i$ and $\tilde{t}_i$, *i* = 1, 2, …, *n*, we have $\omega_{(i)}\tilde{s}_{(i)} \le \omega_{(j)}\tilde{s}_{(j)} \Leftrightarrow \omega_{(i)}\tilde{t}_{(i)} \le \omega_{(j)}\tilde{t}_{(j)}$. Following Eq. (5), we obtain

$$\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \left(s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\alpha_{(j)}}, s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\beta_{(j)}}\right) \le \left(s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\mu_{(j)}}, s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}v_{(j)}}\right) = \mathrm{LIFHWAA}(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$$

Furthermore, we have $\mathrm{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \mathrm{LIFHWGM}(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$.

For (4): From $\tilde{s}^- \le \tilde{s}_i \le \tilde{s}^+$ for any *i* = 1, 2, …, *n*, we have

$$\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \left(s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\alpha_{(j)}}, s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\beta_{(j)}}\right) \ge \left(s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\min_{j\in N}\alpha_{(j)}}, s_{\sum_{j=1}^n \frac{w_j\omega_{(j)}}{\sum_{j=1}^n w_j\omega_{(j)}}\min_{j\in N}\beta_{(j)}}\right) = \left(s_{\min_{j\in N}\alpha_{(j)}}, s_{\min_{j\in N}\beta_{(j)}}\right) = \left(s_{\min_{j\in N}\alpha_j}, s_{\min_{j\in N}\beta_j}\right) = \tilde{s}^-$$

Similarly, we obtain $\mathrm{LIFHWAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \tilde{s}^+$. □

We can further have $\tilde{\alpha}^- \leq \text{LIFHWGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \tilde{\alpha}^+$ as the proof for the LIFHWAA operator.

Remark 1 Because the LIFWAA, LIFWGM, LIFOWAA, and LIFOWGM operators can be seen as special cases of the LIFHWAA and LIFHWGM operators, they all satisfy the properties listed in Theorem 2.

3.2 Linguistic Intuitionistic Fuzzy Aggregation Operators Based on Fuzzy Measures

The aggregation operators defined in Sect. 3.1 are based on the assumption that the weights of elements in a set are independent. However, this conclusion does not hold in many situations [23, 24, 35]. Considering the interactions among the weights of elements, this subsection defines several linguistic intuitionistic fuzzy Shapley aggregation operators based on fuzzy measures. First, we recall the concept of fuzzy measures.

Definition 8 [31] A fuzzy measure μ on the finite set $X = \{x_1, x_2, \ldots, x_n\}$ is a set function μ: $P(X) \to [0, 1]$ satisfying

(1) $\mu(\phi) = 0$, $\mu(X) = 1$,
(2) If $A, B \in P(X)$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$,

where $P(X)$ is the power set of X.

In cooperative game theory, the Shapley function [32] is one of the most important payoff indices that satisfies several desirable properties, such as efficiency, dummy, null, symmetry, and linear property. When we restrict the Shapley function on fuzzy measures, the following expression is obtained:

$$Sh_{x_i}(\mu, X) = \sum_{S \subseteq N \backslash x_i} \frac{(n-s-1)!s!}{n!}(\mu(S \cup x_i) - \mu(S)) \quad \forall x_i \in X \tag{7}$$

where μ is a fuzzy measure on $X = \{x_1, x_2, \ldots, x_n\}$, s, t, and n are the cardinalities of S, T, and X, respectively.

Property 2 *[27] Let μ be a fuzzy measure on $X = \{x_1, x_2, \ldots, x_n\}$, and let Sh be the Shapley function for the fuzzy measure μ on the set X. Then, we have $\sum_{i=1}^{n} Sh_{x_i}(\mu, X) = 1$ and $Sh_{x_i}(\mu, X) \geq 0$ for any $x_i \in X$.*

Property 2 shows that the Shapley function Sh is a weighting vector for the fuzzy measure μ. Using the Shapley function shown in Eq. (7), the linguistic intuitionistic fuzzy Shapley arithmetical averaging (LIFSAA) operator and the linguistic intuitionistic fuzzy Shapley geometric mean (LIFSGM) operator are defined as follows:

Definition 9 Let $\tilde{s}_i = (s_{\alpha_i}, s_{\beta_i})$, $i = 1, 2, \ldots, n$, be a set of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, the LIFSAA operator is defined as:

$$\text{LIFSAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \oplus_{i=1}^{n} Sh_{\tilde{s}_i}(\mu, \tilde{S})\tilde{s}_i = \left(s_{\sum_{i=1}^{n} Sh_{\tilde{s}_i}(\mu,\tilde{S})\alpha_i}, s_{\sum_{i=1}^{n} Sh_{\tilde{s}_i}(\mu,\tilde{S})\beta_i}\right) \tag{8}$$

and the LIFSGM operator is defined as:

$$\text{LIFSGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \otimes_{i=1}^{n} \tilde{s}_i^{Sh_{\tilde{s}_i}(\mu,\tilde{S})} = \left(s_{\Pi_{i=1}^{n} \alpha_i^{Sh_{\tilde{s}_i}(\mu,\tilde{S})}}, s_{\Pi_{i=1}^{n} \beta_i^{Sh_{\tilde{s}_i}(\mu,\tilde{S})}}\right) \tag{9}$$

where $Sh_{\tilde{s}_i}(\mu, \tilde{S})$ is the Shapley value of the LIFV $\tilde{s}_i$, and μ is a fuzzy measure on the set $\tilde{S} = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$.

Similar to the LIFSAA and LIFSGM operators, the linguistic intuitionistic fuzzy ordered Shapley arithmetical averaging (LIFOSAA) operator and the linguistic intuitionistic fuzzy ordered Shapley geometric mean (LIFOSGM) operator are defined as follows:

Definition 10 Let $\tilde{s}_i = (s_{\alpha_i}, s_{\beta_i})$, $i = 1, 2, \ldots, n$, be a set of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, the LIFOSAA operator is defined as:

$$\text{LIFOSAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \oplus_{j=1}^{n} Sh_j(v, N)\tilde{s}_{(j)} = \left(s_{\sum_{j=1}^{n} Sh_j(v,N)\alpha_{(j)}}, s_{\sum_{j=1}^{n} Sh_j(v,N)\beta_{(j)}}\right) \tag{10}$$

and the LIFOSGM operator is defined as:

$$\text{LIFOSGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \otimes_{i=1}^{n} \tilde{s}_{(j)}^{Sh_j(v,N)} = \left(s_{\Pi_{i=1}^{n} \alpha_{(j)}^{Sh_j(v,N)}}, s_{\Pi_{i=1}^{n} \beta_{(j)}^{Sh_j(v,N)}}\right) \tag{11}$$

where $(\cdot)$ is a permutation on $N = \{1, 2, \ldots, n\}$ such that $\tilde{s}_{(1)} \leq \tilde{s}_{(2)} \leq \cdots \leq \tilde{s}_{(n)}$, and $Sh_j(v, N)$ is the Shapley value of the jth ordered position, and v is a fuzzy measure on the set N.

Considering the importance of these two aspects, we further define the linguistic intuitionistic fuzzy hybrid Shapley arithmetical averaging (LIFHSAA) operator and the linguistic intuitionistic fuzzy hybrid Shapley geometric mean (LIFHSGM) operator.

Definition 11 Let $\tilde{s}_i = (s_{\alpha_i}, s_{\beta_i})$, $i = 1, 2, \ldots, n$, be a set of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, the LIFHSAA operator is defined as:

$$\mathrm{LIFHSAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \oplus_{j=1}^{n} \frac{Sh_j(\mu, N)Sh_{\tilde{s}_{(j)}}(\mu, \tilde{S})}{\sum_{j=1}^{n} Sh_j(\mu, N)Sh_{\tilde{s}_{(j)}}(\mu, \tilde{S})} \tilde{s}_{(j)} = \left(s_{\frac{Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}{\sum_{j=1}^{n} Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}\alpha_{(j)}}, s_{\frac{Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}{\sum_{j=1}^{n} Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}\beta_{(j)}} \right) \quad (12)$$

and the LIFHSGM operator is defined as:

$$\mathrm{LIFHSGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \otimes_{i=1}^{n} \tilde{s}_{(j)}^{\frac{Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}{\sum_{j=1}^{n} Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}} = \left(s_{\Pi_{i=1}^{n} \alpha_{(j)}^{\frac{Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}{\sum_{j=1}^{n} Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}}}, s_{\Pi_{i=1}^{n} \beta_{(j)}^{\frac{Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}{\sum_{j=1}^{n} Sh_j(\mu,N)Sh_{\tilde{s}_{(j)}}(\mu,\tilde{S})}}} \right) \quad (13)$$

where $(\cdot)$ is a permutation on N such that $Sh_{\tilde{s}_{(1)}}(\mu, \tilde{S})\tilde{s}_{(1)} \leq Sh_{\tilde{s}_{(2)}}(\mu, \tilde{S})\tilde{s}_{(2)} \leq \cdots \leq Sh_{\tilde{s}_{(n)}}(\mu, \tilde{S})\tilde{s}_{(n)}$, and the other notations as shown in Definitions 9 and 10.

Theorem 3 *Let $\tilde{s}_i = (s_{\alpha_i}, s_{\beta_i})$, i = 1, 2, …, n, be a set of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$. Then, their aggregated values using the LIFHSAA and LIFHSGM operators are still IVIFVs.*

Theorem 4 *Let $\tilde{s}_i = (s_{\alpha_i}, s_{\beta_i})$ and $\tilde{t}_i = (s_{\mu_i}, s_{v_i})$, i = 1, 2, …, n, be any two collections of LIFVs defined on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$.*

(1) ***Commutativity.*** *Let $\tilde{s}'_i = (s_{\alpha'_i}, s_{\beta'_i})$ be a permutation of $\tilde{s}_i$, i = 1, 2, …, n, then*

$$\mathrm{LIFHSAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \mathrm{LIFHSAA}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n)$$
$$\mathrm{LIFHSGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \mathrm{LIFHSGM}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n)$$

(2) ***Idempotency.*** *If all LIFVs $\tilde{s}_i$, i = 1, 2, …, n, are equal, i.e., $\tilde{s}_i = \tilde{s} = (s_\alpha, s_\beta)$ for all i, then*

$$\mathrm{LIFHSAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{\alpha}$$
$$\mathrm{LIFHSGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{\alpha}$$

(3) ***Comonotonicity.*** *If $\tilde{s}_i$ and $\tilde{t}_i$ are comonotonic, namely, $\tilde{s}_{(1)} \leq \tilde{s}_{(2)} \leq \cdots \leq \tilde{s}_{(n)}$ if and only if $\tilde{t}_{(1)} \leq \tilde{t}_{(2)} \leq \cdots \leq \tilde{t}_{(n)}$ for all i, where $(\cdot)$ is a permutation on N = {1, 2, …, n} such that $\tilde{s}_{(j)}$ and $\tilde{t}_{(j)}$ are the jth least values of $\tilde{s}_i$ and $\tilde{t}_i$, i = 1, 2, …, n, respectively. Then,*

$$\mathrm{LIFHSAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \mathrm{LIFHSAA}(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$$
$$\mathrm{LIFHSGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \mathrm{LIFHSGM}(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$$

(4) ***Boundary.*** *Let $\tilde{s}^- = \left(s_{\min_{i \in N} \alpha_i}, s_{\max_{i \in N} \beta_i}\right)$ and $\tilde{s}^+ = \left(s_{\max_{i \in N} \alpha_i}, s_{\min_{i \in N} \beta_i}\right)$, then*

$$\tilde{\alpha}^- \leq \mathrm{LIFHSAA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \tilde{\alpha}^+ \text{ and } \tilde{\alpha}^- \leq \mathrm{LIFHSGM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \tilde{\alpha}^+$$

where N = {1, 2, …, n}.

Proof From Theorem 2 and Property 2, we can easily derive the conclusions. □

4 Programming Models for the Optimal Fuzzy and Additive Measures

In decision-making problems, the weighting information may not be completely known. This section builds several programming models to determine the optimal fuzzy and additive measures on the DM set, on the criteria set, and on their ordered position sets, respectively.

Considering a group decision-making problem, without loss of generality, suppose that there are m alternatives $A = \{a_1, a_2, \ldots, a_m\}$, which are evaluated by q DMs $E = \{e_1, e_2, \ldots, e_q\}$ following n criteria $C = \{c_1, c_2, \ldots, c_n\}$. Let $\tilde{S}^k = \left(\tilde{s}^k_{ij}\right)_{m \times n}$ be the individual linguistic intuitionistic fuzzy decision matrix (LIFDM) offered by the DM e_k, $k = 1, 2, \ldots, q$, where $\tilde{s}^k_{ij} = \left(s_{\alpha^k_{ij}}, s_{\beta^k_{ij}}\right)$ is the LIFV offered by the DM e_k for the alternative $a_i \in A$ with respect to the criterion $c_j \in C$.

First, we introduce the concept of correlation coefficient decision matrices as follows, which is then used to build model for determining the optimal fuzzy and additive measures on the DM set.

Definition 12 Let $\tilde{S}^k = \left(\tilde{s}^k_{ij}\right)_{m \times n}$ and $\tilde{S}^l = \left(\tilde{s}^l_{ij}\right)_{m \times n}$ be any two LIFDMs. Then, their correlation coefficient decision matrix (CCDM)$C^{kl} = (c^{kl}_{ij})_{m \times n}$ is defined as follows:

$$c^{kl}_{ij} = \frac{c(\tilde{s}^k_{ij}, \tilde{s}^l_{ij})}{\sqrt{e(\tilde{s}^k_{ij}) \times e(\tilde{s}^l_{ij})}} \quad (14)$$

for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, where $c(\tilde{s}^k_{ij}, \tilde{s}^l_{ij}) = \frac{1}{2}(\alpha^k_{ij}\alpha^l_{ij} + \beta^k_{ij}\beta^l_{ij})$, $e(\tilde{s}^k_{ij}) = \frac{(\alpha^k_{ij})^2 + (\beta^k_{ij})^2}{2}$ and $e(\tilde{s}^k_{ij}) = \frac{(\alpha^k_{ij})^2 + (\beta^k_{ij})^2}{2}$.

Following the correlation coefficient decision matrix defined in Eq. (14), we build the following programming model to determine the weights of the DMs:

$$\phi^* = \max \sum_{k=1}^{q} \left(\sum_{l=1, l\neq k}^{q} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^{kl} \right) Sh_{e_k}(\mu_E, E)$$
$$s.t. \begin{cases} \mu_E(S) \le \mu_E(T) \ \forall S, T \subseteq E \wedge S \subseteq T \\ \mu_E(e_k) \in H_{e_k}, \quad k = 1, 2, \ldots, q \\ \mu_E(\emptyset) = 0, \quad \mu_E(E) = 1 \end{cases} \quad \text{(M-1)}$$

where H_{e_k} is the known weighting information of the DM e_k, and $Sh_{e_k}(\mu_E, E)$ is the Shapley value of the DM e_k for the fuzzy measure μ_E on the DM set E.

Let $\tilde{S}^{Sh_{e_k},k} = \left(\tilde{s}_{ij}^{Sh_{e_k},k}\right)_{m\times n}$ be the individual weighted LIFDM (WLIFDM) offered by the DM e_k, where $\tilde{s}_{ij}^{Sh_{e_k},k} = \left(s_{Sh_{e_k}(\mu_E,E)\times\alpha_{ij}^k}, s_{Sh_{e_k}(\mu_E,E)\times\beta_{ij}^k}\right)$ for all i = 1, 2, …, m, and all j = 1, 2, …, n, and $Sh_{e_k}(\mu, E)$ is the Shapley value of the DM e_k determined by model (M-1). Furthermore, let $d\left(\tilde{s}_{ij}^{Sh_{e_k},k}\right) = Sh_{e_k}(\mu_E, E) \times \alpha_{ij}^k - Sh_{e_k}(\mu_E, E) \times \beta_{ij}^k$ for all k = 1, 2, …, q, which is the Shapley weighted score of the LIFV $\tilde{s}_{ij}^k$.

For each pair of (i, j), we reorder $d\left(\tilde{s}_{ij}^{Sh_{e_k},k}\right)$ for k = 1, 2, …, q in an increasing order, denoted by $d\left(\tilde{s}_{ij}^{Sh_{e_k},(k)}\right)$, k = 1, 2, …, q, where $d\left(\tilde{s}_{ij}^{Sh_{e_k},(1)}\right) \le d\left(\tilde{s}_{ij}^{Sh_{e_k},(2)}\right) \le \cdots \le d\left(\tilde{s}_{ij}^{Sh_{e_k},(q)}\right)$ for all i = 1, 2, …, m, and all j = 1, 2, …, n.

To reduce the influence of extreme evaluation values offered by the DMs, we construct the following programming model to determine the fuzzy measure μ_{N_E} on the ordered position set N_E = {1, 2, …, q}:

$$\varphi^* = \min\left(\left(\sum_{k=\min(q)+1}^{q}\left(\frac{\sum_{i=1}^m\sum_{j=1}^n d\left(\tilde{s}_{ij}^{Sh_{e_k},(k)}\right)}{m\times n}\right)Sh_k(\mu_{N_E}, N_E)\right) - \left(\sum_{k=1}^{\min(q)}\left(\frac{\sum_{i=1}^m\sum_{j=1}^n d\left(\tilde{s}_{ij}^{Sh_{e_k},(k)}\right)}{m\times n}\right)Sh_k(\mu_{N_E}, N_E)\right)\right)$$
$$s.t. \begin{cases} \mu_{N_E}(S) \le \mu_{N_E}(T) \quad \forall S \subseteq T \subseteq N_E \\ \mu_{N_E}(k) \in W_k \quad k = 1, 2, \ldots, q \\ \mu_{N_E}(k) \ge 0 \quad k = 1, 2, \ldots, q \end{cases} \quad \text{(M-2)}$$

where $\text{mid}(q) = \begin{cases} \frac{q}{2} & q \text{ is an even number} \\ \frac{q+1}{2} & q \text{ is an odd number} \end{cases}$, W_k is the known weighting information of the kth ordered position, and $Sh_k(\mu_{N_E}, N_E)$ is the Shapley value of the kth ordered position for the fuzzy measure μ_{N_E} on the ordered position set N_E.

Note that when there is no interactive, models (M-1) and (M-2) reduce to the following models for the additive weighting vectors on the DM set E and on the ordered position set N_E:

$$f^* = \max \sum_{k=1}^{q} \left(\sum_{l=1, l\neq k}^{q} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^{kl} \right) \omega_{e_k}$$
$$s.t. \begin{cases} \sum_{k=1}^{q} \omega_{e_k} = 1 \\ \omega_{e_k} \in H_{e_k} \wedge \omega_{e_k} \ge 0 \ k = 1, 2, \ldots, q \end{cases} \quad \text{(M-3)}$$

and

$$h^* = \min\left(\left(\sum_{k=\min(q)+1}^{q}\left(\frac{\sum_{i=1}^m\sum_{j=1}^n d\left(\tilde{s}_{ij}^{Sh_{e_k},(k)}\right)}{m\times n}\right)w_k\right) - \left(\sum_{k=1}^{\min(q)}\left(\frac{\sum_{i=1}^m\sum_{j=1}^n d\left(\tilde{s}_{ij}^{Sh_{e_k},(k)}\right)}{m\times n}\right)w_k\right)\right)$$
$$s.t. \begin{cases} \sum_{k=1}^{q} w_k = 1 \\ w_k \in W_k \wedge w_k \ge 0 \ k = 1, 2, \ldots, q \end{cases} \quad \text{(M-4)}$$

where $\omega_E = (\omega_{e_1}, \omega_{e_2}, \ldots, \omega_{e_q})$ is the additive weighting vector on the DM set E, $w_{N_E} = (w_1, w_2, \ldots, w_q)$ is the additive weighting vector on the ordered set N_E and the other notations as shown in models (M-1) and (M-2).

Let $\tilde{S} = (\tilde{s}_{ij})_{m\times n}$ be the comprehensive linguistic intuitionistic fuzzy decision matrix (CLIFDM), and let $\tilde{s}^- = (s_0, s_{2t})$ and $\tilde{s}^+ = (s_{2t}, s_0)$. We define the following correlation coefficient for the LIFV $\tilde{s}_{ij}$:

$$c_{ij}^- = \frac{c(\tilde{s}_{ij}, \tilde{s}^-)}{\sqrt{e(\tilde{s}_{ij}) \times e(\tilde{s}^-)}} \text{ and } c_{ij}^+ = \frac{c(\tilde{s}_{ij}, \tilde{s}^+)}{\sqrt{e(\tilde{s}_{ij}) \times e(\tilde{s}^+)}} \quad (15)$$

where i = 1, 2, …, m, and j = 1, 2, …, n.

To determine the optimal fuzzy measure μ_C on the criteria set C, we establish the following programming model:

$$\psi^* = \max \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{c_{ij}^+}{c_{ij}^- + c_{ij}^+} Sh_{c_j}(\mu_C, C)$$
$$s.t. \begin{cases} \mu_C(S) \le \mu_C(T) \ \forall S, T \subseteq C \wedge S \subseteq T \\ \mu_C(c_j) \in H_{c_j}, \quad j = 1, 2, \ldots, n \\ \mu_C(\emptyset) = 0, \quad \mu_C(C) = 1 \end{cases} \quad \text{(M-5)}$$

where H_{c_j} is the known weighting information of the criterion c_j, and $Sh_{c_j}(\mu_C, C)$ is its Shapley value for the fuzzy measure μ_C on the criteria set C.

Let $\tilde{S}^{Sh} = \left(\tilde{s}_{ij}^{Sh}\right)_{m\times n}$ be the weighted comprehensive linguistic intuitionistic fuzzy decision matrix (WCLIFDM), and let $d\left(\tilde{s}_{ij}^{Sh}\right) = Sh_{c_j}(\mu_C, C) \times \alpha_{ij} - Sh_{c_j}(\mu_C, C) \times \beta_{ij}$ for all each pair of (i, j), which is the Shapley weighted score of the LIFV $\tilde{s}_{ij}$. For each i = 1, 2, …, m, we reorder $d\left(\tilde{s}_{ij}^{Sh}\right)$ in an increasing order for all j = 1, 2, …, n, where $d\left(\tilde{s}_{i(1)}^{Sh}\right) \le d\left(\tilde{s}_{i(2)}^{Sh}\right) \le \cdots \le d\left(\tilde{s}_{i(n)}^{Sh}\right)$ for all i = 1, 2, …, m.

To decrease the influence of extreme evaluation values, we construct the following programming model to determine the fuzzy measure on the ordered position set $N_C = \{1, 2, \ldots, n\}$:

$$\zeta^* = \min\left(\left(\sum_{j=\min(n)+1}^{n}\left(\frac{\sum_{i=1}^{m} d\left(\tilde{s}_{i(j)}^{Sh}\right)}{m}\right)Sh_j(\mu_{N_C}, N_C)\right) - \left(\sum_{j=1}^{\min(n)}\left(\frac{\sum_{i=1}^{m} d\left(\tilde{s}_{i(j)}^{Sh}\right)}{m}\right)Sh_j(\mu_{N_C}, N_C)\right)\right)$$
$$s.t.\begin{cases}\mu_{N_C}(S) \le \mu_{N_C}(T) & \forall S \subseteq T \subseteq N_C\\ \mu_{N_C}(j) \in W_j & j = 1,2,\ldots,n\\ \mu_{N_C}(j) \ge 0 & j = 1,2,\ldots,n\end{cases} \qquad \text{(M-6)}$$

where $\text{mid}(n)=\begin{cases}\frac{n}{2} & n \text{ is an even number}\\ \frac{n+1}{2} & n \text{ is an odd number}\end{cases}$, and W_j is the known weighting information of the jth ordered position, and $Sh_j(\mu_{N_C}, N_C)$ is the Shapley value of the jth ordered position for the fuzzy measure μ_{N_C} on the ordered position set N_C.

Similar to the additive measures on the DM set E and on the ordered position set N_E, when the importance of criteria and the ordered positions have no interaction, models (M-5) and (M-6) reduce to the following programming models:

$$p^* = \max \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{c_{ij}^{+}}{c_{ij}^{-} + c_{ij}^{+}}\omega_{c_j}$$
$$s.t.\begin{cases}\sum_{j=1}^{n}\omega_{c_j} = 1\\ \omega_{c_j} \in H_{c_j} \wedge \omega_{c_j} \ge 0 \quad j = 1,2,\ldots,n\end{cases} \qquad \text{(M-7)}$$

and

$$g^* = \min\left(\left(\sum_{j=\min(n)+1}^{n}\left(\frac{\sum_{i=1}^{m} d\left(\tilde{s}_{i(j)}^{Sh}\right)}{m}\right)w_j\right) - \left(\sum_{j=1}^{\min(n)}\left(\frac{\sum_{i=1}^{m} d\left(\tilde{s}_{i(j)}^{Sh}\right)}{m}\right)w_j\right)\right) \qquad \text{(M-8)}$$
$$s.t.\begin{cases}\sum_{j=1}^{n} w_j = 1\\ w_j \in W_j \wedge w_j \ge 0 \quad j = 1,2,\ldots,n\end{cases}$$

where $\omega_C = (\omega_{c_1}, \omega_{c_2}, \ldots, \omega_{c_n})$ is the additive weighting vector on the criteria set C, $w_{N_C} = (w_1, w_2, \ldots, w_n)$ is the additive weighting vector on the ordered set N_C, and the other notations as shown in models (M-5) and (M-6).

5 A Group Decision-Making Method with Linguistic Intuitionistic Fuzzy Information

This section contains two parts. This first part introduces a group decision-making algorithm based on the defined aggregation operators and the built programming models for the optimal fuzzy and additive measures. The second part selects a practical group decision-making problem to show the specific application of the offered procedure.

5.1 An algorithm

Step 1: Let $\tilde{S}^k = \left(\tilde{s}_{ij}^k\right)_{m \times n}$ be the individual LIFDM offered by the DM e_k, $k = 1, 2, \ldots, q$. Model (M-1) or model (M-3) is used to determine the optimal fuzzy or additive measure on the DM set $E = \{e_1, e_2, \ldots, e_q\}$;
Step 2: Model (M-2) or model (M-4) is adopted to calculate the optimal fuzzy or additive measure on the ordered position set $N_E = \{1, 2, \ldots, q\}$;
Step 3: The LIFHSAA and LIFHSGM operators or the LIFHWAA and LIFHWGM operators are utilized to calculate the comprehensive LIFDM;
Step 4: Model (M-5) or model (M-7) is adopted to calculate the optimal fuzzy or additive measure on the criteria set $C = \{c_1, c_2, \ldots, c_n\}$;
Step 5: Model (M-6) or model (M-8) is used to calculate the optimal fuzzy or additive measure on the ordered position set $N_C = \{1, 2, \ldots, n\}$;
Step 6: The LIFHSAA and LIFHSGM operators or the LIFHWAA and LIFHWGM operators is adopted to calculate the comprehensive LIFVs $\tilde{s}_i = (s_{\alpha_i}, s_{\beta_i})$, $i = 1, 2, \ldots, m$;
Step 7: The score and accuracy functions are utilized to calculate the score and accuracy values of the comprehensive LIFVs $\tilde{s}_i$, $i = 1, 2, \ldots, m$;
Step 8: We rank objects x_i, $i = 1, 2, \ldots, m$, based on the order relationship of $\tilde{s}_i$, $i = 1, 2, \ldots, m$.

5.2 A Case Study

We consider the decision-making problem of assessing engines (adapted from Ref. [12]). There are four brands of engines (alternatives) $A = \{A_1, A_2, A_3, A_4\}$ that are assessed using the linguistic term set $S = \{s_0$: extremely poor, s_1: very poor, s_2: poor, s_3: slight poor, s_4: indifferent, s_5: slight good, s_6: good, s_7: very good, s_8: extremely good$\}$ with respect to four criteria: C_1: responsiveness, C_2: fuel economy, C_3: vibration, and C_4: starting. The individual LIFDMs offered by four DMs $E = \{e_1, e_2, e_3, e_4\}$ are listed as shown in Tables 1, 2, 3, and 4.

Table 1 Individual LIFDM $\tilde{S}^1$ offered by the DM e_1

	C_1	C_2	C_3	C_4
A_1	(s_2, s_4)	(s_6, s_1)	(s_3, s_5)	(s_5, s_2)
A_2	(s_3, s_4)	(s_2, s_5)	(s_4, s_2)	(s_7, s_1)
A_3	(s_6, s_2)	(s_7, s_1)	(s_6, s_2)	(s_4, s_3)
A_4	(s_2, s_5)	(s_5, s_2)	(s_3, s_5)	(s_2, s_5)

Table 2 Individual LIFDM $\tilde{S}^2$ offered by the DM e_2

	C_1	C_2	C_3	C_4
A_1	(s_3, s_5)	(s_2, s_5)	(s_5, s_2)	(s_6, s_1)
A_2	(s_4, s_2)	(s_3, s_4)	(s_7, s_1)	(s_2, s_5)
A_3	(s_6, s_2)	(s_2, s_5)	(s_4, s_3)	(s_7, s_1)
A_4	(s_3, s_5)	(s_4, s_2)	(s_2, s_5)	(s_5, s_2)

Table 3 Individual LIFDM $\tilde{S}^3$ offered by the DM e_3

	C_1	C_2	C_3	C_4
A_1	(s_2, s_5)	(s_3, s_5)	(s_4, s_3)	(s_4, s_3)
A_2	(s_3, s_4)	(s_7, s_1)	(s_2, s_5)	(s_7, s_1)
A_3	(s_2, s_5)	(s_6, s_2)	(s_4, s_2)	(s_6, s_1)
A_4	(s_4, s_2)	(s_5, s_2)	(s_7, s_1)	(s_3, s_4)

Table 4 Individual LIFDM $\tilde{S}^4$ offered by the DM e_4

	C_1	C_2	C_3	C_4
A_1	(s_6, s_1)	(s_5, s_2)	(s_3, s_5)	(s_2, s_5)
A_2	(s_2, s_5)	(s_7, s_1)	(s_7, s_1)	(s_3, s_4)
A_3	(s_7, s_1)	(s_4, s_3)	(s_6, s_2)	(s_2, s_5)
A_4	(s_5, s_2)	(s_2, s_5)	(s_5, s_2)	(s_4, s_2)

The known weighting information of the experts is defined as:

$$\omega_{e_1} = [0.1, 0.3], \omega_{e_2} = [0.3, 0.5],$$
$$\omega_{e_3} = [0.1, 0.2], \omega_{e_4} = [0.2, 0.4],$$

and the known weighting information of the criteria is offered as:

$$\omega_{c_1} = [0.1, 0.2], \omega_{c_2} = [0.3, 0.4],$$
$$\omega_{c_3} = [0.2, 0.3], \omega_{c_4} = [0.1, 0.2]$$

Furthermore, the known weighting information of the ordered positions is offered as:

$$w_1 = [0.1, 0.3], w_2 = [0.2, 0.4], w_3 = [0.2, 0.4], w_4 = [0.1, 0.3].$$

To rank these four brands of engines, the following procedure is needed:

Step 1: Following model (M-1), the optimal fuzzy measure on the DM set E is

$$\begin{aligned}\mu_E(e_1) = \mu_E(e_3) = \mu_E(e_4) = \mu_E(e_1, e_4) \\ = \mu_E(e_3, e_4) = 0.2,\ \mu_E(e_2) = \mu_E(e_2, e_4) \\ = \mu_E(e_1, e_2, e_4) = 0.3, \mu_E(e_1, e_3) \\ = \mu_E(e_2, e_3) = \mu_E(e_1, e_2, e_3) \\ = \mu_E(e_1, e_3, e_4) = \mu_E(e_2, e_3, e_4) \\ = \mu_E(E) = 1.\end{aligned}$$

Following Eq. (7), the Shapley values of the DMs are

$$\begin{aligned}Sh_{e_1}(\mu_E, E) = 0.183, Sh_{e_2}(\mu_E, E) = 0.233, Sh_{e_3}(\mu_E, E) \\ = 0.533, Sh_{e_4}(\mu_E, E) = 0.05.\end{aligned}$$

Step 2: On the basis of model (M-2), the optimal fuzzy measure on the ordered position set N_E is

$$\begin{aligned}\mu_{N_E}(1) = \mu_{N_E}(4) = \mu_{N_E}(1, 4) = 0.1, \mu_{N_E}(3) = \mu_{N_E}(1, 3) \\ = \mu_{N_E}(3, 4) = \mu_{N_E}(1, 3, 4) = 0.2, \mu_{N_E}(2) \\ = \mu_{N_E}(2, 4) = 0.4, \mu_{N_E}(1, 2) \\ = \mu_{N_E}(2, 3) = \mu_{N_E}(1, 2, 3) = \mu_{N_E}(1, 2, 4) \\ = \mu_{N_E}(2, 3, 4) = \mu_{N_E}(N_E) = 1.\end{aligned}$$

Following Eq. (7), the Shapley values of the ordered positions are

$$\begin{aligned}Sh_1(\mu_{N_E}, N_E) = 0.125, Sh_2(\mu_{N_E}, N_E) = 0.675, Sh_3(\mu_{N_E}, N_E) \\ = 0.175, Sh_4(\mu_{N_E}, N_E) = 0.025.\end{aligned}$$

Step 3: Using the LIFHSAA operator, the comprehensive LIFDM is shown in Table 5.

Step 4: Model (M-5) is used to calculate the optimal fuzzy measure on the criteria set C, where

$$\begin{aligned}\mu_C(c_1) = 0.1, \mu_C(c_3) = \mu_C(c_4) = \mu_C(c_1, c_2) = \mu_C(c_1, c_3) \\ = 0.2, \mu_C(c_2) = \mu_C(c_2, c_3) = \mu_C(c_1, c_2, c_3) \\ = 0.3, \mu_C(c_1, c_4) \\ = \mu_C(c_2, c_4) \\ = \mu_C(c_3, c_4) = \mu_C(c_1, c_2, c_4)\mu_C(c_1, c_3, c_4) \\ = \mu_C(c_2, c_3, c_4) = \mu_C(C) = 1.\end{aligned}$$

Following Eq. (7), the Shapley values of the criteria are

$$\begin{aligned}Sh_{c_1}(\mu_C, C) = 0.083, Sh_{c_2}(\mu_C, C) = 0.167, Sh_{c_3}(\mu_C, C) \\ = 0.133, Sh_{c_4}(\mu_C, C) = 0.617.\end{aligned}$$

Step 5: Model (M-6) is adopted to calculate the optimal fuzzy measure on the ordered position set N_C, where

Table 5 Comprehensive LIFDM $\tilde{S}$

	C_1	C_2	C_3	C_4
A_1	$(s_{2.631}, s_{4.856})$	$(s_{2.7668}, s_{4.2969})$	$(s_{4.2969}, s_{3.6898})$	$(s_{4.1278}, s_{2.8722})$
A_2	$(s_{2.9573}, s_{3.7736})$	$(s_{2.8837}, s_{4.1892})$	$(s_{3.8949}, s_{3.1597})$	$(s_{6.6868}, s_{1.242})$
A_3	$(s_{4.3162}, s_{3.2018})$	$(s_{6.4041}, s_{1.5247})$	$(s_{4.1123}, s_{2.663})$	$(s_{5.9626}, s_{1.1098})$
A_4	$(s_{2.6939}, s_{4.7813})$	$(s_{4.2264}, s_{2.1107})$	$(s_{3.2385}, s_{4.5447})$	$(s_{3.8347}, s_{2.8605})$

Table 6 Ranking values and orders obtained from different aggregation operators

Aggregation operators	Ranking values of x_1	Ranking values of x_2	Ranking values of x_3	Ranking values of x_4	Ranking orders
The LIFHSAA operator	− 1.3563	− 0.2027	2.0365	0.2239	$x_3 \succ x_4 \succ x_2 \succ x_1$
The LIFHSGM operator	− 1.2418	− 0.1451	2.0653	− 0.7949	$x_3 \succ x_2 \succ x_4 \succ x_1$
The LIFHWAA operator	0.5031	1.8664	2.9291	1.2945	$x_3 \succ x_2 \succ x_4 \succ x_1$
The LIFHWGM operator	0.7585	1.9274	2.6085	1.3815	$x_3 \succ x_2 \succ x_4 \succ x_1$

$$\begin{aligned}\mu_{N_C}(4) &= 0.1, \mu_{N_C}(1) = \mu_{N_C}(3) = \mu_{N_C}(1,3) \\ &= \mu_{N_C}(1,4) = \mu_{N_C}(3,4) \\ &= \mu_{N_C}(1,3,4) = 0.2, \mu_{N_E}(2) = \mu_{N_C}(2,3) \\ &= \mu_{N_C}(2,4) = \mu_{N_C}(2,3,4) \\ &= 0.4, \mu_{N_C}(1,2) = \mu_{N_C}(1,2,3) \\ &= \mu_{N_C}(1,2,4) = \mu_{N_C}(N_C) = 1.\end{aligned}$$

Following Eq. (7), the Shapley values of the ordered positions are

$$\begin{aligned}Sh_1(\mu_{N_C}, N_C) &= 0.358, Sh_2(\mu_{N_C}, N_C) \\ &= 0.558, Sh_3(\mu_{N_C}, N_C) \\ &= 0.058, Sh_4(\mu_{N_C}, N_C) = 0.025.\end{aligned}$$

Step 6: Again using the LIFHSAA operator, the comprehensive LIFVs are

$$\tilde{s}_1 = (s_{2.9347}, s_{4.291}), \tilde{s}_2 = (s_{3.4243}, s_{3.6271}),$$
$$\tilde{s}_3 = (s_{4.5525}, s_{2.516}), \tilde{s}_4 = (s_{3.6641}, s_{3.4402}).$$

Step 7: Following the comprehensive LIFVs, the linguistic scores are

$$S(\tilde{s}_1) = -1.3563, S(\tilde{s}_2) = -0.2027, S(\tilde{s}_3) = 2.0365,$$
$$S(\tilde{s}_4) = 0.2239.$$

Thus, the ranking order of objects is $A_3 \succ A_4 \succ A_2 \succ A_1$, and the third brand of engines is the best.

In this example, ranking values and orders obtained from different aggregation operators are derived as shown in Table 6.

Table 6 shows that different ranking orders are obtained for different aggregation operators. However, all ranking orders show that the third brand of engines is the best. In practical decision-making examples, we suggest the DMs to apply the aggregation operators based on fuzzy measures. However, when there is an explanation that the importance of elements in a set is independent. It is sufficient to use the aggregation operators based on additive measures. Because the hybrid aggregation operators consider more weighting information than the arithmetical averaging and geometric mean operators, we recommend the DMs to adopt the hybrid aggregation operators to calculate the comprehensive ranking values of alternatives.

In this example, when Chen et al's method [8] is adopted, the ranking scores of objects are

$$S(\tilde{s}_1) = 4.0764, S(\tilde{s}_2) = 7.8481, S(\tilde{s}_3) = 7.8354,$$
$$S(\tilde{s}_4) = 1.4066,$$

by which the ranking is $x_2 \succ x_3 \succ x_1 \succ x_4$.

On the other hand, when Chen et al's method [8] based on the linguistic intuitionistic fuzzy weighted geometric (LIFWG) operator and the linguistic intuitionistic fuzzy hybrid geometric (LIFHG) operator is used, the ranking scores of objects are

$$S(\tilde{s}_1) = -0.4083, S(\tilde{s}_2) = 0.4380, S(\tilde{s}_3) = 1.4870,$$
$$S(\tilde{s}_4) = -0.1654,$$

by which the ranking is $x_3 \succ x_2 \succ x_4 \succ x_1$.

Furthermore, when Garg and Kumar's method [10] is applied, the ranking scores of objects are

$$S(\tilde{s}_1) = 4.0927, S(\tilde{s}_2) = 4.2550, S(\tilde{s}_3) = 4.4913,$$
$$S(\tilde{s}_4) = 4.1359,$$

by which the ranking is $x_3 \succ x_2 \succ x_4 \succ x_1$.

Except for Chen et al's method [8] based on the linguistic intuitionistic fuzzy arithmetical weighted geometric (LIFAW) operator and the linguistic intuitionistic fuzzy hybrid arithmetical weighted (LIFHAW) operator, all other methods show that the third brand of engines is the best.

However, their ranking orders and ranking values are different.

Remark 2 Methods in [19, 45] cannot be applied in this example, which did not consider group decision-making situation. Notably, Liu and Liu's method [19] needs to divide attributes into different categories. The hybrid linguistic intuitionistic fuzzy aggregation operators in [8, 10] do not satisfy *idempotency* and *boundary*, which is caused by the balancing coefficient. Furthermore, methods in [8, 10] are based on the assumption that the weight information is completely known. Moreover, none of them can address the situation where there are interactive characteristics.

6 Conclusions

To denote the qualitative preferred and non-preferred information of the DMs rather than quantitative ones, this paper applied LIFVs [8] to develop a linguistic intuitionistic fuzzy group decision-making method. The main contributions include: (1) several new linguistic intuitionistic fuzzy operations are defined; (2) two types of aggregation operators are defined that satisfy several desirable properties; (3) models for determining the optimal fuzzy and additive measures on the DM set, on the criteria set and on their ordered position sets are built, respectively; and (4) a practical group decision-making problem about evaluating engines is provided to show the specific application of the developed theoretical results.

This paper focuses on linguistic intuitionistic fuzzy group decision making based on aggregation operators. In future, we shall continue to research linguistic intuitionistic fuzzy group decision making using other measures, including distance measure, entropy, and similarity measure. Furthermore, we shall study the application of linguistic intuitionistic fuzzy decision making in some other fields, including engineering project management, medical recommendation, software quality assurance management, and selecting cooperative partner. Notably, most of the current researches for decision making with fuzzy information are restricted to concrete problems rather than in the setting of dynamical systems. Thus, we will pay more attention to research methods for dynamical decision making in a similar way as the literature [33, 34].

Acknowledgements This work was supported by the National Natural Science Foundation of China (Nos. 71571192 and 71671188), the Innovation-Driven Project of Central South University (No. 2018CX039), the Beijing Intelligent Logistics System Collaborative Innovation Center (No. 2018KF-06), the Fundamental Research Funds for the Central Universities of Central South University (No. 2018zzts094), and the State Key Program of National Natural Science of China (No. 71431006).

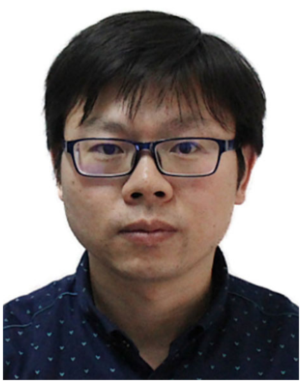
References

1. Atanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**(3), 87–96 (1986)
2. Atanassov, K., Gargov, G.: Interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst. **31**(3), 343–349 (1989)
3. Burillo, P., Bustince, H.: Entropy on intuitionistic fuzzy sets and on interval valued fuzzy sets. Fuzzy Sets Syst. **78**(3), 305–316 (1996)
4. Bao, T.T., Xie, X.L., Long, P.Y., Wei, Z.K.: MADM method based on prospect theory and evidential reasoning approach with unknown attribute weights under intuitionistic fuzzy environment. Expert Syst. Appl. **88**, 305–317 (2017)
5. Buyukozkan, G., Gocer, F.: Application of a new combined intuitionistic fuzzy MCDM approach based on axiomatic design methodology for the supplier selection problem. Appl. Soft Comput. **52**, 1222–1238 (2017)
6. Chen, C.T., Tai, W.S.: Measuring the intellectual capital performance based on 2-tuple fuzzy linguistic information. In: Proceedings of the 10th Annual Meeting Asia-Pacific Region Decision Sciences Institute, Taiwan (2005)
7. Cheng, H., Meng, F.Y., Chen, K.: Several generalized interval-valued 2-Tuple linguistic weighted distance measures and their application. Int. J. Fuzzy Syst. **19**(4), 967–981 (2017)
8. Chen, Z.C., Liu, P.H., Pei, Z.: An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. Int. J. Comput. Intell. Syst. **8**(4), 747–760 (2015)
9. Dong, Y.H., Xu, Y.F.H., Li, Y., Feng, B.: The OWA-based consensus operator under linguistic representation models using position indexes. Eur. J. Oper. Res. **203**(2), 455–463 (2010)
10. Garg, H., Kumar, K.: Some aggregation operators for linguistic intuitionistic fuzzy set and its application to group decision-making process using the set pair analysis. Arab. J. Sci. Eng. **43**, 3213–3227 (2018)
11. Herrera, F., Martinez, L.: A 2-tuple linguistic representation model for computing with words. IEEE Trans. Fuzzy Syst. **8**(6), 746–752 (2000)
12. Isitt, T.: The sports tourers. Motor Cycle Int. **64**(1), 19–27 (1990)
13. Krishankumar, R., Ravichandran, K.S., Saeid, A.B.: A new extension to PROMETHEE under intuitionistic fuzzy environment for solving supplier selection problem with linguistic preferences. Appl. Soft Comput. **60**, 564–576 (2017)
14. Li, D.F., Cheng, C.T.: New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern Recogn. Lett. **23**(1–3), 221–225 (2002)
15. Liu, P.D., Li, Y.H., Chen, Y.B.: Some generalized Einstein aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. Sci. Iran. **22**(6), 2684–2701 (2015)
16. Liu, P.D.: Multiple attribute group decision making method based on interval-valued intuitionistic fuzzy power Heronian aggregation operators. Comput. Ind. Eng. **108**, 199–212 (2017)
17. Liu, P.D., Chen, S.M., Liu, J.L.: Multiple attribute group decision making based on intuitionistic fuzzy interaction partitioned Bonferroni mean operators. Inf. Sci. **411**, 98–121 (2017)
18. Liu, P.D.: Some geometric aggregation operators based on interval intuitionistic uncertain linguistic variables and their application to group decision making. Appl. Math. Model. **37**(4), 2430–2444 (2013)
19. Liu, P.D., Liu, W.Q.: Scaled prioritized operators based on the linguistic intuitionistic fuzzy numbers and their applications to multi-attribute decision making. Int. J. Fuzzy Syst. **20**(5), 1539–1550 (2018)
20. Liang, Z.Z., Shi, P.F.: Similarity measures on intuitionistic fuzzy sets. Pattern Recogn. Lett. **24**(15), 2687–2693 (2003)

21. Meng, F.Y., Tan, C.Q.: A method for multi-attribute group decision making based on generalized interval-valued intuitionistic fuzzy Choquet integral operators. Int. J. Uncertain. Fuzz. Knowl. Based Syst. **25**(5), 821–849 (2017)
22. Meng, F.Y., Chen, X.H., Zhang, Q.: Multi-attribute decision analysis under a linguistic hesitant fuzzy environment. Inf. Sci. **267**, 287–305 (2014)
23. Meng, F.Y., Tang, J., Li, C.L.: Uncertain linguistic hesitant fuzzy sets and their application in multi-attribute decision making. Int. J. Intell. Syst. **33**(3), 586–614 (2018)
24. Meng, F.Y., Chen, X.H.: Correlation coefficient of interval-valued intuitionistic uncertain linguistic sets and its application. Cyber. Syst. **48**(1), 114–135 (2017)
25. Meng, F.Y., Yuan, Y.G., Chen, X.H.: Some generalized interval-valued 2-tuple linguistic interval distance measures and their application. Int. J. Uncertain. Fuzz. Knowl. Based Syst. **25**(5), 759–786 (2017)
26. Meng, F.Y., Tang, J., Xu, Z.S.: Exploiting the priority weights from interval linguistic fuzzy preference relations. Soft. Comput. (2017). https://doi.org/10.1007/s00500-017-2878-y
27. Meng, F.Y., Tang, J.: Interval-valued intuitionistic fuzzy multi-criteria group decision making based on cross entropy and Choquet integral. Int. J. Intell. Syst. **28**(12), 1172–1195 (2013)
28. Rani, D., Gulati, T.R.: Application of intuitionistic fuzzy optimization technique in transportation models. Opsearch **53**(4), 761–777 (2016)
29. Rodríguez, R.M., Martínez, L., Herrera, F.: Hesitant fuzzy linguistic term sets for decision making. IEEE Trans. Fuzzy Syst. **20**(1), 109–119 (2012)
30. Szmidt, E., Kacprzyk, J.: Entropy for intuitionistic fuzzy sets. Fuzzy Sets Syst. **118**(3), 467–477 (2001)
31. Sugeno, M.: Theory of fuzzy integral and its application. Doc. Dissert. Tokyo Institute of Technology (1974)
32. Shapley, L.S.: A value for n-person game. In: Kuhn, H., Tucker, A. (eds.) Contributions to the Theory of Games. Princeton University Press, Princeton (1953)
33. Tang, Z., Park, J.H., Zheng, W.X.: Distributed impulsive synchronization of Lur'e dynamical networks via parameter variation methods. Int. J. Robust Nonlinear Control **28**(3), 1001–1015 (2018)
34. Tang, Z., Park, J.H., Feng, J.W.: Novel approaches to pin cluster synchronization on complex dynamical networks in Lur'e forms. Commun. Nonlinear Sci. Numer. Simul. **57**, 422–438 (2018)
35. Wang, S.F.: Interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and their calculations. J. Comput. Anal. Appl. **23**(4), 703–712 (2017)
36. Xian, S.D., Dong, Y.F., Yin, Y.B.: Interval-valued intuitionistic fuzzy combined weighted averaging operator for group decision making. J. Oper. Res. Soc. **68**(1), 1–11 (2017)
37. Xu, Z.S.: Choquet integrals of weighted intuitionistic fuzzy information. Inf. Sci. **180**(5), 726–736 (2010)
38. Xu, Z.S., Liao, H.C., Zeng, X.J.: A thermodynamic method of intuitionistic fuzzy MCDM to assist the hierarchical medical system in China. Inf. Sci. **420**, 490–504 (2017)
39. Xu, Z.S.: EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations. Int. J. Uncertain. Fuzz. Knowl. Based Syst. **12**(6), 791–810 (2004)
40. Xu, Y.J., Wang, H.M.: Approaches based on 2–tuple linguistic power aggregation operators for multiple attribute group decision making under linguistic environment. Appl. Soft Comput. **11**(5), 3988–3997 (2011)
41. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–353 (1965)
42. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning Part I. Inf. Sci. **8**(3), 199–249 (1975)
43. Zhang, Z.M.: Multi-criteria group decision-making methods based on new intuitionistic fuzzy Einstein hybrid weighted aggregation operators. Neural Comput. Appl. **28**(12), 3781–3800 (2017)
44. Zhang, Z.M.: Interval-valued intuitionistic fuzzy Frank aggregation operators and their applications to multiple attribute group decision making. Neural Comput. Appl. **28**(6), 1471–1501 (2017)
45. Zhang, H.Y., Peng, H.G., Wang, J., Wang, J.Q.: An extended outranking approach for multi-criteria decision-making problems with linguistic intuitionistic fuzzy numbers. Appl. Soft Comput. **59**, 462–474 (2017)
46. Zeng, S.Z., Merigo, J.M., Su, W.H.: The uncertain probabilistic OWA distance operator and its application in group decision making. Math. Comput. Model. **37**(9), 6266–6275 (2013)
47. Zhou, L.G., Wu, J.X., Chen, H.Y.: Linguistic continuous ordered weighted distance measure and its application to multiple attributes group decision making. Appl. Soft Comput. **25**(C), 266–276 (2014)

Ruiping Yuan is an associate professor in School of Information, Beijing Wuzi University. She received her bachelor degree and Ph.D. degree in School of Management and Economics, Beijing Institute of Technology, China, in 2004 and 2009, respectively. Her research interests include intelligent logistics, supply chain management, and decision making. She has contributed over 20 papers to professional journals and got 10 patents in related research fields.

Jie Tang is carrying out her Ph.D. degree in School of Business, Central South University. She received her master degree in School of Management, Qingdao Technological University, China in 2008. Currently, she has contributed 16 journal articles to professional journals. Her research interests include uncertain multi-attribute decision making, cluster analysis, and pattern recognition.



Fanyong Meng is a Professor in School of Business, Central South University. He received his master degree in School of Mathematics and Information Science, Guangxi University, China in 2008, and earned his Ph.D. degree in School of Management and Economics, Beijing Institute of Technology, China in 2011. He finished his postdoctoral work in School of Business, Central South University, China in 2016. Currently, he has contributed over 100 journal articles to professional journals. His research interests include fuzzy mathematics, decision making, and games theory.